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Mobility analysis and structural synthesis of a class of spatial mechanisms with coupling chains

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SUMMARY
This paper presents a systematic method for dealing with mobility analysis and structural synthesis of a class of important spatial mechanisms with coupling chains, which involve more complex coupling relations than spatial parallel mechanisms. First, an approach to the establishment of the motion screw equation of the class of mechanisms is derived. Then, a general methodology for mobility analysis along with detection of rigid substructures is proposed based on the motion screw equation. Third, the principle of structural synthesis of the class of mechanisms is established on the basis of the method of mobility analysis. Finally, some novel spatial mechanisms with coupling chains are synthesized, illustrating the effectiveness of the method. The study of the paper will benefit structural analysis and synthesis of more complex spatial mechanisms with coupling chains.

KEYWORDS: Spatial mechanism; Coupling chain; Mobility analysis; Rigid substructure; Structural synthesis.

Nomenclature

$\boldsymbol{S}$:	direction vector of a screw
$\boldsymbol{r}$:	position vector of a screw
$\$$:	screw (motion screw)
p:	pitch of a screw
$\boldsymbol{S}_0$:	dual unit of a screw
$\$^r$:	constraint screw
$\subseteq$:	contained relation
$\cap$:	intersecting operation
$\cup$:	union operation
v:	prismatic joint velocity
ω:	revolute joint velocity
$\$^{\parallel}_{MP1}$:	motion screw system of MP1 as the output link of substructure II
$\$^{\parallel}_{MP2}$:	motion screw system of MP2 as the output link of substructure II
$\$_{Ci}$:	motion screw system of the ith chain
M :	mobility of a mechanism
M_e :	mobility of the end platform
M_0:	mobility (initial mobility) of serial part
M_{para}:	mobility of the equivalent sub parallel mechanism

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M_{idle}: idle mobility
$\{M\}$: subgroup of the moving platform
$\{L_i\}$: subgroup of the ith limb
R^1_{mp1}: the first DOF(which is a rotation) of MP1
T^2_{mp1}: the second DOF(which is a translation) of MP1
$\$^0_{EP}$: initial motion screw system
$\$^{\varphi}_{MP1}$: complement set of $\$^0_{EP}$ under $\$^{\|}_{MP1}$
$\$^{\varphi}_{MP2}$: complement set of $\$^0_{EP}$ under $\$^{\|}_{MP2}$
f^{i+1}_{mp1}: the $(i+1)$th joint velocity (which is a dependent parameter) of MP1
f^{j+1}_{mp2}: the $(j+1)$th joint velocity (which is a dependent parameter) of MP2

1. Introduction

Mobility analysis is the basis of structural synthesis of mechanisms and robots. In IFToMM terminology, the mobility or degree-of-freedom (DOF) of a mechanism is defined as the number of independent parameters required to define completely the configuration of the mechanism at any instant of time.[1] For metamorphic mechanisms,[2] their mobility should be considered respectively according to different motion stages. Mobility analysis of a mechanism involves the following tasks: (a) analyzing mobility of the mechanism, (b) analyzing mobility of the end-effector, (b) determining mobility property of the end-effector.

Grübler–Kutzbach(G–K)[3,4] mobility criterion can be used to correctly analyze planar mechanisms and some spatial mechanisms. However, with the development of mechanisms, more and more over-constrained spatial parallel mechanisms[5] to which the traditional G–K mobility criterion is not applicable have been found. Recently many improved G–K mobility criteria have been proposed mainly based on screw theory[6–8] and displacement group theory.[9–11]

Structural synthesis of spatial parallel mechanisms has attracted many researchers in the past several decades. Using the constraint synthesis method based on screw theory, Huang and Li,[6] Kong and Gosselin,[8] and Fang and Tsai[12] synthesized lower mobility parallel manipulators systematically. Using the method based on group theory, Herve,[13] Fanghella and Galletti[14] synthesized some spatial parallel mechanisms satisfying algebra structures of displacement group. Further, refs. [15] and [16] synthesized some parallel mechanisms satisfying algebra structures of differential manifold. Gao *et al.*[17,18] proposed a concept of G_f set to synthesis parallel mechanisms. Gogu[19] developed a structural synthesis method based on linear transformation. Cheng *et al.*[20] analyzed the kinematic calibrations of a hip joint parallel simulator. In addition, some spatial metamorphic mechanisms have been synthesized.[21–23]

However, spatial mechanisms with coupling chains are concerned in relatively little literature, comparing with spatial parallel mechanisms. Dai *et al.*[24] analyzed the mobility of a complex structure ball based on constraint screws with some logical derivations. Zoppi *et al.*[25] analyzed the mobility and velocity of two spatial mechanisms with coupling chains by complementing additional constraints from logical derivations. Zeng and Fang[26,27] synthesized some spatial mechanisms with coupling chains based on displacement group theory and logical proposition. Up to now, mobility analysis and structural synthesis of this kind of spatial mechanisms have not been studied systematically without logical proposition.

This paper aims to develop a systematic method for mobility analysis and structural synthesis of a class of important spatial mechanisms with coupling chains, called *two-layer and two-loop* (TLTL) *mechanisms*, which are basic units for constructing complex multi-layer and multi-loop mechanisms. For example, a typical TLTL mechanism is shown in Fig. 1. The following issues are concerned in the paper, (a) analyzing mobility of TLTL mechanisms; (b) analyzing mobility of the end platforms (EPs) and their mobility properties; (c) synthesizing TLTL mechanisms.

The paper is organized as follows. In Section 2, some basic concepts are recalled briefly. In Section 3, an approach to the establishment of the motion screw equation of TLTL mechanisms is derived. In Section 4, detection of rigid substructures is studied. In Section 5, a method for mobility analysis of TLTL mechanisms is presented. In Section 6, the principle of structural synthesis is proposed and some novel TLTL mechanisms are synthesized.

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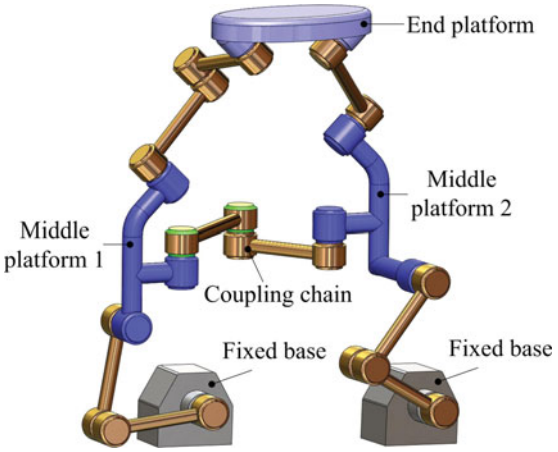


Fig. 1. A two-layer and two-loop (TLTL) mechanism.

2. Basic Concepts

2.1. Screw theory

In screw theory,[7,28] a unit screw $\$$ is defined as

$$\$ = \begin{pmatrix} \boldsymbol{S} \\ \boldsymbol{S}_0 \end{pmatrix} = \begin{cases} \begin{pmatrix} \boldsymbol{S} \\ \boldsymbol{r} \times \boldsymbol{S} + p\boldsymbol{S} \end{pmatrix} \text{if } p \text{ is finite} \\ \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{S} \end{pmatrix} \text{if } p \text{ is infinite} \end{cases}, \tag{1}$$

where $\boldsymbol{S}$ is a unit vector specifying the direction of the screw axis, $\boldsymbol{r}$ is the position vector of any point on the screw axis with respect to a reference frame, and p is called the pitch. $\$$ has six components, called the Plücker coordinates. A unit screw associated with a prismatic pair or a constraint couple can be denoted as $(\boldsymbol{0}, \boldsymbol{S})^T$. A unit screw associated with a revolute pair or a constraint force is denoted as $(\boldsymbol{S}, \boldsymbol{r} \times \boldsymbol{S})^T$.

Reciprocal product of two screws $\$$ and $\$^r$ is

$$\$ \circ \$^r = \boldsymbol{S} \bullet \boldsymbol{S}_0^r + \boldsymbol{S}_0 \bullet \boldsymbol{S}^r, \tag{2}$$

where "$\circ$" denotes reciprocal product. If $\$ \circ \$^r$ is equal to zero, $\$$ denotes a twist (motion screw), and $\$^r$ denotes a wrench (constraint screw).

For a closed-loop mechanism, a motion screw equation can be established for mobility analysis. For example, the motion screw equation of a single loop mechanism is[7,29]

$$\omega_{11}\$_{11} + \cdots + v_{1i}\$_{1i} + \cdots + \omega_{1n}\$_{1n} = \omega_{21}\$_{21} + \cdots + v_{2i}\$_{2i} + \cdots + \omega_{2k}\$_{2k}, \tag{3}$$

where $\$_{1i}$ denotes the unit motion screw associated with the ith single DOF pair of limb 1, v_{1i} and ω_{1n} denote prismatic joint velocity and revolute joint velocity. Equation (3) can be represented as the system of linear equations

$$\begin{pmatrix} \$_{11} \cdots \$_{1n} & -\$_{21} \cdots -\$_{2k} \end{pmatrix} \begin{pmatrix} \omega_{11} \\ \cdots \\ \omega_{1n} \\ \omega_{21} \\ \cdots \\ \omega_{2k} \end{pmatrix} = \boldsymbol{0}, \tag{4}$$

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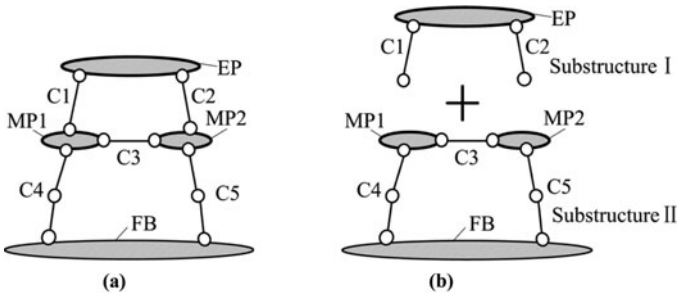


Fig. 2. (a) The topology of TLTL mechanisms, (b) structural decomposition.

where the coefficient matrix consists of unit motion screws of kinematic pairs, and the unknowns are joint velocities. The DOF of a single loop mechanism is equal to the dimension of the solution space of its motion screw equation.

3. Motion Screw Equation

The topology of a TLTL mechanism is shown in Fig. 2(a), with an EP, a fixed base (FB), two middle platforms (MP1 and MP2), and five serial chains (C1, C2, C3, C4 and C5). Each TLTL mechanism can be decomposed into two substructures, shown in Fig. 2(b). Substructure I consists of EP, C1 and C2, and substructure II is a single loop mechanism consisting of MP1, MP2, C3, C4, C5 and FB.

Because both C1 and C2 connect with EP, from Fig. 2(b), the motion screw equation of a TLTL mechanism can be established according to the following relation

$$\$^{\parallel}_{MP1} \cup \$_{C1} = \$^{\parallel}_{MP2} \cup \$_{C2}, \tag{5}$$

where $\cup$ denotes the union operation, $\$_{C1}$ and $\$_{C2}$ denote the motion screw systems of chains C1 and C2, respectively, $\$^{\parallel}_{MP1}$ and $\$^{\parallel}_{MP2}$denotes the motion screw systems of MP1 and MP2 as output links of substructure II, respectively. $\$_{C1}$ and $\$_{C2}$ can be easily determined according to the structures of chains C1 and C2, respectively

$$\begin{cases} \$_{C1} = \omega^1_{c1}\$^1_{c1} + \cdots + \omega^n_{c1}\$^n_{c1} \\ \$_{C2} = \omega^1_{c2}\$^1_{c2} + \cdots + \omega^k_{c2}\$^k_{c2}. \end{cases} \tag{6}$$

Without loss of generality, assuming that MP1 outputs u rotational DOFs and h translational DOFs, and MP2 outputs m rotational DOFs and g translational DOFs, $\$^{\parallel}_{MP1}$ and $\$^{\parallel}_{MP2}$ can be obtained according those mobility properties

$$\begin{cases} \$^{\parallel}_{MP1} = \omega^1_{mp1}\$^1_{mp1} + \cdots + \omega^u_{mp1}\$^u_{mp1} + v^{u+1}_{mp1}\$^{u+1}_{mp1} + \cdots + v^{u+h}_{mp1}\$^{u+h}_{mp1} \\ \$^{\parallel}_{MP2} = \omega^1_{mp2}\$^1_{mp2} + \cdots + \omega^m_{mp2}\$^m_{mp2} + v^{m+1}_{mp1}\$^{m+1}_{mp1} + \cdots + v^{m+g}_{mp1}\$^{m+g}_{mp1}. \end{cases} \tag{7}$$

3.1. Dependence of motion screw systems of the middle platforms

The number of independent parameters of substructure II is equal to its DOF. However, generally, the sum of the DOFs of MP1 and MP2 as the output links of substructure II is greater than the DOF of substructure II. As a result, only some of the DOFs of MP1 and MP2 are independent, called *independent DOFs*, and the others are dependent, called *dependent DOFs*.

For example, for substructure II shown in Fig. 3, its DOF is 2. MP1 outputs one rotational DOF R^1_{mp1} and one translational DOF T^2_{mp1}. MP2 also outputs one rotational DOF R^1_{mp2} and one translational DOF T^2_{mp2}. R^1_{mp1} and T^2_{mp1} can be chosen as independent DOFs, then R^1_{mp2} and T^2_{mp2} are dependent ones.

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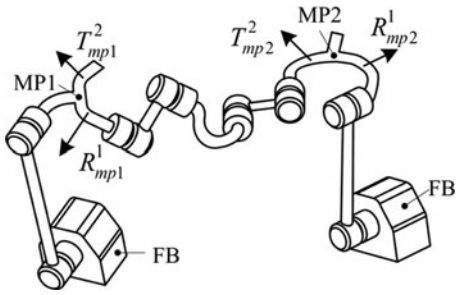


Fig. 3. DOFs of the middle platforms.

For motion screw systems of MP1 and MP2, the joint velocities corresponding to independent DOFs are regarded as independent parameters, and the remaining joint velocities are regarded as dependent parameters.

Without loss of generality, assume that joint velocities $\omega^1_{mp1},\ldots,\omega^i_{mp1}$ and $\omega^1_{mp2},\ldots,\omega^j_{mp2}$ are independent parameters in Eq. (7). Then, each dependent joint velocity can be represented as the function of the independent ones, such as $\omega^{i+1}_{mp1}=f^{i+1}_{mp1}(\omega^1_{mp1},\cdots,\omega^i_{mp1},\omega^1_{mp2},\cdots,\omega^j_{mp2})$ and $\omega^{j+1}_{mp2}=f^{j+1}_{mp2}(\omega^1_{mp1},\cdots,\omega^i_{mp1},\omega^1_{mp2},\cdots,\omega^j_{mp2})$. Then, Eq. (7) will be changed to

$$\begin{cases} \$^{\parallel}_{MP1}=\omega^1_{mp1}\$^1_{mp1}+\cdots+\omega^i_{mp1}\$^i_{mp1}+f^{i+1}_{mp1}(\omega^1_{mp1},\cdots,\omega^i_{mp1},\omega^1_{mp2}\cdots,\omega^j_{mp2})\$^{i+1}_{mp1} \\ \quad +\cdots+f^{u+h}_{mp1}(\omega^1_{mp1},\cdots,\omega^i_{mp1},\omega^1_{mp2}\cdots,\omega^j_{mp2})\$^{u+h}_{mp1} \\ \$^{\parallel}_{MP2}=\omega^1_{mp2}\$^1_{mp2}+\cdots+\omega^j_{mp2}\$^j_{mp2}+f^{j+1}_{mp2}(\omega^1_{mp1},\cdots,\omega^i_{mp1},\omega^1_{mp2}\cdots,\omega^j_{mp2})\$^{j+1}_{mp2} \\ \quad +\cdots+f^{m+g}_{mp2}(\omega^1_{mp1},\cdots,\omega^i_{mp1},\omega^1_{mp2}\cdots,\omega^j_{mp2})\$^{m+g}_{mp2}. \end{cases} \tag{8}$$

3.2. Motion screw equation

Substituting Eqs. (6) and (8) into (5), the motion screw equation of a TLTL mechanism can be yielded

$$\begin{aligned} &\left(\omega^1_{mp1}\$^1_{mp1}+\cdots+f^{u+h}_{mp1}\$^{u+h}_{mp1}\right)+\left(\omega^1_{c1}\$^1_{c1}+\cdots+\omega^n_{c1}\$^n_{c1}\right) \\ &\quad=\left(\omega^1_{mp2}\$^1_{mp2}+\cdots+f^{m+g}_{mp2}\$^{m+g}_{mp2}\right)+\left(\omega^1_{c2}\$^1_{c2}+\cdots+\omega^k_{c2}\$^k_{c2}\right). \end{aligned} \tag{9}$$

Equation (9) can be changed to the system of linear equations

$$\left(\$^1_{mp1}\cdots\$^{u+h}_{mp1}\cdots-\$^k_{c2}\right)\begin{pmatrix}\omega^1_{mp1}\\ \cdots\\ f^{u+h}_{mp1}\\ \cdots\\ \omega^k_{c2}\end{pmatrix}=\mathbf{0}. \tag{10}$$

Comparing with the motion screw equation (i.e., Eq. (4)) of a single loop mechanism, the motion screw equation (i.e., Eq. (10)) of a TLTL mechanism contains some dependent unknowns (joint velocities) which should be identified before solving the equation. From view of linear algebra, generally, those unknowns and corresponding unit motion screws of coefficient matrix have no effect on the dimension of the solution space of the equation.

4. Detection of Rigid Substructures

A closed loop mechanism may contain some rigid substructures, such as rigid loops and rigid serial chains. Generally, those substructures are unreasonable and should be detected to exclude them. Here, a method is proposed to deal with detection of rigid substructures of a TLTL mechanism as follows.

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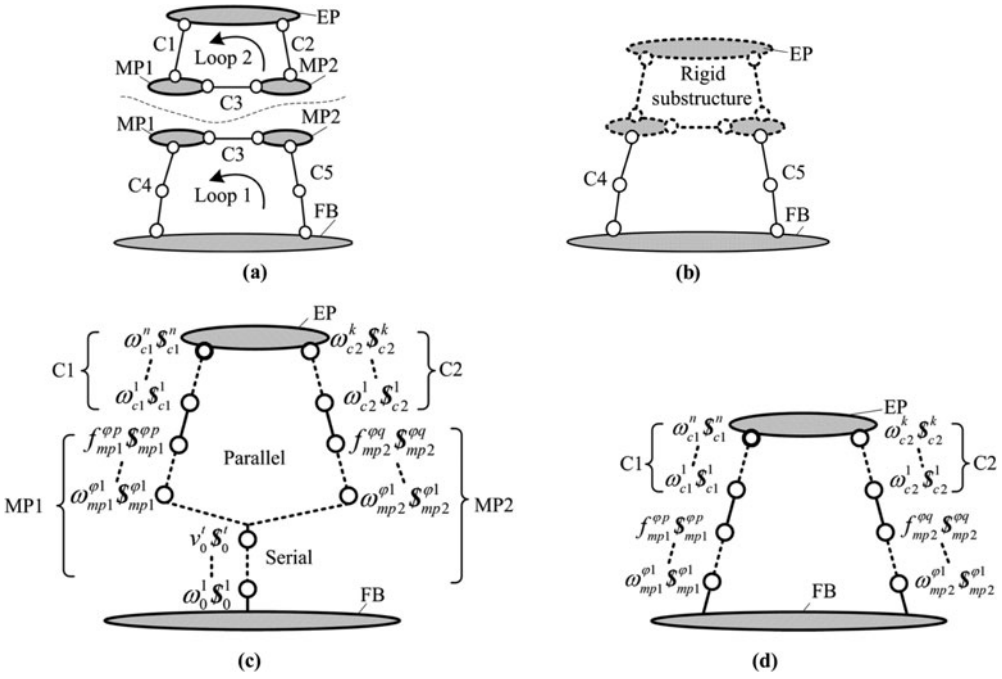


Fig. 4. (a) Two basic loops, (b) initial parallel mechanism, (c) the equivalent mechanism, (d) the equivalent sub parallel mechanism.

Step 1: Detect whether two basic loops shown in Fig. 4(a) are rigid substructures base on the motion screw equation of single loop mechanisms. If none is rigid, go to next step. If only one loop is rigid, the mechanism is a parallel mechanism, whose rigid substructures can be detected by the existing methods based on constraint screws or displacement subgroups. Otherwise, the mechanism is rigid.

Step 2: Analyze the DOF and motion screw system of the EP under assuming that loop 2 is a rigid substructure. When loop 2 is assumed to be rigid, a parallel mechanism, called *initial parallel mechanism*, can be determined, shown in Fig. 4(b), in which EP of the TLTL mechanism can be regarded as the moving platform, and serial chains C4 and C5 can be regarded as limbs. In the parallel mechanism, the DOF and motion screw system of the EP are called *initial DOF* and *initial motion screw system*, respectively. Thus, the *initial motion screw system* $\$^0_{EP}$ satisfies

$$\$^0_{EP} = \$_{C4} \cap \$_{C5}, \tag{11}$$

where $\cap$ denotes the intersecting operation.

Step 3: Establish the relation between $\$^{\|}_{MP1}$, $\$^{\|}_{MP2}$ and $\$^0_{EP}$. According to substructure II, $\$^{\|}_{MP1}$ and $\$^{\|}_{MP2}$ satisfy

$$\begin{cases} \$^{\|}_{MP1} = \$_{C4} \cap (\$_{C3} \cup \$_{C5}) \\ \$^{\|}_{MP2} = \$_{C5} \cap (\$_{C3} \cup \$_{C4}). \end{cases} \tag{12}$$

From Eqs. (11) and (12), $\$^0_{EP}$ is a subset of both $\$^{\|}_{MP1}$ and $\$^{\|}_{MP2}$, namely,

$$\begin{cases} \$^0_{EP} \subseteq \$^{\|}_{MP1} \\ \$^0_{EP} \subseteq \$^{\|}_{MP2}, \end{cases} \tag{13}$$

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where $\subseteq$ denotes the contained relation. $\$^0_{EP}$ can be easily obtained according to the property of initial mobility,

$$\$^0_{EP} = \omega^1_0\$^1_0 + \cdots + v^t_0\$^t_0. \tag{14}$$

Assume that $\$^{\varphi}_{MP1}$ and $\$^{\varphi}_{MP2}$ are the complement sets of $\$^0_{EP}$ under $\$^{\|}_{MP1}$ and $\$^{\|}_{MP2}$, respectively, namely

$$\begin{cases} \$^{\|}_{MP1} = \$^0_{EP} \cup \$^{\varphi}_{MP1} \\ \$^{\|}_{MP2} = \$^0_{EP} \cup \$^{\varphi}_{MP2}. \end{cases} \tag{15}$$

Generally, $\$^{\varphi}_{MP1}$ and $\$^{\varphi}_{MP2}$ can be represented as

$$\begin{cases} \$^{\varphi}_{MP1} = \omega^{\varphi 1}_{mp1}\$^{\varphi 1}_{mp1} + \cdots + v^{\varphi p}_{mp1}\$^{\varphi p}_{mp1} \\ \$^{\varphi}_{MP2} = \omega^{\varphi 1}_{mp2}\$^{\varphi 1}_{mp2} + \cdots + v^{\varphi q}_{mp2}\$^{\varphi q}_{mp2}. \end{cases} \tag{16}$$

Therefore, from Eqs. (14), (15) and (16), $\$^{\|}_{MP1}$ and $\$^{\|}_{MP2}$ can be represented as

$$\begin{cases} \$^{\|}_{MP1} = (\omega^1_0\$^1_0 + \cdots + v^t_0\$^t_0) + \left(\omega^{\varphi 1}_{mp1}\$^{\varphi 1}_{mp1} + \cdots + v^{\varphi p}_{mp1}\$^{\varphi p}_{mp1}\right) \\ \$^{\|}_{MP2} = (\omega^1_0\$^1_0 + \cdots + v^t_0\$^t_0) + \left(\omega^{\varphi 1}_{mp2}\$^{\varphi 1}_{mp2} + \cdots + v^{\varphi q}_{mp2}\$^{\varphi q}_{mp2}\right). \end{cases} \tag{17}$$

Step 4: Determine independent joint velocities in $\$^{\|}_{MP1}$ and $\$^{\|}_{MP2}$. Based on mobility analysis of substructure II, joint velocities $\omega^1_0,\ldots, v^t_0$ are preferred as independent parameters. Assume that the remaining independent parameters are joint velocities $\omega^{\varphi 1}_{mp1},\ldots, \omega^{\varphi i}_{mp1}$ and $\omega^{\varphi 1}_{mp2},\ldots, \omega^{\varphi j}_{mp2}$. Then, each dependent joint velocity is represented as the function of independent ones. As a result, Eq. (17) is changed to

$$\begin{cases} \$^{\|}_{MP1} = \left(\omega^1_0\$^1_0 + \cdots + v^t_0\$^t_0\right) + \left(\omega^{\varphi 1}_{mp1}\$^{\varphi 1}_{mp1} + \cdots + \omega^{\varphi i}_{mp1}\$^{\varphi i}_{mp1}\right) + \left(f^{\varphi(i+1)}_{mp1}\$^{\varphi(i+1)}_{mp1} + \cdots + f^{\varphi p}_{mp1}\$^{\varphi p}_{mp1}\right) \\ \$^{\|}_{MP2} = \left(\omega^1_0\$^1_0 + \cdots + v^t_0\$^t_0\right) + \left(\omega^{\varphi 1}_{mp2}\$^{\varphi 1}_{mp2} + \cdots + \omega^{\varphi j}_{mp2}\$^{\varphi j}_{mp2}\right) + \left(f^{\varphi(j+1)}_{mp2}\$^{\varphi(j+1)}_{mp2} + \cdots + f^{\varphi q}_{mp2}\$^{\varphi q}_{mp2}\right). \end{cases} \tag{18}$$

Step 5: Obtain the kinematically equivalent mechanism of the TLTL mechanism. If the unit motion screws in Eq. (18) are replaced with the kinematically equivalent joints, those joints and substructure I constitute an *equivalent serial-parallel mechanism*, shown in Fig. 4(c), in which the serial part constituted by the equivalent joints of the unit motion screws of the initial screw system in Eq. (14).

Step 6: Establish the motion screw equation of the *equivalent sub parallel mechanism*, which is obtained by deleting the serial part from the equivalent serial-parallel mechanism, shown in Fig. 4(d)

$$\begin{aligned} &\omega^{\varphi 1}_{mp1}\$^{\varphi 1}_{mp1} + \cdots + f^{\varphi p}_{mp1}\$^{\varphi p}_{mp1} + \omega^1_{c1}\$^1_{c1} + \cdots + \omega^n_{c1}\$^n_{c1} \\ &\quad = \omega^{\varphi 1}_{mp2}\$^{\varphi 1}_{mp2} + \cdots + f^{\varphi q}_{mp2}\$^{\varphi q}_{mp2} + \omega^1_{c2}\$^1_{c2} + \cdots + \omega^k_{c2}\$^k_{c2}. \end{aligned} \tag{19}$$

Step 7: Perform elementary row transformations of the coefficient matrix of Eq. (19), and analyze the characteristics of the solution.

If Eq. (19) only has zero solution, chains C1, C2 and C3 are rigid substructures. Further, if there is no initial mobility, chains C4 and C5 are also rigid substructures.

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If both $\omega_{mp1}^{\varphi 1}, \cdots, f_{mp1}^{\varphi p}$ and $\omega_{mp2}^{\varphi 1}, \cdots, f_{mp2}^{\varphi q}$ have only zero solution, chain C3 is rigid substructure. Further, if there is no initial mobility, chains C4 and C5 are also rigid substructures.

If joint velocities $\omega_{c1}^{1}, \cdots, \omega_{c1}^{n}$ have only zero solution, chain C1 is rigid substructure.

If joint velocities $\omega_{c2}^{1}, \cdots, \omega_{c2}^{k}$ have only zero solution, chain C2 is rigid substructure.

5. Mobility Analysis

5.1. Calculation of mobility

Since a TLTL mechanism is kinematically equivalent to a serial-parallel mechanism, as shown in Fig. 4(d), its mobility can be calculated as

$$\begin{cases} M = M_0 + M_{\text{para}} + M_{\text{idle}} \\ M_e = M_0 + M_{\text{para}}, \end{cases} \tag{20}$$

where M denotes the mobility of a mechanism, M_e denotes the mobility of the EP, M_0 denotes the mobility of serial part, i.e., initial mobility, M_{para} denotes the mobility of the equivalent sub parallel mechanism, and M_{idle} denotes idle mobility (generally $M_{\text{idle}} = 0$).

The detailed steps of mobility analysis with rigid substructure detection are performed as follows.

Step 1: Identify whether there exists one rigid basic loop. If so, use the existing methods to analyze mobility. Otherwise, go to the next step.
Step 2: Analyze the initial mobility and the initial motion screw system of the EP.
Step 3: Decompose the kinematic structure into two substructures.
Step 4: Analyze motion screw systems of the two middle platforms in substructure II and establish the relation between the two screw systems and the initial motion screw system.
Step 5: Determine independent joint velocities of the motion screw systems of the two middle platforms. Here, the joint velocities of the initial motion screw system are preferred as the independent ones.
Step 6: Determine the equivalent sub parallel mechanism and obtain its motion screw equation.
Step 7: Perform elementary row transformations of the coefficient matrix to obtain mobility and identify rigid substructures.

Example 1: For the TLTL mechanism shown in Fig. 5(a), revolute pairs in each serial chain are parallel to each other, revolute pairs between different serial chains are not parallel to each other, and prismatic pairs are not parallel to each other.

It is easily found that none of loops 1 and 2 is rigid substructure. The initial parallel mechanism is shown in Fig. 5(b). Based on the existing constraint screw method[7] or displacement group method[11] for mobility analysis of spatial parallel mechanisms, it is easily obtained that the initial mobility is one translation, T_0^1, along the intersecting line of the normal planes of R_{c4}^1 and R_{c5}^1. So, $M_0 = 1$, the initial motion screw system is represented as

$$\$_{EP}^{0} = v_0^1 \$_0^1. \tag{21}$$

The mechanism can be decomposed into two substructures shown in Fig. 5(c). In substructure II shown in Fig. 5(d), MP1 has two translational DOFs, T_{mp1}^1 and T_{mp1}^2, parallel to the normal plane of R_{c4}^1. Similarly, MP2 has two translational DOFs, T_{mp2}^1 and T_{mp2}^2, parallel to the normal plane of R_{c5}^1.

So, the motion screw systems of the two middle platforms in substructure II are represented as

$$\begin{cases} \$_{MP1}^{\parallel} = v_0^1 \$_0^1 + v_{mp1}^{\phi 1} \$_{mp1}^{\phi 1} \\ \$_{MP2}^{\parallel} = v_0^1 \$_0^1 + v_{mp2}^{\varphi 1} \$_{mp2}^{\varphi 1}. \end{cases} \tag{22}$$

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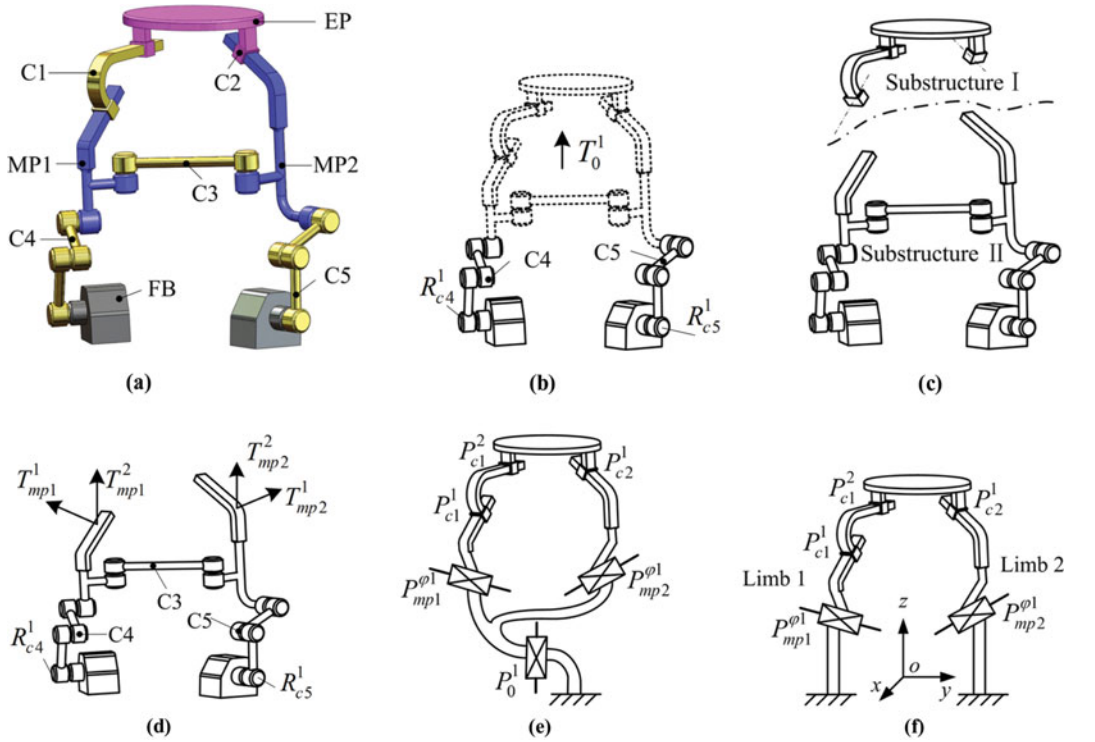


Fig. 5. (a) A TLTL mechanism, (b) structural decomposition, (c) initial parallel mechanism, (d) substructure II, (e) the equivalent mechanism, (f) the equivalent sub parallel mechanism.

The DOF of substructure II is 2. Joint velocities v_0^1 and $v_{mp1}^{\varphi 1}$ are chosen as independent parameters. As a result, dependent parameter $v_{mp2}^{\varphi 1}$ can be represented as

$$v_{mp2}^{\varphi 1} = f_{mp2}^{\varphi 1}\left(v_0^1, v_{mp1}^{\varphi 1}\right). \tag{23}$$

The equivalent serial-parallel mechanism can be obtained, shown in Fig. 5(e), in which pairs P_0^1, $P_{mp1}^{\varphi 1}$ and $P_{mp2}^{\varphi 1}$ are the equivalent joints of unit motion screws $\$_0^1$, $\$_{mp1}^{\varphi 1}$ and $\$_{mp2}^{\varphi 1}$, respectively. The corresponding equivalent sub parallel mechanism is shown in Fig. 5(f) and its motion screw equation is

$$v_{c1}^1\$_{c1}^1 + v_{c1}^2\$_{c1}^2 - v_{c2}^1\$_{c2}^1 + v_{mp1}^{\varphi 1}\$_{mp1}^{\varphi 1} - v_{mp2}^{\varphi 1}\$_{mp2}^{\varphi 1} = 0, \tag{24}$$

where $v_{mp2}^{\varphi 1}$ can be denoted as $v_{mp2}^{\varphi 1} = f_{mp2}^{\varphi 1}(v_{mp1}^{\varphi 1})$ since v_0^1 has no effect on Eq. (24).

Equation (24) can be changed to the system of linear equations

$$\begin{pmatrix} \$_{c1}^1 & \$_{c1}^2 & \$_{c2}^1 & \$_{mp1}^{\varphi 1} & \$_{mp2}^{\varphi 1} \end{pmatrix} \begin{pmatrix} v_{c1}^1 \\ v_{c1}^2 \\ v_{c2}^1 \\ v_{mp1}^{\varphi 1} \\ v_{mp2}^{\varphi 1} \end{pmatrix} = \mathbf{0}. \tag{25}$$

A reference fame is set up, shown in Fig. 5(f), with z-axis perpendicular to the FB plane, x-axis and y-axis on FB plane.

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The direction vector $\boldsymbol{S}_{c1}^{1}$ of pair P_{c1}^{1} can be denoted as $(a_{c1}^{1}, b_{c1}^{1}, c_{c1}^{1})^{T}$, where a_{c1}^{1}, b_{c1}^{1}, c_{c1}^{1} are the x, y, and z components, respectively. So, the unit motion screw $\$_{c1}^{1}$ of pair P_{c1}^{1} is $(\mathbf{0},\boldsymbol{S}_{c1}^{1})^{T} = (0, 0, 0, a_{c1}^{1}, b_{c1}^{1}, c_{c1}^{1})^{T}$, according to the above-mentioned definition of a screw.

Likewise, other unit motion screws $\$_{c1}^{2}$, $\$_{c2}^{1}$, $\$_{mp1}^{\varphi 1}$ and $\$_{mp2}^{\varphi 1}$ in Eq. (25) can be obtained. Then, Eq. (25) is represented as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{c1}^{1} & a_{c1}^{2} & -a_{c2}^{1} & a_{mp1}^{\varphi 1} & -a_{mp2}^{\varphi 1} \\ b_{c1}^{1} & b_{c1}^{2} & -b_{c2}^{1} & b_{mp1}^{\varphi 1} & -b_{mp2}^{\varphi 1} \\ c_{c1}^{1} & c_{c1}^{2} & -c_{c2}^{1} & c_{mp1}^{\varphi 1} & -c_{mp2}^{\varphi 1} \end{pmatrix} \begin{pmatrix} v_{c1}^{1} \\ v_{c1}^{2} \\ v_{c2}^{1} \\ v_{mp1}^{\varphi 1} \\ v_{mp2}^{\varphi 1} \end{pmatrix} = \mathbf{0}. \tag{26}$$

Performing elementary row transformations of the coefficient matrix of Eq. (26), the type of simplest row of the matrix is obtained

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & A_1 & A_2 \\ 0 & 1 & 0 & B_1 & B_2 \\ 0 & 0 & 1 & C_1 & C_2 \end{pmatrix} \begin{pmatrix} v_{c1}^{1} \\ v_{c1}^{2} \\ v_{c2}^{1} \\ v_{mp1}^{\varphi 1} \\ v_{mp2}^{\varphi 1} \end{pmatrix} = \mathbf{0}, \tag{27}$$

where A_i, B_i and C_i denote non-zero parameters. Multiplying the sixth row with the unknowns in Eq. (27), yields

$$v_{c2}^{1} + C_1 v_{mp1}^{\varphi 1} + C_2 v_{mp2}^{\varphi 1} = 0. \tag{28}$$

Note that $v_{mp2}^{\varphi 1} = f_{mp2}^{\varphi 1}(v_{mp1}^{\varphi 1})$, Thus, from Eq. (28), v_{c2}^{1} can be represented as

$$v_{c2}^{1} = F_{c2}^{1}\left(v_{mp1}^{\varphi 1}\right). \tag{29}$$

Multiplying the fifth row with the unknowns in Eq. (27), we obtain

$$v_{c1}^{2} + B_1 v_{mp1}^{\varphi 1} + B_2 v_{mp2}^{\varphi 1} = 0. \tag{30}$$

Thus, v_{c1}^{2} can be represented as

$$v_{c1}^{2} = F_{c1}^{2}\left(v_{mp1}^{\varphi 1}\right). \tag{31}$$

Similarly, $v_{c1}^{1} = F_{c1}^{1}(v_{mp1}^{\varphi 1})$.

Finally, the solution of Eq. (26) is represented as

$$\begin{cases} v_{mp1}^{\varphi 1} = v_{mp1}^{\varphi 1}; v_{mp2}^{\varphi 1} = f_{mp2}^{\varphi 1}\left(v_{mp1}^{\varphi 1}\right); v_{c2}^{1} = F_{c2}^{1}\left(v_{mp1}^{\varphi 1}\right); \\ v_{c1}^{2} = F_{c1}^{2}\left(v_{mp1}^{\varphi 1}\right); v_{c1}^{1} = F_{c1}^{1}\left(v_{mp1}^{\varphi 1}\right). \end{cases} \tag{32}$$

From Eq. (32), the dimension of the solution space of Eq. (26) is 1, so the mobility of the equivalent sub parallel mechanism is $M_{\text{para}} = 1$. There is no idle mobility, i.e., $M_{\text{idle}} = 0$. Finally, the mobility of the mechanism is

$$M = M_0 + M_{\text{para}} + M_{\text{idle}} = 1 + 1 + 0 = 2. \tag{33}$$

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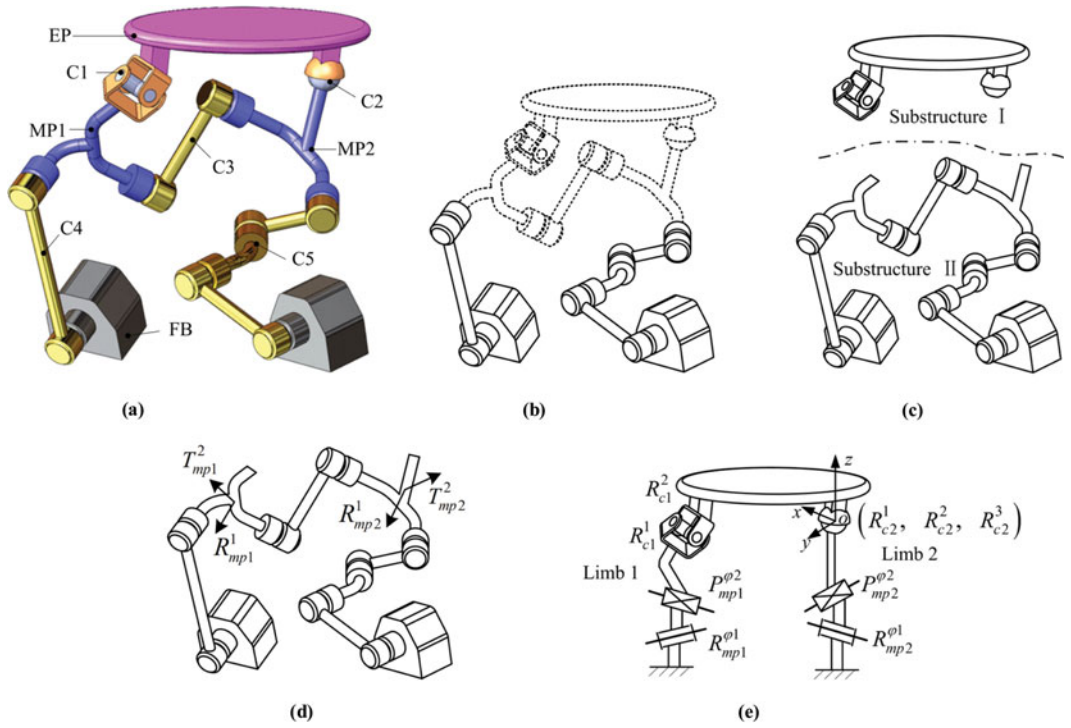


Fig. 6. (a) Another TLTL mechanism, (b) initial parallel mechanism, (c) structural decomposition, (d) substructure II, (e) the equivalent sub parallel mechanism.

The mobility of the EP is $M_e = 2$. It is pointed out that the type of the coefficient matrix in Eq. (26) is not changed in continual motion, so the mobility is full-cycle. Since each joint velocity has non-zero solution, there is no rigid substructure.

Example 2: For the mechanism as shown in Fig. 6(a), serial chain C1 includes a universal pair. C2 includes a spherical pair. Both C3 and C4 include two parallel revolute pairs. C5 includes four revolute pairs, the former two of which are parallel to each other and the latter two of which are parallel to each other. Revolute joint axes between different chains are not parallel to each other.

It is easily found that none of loops 1 and 2 is rigid substructure. The initial parallel mechanism is shown in Fig. 6(b) and there is no initial mobility.

The mechanism can be decomposed into two substructures, shown in Fig. 6(c). In substructure II shown in Fig. 6(d), MP1 has one rotational mobility R^1_{mp1} and one translational mobility T^2_{mp1}. MP2 has also one rotational mobility R^1_{mp2} and one translational mobility T^2_{mp2}.

$$\begin{cases} \$^{\parallel}_{MP1} = \omega^{\varphi 1}_{mp1}\$^{\varphi 1}_{mp1} + v^{\varphi 2}_{mp1}\$^{\varphi 2}_{mp1} \\ \$^{\parallel}_{MP2} = \omega^{\varphi 1}_{mp2}\$^{\varphi 1}_{mp2} + v^{\varphi 2}_{mp2}\$^{\varphi 2}_{mp2}. \end{cases} \tag{34}$$

The mobility of substructure II is 2. Joint velocities $\omega^{\varphi 1}_{mp1}$ and $v^{\varphi 2}_{mp1}$ are chosen as independent parameters. Dependent parameters $\omega^{\varphi 1}_{mp2}$ and $v^{\varphi 2}_{mp2}$ can be represented as

$$\begin{cases} \omega^{\varphi 1}_{mp2} = f^{\varphi 1}_{mp2}\left(\omega^{\varphi 1}_{mp1}, \quad v^{\varphi 2}_{mp1}\right) \\ v^{\varphi 2}_{mp2} = f^{\varphi 2}_{mp2}\left(\omega^{\varphi 1}_{mp1}, \quad v^{\varphi 2}_{mp1}\right). \end{cases} \tag{35}$$

Because there is no initial mobility, the equivalent mechanism of the TLTL mechanism is just the sub parallel mechanism, shown in Fig. 6(e), in which $R^{\varphi 1}_{mp1}$, $P^{\varphi 2}_{mp1}$, $R^{\varphi 1}_{mp2}$ and $P^{\varphi 2}_{mp2}$ are the equivalent

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joints of unit motion screws, $\$_{mp1}^{\varphi 1}$, $\$_{mp1}^{\varphi 2}$, $\$_{mp2}^{\varphi 1}$ and $\$_{mp2}^{\varphi 2}$, respectively. The motion screw equation of the mechanism is

$$\omega_{c1}^{1}\$_{c1}^{1}+\omega_{c2}^{1}\$_{c2}^{1}-\omega_{c2}^{1}\$_{c2}^{1}-\omega_{c2}^{2}\$_{c2}^{2}-\omega_{c2}^{3}\$_{c2}^{3}-\omega_{mp2}^{\varphi 1}\$_{mp2}^{\varphi 1}-v_{mp2}^{\varphi 2}\$_{mp2}^{\varphi 2}+\omega_{mp1}^{\varphi 1}\$_{mp1}^{\varphi 1}+v_{mp1}^{\varphi 2}\$_{mp1}^{\varphi 2}=0. \tag{36}$$

A reference fame is established, shown in Fig. 6(e), with the origin coincides with the center of the spherical pair, z-axis perpendicular to the FB plane and x-axis parallel to R_{c1}^{2}. The corresponding system of linear equations is

$$\begin{pmatrix} 0 & 1 & -a_{c2}^{1} & -a_{c2}^{2} & -a_{c2}^{3} & -a_{mp2}^{\varphi 1} & 0 & a_{mp1}^{\varphi 1} & 0 \\ b_{c1}^{1} & 0 & -a_{c2}^{1} & -b_{c2}^{2} & -b_{c2}^{3} & -b_{mp2}^{\varphi 1} & 0 & b_{mp1}^{\varphi 1} & 0 \\ c_{c1}^{1} & 0 & -c_{c2}^{1} & -c_{c2}^{2} & -c_{c2}^{3} & -c_{mp2}^{\varphi 1} & 0 & c_{mp1}^{\varphi 1} & 0 \\ d_{c1}^{1} & d_{c1}^{2} & 0 & 0 & 0 & -d_{mp2}^{\varphi 1} & -a_{mp2}^{\varphi 2} & d_{mp1}^{\varphi 1} & a_{mp1}^{\varphi 2} \\ e_{c1}^{1} & e_{c1}^{2} & 0 & 0 & 0 & -e_{mp2}^{\varphi 1} & -b_{mp2}^{\varphi 2} & e_{mp1}^{\varphi 1} & b_{mp1}^{\varphi 2} \\ f_{c1}^{1} & f_{c1}^{2} & 0 & 0 & 0 & -f_{mp2}^{\varphi 1} & -c_{mp2}^{\varphi 2} & f_{mp1}^{\varphi 1} & c_{mp1}^{\varphi 2} \end{pmatrix} \begin{pmatrix} \omega_{c1}^{1} \\ \omega_{c1}^{2} \\ \omega_{c2}^{1} \\ \omega_{c2}^{2} \\ \omega_{c2}^{3} \\ \omega_{mp2}^{\varphi 1} \\ v_{mp2}^{\varphi 2} \\ \omega_{mp1}^{\varphi 1} \\ v_{mp1}^{\varphi 2} \end{pmatrix} = \mathbf{0}. \tag{37}$$

Using elementary row transformation of the coefficient matrix, yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & A_3 & A_4 & A_5 \\ 0 & 1 & 0 & 0 & 0 & 0 & B_3 & B_4 & B_5 \\ 0 & 0 & 1 & 0 & 0 & 0 & C_3 & C_4 & C_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & D_3 & D_4 & D_5 \\ 0 & 0 & 0 & 0 & 1 & 0 & E_3 & E_4 & E_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & G_3 & G_4 & G_5 \end{pmatrix} \begin{pmatrix} \omega_{c1}^{1} \\ \omega_{c1}^{2} \\ \omega_{c2}^{1} \\ \omega_{c2}^{2} \\ \omega_{c2}^{3} \\ \omega_{mp2}^{\varphi 1} \\ v_{mp2}^{\varphi 2} \\ \omega_{mp1}^{\varphi 1} \\ v_{mp1}^{\varphi 2} \end{pmatrix} = \mathbf{0}. \tag{38}$$

Combining Eq. (35) with (38), and multiplying the sixth row with the unknown vector in Eq. (38), a new equation can be obtained

$$f_{mp2}^{\varphi 1}\left(\omega_{mp1}^{\varphi 1},\quad v_{mp1}^{\varphi 2}\right)+G_3 f_{mp2}^{\varphi 2}\left(\omega_{mp1}^{\varphi 1},\quad v_{mp1}^{\varphi 2}\right)+G_4\omega_{mp1}^{\varphi 1}+G_5 v_{mp1}^{\varphi 2}=0. \tag{39}$$

From Eq. (39), we can obtain

$$\omega_{mp1}^{\varphi 1}=F_{mp1}^{\varphi 1}\left(v_{mp1}^{\varphi 2}\right). \tag{40}$$

Similarly, other joint velocities can be determined. Finally, the solution of Eq. (37) is represented as

$$\begin{cases} v_{mp1}^{\varphi 2}=v_{mp1}^{\varphi 2};\omega_{mp1}^{\varphi 1}=F_{mp1}^{\varphi 1}\left(v_{mp1}^{\varphi 2}\right);\omega_{mp2}^{\varphi 1}=F_{mp2}^{\varphi 1}\left(v_{mp1}^{\varphi 2}\right);v_{mp2}^{\varphi 2}=F_{mp2}^{\varphi 2}\left(v_{mp1}^{\varphi 2}\right); \\ \omega_{c1}^{1}=F_{c1}^{1}\left(v_{mp1}^{\varphi 2}\right);\omega_{c1}^{2}=F_{c1}^{2}\left(v_{mp1}^{\varphi 2}\right);\omega_{c2}^{1}=F_{c2}^{1}\left(v_{mp1}^{\varphi 2}\right);\omega_{c2}^{2}=F_{c2}^{2}\left(v_{mp1}^{\varphi 2}\right);\omega_{c2}^{3}=F_{c2}^{3}\left(v_{mp1}^{\varphi 2}\right). \end{cases} \tag{41}$$

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From Eq. (41), the dimension of the solution space is 1, so the mobility of the equivalent sub parallel mechanism is 1. The idle mobility is 0. Hence, the mobility of the TLTL mechanism is

$$M = M_0 + M_{\text{para}} + M_{\text{idle}} = 0 + 1 + 0 = 1. \tag{42}$$

The mobility of the EP is also $M_e = 1$. It is pointed out that the type of the coefficient matrix in Eq. (37) is not changed in continual motion, so the mobility is full-cycle. Since each joint velocity has non-zero solution, there is no rigid substructure.

5.2. Mobility property of the end platform

Because a TLTL mechanism is kinematically equivalent to a serial-parallel mechanism, the mobility property of the serial part of the equivalent mechanism is obtained directly, and the key problem is to determine the mobility property of the moving platform of the equivalent sub parallel mechanism. In fact, that is to determine how many independent linear vectors (rotations), couples (translations) and general screws (screw motions) in a set of bases spanning the motion screw system of the moving platform.

In screw theory,[7,28] the combinations of rotations and/or translations include the following cases.

Case 1: The combinations of two intersecting rotations can only generate new rotations about the intersecting point.
Case 2: The combinations of two translations can only generate new translations.
Case 3: The combinations of two parallel rotations, or of a rotation and a translation perpendicular to the rotation, can only generate new rotations parallel to the given rotation.
Case 4: The combinations of three motions of a 3-DOF planar sub-chain can only generate new rotations parallel to the rotation of the sub-chain. Further, any rotation perpendicular to the motion plane of the sub-chain can be obtained by the linear combination of the three motions.

Based on the above cases, the mobility property of the moving platform of the equivalent sub parallel mechanism can be identified from kinematic structure as follows.

Step 1: Identify whether one limb only consists of revolute pairs intersecting at a common point. If so, the moving platform only has rotational DOFs about the point. That can be verified by the above case 1. Otherwise, go to the next step.
Step 2: Identify whether one limb only consists of prismatic pairs. If so, the moving platform only has translational DOFs. That can be verified by the above case 2. Otherwise, go to the next step.
Step 3: Identify whether the number of single DOF pairs of a limb motion screw system is equal to the mobility of the moving platform (Here, a composite pair is regarded as the combination of single DOF pairs). If so, the moving platform outputs rotational and/or translational DOFs provided by the joints of the limb. Otherwise, go to the next step.
Step 4: Analyze first the translational DOFs according to the *translational motion screw equation*, which is obtained by only considering the translations provided by the limbs. Then, the properties of other DOFs of the moving platform are determined as follows:
If some revolute pairs in all limbs intersect at a point and the rest revolute pairs in each limb constitute a 3-DOF planar or spherical sub-chain, the moving platform outputs rotations beside translations, if any. That can be verified by the above cases 1 and 4. Otherwise, go to the next step.
If translational DOFs of the moving platform are equal to the number of the prismatic pairs of a limb and all the revolute pairs of the limb intersect at a point, the moving platform outputs rotations beside translations. Otherwise, go to the next step.
In a limb, if all revolute pairs are parallel to each other and all prismatic pairs are perpendicular to the revolute pairs, the moving platform outputs instantaneous rotations

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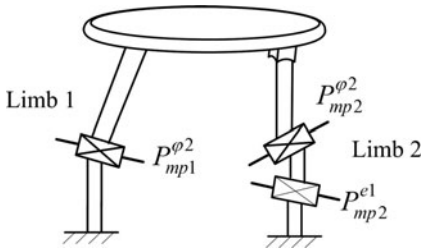


Fig. 7. A new mechanism obtained under only considering translations.

beside translations, if any. That can be verified by the above cases 3. Otherwise, go to the next step.
The moving platform outputs general screw motions beside translations, if any.

Example 3: For the equivalent sub parallel mechanism shown in Fig. 6(e), the mobility of the moving platform is equal to 1. It doesn't satisfy the above steps 1, 2 and 3.

Limb 1 contains one prismatic pair $P_{mp1}^{\varphi 2}$, and limb 2 contains one prismatic pair $P_{mp2}^{\varphi 2}$. Further, revolute pair R_{mp2}^{1} of limb 2 is kinematically equivalent to a prismatic pair P_{mp2}^{e1}, because the limb contains a spherical pair.

So, when only considering the translations provided by the limbs, the equivalent sub parallel mechanism can be changed into the mechanism shown in Fig. 7. The corresponding translational motion screw equation is

$$v_{mp1}^{\varphi 1}\$_{mp1}^{\varphi 1} - v_{mp2}^{\varphi 1}\$_{mp2}^{\varphi 1} - v_{mp2}^{e1}\$_{mp2}^{e1} = 0. \tag{43}$$

Equation (43) has only zero solution. So, the moving platform has no translations. Further, other conditions of step 4 cannot be satisfied. As a result, the moving platform outputs one general screw motion.

6. Structural Synthesis of TLTL Mechanisms

Reverse the process of mobility analysis and a structural synthesis method for TLTL mechanisms is obtained.

Step 1: Specify desired mobility property of the end platform.
Step 2: Synthesize a single loop parallel mechanism which can output the desired mobility property. The mechanism is called *the first layer parallel mechanism*.
Step 3: Choose the motions of some joints of the first layer parallel mechanism as *the first layer separated mobility property*. By removing those joints and the FB from the first layer parallel mechanism, the remaining structure serves as substructure I.
Step 4: Synthesize another single loop parallel mechanism which can output the first layer separated mobility property. The mechanism is called *the second layer parallel mechanism*.
Step 5: Divide one limb of the second layer parallel mechanism into two serial sub-chains, and the remaining limb serves as a serial chain. Consequently, substructure II is determined.
Step 6: A TLTL mechanism can be obtained by connecting substructure I to substructure II. Further, verify whether the synthesized mechanism can generate the desired mobility property according to the proposed principle of mobility analysis. If not, the mechanism is invalid. Repeat the procedures to find valid ones.

For the above steps 2 and 4, comparing with the constraint synthesis method based on screw theory,[6,7] the method based on displacement group theory[13,15] is more convenient for synthesizing single loop mechanisms. Some typical subgroups are listed in Table I.

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Table I. Some typical subgroups..

Subgroup	Generator	Annotation
$\{\boldsymbol{R}(N,\mathbf{u})\}$	Revolute pair	Point *N,* unit vector $\boldsymbol{u}$
$\{\boldsymbol{C}(N,v)\}$	Cylindrical pair	Point *N,* unit vector $\boldsymbol{v}$
$\{\boldsymbol{G}(u)\}$	Planar pair	The normal $\boldsymbol{u}$
$\{G_2(u)\}$	2-DOF planar pair	The normal $\boldsymbol{u}$
$\{P(v)\}$	Prismatic pair	unit vector $\boldsymbol{v}$
$\{T\}$	Three prismatic pairs	
$\{T_2(v)\}$	Two prismatic pairs	
$\{S(N)\}$	Spherical pair	Point *N*
$\{S_2(N)\}$	2-DOF spherical pair	Point *N*
$\{D\}$	Free rigid body	

Here, the structural synthesis of TLTL mechanisms with two kinds of typical mobility properties is used to illustrate the proposed method.

6.1. Mobility property: Two-translation (2T)

First, according to the subgroup operation

$$\{T\,(\boldsymbol{P}_{vw})\} = \{L_1\} \cap \{L_2\} = T\,(\boldsymbol{P}_{vw}) \cap T, \tag{44}$$

the first layer parallel mechanism can be obtained, in which one limb consists of two independent prismatic pairs, the other consists of three independent ones, shown in Fig. 8(a).

Second, prismatic pair P_{a21} is chosen as the joint generating the first layer separated mobility property, shown in Fig. 8(b). By removing P_{a21} and the FB from the first layer parallel mechanism, the remaining structure serves as substructure I, shown in Fig. 8(c), in which chain C1 consists of prismatic pairs P_{a11} and P_{a12} and chain C2 consists of prismatic pairs P_{a22} and P_{a23}, and EP is the moving platform of the first layer parallel mechanism.

Third, since the first layer separated mobility property is one translation, according to

$$\{T\,(\boldsymbol{u})\} = \{L_1\} \cap \{L_2\} = T\,(\boldsymbol{u}) \cap T, \tag{45}$$

the second layer parallel mechanism is obtained, in which limb 1 consists of three prismatic pairs and limb 2 consists of one prismatic pair, shown in Fig. 8(d).

Fourth, divide limb 1 into two sub-chains C3 and C4. Let C3 consist of prismatic pairs P_{b12} and P_{b13}. Let C4 consist of prismatic pair P_{b11}. Substructure II is thus obtained, shown in Fig. 8(e), in which limb 2 serves as chain C5, the link connecting C3 to C4 serves as MP1, and the moving platform serves as the MP2.

Finally, through fixing P_{a11} and P_{a22} of substructure I to MP1 and MP2 of substructure II, a TLTL mechanism is obtained, shown in Fig. 8(f), whose mobility property is two translations through analysis.

Similarly, more mechanisms can be synthesized, such as the ones shown from Fig. 8(g) to (j).

6.2. Mobility property: two rotations (2R)

First, according to the following relation

$$\{\boldsymbol{S}_2(N)\} = \{L_1\} \cap \{L_2\} = \{\boldsymbol{S}_2(N)\} \cap \{\boldsymbol{S}_2(N)\}\,.\,\{G(\boldsymbol{u})\}\,, \tag{46}$$

the first layer parallel mechanism is obtained, in which one limb consists of two intersecting revolute pairs, the other consists of five revolute pairs, shown in Fig. 9(a), in which two pairs intersecting with the ones of the former limb, and other three pairs are parallel.

Second, revolute pairs R_{a11} and R_{a12} are chosen as the joints generating the first layer separated mobility property, shown in Fig. 9(b). By removing R_{a11}, R_{a12} and the FB from the first layer parallel mechanism, the remaining structure serves as substructure I, shown in Fig. 9(c), in which chain C1

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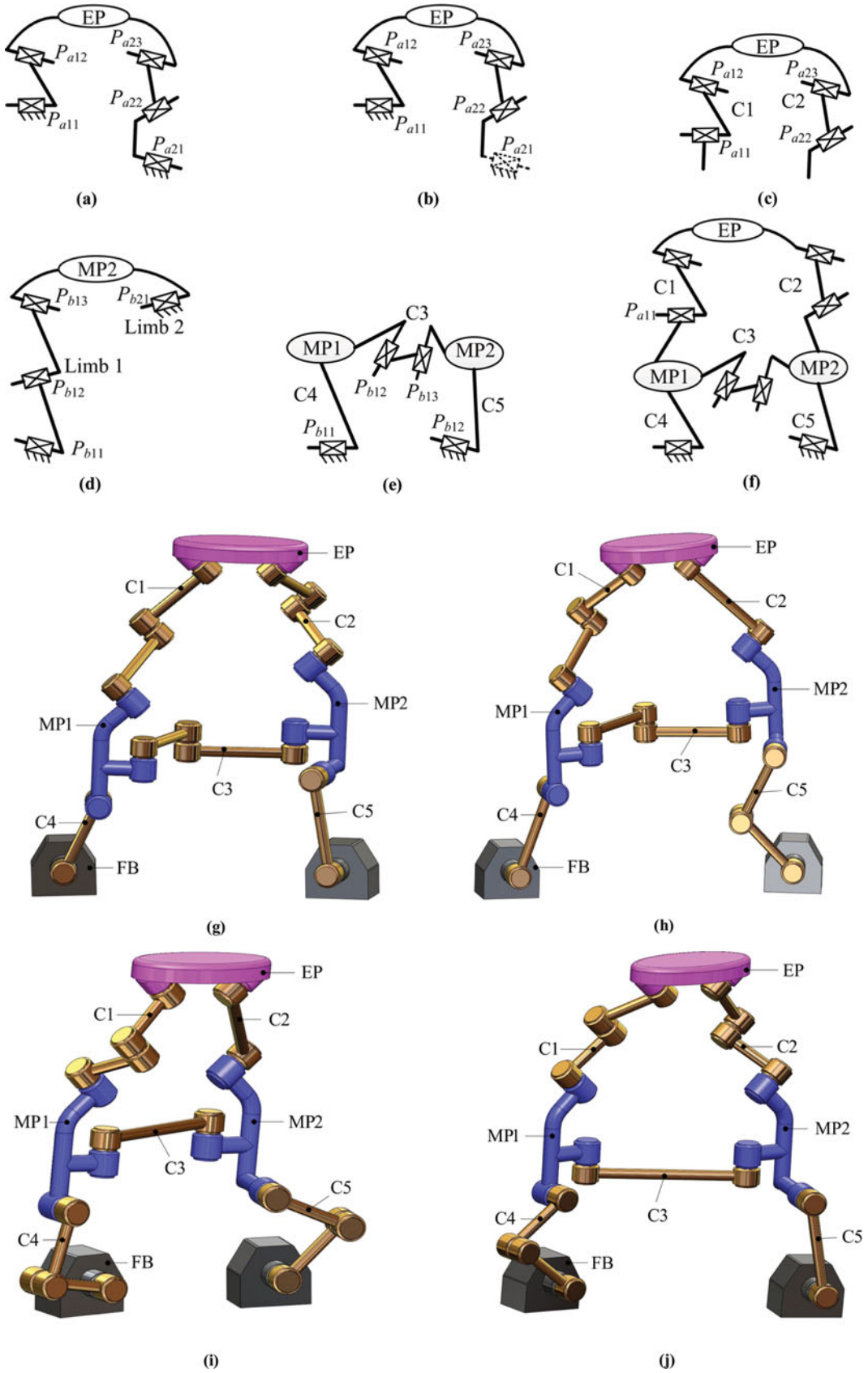


Fig. 8. (a) The first layer parallel mechanism, (b) the first layer separated mobility property, (c) substructure I, (d) the second layer parallel mechanism, (e) substructure II, (f) a new TLTL mechanism, (g) (j) other new mechanisms.

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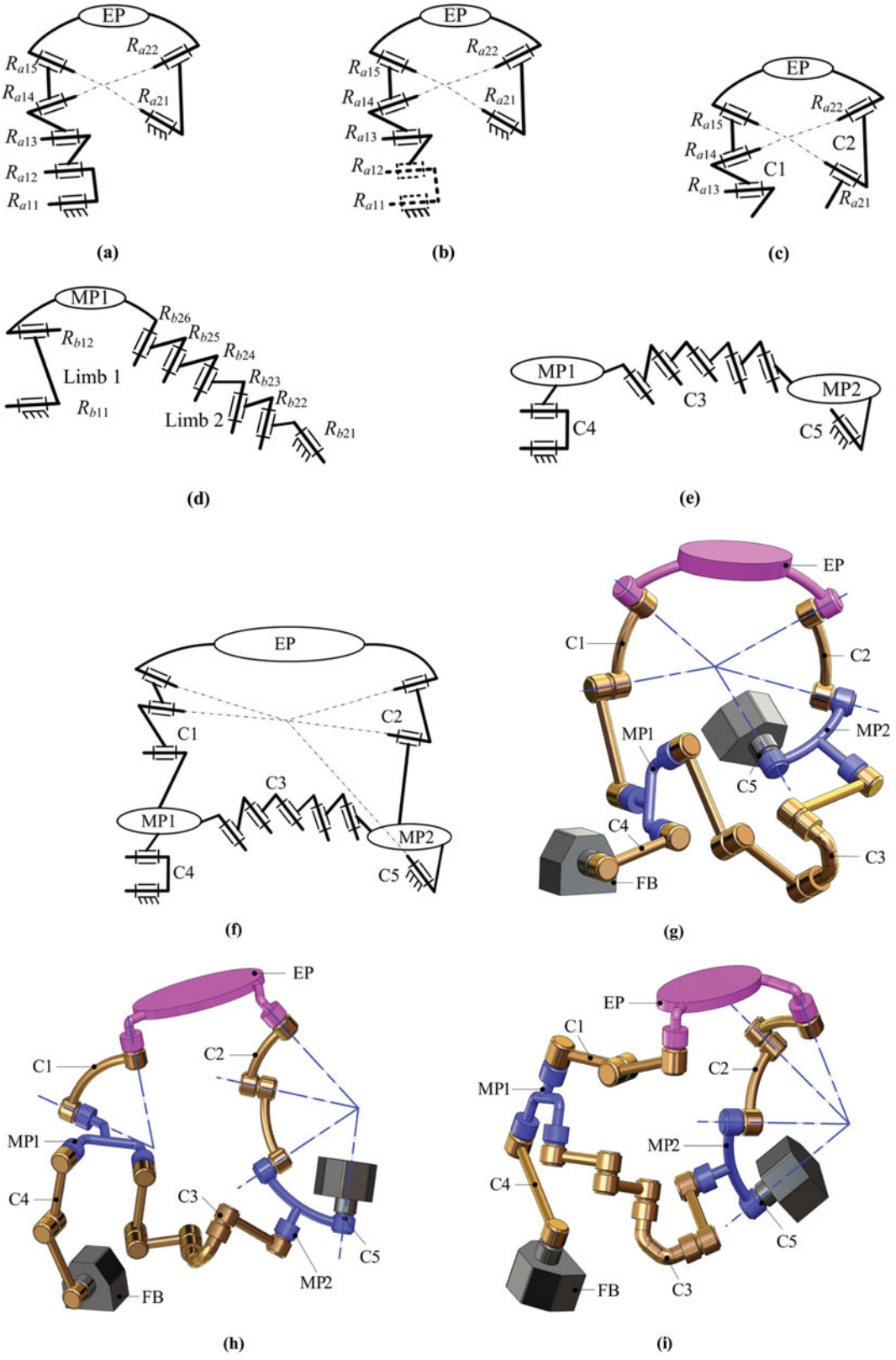


Fig. 9. (a) The first layer parallel mechanism, (b) the first layer separated mobility property, (c) substructure I, (d) the second layer parallel mechanism, (e) substructure II, (f) a new TLTL mechanism, (g) the three-dimensional model of the new mechanism, (h) (i) other synthesized new mechanisms.

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consists of revolute pairs R_{a13}, R_{a14} and R_{a15}, and chain C2 consists of revolute pairs R_{a21} and R_{a22}, and EP is the moving platform of the first layer parallel mechanism.

Third, the first layer separated mobility is 2-DOF planar motions. According to the following relation

$$\{G_2(\boldsymbol{w})\} = \{L_1\} \cap \{L_2\} = \{G_2(\boldsymbol{w})\} \cap \{D\}\,, \tag{47}$$

the second layer parallel mechanism is obtained in which limb 1 consists of two parallel revolute pairs and limb 2 consists of six independent revolute pairs, shown in Fig. 9(d).

Fourth, divide limb 2 into two sub-chains C3 and C5. Let C3 consist of five revolute pairs from R_{b22} to R_{b26}. Let C5 consist of revolute pair R_{b21}. Substructure II is obtained, shown in Fig. 9(e), in which limb 1 serves as chain C4, the link connecting C3 with C5 serves as MP2, and the moving platform serves as MP1.

Finally, a TLTL is obtained through connecting R_{a13} and R_{a21} of substructure I to MP1 and MP2 of substructure II, respectively, shown in Fig. 9(f). Its three-dimensional model is shown in Fig. 9(g). The new mechanism can output two rotations through analysis. Similarly, more mechanisms can be synthesized, such as the ones shown from Fig. 9(h) to (i).

7. Conclusions

This paper focuses on mobility analysis and structural synthesis of a class of spatial mechanisms with coupling chains, called TLTL mechanisms. A TLTL mechanism can be decomposed into two substructures, in which sub-substructure II is a single loop mechanism. The DOFs of two middle platforms of substructure II are divided into two parts, namely, independent DOFs and dependent DOFs. Finally, a TLTL mechanism is kinematically equivalent to a special serial-parallel mechanism.

The motion screw equation of a TLTL mechanism contains some dependent joint velocities which should be identified before solving the equation. Mobility analysis along with detection of rigid substructures is performed for TLTL mechanisms based on the motion screw equation.

On the basis of the principle of mobility analysis, some novel TLTL mechanisms have been synthesized. The related research benefits the design of more complex spatial mechanisms with coupling chains.

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