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Paranormed sequence spaces generated by infinite matrices

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A paranormed space $X = (X, g)$ is a topological linear space in which the topology is given by paranorm g—a real subadditive function on X such that $g(\theta) = 0$, $g(x) = g(-x)$ and such that multiplication is continuous. In the above, θ is the zero in the complex linear space X and continuity of multiplication means that $\lambda_n \to \lambda$, $x_n \to x$ (i.e. $g(x_n - x) \to 0$) imply $\lambda_n x_n \to \lambda x$, for scalars λ and vectors x. We shall use the term semimetric function to describe a real subadditive function g on X such that $g(\theta) = 0$, $g(x) = g(-x)$. Two familiar paranormed sequence spaces, which have been extensively studied (3), are $l(p)$ and $m(p)$. For a given sequence $p = (g_k)$ of strictly positive numbers, $l(p)$ is the set of all complex sequences $x = (x_k)$ such that $\Sigma |x_k|^{p_k} < \infty$ and $m(p)$ is the set of x such that $\sup |x_k|^{p_k} < \infty$. Throughout, sums and suprema without limits are taken from 1 to ∞. Simons (3) considered only the case in which $0 < p_k \leqslant 1$ so that natural paranorms would seem to be $g_1(x) = \Sigma |x_k|^{p_k}$ in $l(p)$ and $g_2(x) = \sup |x_k|^{p_k}$ in $m(p)$. In fact Simons showed that g_1 was a paranorm for $l(p)$, but that g_2 did not satisfy the continuity of multiplication axiom.

Such sequence spaces as $l(p)$ and $m(p)$, and indeed nearly all those sequence spaces which have so far been studied, can be thought of as generated by infinite matrices $A = (a_{nk})$ $(n, k = 1, 2. \ldots)$. Thus if we define

$$A_n(x) = \Sigma_k a_{nk} |x_k|^{p_k},$$

for any matrix of complex numbers a_{nk} and any positive sequence p, we can consider $[A, p]_\infty$, the set of x such that $A_n(x)$ exists for each n and $A_n(x) = O(1)$. (We also say that x is strongly bounded by A when $A_n(x) = O(1)$.) Similarly, $[A, p]_0$ denotes the set of x such that $A_n(x) = o(1)$ and $[A, p]$ the set of x which are strongly summable by A, i.e. those x for which a number l exists such that $A_n(x - le)$ exists for each n and tends to zero as $n \to \infty$. Here and elsewhere $e = (1, 1, 1, \ldots)$. It is clear that for arbitrary A and positive p_k the sets $[A, p]_\infty$, $[A, p]$, $[A, p]_0$ contain the origin $(0, 0, \ldots)$. Also, for any non-negative A and positive p_k the sets are absolutely convex subsets of the space of all sequences. For, whenever $A_n(x - le)$, $A_n(y - me)$ exist and $|\lambda| + |\mu| \leqslant 1$ it is easy to see that

$$A_n(\lambda x + \mu y - (\lambda l + \mu m) e) \leqslant A_n(x - le) + A_n(y - me).$$

We remark that the absolute convexity holds without any restriction such a boundedness on p. The linearity of the sets $[A, p]_\infty$, etc., and the problem of uniqueness of strong limits has recently been discussed in (2).

In the present note we make some remarks in connexion with paranorming the spaces $[A, p]_\infty$, $[A, p]_0$. Also we discuss some of the problems that arise in connexion with the

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determination of the continuous duals of certain of the special spaces. For reference we now list the special spaces considered. The unit matrix I generates the spaces $m(p) = [I,p]_\infty$, $c(p) = [I,p]$ and $c_0(p) = [I,p]_0$. In the case when $p_k = p' > 0$ for all k these are just the spaces of bounded sequences, convergent sequences and null sequences. The space $l(p)$ can be thought of as both an $[A,p]_\infty$ space and as a $[B,p]_0$ space. For if $a_{nk} = 1\,(1 \leqslant k \leqslant n), a_{nk} = 0\,(k > n)$ and $b_{nk} = 0\,(1 \leqslant k < n), b_{nk} = 1\,(k \geqslant n)$ then $l(p) = [A,p]_\infty = [B,p]_0$. Finally, if A is the matrix of Cesàro means of order 1 we have the spaces $[C,l,p]$, etc., which we denote by $w(p)$, etc., for simplicity. Thus, for example, $x \in w(p)$ means that there exists l such that

$$\frac{1}{n}\sum_{k=1}^{n}|x_k - l|^{p_k} \to 0.$$

For the following we shall restrict ourselves to non-negative matrices A and suppose throughout that $p = (p_k)$ is a strictly positive sequence. It was shown in (2) that the boundedness of p was sufficient for $[A,p]$ and $[A,p]_\infty$ to be linear spaces. The argument of (2) shows also that $[A,p]_0$ is linear. It was also noted in (2) that $p_k = O(1)$ is necessary for the linearity of the spaces $l(p)$ and $w(p)$. We now show that $c(p)$ is a linear space only if $p_k = O(1)$. For suppose $p_k \neq O(1)$. Then we can find positive integers k_i such that $k_{i+1} > k_i + 1$ and such that $p_{k_i} > i$, $i = 1, 2, \ldots$. Define $x \in c(p)$ by $x_k = i^{-1/p_k}$ $(k = k_i)$, $x_k = 0$ $(k \neq k_i)$. Then $2x_{k_i} \to 2$, but we shall show that $2x \notin c(p)$, contradicting the linearity of $c(p)$. For suppose there exists l such that $|2x_k - l|^{p_k} \to 0$. Then for all sufficiently large $k \neq k_i$, $|l|^{p_k} < 1$ and so $|l| < 1$. Also $|2x_{k_i} - l|^{p_{k_i}} \to 0$ $(i \to \infty)$ and since $p_{k_i} \to \infty$ we must have $|2 - l| \leqslant 1$, but this contradicts $|l| < 1$.

In general $p_k = O(1)$ is not necessary for $[A,p]_\infty$, etc., to be linear spaces. For we may take A to have infinitely many zero columns and allow the p_k corresponding to these to be arbitrary, the other p_k forming a bounded sequence. An example is the Toeplitz matrix which maps $(|x_k|^{p_k})$ to $(|x_{2k+1}|^{p_{2k+1}})$, where $p_{2k} = k$, $p_{2k+1} = 1$. Since (p_k) bounded is sufficient for linearity and since it is also necessary in most of the important cases we shall henceforth suppose that it is satisfied.

In the case $0 < p_k \leqslant 1$, the inequality $|x_k + y_k|^{p_k} \leqslant |x_k|^{p_k} + |y_k|^{p_k}$ suggests the natural semimetric function

$$g(x) = \sup_n A_n(x) \tag{1}$$

for the spaces $[A,p]_\infty$ and $[A,p]_0$. In general $[A,p]$ is not a subset of $[A,p]_\infty$ so that (1) is not suitable for $[A,p]$. In the more general case when $p_k = O(1)$, a suitable semimetric for $[A,p]_\infty$ and $[A,p]_0$ is

$$g(x) = \sup_n (\Sigma_k a_{nk}|x_k|^{p_k})^{1/M}, \tag{2}$$

where $M = \max(1, \sup p_k)$, which gives (1) when $0 < p_k \leqslant 1$. We need only check subadditivity:

$$A_n(x+y) \leqslant \Sigma_k a_{nk}(|x_k|^{p_k/M} + |y_k|^{p_k/M})^M,$$

since $p_k \leqslant M$, whence, since $M \geqslant 1$, we see by Minkowski's inequality that g is subadditive.

It is perhaps worth noting that g given by (2) satisfies the stronger condition that $|\lambda| \leqslant 1$ implies $g(\lambda x) \leqslant g(x)$. A function g satisfying this condition, which is subadditive and such that $g(x) = 0$ if and only if $x = \theta$, is sometimes called a pseudo-norm. Thus

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in an obvious sense, our g is a semi-pseudo-norm, but the term is extremely ugly and we avoid it.

We now show that $[A,p]_0$ is a topological linear space.

THEOREM 1. *For any non-negative matrix A and any bounded sequence, p, $[A,p]_0$ is paranormed by* (2).

Proof. In view of the earlier remarks we have only to check continuity of multiplication. Now for any complex λ we have

$$|\lambda|^{p_k} \leqslant \max(1, |\lambda|^H), \quad \text{where} \quad H = \sup p_k < \infty, \quad \text{so} \quad g(\lambda x) \leqslant (\sup |\lambda|^{p_k})^{1/M} g(x)$$

on $[A, p]_0$, whence $\lambda \to 0$, $x \to 0$, $x \to \theta$ imply $\lambda x \to \theta$ and also $x \to \theta$, λ fixed imply $\lambda x \to \theta$. Now let $\lambda \to 0$, x fixed. For $|\lambda| < 1$ we have $A_n(\lambda x) < \epsilon$ for $n > N$. And for $1 \leqslant n \leqslant N$, since $\Sigma_k a_{nk}|x_k|^{p_k} < \infty$, there exists k_0 such that

$$\sum_{k_0}^{\infty} a_{nk}|\lambda x_k|^{p_k} < \epsilon.$$

Taking λ small enough we then have $A_n(\lambda x) < 2\epsilon$ for all n, i.e. $g(\lambda x) \to 0$ as $\lambda \to 0$.

COROLLARY 1. *Since $l(p)$ can be regarded as an $[A,p]_0$ space, Theorem* 1 *extends part of Lemma* 1 (3).

COROLLARY 2. *If A is non-negative and $0 < \inf p_k \leqslant \sup p_k < \infty$, then $[A,p]_\infty$ is paranormed by* (2).

For we need only check that $\lambda \to 0$, x fixed imply $\lambda x \to \theta$, and this follows from

$$g^M(\lambda x) \leqslant |\lambda|^m g^M(x),$$

which holds when $|\lambda| < 1$, $m = \inf p_k > 0$.

We remark that in general $\inf p_k > 0$ is not necessary for $[A,p]_\infty$ to be paranormed by (2). For example, we may take A to have infinitely many zero columns and suitably define p with $\inf p_k = 0$ so that $[A,p]_\infty$ is paranormed by (2). Also, A can be a Toeplitz matrix, if required. However, in the case $A = I$, $0 < p_k \leqslant 1$, it was shown in (3) that $[I,p]_\infty = m(p)$ was paranormed by (2) only if $\inf p_k > 0$. We now show that this type of result is also true for the space $[C,1,p]_\infty = w_\infty(p)$. The problem of finding those A such that $[A,p]_\infty$ paranormed by (2) implies $\inf p_k > 0$ remains open.

THEOREM 2. *$w_\infty(p)$ is paranormed by* (2) *if and only if $0 < \inf p_k \leqslant \sup p_k < \infty$.*

Proof. The sufficiency is covered in Corollary 2 above. Now if $w_\infty(p)$ is paranormed by (2) then $w(p)$ is a linear space and so $\sup p_k < \infty$ (see (2) and earlier remarks). Also we have $\lambda \to 0$, x fixed imply $\lambda x \to \theta$. This last property implies $\inf p_k > 0$. For suppose not. Then there exists $k_1 < k_2 < \ldots$ such that $p_{k_i} < 1/i$, $(i = 1, 2, \ldots)$. Also, we may choose the k_i such that k_1 lies in the interval $2^{r_1} \leqslant k < 2^{r_1+1}$, k_2 lies in the interval $2^{r_2} \leqslant k < 2^{r_2+1}, \ldots$, where $r_1 < r_2 < \ldots$. Now define $\bar{x}_k = 2^{r_i/p_k}$ for $k = k_i$ and $\bar{x}_k = 0$ otherwise. Then if we write

$$h(x) = \sup_r \{(1/2^r) \Sigma_r |x_k|^{p_k}\}^{1/M}$$

for all $x \in w_\infty(p)$, where Σ_r is the sum over $2^r \leqslant k < 2^{r+1}$, we have

$$\tfrac{1}{2} g(x) \leqslant h(x) \leqslant 2g(x), \tag{3}$$

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where g is defined by (2) with $A = (C, 1)$. Now $h(\bar{x}) = 1$, but for $r = r_i$ and $0 < |\lambda| \leqslant 1$,

$$(1/2^r)\,\Sigma_r |\lambda \bar{x}_k|^{p_k} = |\lambda|^{p_{k_i}} \geqslant |\lambda|^{1/i} \to 1$$

as $i \to \infty$. Hence, for $0 < |\lambda| \leqslant 1$ we have $h(\lambda\bar{x}) = 1$ and so $g(\lambda\bar{x}) \geqslant \frac{1}{2}$ by (3). But this contradicts the fact that $\lambda \to 0$, $\bar{x} \in w_\infty(p)$ imply $\lambda\bar{x} \to \theta$.

Whilst we are considering properties which depend on whether or not $\inf p_k > 0$, we take up a remark made in (2) which concerns the continuous dual $w_0^*(p)$ of $w_0(p)$. In the case $0 < \inf p_k \leqslant p_k \leqslant 1$, it was proved in (2) that f is a continuous linear functional on $w_0(p)$ if and only if

$$f(x) = \Sigma a_k x_k, \tag{4}$$

for all x in $w_0(p)$, where a is in $\mathscr{M}(p)$—the set of all sequences a such that

$$\sum_{r=0}^{\infty} \max_r |2^{r/p_k} a_k| < \infty,$$

where $\max_r$ denotes the maximum over the range $2^r \leqslant k < 2^{r+1}$. It was pointed out in (2) that (4) defines an element of $w_0^*(p)$ without the restriction $\inf p_k > 0$, when $a \in \mathscr{M}(p)$. Also, every $f \in w_0^*(p)$ can be written as in (4) but it was shown that this implied $a \in \mathscr{M}(p)$ only with the restriction $\inf p_k > 0$. We now show that the case $\inf p_k = 0$ is essentially different. We have not been able to deal with the general problem but the following theorem covers a certain class Q of null sequences p. Loosely speaking Q is the set of p which tend to zero fairly quickly, at any rate quicker than

$$p_k = 1/\log\log k \ (k \geqslant 3).$$

Precisely, Q is the set of p for which a number $N = N(p) > 1$ exists such that

$$\Sigma N^{-1/p_k} < \infty.$$

The referee observes another way of describing the set Q. It is easy to see that $\limsup (p_k \log k) < \infty$ implies $p \in Q$ and that $p \in Q$ implies $\liminf (p_k \log k) < \infty$. Also, if p_k decreases then $p \in Q$ implies $\limsup (p_k \log k) < \infty$. It follows readily that Q is the set of all p such that there exists a rearrangement q of p for which $q_k > 0$ and $\limsup (q_k \log k) < \infty$.

THEOREM 3. *Let $p \in Q$. Then $w_0^*(p)$ can be identified with $S(p)$ the set of all sequences $a = (a_k)$ such that $\sup_{r \geqslant 0} 2^r \max_r |a_k|^{p_k} < \infty$, where the maximum is over $[2^r, 2^{r+1})$.*

Proof. First, we note that $p \in Q$ implies $p_k \to 0$ and that $\mathscr{M}(p) \subset S(p)$ strictly. Now for the necessity, if $f \in w_0^*(p)$ then $f(x) = \Sigma a_k x_k$, where $a_k = f(e_k)$, $e_k = (0, 0, \ldots, 1, 0, \ldots)$ with 1 in the kth place. Hence $\Sigma a_k x_k$ converges whenever $x \in w_0(p)$. In fact this last assertion implies $a \in S(p)$ without any restriction on p. For suppose $M_r = 2^r \max_r |a_k|^{p_k}$ is not bounded. Then $M_{r_i} > i$, $r_1 < r_2 < \ldots$, and if the maximum occurs at $k_i \in [2^{r_i}, 2^{r_i+1})$ we define $x_{k_i} = 1/a_{k_i}$, $x_k = 0$ $(k \neq k_i)$. Then $x \in w_0(p)$ but $\Sigma a_k x_k$ diverges. Thus every $f \in w_0^*(p)$ can be represented in the form $f(x) = \Sigma a_k x_k$ where $x \in w_0(p)$, $a \in S(p)$.

Conversely, when $x \in w_0(p)$, $a \in S(p)$ and $p \in Q$, $f(x) = \Sigma a_k x_k$ is linear. For, whenever r is sufficiently large

$$\Sigma_r |x_k|^{p_k} < \frac{2^r}{NH},$$

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where $\Sigma N^{-1/p_k} < \infty$ and $\sup M_r = H$. Here and elsewhere Σ_r denotes a sum over $2^r \leqslant k < 2^{r+1}$. Hence for all sufficiently large r we have

$$\Sigma_r |a_k x_k| \leqslant \Sigma_r N^{-1/p_k},$$

so that $\Sigma |a_k x_k| < \infty$. Consequently f is linear on $w_0(p)$. Finally, we show that f is continuous at the origin, from which it follows by linearity that f is continuous everywhere. Let $x^{(n)} \to (0, 0, \ldots)$; $x^{(n)} \in w_0(p)$ and $\sup M_r = H, \sup p_k = M$. Then for all r and all sufficiently large n, $\Sigma_r |a_k x_k^{(n)}| \leqslant \Sigma_r N^{-1/p_k}$.

Now choose r_0 so as to make $\sum_{r_0+1}^{\infty} \Sigma_r N^{-1/p_k}$ as small as we please and observe that

$$\sum_{r=0}^{r_0} \Sigma_r |a_k x_k^{(n)}| \leqslant \sum_{r=0}^{r_0} \Sigma_r H^{1/p_k} [g(x^{(n)})]^{M/p_k},$$

where $$[g(x^{(n)})]^M = \sup_r \frac{1}{2^r} \Sigma_r |x_k^{(n)}|^{p_k} \to 0 \quad (n \to \infty).$$

Hence $$f(x^{(n)}) \to 0 \quad \text{as} \quad g(x^{(n)}) \to 0.$$

In the case when $0 < p_k \leqslant 1$ it is now seen that the characterization of $w_0^*(p)$ is reasonably satisfactory, though such cases as $p_k = 1/(\log \log k)$ $(k \geqslant 3)$ escape. When $p_k > 1$ for every k, I have not been able to obtain anything satisfactory, except in the case $p_k = p' > 1$ for every k. One can then identify $w_0^*(p)$ with the set of all a such that

$$\sum_{r=0}^{\infty} 2^{r/p'} (\Sigma_r |a_r|^{q'})^{1/q'} < \infty,$$

where $1/p' + 1/q' = 1$. For details see (1). However, if we look at the rather simpler space $l(p)$ we can give a fairly complete picture of $l^*(p)$. The case $0 < p_k \leqslant 1$ is very easy and in (3) it is shown that $l^*(p)$ can be identified with $m(p)$. We now deal with the case $\inf p_k > 1$; there appear to be special problems connected with the case $p_k > 1$ and $\inf p_k = 1$ (a few remarks are made after Theorem 4).

THEOREM 4. *Let* $1 < \inf p_k \leqslant \sup p_k < \infty$ *and let* $(1/p_k) + (1/q_k) = 1$. *Then* $l^*(p)$ *can be identified with* $l(q)$.

Proof. Let $x \in l(p)$, $a \in l(q)$. Then $\Sigma a_k x_k$ converges absolutely, actually with no restriction on p except $p_k > 1$. For we have the well-known inequality

$$|a_k x_k| \leqslant \frac{1}{q_k} |a_k|^{q_k} + \frac{1}{p_k} |x_k|^{p_k} \leqslant |a_k|^{q_k} + |x_k|^{p_k}.$$

Hence $f(x) = \Sigma a_k x_k$ is linear on $l(p)$. Now f is continuous at $(0, 0, \ldots,)$. For suppose $0 < g(x) \leqslant 1$, where $$g(x) = (\Sigma |x_k|^{p_k})^{1/M},$$

$1 < M = \sup p_k < \infty$. Then $\{g(x)\}^{M/p_k} \leqslant g(x)$ and so

$$\Sigma |a_k| \frac{|x_k|}{g(x)} \leqslant \Sigma |a_k|^{q_k} + \Sigma \frac{|x_k|^{p_k}}{g^M(x)},$$

$$\Sigma |a_k x_k| \leqslant (\Sigma |a_k|^{q_k} + 1)\, g(x),$$

whence f is continuous at the origin. We may note that the argument is valid as long as $1 < p_k \leqslant \sup p_k < \infty$.

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Consequently, whenever $a \in l(q)$, f is a continuous linear functional on $l(p)$.

Conversely, let $f \in l^*(p)$. Then $f(x) = \Sigma a_k x_k$ for all $x \in l(p)$, and when $\inf p_k = m > 1$ this implies $a \in l(q)$. For suppose $a \notin l(q)$. Then determine $k_1 < k_2 < \ldots$ such that

$$M_i = \sum_{k_i+1}^{k_{i+1}} |a_k|^{q_k} > i^{2m/(m-1)}, \quad (i = 1, 2, \ldots).$$

Define $$x_k = \operatorname{sgn} a_k |a_k|^{q_k - 1} i^{-2} M_i^{-1/p_k} \quad \text{for} \quad k_i + 1 \leqslant k \leqslant k_{i+1}.$$

Then $x \in l(p)$, since $p_k > 1$, but $M_i \geqslant 1$ and $p_k \geqslant m$ imply $M_i^{1/m} \geqslant M_i^{1/p_k}$, so that

$$\sum_k a_k x_k \geqslant \sum_i i^{-2} M_i^{1-1/m} = \infty.$$

Hence we must have $a \in l(q)$.

We now note something about the case $p_k > 1$ but $\inf p_k = 1$. In this case it is possible to have $l(p) = l_1$ so that $l^*(p) = l_1^* = l_\infty$, and l_∞ is strictly larger than $l(q)$ for every q. In fact we can say exactly which sequences $p_k \geqslant 1$ are such that $l(p) = l_1$. Using Theorem 3 (3) it is easy to show that $l(p) = l_1$ if and only if there is a number $N > 1$ such that

$$\Sigma N^{s_k} < \infty,$$

where $s_k = (1 - p_k)^{-1}$, and we define $N^{s_k} = 0$ when $p_k = 1$. For $l(p) = l_1$ is equivalent to $l(p') = l_1$, where $p_k' = p_k^{-1}$, and we apply Theorem 3 (3) with the p_k there replaced by our p_k'.

Thus, for example,

$$l(p) = l_1 \quad \text{when} \quad p_k = 1 + \frac{1}{k} \quad \text{or} \quad p_k = 1 + \frac{1}{\log k},$$

but not when $$p_k = 1 + \frac{1}{\log \log k}.$$

Finally, we note that a problem similar to that for $w_0^*(p)$ ($l^*(p)$), when $\inf p_k = 0$ ($\inf p_k = 1$), arises for $c_0^*(p)$ when $\inf p_k = 0$. It may be easier to deal with this than $w_0^*(p)$. All we can say at the moment is that $c_0^*(p) = l_1$ when $\inf p_k > 0$ (which is of course trivial) and that $c_0^*(p) = m(p) \neq l_1$ when $p \in Q$ (see Theorem 3 above). We may also note, that for

$$p_k = \frac{1}{\log \log k} \notin Q,$$

the condition $a \in m(p)$ is not sufficient for $\Sigma a_k x_k$ to converge whenever $x \in c_0(p)$. We may, for example, take $(a_k) = e$, $x_k = 1/k$. Even if the characterization of $c_0^*(p)$ is put aside there is still the interesting convergence factor problem of finding necessary and sufficient conditions on (a_k) such that $\Sigma a_k x_k$ converges (or converges absolutely) whenever $|x_k|^{p_k} \to 0$. Of course the absolute convergence problem amounts to finding the Köthe–Toeplitz dual of $c_0(p)$.

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