J. Plasma Physics (2004), *vol.* 70, *part* 2, *pp.* 237–250. © 2004 Cambridge University Press
DOI: 10.1017/S0022377803002496 Printed in the United Kingdom

Gas-dynamic description of electrostatic solitons

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(Received 25 March 2003; revised 4 April 2003; accepted 25 April 2003)

Abstract. The nonlinear propagation of electrostatic solitary structures in unmagnetized multispecies plasmas is studied in the wave frame, where they are stationary, via the recently developed McKenzie approach as an alternative to the more usual Sagdeev pseudo-potential method. This way of looking at the problem brings out the gas-dynamic aspects, which then allow a straightforward characterization of the solitary wave possibilities in terms of the species' own sonic points and of the global charge neutral points. A qualitative discussion of ion-, dust- and electron-acoustic solitary waves is given in terms of these concepts and the results are contrasted with those obtained by other methods. Ion-acoustic solitons can be shown to always be compressive, without invoking simplifying assumptions such as cold ions or Boltzmann electrons. Beam-plasmas can also be studied, as in the electron-acoustic solitary wave model for the spiky structures of the broadband electrostatic noise observed in the auroral regions of the Earth's magnetosphere. Such solitons always show a potential dip.

1. Introduction

The study of genuinely nonlinear solitary and periodic waves in different physical systems has a long history, given the obvious statement that the basic equations describing such systems are almost all intrinsically nonlinear. In particular, in plasma physics a rich variety of nonlinear structures are possible and these have been widely studied. It is therefore impossible to give all but the most essential references for the specific examples illustrated below. For further details about nonlinear waves in general, the reader is referred to excellent textbooks and monographs such as, e.g., those by Infeld and Rowlands [1] and Ablowitz and Clarkson [2]. Again, these are not the only valuable introductions or overviews, but we use them to access much of the original literature.

Two of the most successfully applied methods are reductive perturbation techniques [3] in various guises (which are however limited to not too large nonlinearities) and the so-called Sagdeev or pseudo-potential approach [4, 5], which gives exact solutions for arbitrary amplitudes but cannot deal with as many problems as we would like. The reductive perturbation method has given us standard nonlinear paradigms like the Korteweg–de Vries (KdV) and (derivative) nonlinear Schrödinger equations [1, 2].

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The Sagdeev potential method, on the other hand, has been shown to be particularly suited to dealing with electrostatic nonlinear modes in various plasmas, the description of which brings out the motion of the nonlinear structure in an inertial frame of reference. We essentially eliminate the velocities between the continuity and momentum equations per species by using some form of polytropic equations of state, so that the densities are expressed as functions of the electrostatic potential of the wave. In this way, Poisson's law becomes a second-order differential equation for this electrostatic potential, which after integration gives rise to a kind of energy integral. The solution to the problem at hand is then discussed in terms of the properties of one-dimensional motion in an equivalent potential energy, the Sagdeev pseudo-potential [1, 2].

Although the Sagdeev approach has many followers and successes to its credit, it obscures some of the physical mechanisms that lead to the existence or not of the solitary structures. In this respect, the recently developed approach of McKenzie and Doyle [6–9] is the ideal complement to the Sagdeev treatment. Both techniques ultimately lead to the same exact solutions for a given problem, but differ in the degree of physical insight. The McKenzie method uses a reference frame that moves with the nonlinear structure and relies on various Bernoulli-like invariants to nail down as many physical constraints as possible on the existence of nonlinear modes. The interpretation of such constraints brings out the gas-dynamic aspects of the modes. While it works very well for electrostatic modes, as we recall in the following, the application to electromagnetic waves in multispecies plasmas is fraught with additional complexities which come from the magnetic part of the Lorentz force in the equations of motion. These will need separate investigation, despite recent advances for rather limited configurations [10, 11].

Inspired by McKenzie and Doyle's work [6–9], in this paper we aim to give a general multispecies discussion of nonlinear electrostatic plasma structures and to extend the treatment to beam-plasma models. Afterwards, we recall some of the well-known plasma modes, such as ion-, dust- and electron-acoustic solitary waves. For the latter we discuss a three-electron component plasma, including cool, hot and beam electrons, which has been proposed for modelling broadband electrostatic noise in the auroral regions [12–16].

2. Theoretical analysis

2.1. Basic formalism and mass flux conservation

In the following, we adapt the recent treatment of McKenzie and Doyle [6–9] for stationary progressive waves of the form $f(x+Vt)$, by giving a truly multispecies description and by including the possibility that beam species are present. Such waves are described in the wave frame in which the structures appear steady ($\partial/\partial t=0$), in contrast to the Sagdeev pseudo-potential approach where the nonlinear waves are viewed in an inertial (laboratory) frame. Before focusing on some typical electrostatic solitary waves, in beam-plasma systems where plasma components are allowed to stream relative to each other, we also want to first give a systematic exposé that essentially parallels McKenzie and Doyle's description [6–9], but ordered in a way that we find advantageous.

Plasma species will stream along the x-axis in the wave frame, with a species' reference speed v_{j0} at $x=-\infty$, where the subscript j refers to the different plasma

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species (to be specified later). For the particles that in the absence of wave disturbances would be at rest in an inertial frame, obviously now $v_{j0} = V$ in the wave frame, whereas beam species will have a different reference speed $v_{j0} = V - V_j$. Here V_j is the beam speed measured initially in an inertial frame. Because of the applications to beam-plasma systems we have in mind, we prefer not to use dimensionless variables yet. The standard set of basic equations for electrostatic modes includes the per species continuity and momentum equations,

$$\begin{aligned} \frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j v_j) &= 0, \\ \frac{\partial v_j}{\partial t} + v_j \frac{\partial v_j}{\partial x} + \frac{1}{n_j m_j}\frac{\partial p_j}{\partial x} &= -\frac{q_j}{m_j}\frac{\partial \phi}{\partial x}, \end{aligned} \tag{1}$$

and the system is then closed by Poisson's law,

$$\varepsilon_0 \frac{\partial^2 \phi}{\partial x^2} = -\sum_j n_j q_j. \tag{2}$$

Here n_j, v_j and p_j refer to the number density, fluid velocity and pressure, respectively, of each plasma species with charge q_j and mass m_j, and ϕ is the electrostatic potential.

Since in the wave frame there are no time derivatives, these equations can easily be integrated, starting with the continuity equation which expresses conservation of the mass flux

$$n_j m_j v_j = n_{j0} m_j v_{j0}. \tag{3}$$

2.2. Bernoulli integrals

We will assume that the pressures are polytropic, with index γ_j, so that $p_j \propto n_j^{\gamma_j} \propto v_j^{-\gamma_j}$. This includes the usual adiabatic flow for $\gamma_j = 3$ and the last proportionality comes from the conservation of mass flux. If necessary, the special case of isothermal pressures ($\gamma_j = 1$) can easily be included by adapting the general expressions. The equations of motion thus become

$$v_j \left(1 - \frac{v_{j0}^{\gamma_j + 1}}{v_j^{\gamma_j + 1} M_j^2}\right) \frac{dv_j}{dx} = -\frac{q_j}{m_j}\frac{d\phi}{dx} = \frac{q_j}{m_j} E, \tag{4}$$

where (3) has been used and $E = -d\phi/dx$ is the electric field of the nonlinear structure. The Mach number M_j and thermal velocity c_{tj} of each component are defined by

$$M_j = \frac{v_{j0}}{c_{tj}}, \quad c_{tj}^2 = \frac{\gamma_j p_{j0}}{n_{j0} m_j}. \tag{5}$$

In the gas-dynamic description that we are following, the sonic points, defined as

$$\frac{v_{js}}{v_{j0}} = \frac{1}{M_j^{2/(\gamma_j + 1)}}, \tag{6}$$

correspond to where the species' local flow speeds match their local acoustic speeds $(\gamma_j p_j / n_j m_j)^{1/2}$. Hence (4) indicates that these sonic points play a crucial role in limiting the amplitude of the wave, by choking the flow ($dv_j/dx \to +\infty$) before a possible equilibrium point can be reached from the initial values $v_j = v_{j0}$, the beginning of the wave.

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After integration of the equations of motion (4), a Bernoulli type integral is obtained per species,

$$\mathscr{E}_j \equiv \frac{1}{2}\left(v_j^2 - v_{j0}^2\right) + \frac{c_{tj}^2}{\gamma_j - 1}\left(\frac{v_{j0}^{\gamma_j - 1}}{v_j^{\gamma_j - 1}} - 1\right) = -\frac{q_j}{m_j}\phi. \tag{7}$$

Here the first term refers to changes in kinetic energy, whereas the second term gives changes in enthalpy. This is clear from the alternative way in which this expression can be written,

$$\frac{1}{2}m_j v_j^2 + \frac{\gamma_j}{\gamma_j - 1}\frac{p_j}{n_j} + q_j\phi = \frac{1}{2}m_j v_{j0}^2 + \frac{\gamma_j}{\gamma_j - 1}\frac{p_{j0}}{n_{j0}}. \tag{8}$$

Since a consequence of (7) is that all $m_j\mathscr{E}_j/q_j$ are equal, the different species' velocities can be eliminated in function of one of them and the corresponding equation of motion (4) then expresses E as a function of the same velocity. When we now compute

$$\frac{d\mathscr{E}_j}{dv_j} = v_j\left(1 - \frac{v_{j0}^{\gamma_j + 1}}{v_j^{\gamma_j + 1}M_j^2}\right), \tag{9}$$

we first note that at the sonic point given by (6), $\mathscr{E}_j$ reaches a minimum negative value. Initially $\mathscr{E}_j$ is zero and its derivative is then given by

$$\left.\frac{d\mathscr{E}_j}{dv_j}\right|_{v_j = v_{j0}} = v_{j0}\left(1 - \frac{1}{M_j^2}\right). \tag{10}$$

Thus $\mathscr{E}_j$ has a slope at the initial point which is positive if the incoming flow is supersonic ($M_j > 1$) and negative for subsonic flows ($M_j < 1$). If we first discuss positive particles, we learn from the behaviour of the Bernoulli relations that positive particles are decelerated ($v_j < v_{j0}$) and driven towards their sonic point in a potential hill ($\phi > 0$) if the flow is supersonic ($M_j > 1$), whereas subsonic, hotter positive species (with $M_j < 1$) are accelerated ($v_j > v_{j0}$) while being driven towards their sonic point. Because of mass flux conservation (3), this corresponds for supersonic, cooler positive species to a compression and for subsonic, hotter positive species to a rarefaction.

These conclusions are reversed for negative particles, so that supersonic, cooler negative species are rarefied, whereas subsonic, hotter negative species are compressed, while being driven away from their sonic points. In a potential dip ($\phi < 0$), all these conclusions are reversed again. This is illustrated schematically in Fig. 1. Consequently, if a given plasma species (with index ℓ) is to be driven towards its sonic point, the potential has to obey $q_\ell\phi > 0$. Thus for a positive species, a potential hill ($\phi > 0$) is needed, whereas a negatively charged species would require a potential dip or valley ($\phi < 0$). This will be of importance when we discuss specific solitary waves.

Owing to charge neutrality in the undisturbed state, we also see from (7) that

$$\sum_j n_{j0}m_j\mathscr{E}_j = -\phi\sum_j n_{j0}q_j = 0, \tag{11}$$

a result that we will find useful later on.

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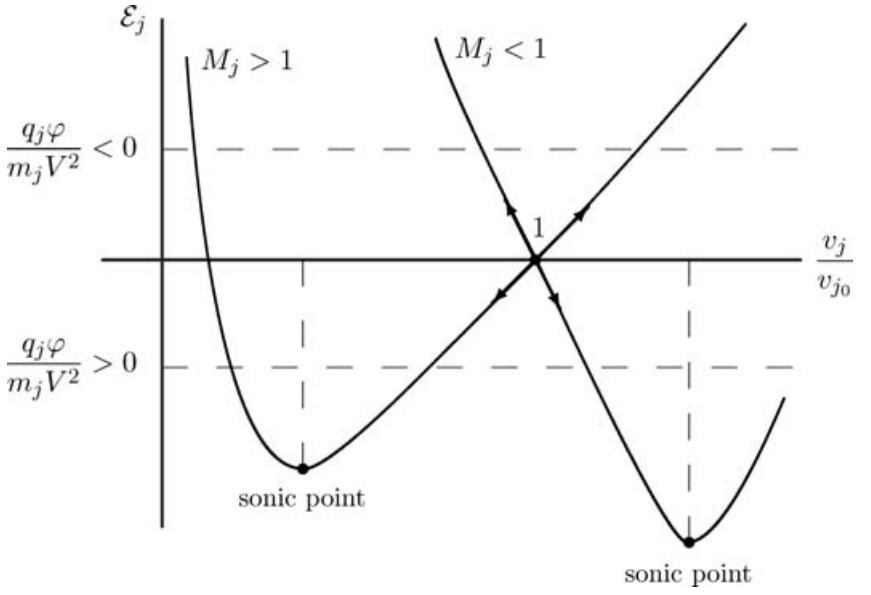


Figure 1. Bernoulli considerations for supersonic and subsonic species, with arrows indicating the possible departures from initial conditions.

2.3. Global pressure invariant

Finally, the Poisson equation can be integrated after multiplying it by $d\phi/dx$, yielding a global pressure function in the direction of propagation,

$$\sum_j n_{j0} m_j \mathscr{P}_j = \frac{\varepsilon_0}{2}\left(\frac{d\phi}{dx}\right)^2, \tag{12}$$

where the (normalized) particle pressure functions $\mathscr{P}_j$ for each species are given by

$$\mathscr{P}_j \equiv v_{j0}(v_j - v_{j0}) + \frac{c_{ij}^2}{\gamma_j}\left(\frac{v_{j0}^{\gamma_j}}{v_j^{\gamma_j}} - 1\right). \tag{13}$$

The first term relates to changes in dynamic pressure and the second term to changes in thermal pressure. Again, (12) is obviously

$$\sum_j \left(n_j m_j v_j^2 - n_{j0} m_j v_{j0}^2\right) + \sum_j (p_j - p_{j0}) = \frac{\varepsilon_0}{2} E^2. \tag{14}$$

Subtracting (11) from (13) gives an alternative expression for the square of the electric field, namely

$$\frac{\varepsilon_0}{2} E^2 = \sum_j n_{j0} m_j (\mathscr{P}_j - \mathscr{E}_j), \tag{15}$$

which allows an easier interpretation of the conditions needed to guarantee the existence of solitary wave solutions. As already pointed out, all $m_j\mathscr{E}_j/q_j$ are equal and the different species' velocities can be eliminated in function of one of them, so that (15) can, in principle, be reduced to a first-order differential equation in one of the velocities, by using (4) to express E in that particular velocity.

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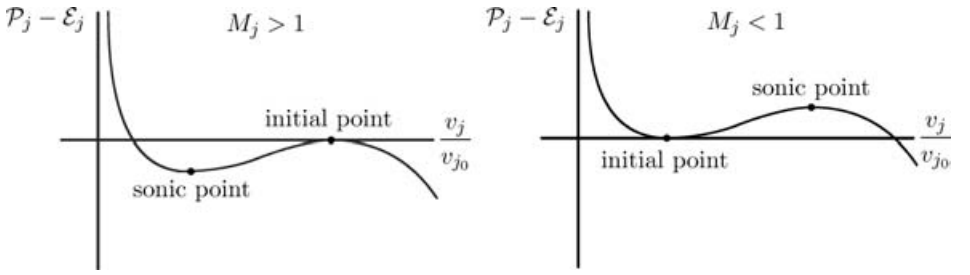


Figure 2. Behaviour of pressure-Bernoulli difference functions.

For later use we need more information about the different terms in (15). One can easily compute that

$$\frac{d}{dv_j}(\mathscr{P}_j - \mathscr{E}_j) = (v_{j0} - v_j)\left(1 - \frac{v_{j0}^{\gamma_j+1}}{v_j^{\gamma_j+1}M_j^2}\right), \tag{16}$$

and consequently all pressure-Bernoulli difference functions $\mathscr{P}_j - \mathscr{E}_j$ have a turning point at the respective sonic point, so that for each species its sonic point is placed between a zero of $\mathscr{P}_j - \mathscr{E}_j$ and the double zero at the initial point. Moreover, computation of the second derivative of these pressure-Bernoulli difference functions in the initial point means that

$$\left.\frac{d^2}{dv_j^2}(\mathscr{P}_j - \mathscr{E}_j)\right|_{v_j = v_{j0}} = \frac{1}{M_j^2} - 1, \tag{17}$$

so that for supersonic ($M_j^2 > 1$) cooler species $\mathscr{P}_j - \mathscr{E}_j$ has a local (zero) maximum in the initial point and reaches a local (negative) minimum at the sonic point. This conclusion is reversed for subsonic ($M_j^2 < 1$) hotter species for which $\mathscr{P}_j - \mathscr{E}_j$ increases from a local (zero) minimum in the initial point to a local (positive) maximum at the sonic point. This is shown in Fig. 2.

Before going to the existence conditions for solitary waves, we briefly digress on how heavier species might be eliminated from the description. In the traditional Sagdeev approach heavy species are only supposed to contribute charge but no pressure to the wave in the inertial frame and hence have $v_j = v_{j0} = V$ in the wave frame. In contrast to the classical picture, we assume that such heavy species are not quite immobile in the inertial frame, but move very slowly with respect to it and hence their densities will not change very much. Given that for those species the normalized velocity v_j stays close to $v_{j0} = V$, the pressure and Bernoulli functions may be approximated by the linear term in their Taylor expansions, yielding

$$\mathscr{P}_j \simeq V(v_j - V)\left(1 - \frac{1}{M_j^2}\right) \simeq \mathscr{E}_j. \tag{18}$$

Consequently, when we split the sums over species into those over heavy and over light elements, we find that (15) is rewritten in terms of the parameters for the lighter species only,

$$\sum_{\text{light}} n_{j0} m_j (\mathscr{P}_j - \mathscr{E}_j) = \frac{\varepsilon_0}{2} E^2. \tag{19}$$

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2.4. Existence conditions for solitary waves

Now we come to the conditions for the existence of a solitary wave structure. First, near the initial point we put $v_j = v_{j0} + \Delta_j$ and suppose that $|\Delta_j| \ll |v_{j0}|$, so that with the help of the linearized form of the Bernoulli relations (7),

$$\frac{m_j \Delta_j v_{j0}}{q_j} \left(1 - \frac{1}{M_j^2}\right) = \frac{m_\ell \Delta_\ell v_{\ell 0}}{q_\ell} \left(1 - \frac{1}{M_\ell^2}\right), \tag{20}$$

we can linearize (4) to

$$\frac{d\Delta_\ell}{dx} = \pm \Delta_\ell \left[\sum_j \frac{\omega_{\mathrm{p}j}^2}{c_{tj}^2 - v_{j0}^2}\right]^{1/2}. \tag{21}$$

The various plasma frequencies $\omega_{\mathrm{p}j}$ are defined through $\omega_{\mathrm{p}j}^2 = n_{j0} q_j^2 / \varepsilon_0 m_j$.

In these results, the index ℓ can refer to any of the species and (21) possesses real, exponential solutions of the form $\Delta_\ell \propto \exp(\pm \kappa x)$, provided the exponent κ is real or, in other words,

$$\kappa^2 = \sum_j \frac{\omega_{\mathrm{p}j}^2}{c_{tj}^2 - v_{j0}^2} > 0. \tag{22}$$

The condition (22) will determine the different regimes for the speed V of the nonlinear wave. It follows from the condition on κ^2 that not all species can be supersonic ($c_{tj}^2 < v_{j0}^2$). As we will see later, all species can also not be subsonic ($v_{j0}^2 < c_{tj}^2$) if we want a proper solitary wave to exist.

For the second condition, we start by noting that E has a double root at the initial point, since $dE/dx = -d^2\phi/dx^2$ also vanishes initially, as follows from the Poisson equation (2) at equilibrium. Moreover, the location of the zeros of E (or of E^2) is assisted by computing, with the help of (4), its derivative in the form

$$\frac{\varepsilon_0}{2} \frac{dE^2}{dv_\ell} = \frac{m_\ell v_\ell}{q_\ell} \left(1 - \frac{v_{\ell 0}^{\gamma_\ell + 1}}{v_\ell^{\gamma_\ell + 1} M_\ell^2}\right) \sum_j n_j q_j, \tag{23}$$

where it is again understood that all velocities (or densities) have formally been expressed as functions of a chosen v_ℓ with the help of mass flux conservation (3) and the Bernoulli relations (7). Because it is usually the heavier species that provide the inertia to sustain the waves, we will choose (one of) the heavier, supersonic species to discuss the behaviour of E^2 and its first derivative.

The zeros of E^2 are intertwined with the zeros of the derivative of E^2, the latter occurring at the charge neutral points, defined as $\sum n_j q_j = \sum n_{j0} q_j = 0$, and at the sonic point in question. A solitary wave is only possible if there is a zero of E^2 located between the charge neutral point (outside the initial point) and the sonic point. In this charge neutral point E^2 reaches a local maximum. Then the zero of E^2 located between the charge neutral point and the heavy species' sonic point determines the solitary wave amplitude, because $E = 0$ corresponds to an extremum of ϕ. This is illustrated schematically in Fig. 3. In the special case where the zero of E^2 should fall in the sonic point rather than before it, we can also include double layers in the present gas-dynamic description. At the same time, this is the maximum possible amplitude for the solitary wave structures discussed in general. Note that we have to reverse the graph left–right when discussing subsonic species.

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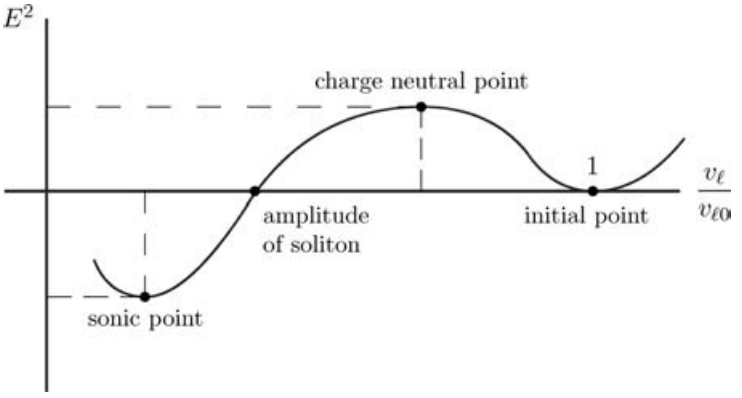


Figure 3. Characteristic points of E^2.

From the Bernoulli integrals (7) it follows that if all species are subsonic ($M_j < 1$), positive species in a potential hill ($\phi > 0$) are accelerated and rarefied, whereas negative species are decelerated and compressed, so that a charge neutral point outside the initial point cannot be reached before one of the positive species reaches its sonic point, where its flow is choked. In a potential dip, the rarefactions and compressions are interchanged, but the conclusion about the unattainability of the charge neutral point remains. An analogous reasoning can also be made when all species would be supersonic ($M_j > 1$), giving an alternative explanation to the one following from discussing the κ condition. Thus, for the proper existence of a solitary wave, at least one supersonic and one subsonic species is necessary.

3. Applications

3.1. Ion-acoustic solitary waves

We now briefly discuss some of the typical applications that have been covered in the literature by various methods, although it is not our aim to recall or rederive all the possible details, which are adequately referenced in the literature. The oldest and most typical example is that of ion-acoustic modes in ordinary plasmas, which are essentially sustained by ion inertia and electron pressure [17]. Using the indices e and i to denote the electrons and a single species of ions, respectively, we rewrite (22) as

$$\kappa^2 = \frac{\omega_{\rm pe}^2}{c_{te}^2 - V^2} + \frac{\omega_{\rm pi}^2}{c_{ti}^2 - V^2} > 0, \tag{24}$$

there being no drift velocities so that $v_{\rm e0} = v_{\rm i0} = V$. Supposing the electrons to be hotter than the protons, expressed as $c_{ti}^2 < c_{te}^2$, in principle (24) admits two velocity regimes,

$$\begin{aligned} c_{\rm ia}^2 < V^2 < c_{te}^2, \\ V^2 < c_{ti}^2, \end{aligned} \tag{25}$$

where the ion-acoustic (linear phase) velocity $c_{\rm ia}$ is defined as

$$c_{\rm ia}^2 = \frac{m_{\rm e} c_{te}^2 + m_{\rm i} c_{ti}^2}{m_{\rm i} + m_{\rm e}}. \tag{26}$$

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This reduces to the textbook definition of the ion-acoustic velocity when $p_{i0} \ll p_{e0}$, a condition needed anyway to avoid strong Landau damping of the waves in the linear regime. From the gas-dynamic interpretation we have given in the general discussion, it follows that the lower velocity regime ($V^2 < c_{ti}^2$) in (25) cannot lead to proper solitary wave formation. Indeed, in this regime both electrons and ions are subsonic and there is no charge neutral point outside the initial condition before a sonic point is reached and the corresponding flow is choked.

Consequently, for ion-acoustic solitary waves to exist, the ions have to be supersonic and the electrons subsonic, which corresponds to the first regime ($c_{ia}^2 < V^2 < c_{te}^2$) in (25). We can now give a general proof that these ion-acoustic solitons are always compressive. As we have seen, one of the necessary conditions for the existence of a soliton is that another charge neutral point exists outside the initial point. The condition for that being $n_e = n_i$, it follows from mass flux conservation (3) that $n_e v_e = n_0 V = n_i v_i$, using $n_{e0} = n_{i0} = n_0$ and, hence, at a charge neutral point $v_e = v_i = v$. The Bernoulli integrals (7) tell us that $m_e \mathscr{E}_e + m_i \mathscr{E}_i = 0$, which reduces in a charge neutral point to

$$f(v) \equiv \frac{m_i + m_e}{2}(v^2 - V^2) + \frac{m_i c_{ti}^2}{\gamma_i - 1}\left(\frac{V^{\gamma_i - 1}}{v^{\gamma_i - 1}} - 1\right) + \frac{m_e c_{te}^2}{\gamma_e - 1}\left(\frac{V^{\gamma_e - 1}}{v^{\gamma_e - 1}} - 1\right) = 0. \quad (27)$$

It is obvious that $\lim_{v \to 0+} f(v) \to +\infty$, $f(V) = 0$ and $\lim_{v \to +\infty} f(v) \to +\infty$. To see where another possible zero of $f(v)$ lies, we investigate $f(v)$ around $v = V$ by putting $v = V + \Delta$ and developing in a Taylor series up to first order in Δ. This gives

$$f(v) = (m_i + m_e) V \left(1 - \frac{c_{ia}^2}{V^2}\right) \Delta + \cdots, \quad (28)$$

which has the sign of Δ, as $c_{ia} < V$. Remembering that $\lim_{v \to 0+} f(v) \to +\infty$ and noting that $f(v) < 0$ for values $v < V$ in the immediate neighbourhood of V, we conclude that $f(v)$ has to cross from positive to negative values for some interior point $0 < \overline{v} < V$ and, hence, there exists a charge neutral point outside V. To find out whether more charge neutral points might exist, we compute

$$\frac{df(v)}{dv} = (m_i + m_e) V \left[\frac{v}{V} - \frac{m_i c_{ti}^2}{(m_i + m_e) V^2}\left(\frac{V}{v}\right)^{\gamma_i} - \frac{m_e c_{te}^2}{(m_i + m_e) V^2}\left(\frac{V}{v}\right)^{\gamma_e}\right]. \quad (29)$$

Putting $\gamma = \min\{\gamma_i, \gamma_e\}$, we see that $(V/v)^{\gamma_{e,i}} \leqslant (V/v)^{\gamma}$ for all $v > V$ and, consequently,

$$\frac{df(v)}{dv} \geqslant (m_i + m_e) V \left[\frac{v}{V} - \frac{c_{ia}^2}{V^2}\left(\frac{V}{v}\right)^{\gamma}\right] > 0. \quad (30)$$

Hence, $f(v)$ is monotonously increasing in $[V, +\infty[$ and there is only one charge neutral point outside $v = V$, $\overline{v}$ in $]0, V[$. This is borne out by the graph of the second derivative,

$$\frac{d^2 f(v)}{dv^2} = m_i + m_e + \frac{m_i c_{ti}^2 \gamma_i}{V^2}\left(\frac{V}{v}\right)^{\gamma_i + 1} + \frac{m_e c_{te}^2 \gamma_e}{V^2}\left(\frac{V}{v}\right)^{\gamma_e + 1} > 0, \quad (31)$$

shown to be positive for all $v > 0$.

We would like to emphasize that nowhere have we used a condition like $m_e \ll m_i$, but that the whole proof rests on $c_{ti}^2 < c_{te}^2$. Neither was it necessary to assume cold ions or Boltzmann electrons, as is often argued, although the latter can easily be incorporated in a similar reasoning as given here. We can also turn the argument

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around and deduce from the existence of $\overline{v} \neq V$, obeying $f(\overline{v}) = 0$, that the Bernoulli combination $m_e\mathscr{E}_e + m_i\mathscr{E}_i = 0$ has a solution with $v_e = v_i \neq V$. Consequently, $n_i = n_e$ and a charge neutral point outside the initial point exists.

To conclude, it is the supersonic species that needs to be driven towards its sonic point, resulting in a deceleration and compression. It turns out that the subsonic species is also decelerated, but driven away from its sonic point, so that both species are ultimately compressed and a charge neutral point outside the initial point is possible, as we have shown. In the present case, the positive ions are the supersonic species and, hence, a potential hill is required. For a hypothetical plasma composed of antiprotons and positrons, a potential dip would occur. *Mutatis mutandis*, analogous remarks can be made for the other solitary waves discussed in the following sections.

Having ensured that E^2 has a double zero at the initial point which is a local minimum, the existence of a compressive solitary wave furthermore requires that E^2 has another zero located between the ion sonic point and the other charge neutral point. Unfortunately, the latter condition is not simple to work out analytically and requires numerical computations or simplifying assumptions like cold ions ($M_i \to \infty$) and massless (or Boltzmann) electrons ($M_e \to 0$). For further details, we refer to McKenzie and Doyle's results [6, 9]. Similar problems occur with the Sagdeev pseudo-potential approach [18].

3.2. *Dust-acoustic solitary waves*

Acoustic modes also occur in dusty plasmas, where charged dust grains are present in addition to the traditional electron and ion ingredients of standard plasmas. These modes can be essentially sustained by ion inertia and electron pressure, when the charged dust is too sluggish to follow the waves and acts as a neutralizing background or when the dust mainly provides the inertia and the electrons and ions together provide the thermal pressure. In the latter case, we find waves with very low frequencies, because the charged dust is much heavier than protons and has typically very much smaller charge-to-mass ratios than plasma ions, even though the charges themselves can be quite high in absolute value compared to ionic charges. For more details about the properties of such dusty or complex plasmas we refer the reader to recent monographs [19–21].

Now using now the supplementary index d to denote the (negatively) charged dust, we find that (22) becomes

$$\kappa^2 = \frac{\omega_{pe}^2}{c_{te}^2 - V^2} + \frac{\omega_{pi}^2}{c_{ti}^2 - V^2} + \frac{\omega_{pd}^2}{c_{td}^2 - V^2} > 0. \tag{32}$$

The regimes compatible with this requirement are

$$\begin{aligned} c_{ia}^2 < V^2 < c_{te}^2, \\ c_{da}^2 < V^2 < c_{ti}^2, \\ V^2 < c_{td}^2, \end{aligned} \tag{33}$$

provided the usual ordering $c_{td}^2 < c_{ti}^2 < c_{te}^2$ exists. The ion-acoustic velocity (squared) in such a dusty plasma is redefined to a fair degree of accuracy as

$$c_{ia}^2 \simeq \frac{n_{e0}m_i c_{ti}^2 + n_{i0}m_e c_{te}^2}{n_{e0}m_i + n_{i0}m_e}, \tag{34}$$

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whereas the dust-acoustic velocity $c_{\rm da}$ is introduced through

$$c_{\rm da}^2 \simeq \frac{\omega_{\rm pd}^2 c_{ti}^2 c_{te}^2}{\omega_{\rm pe}^2 c_{ti}^2 + \omega_{\rm pi}^2 c_{te}^2} = \lambda_{\rm D}^2 \omega_{\rm pd}^2. \tag{35}$$

Here the global plasma Debye length $\lambda_{\rm D}$ is determined from $\lambda_{\rm D}^{-2} = \lambda_{\rm De}^{-2} + \lambda_{\rm Di}^{-2}$, where the electron and ion Debye lengths are defined through $\lambda_{{\rm D}j}^2 = c_{tj}^2/\omega_{{\rm p}j}^2$ [22]. Because the charged dust creates an imbalance between the electron and ion densities, the ion-acoustic regime ($c_{\rm ia}^2 < V^2 < c_{te}^2$) can differ quite a bit from what occurs in a standard plasma and is sometimes called the dust–ion-acoustic regime [23]. The dust grains are then so sluggish that they can be left out of the discussion, as explained at the end of Sec. 2.3. It is the supersonic ions that are driven towards their sonic point, resulting in a potential hill and compressive solitary waves.

For dust-acoustic solitary waves [24] in the lower velocity regime ($c_{\rm da}^2 < V^2 < c_{ti}^2$), the cooler supersonic dust needs to be driven towards its sonic point, which requires a potential dip, because the dust is supposed to be negative in the simplest charging model [19–21]. We will come back to the special case when positive dust is also present later. In a potential dip, the subsonic electrons are accelerated towards their sonic point, but the subsonic plasma ions are decelerated and driven away from their sonic point. Consequently, the negative dust and the plasma ions are compressed, while the electrons are rarefied.

Having ensured that E^2 has a double zero at the initial point which is a local minimum, the existence of a solitary wave furthermore requires that E^2 has another zero before the dust sonic point is reached. Unfortunately, the latter condition is even harder to work out analytically here and needs numerical computations or simplifying assumptions like cold dust ($M_{\rm d} \to +\infty$) and massless (or Boltzmann) electrons and ions ($M_{\rm e}, M_{\rm i} \to 0$) [7, 9]. A similar problem occurs with the Sagdeev pseudo-potential approach [25].

To conclude this section, we can briefly address the case where lighter positively charged dust coexists with heavier negatively charged dust and most of the electrons and ions have been accreted onto the dust grains [26]. The lighter dust is supposed to be mainly charged by photoionization and the heavier dust by the standard orbital motion limited primary electron current to the grains [20, 21]. This model can be treated in the same way as the ion-acoustic modes in a hydrogen plasma, with the lighter dust now playing the role of the subsonic electrons and the heavier dust that of the supersonic ions. The only intrinsic difference resides in the reversal of the sign of the charges, compared to ion-acoustic solitons, so that this novel type of dust-acoustic solitary waves leads to a potential dip. This conclusion, however, is unaltered from that obtained for the usual dust-acoustic solitary waves we discussed earlier in this section, because it is still the (negative) heavier dust that needs to be driven towards its sonic point.

3.3. Electron-acoustic solitary waves

We now apply the gas-dynamic description to electron-acoustic solitary waves, for the existence of which we need at least two electron species, one hotter and one cooler. The applications to broadband electrostatic noise in auroral regions we have in mind also need the additional presence of an electron beam [12–16]. Since the proton dynamics are not crucial to the discussion, we use the approximation that the ions can be eliminated from the treatment and start from the integral (19). With the indices c, h and b denoting the cool, hot and beam electrons, respectively,

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we rewrite (22) as

$$\kappa^2 = \frac{\omega_{\rm pc}^2}{c_{tc}^2 - V^2} + \frac{\omega_{\rm ph}^2}{c_{th}^2 - V^2} + \frac{\omega_{\rm pb}^2}{c_{tb}^2 - (V - V_{\rm b})^2} > 0, \tag{36}$$

and in this expression $V_{\rm b}$ is the beam velocity as originally measured in the inertial frame, so that $v_{\rm b0} = V - V_{\rm b}$.

In the absence of a beam ($\omega_{\rm pb}^2 = 0$) and with the ordering $c_{tc}^2 < c_{th}^2$, the reduced condition

$$\kappa^2 = \frac{\omega_{\rm pc}^2}{c_{tc}^2 - V^2} + \frac{\omega_{\rm ph}^2}{c_{th}^2 - V^2} > 0 \tag{37}$$

yields the allowable regime $c_{\rm ea}^2 < V^2 < c_{th}^2$, since both electron species cannot be subsonic together. The electron-acoustic velocity $c_{\rm ea}$ is defined as

$$c_{\rm ea}^2 = \frac{n_{\rm h0} c_{tc}^2 + n_{\rm c0} c_{th}^2}{n_{\rm c0} + n_{\rm h0}}. \tag{38}$$

In a potential dip, the cooler electrons are compressed and the hotter electrons rarefied, both being driven towards their sonic points, the cooler ones from above, the hotter ones from below. This transonic feature gives rise to the solitary wave, if the other conditions on the roots of E^2 can also be fulfilled. This implies a judicious mixture of hot and cool electrons, as discussed by McKenzie [8] and in many other papers, e.g. [27].

For the case when a beam is present, the conclusion about the potential dip is unaltered, because it has to be one of the electron species (either the cooler one or the beam species) that must be driven towards its sonic point, which always requires a potential dip. In contrast to what is reported in the literature [14], the inclusion of beam electrons cannot generate potential hill solitary waves, a conclusion that presumably comes from using Boltzmann hot electrons and a weak amplitude KdV-like reductive perturbation approach. Both these restrictions can lead to error about the correct range of validity [27]. In particular, Berthomier et al. [14] have studied parameter regimes where the undisturbed cool electron density $n_{\rm c0}$ is of the order of the undisturbed hot electron density $n_{\rm h0}$, so that $c_{\rm ea}$ is then close to c_{th} and a Boltzmann distribution for the hot electrons is not justified, since that would require $V^2 \ll c_{th}^2$. A fully nonlinear study of electron-acoustic solitary waves in the presence of a beam is underway and will be given elsewhere.

4. Conclusions

In this paper we have given a unified treatment of nonlinear electrostatic solitary wave structures in multispecies plasmas, based on the recently advocated gas-dynamic McKenzie approach. To be completely general, we have also included possible beam species. McKenzie's approach does not claim to be simpler than the traditional Sagdeev treatment, nor to yield fundamentally different results, but the gas-dynamic interpretation does, however, offer far better and clearer insight in the limiting factors and existence conditions for electrostatic solitons. It does so by advantageously using the Bernoulli integrals per species and a global pressure invariant, and by focusing on characteristic points like the species' own sonic points and a global charge neutral point outside the undisturbed initial conditions.

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The solitary wave conditions indicate that the plasma species cannot all be supersonic or subsonic. Furthermore, a charge neutral point (other than the initial point) must be reached before one of the plasma species is driven to its sonic point, where its flow is choked, with the consequence that no smooth solitary wave profile can be constructed. The determination of the maximum amplitude of the solitary wave usually has to be done numerically.

We then applied these general ideas to some typical solitary waves such as ion-, dust- and electron-acoustic modes, and recovered some of their characteristic properties, without recalling all the details that were already studied elsewhere. For the ion-acoustic solitons we have shown in all generality that these are always compressive, because it is the supersonic ions that need to be driven towards their sonic point, a result obtained without recourse to simplifying assumptions like cold ions or Boltzmann electrons. We have also argued that the inclusion of an electron beam, in addition to the cool and hot electron components, cannot alter the conclusion that electron-acoustic solitary structures always require a dip in the electrostatic potential, in contrast to claims in the literature purporting to obtain potential hills. These results are of importance for various space and astrophysical plasmas, all characterized by the presence of different electron and/or ion populations.

Acknowledgements

Interesting discussions with M. A. Hellberg and J. F. McKenzie are gratefully acknowledged. FV and TC thank the Fonds voor Wetenschappelijk Onderzoek (Vlaanderen) for a research grant. In addition, FV is grateful for the kind hospitality provided by the Indian Institute of Geomagnetism during visits when part of this work was carried out.

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