Multiple canonical surfaces. By D. W. BABBAGE, Ph.D., Magdalene College.

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1. If the canonical series of an algebraic curve of genus p is compounded of an involution of sets of points on the curve then the involution must be rational and of order two, and the canonical model is a repeated rational normal curve of order $p-1$ with $2p+2$ branch points. An analogous question suggests itself for algebraic surfaces. Under what conditions is the canonical model of a surface, whose geometric genus is not less than two, a multiple surface, and what in this case are the properties of the simple surface on which the multiple surface is based? In this paper we give particular examples of multiple canonical surfaces and attempt to go some way towards the solution of the general problem.

2. Illustrations of double canonical surfaces can easily be given: for example, the double plane with an octavic branch curve, the double quadric surface with a (6, 6) branch curve, and the double cubic surface in [3] branching along its intersection with a quartic surface. Simple surfaces in [3] of which these are the canonical models are respectively a quintic surface with a triple point, a sextic surface with two double lines, and a sextic surface with six triple points which is self-corresponding in the Geiser involution determined by quadrics through these six points†.

An example of a triple canonical surface is the triple plane branching along a curve of order twelve with twenty-four cusps which are its complete intersection with a quartic, the cusps being winding-places of three sheets of the surface‡: a simple non-canonical model of this surface is the sextic surface in [3] with a tritacnode. This is, I believe, the only known example of a triple canonical surface.

We shall be mainly concerned in this paper with double canonical surfaces.

3. In the examples which we have given of double canonical surfaces it will be observed that the simple surfaces which are in (1, 2) correspondence with these, namely a plane, a quadric, and a cubic surface, are all rational normal surfaces. The condition of normality is obviously essential, for if the simple surface can be obtained by projection from a surface of the same order in higher

† If the quadrics are put in correspondence with the planes of a solid S, then the sextic surface corresponds to a general cubic surface in S.

‡ Enriques, *Lezioni sulla teoria delle superficie algebriche* (Padova, 1932), p. 346.

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space the same is true of the double surface, which cannot therefore be the model of a complete canonical system.

It is not however essential for the existence of a double canonical surface F^* that the corresponding simple surface F should be rational. Consider a surface which contains an irrational pencil Λ of genus π of hyperelliptic curves of genus p. The curves of Λ have no characteristic series, so that a curve of the canonical system cuts each curve of Λ in a canonical set, consisting of $p-1$ pairs of the g_2^1 on this curve. The canonical model F^* is thus a double surface for which the corresponding simple surface F contains an irrational pencil of genus π of rational curves of order $p-1$: that is, F is referable to a scroll of genus π, and has zero geometric genus and arithmetic genus $-\pi$.

We shall shew that if a double surface F^* is to be canonical then *it is essential that the corresponding simple surface F should have geometric genus $p_g = 0$.*

4. Let F^n be a normal surface of order n in $[r]$ with prime sections of genus p, and suppose that $F^{2n}{}^*$ is a double surface consisting of two sheets superimposed on F^n and branching along a curve β, which is such that the primes of $[r]$ cut out sets of the canonical system $|K|$ of $F^{2n}{}^*$ (not necessarily the complete system $|K|$). We denote by $|C|$, $|C'|$ respectively the system of prime sections of F and the system adjoint to this, and by $|C^*|$, $|C'^*|$ the double curves of F^* superimposed on these (in neither case necessarily complete systems on F^*).

A particular curve of F^* adjoint to its prime sections consists of a curve C'^* taken together with the branch curve β. Hence on F^* we have the equivalence $C^* \equiv C'^* + \beta - C^*$, i.e.

$$\beta \equiv 2C^* - C'^*.$$

On F we have the similar equivalence

$$\beta \equiv 4C - 2C'.\dagger$$

The canonical system $|K|$ of F^* has genus $2n+1$ and grade $2n$. We notice that, since $(\beta, C) \geqslant 0$, we must have $n \geqslant p-1$, the branch curve being non-existent in the case $n = p-1$: this will happen, for example, if F is a canonical surface.

If F is rational, we can take for β any curve of $|4C - 2C'|$, if this system exists, and then the corresponding F^* will exist and be canonical. If the prime sections of F are mapped in a plane by a complete linear system Σ of curves of order $\mu - 3$ with a number of $(i-1)$-ple base points, β will correspond to a curve of order 2μ with $2i$-ple points at the $(i-1)$-ple base points of Σ.‡

† Severi, "Sulle relazioni che legano i caratteri invarianti di due superficie in corrispondenza algebrica", *Rend. Ist. Lomb.* (2), 36 (1903), 495–511.

‡ Enriques, *loc. cit.* § 60.

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In the general case in order to obtain a surface birationally equivalent to a surface of the type F^* we might proceed as follows. Take the surface ϕ whose prime sections are mapped on F by the system $|2C - C'|$†, that is one half of the system $|\beta| \equiv |4C - 2C'|$, and then the quadric section Φ of the cone projecting ϕ from a point O of external space. Φ clearly projects from O into a surface birationally equivalent to a surface of the type F^*. It does not however follow that this last surface will be the canonical model of Φ. It may, for example, not be normal and map only a sub-system of the canonical system of Φ: further, as we shall see later in an example, the system $|\beta| \equiv |4C - 2C'|$ on F may not be uniquely divisible by two, so that by taking a different half-system we may arrive at a different surface Φ.

We emphasise here an important point, namely that a double surface is not necessarily uniquely defined by assigning a branch curve β on the corresponding simple surface F. This is comparable with the case of a double curve. There are 2^{2p} topologically distinct double curves superimposed on a given curve of genus p with an assigned set of branch points‡. Thus consider the canonical sextic of genus four in [3] which is the intersection of an elliptic cubic cone with a quadric surface. This projects from the vertex of the cone into a double curve $\mathfrak{D}$ superimposed on a plane cubic γ and branching at the intersections of γ with a conic. The canonical sets of $\mathfrak{D}$ contained in its elliptic involution correspond to collinear sets on γ. A transformation of γ (and hence of $\mathfrak{D}$) into itself, obtained by adding a half-period to the elliptic argument of each point of γ, although it carries the six branch points into six points on a conic, does not carry collinear sets of γ into collinear sets. Thus, given a sextic of genus four which contains an elliptic involution and which is represented by a double cubic branching at six points on a conic, the canonical sets that belong to the involution may be superimposed on the collinear sets of the cubic or on any of the three series of sets of three points whose parameter sum differs from that of a collinear set by a half-period. Further definition is needed to fix the double curve uniquely. The number 2^{2p}, in this case four, of topologically distinct double curves is equal to the number of half-series on the simple curve of the linear series defined on this curve by the set of assigned branch points. Thus, given a double cubic γ with any six branch points B, we can obtain the four corresponding canonical models as quadric sections of cubic cones whose cross-sections are the images of γ determined by the four g_3^2 half-systems of $|B|$. I would suggest analogously as very

† Supposing this system to exist, to be irreducible, and to have freedom at least two.

‡ Hurwitz, "Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten", *Math. Ann.* 39 (1891), 1–61 (57); *Math. Werke* (Basel, 1932), 1, 379.

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probable that, given a surface F and on it a curve β necessarily meeting every curve of F in an even number of points, the number of distinct surfaces consisting of two sheets superimposed on F and branching along β is equal to the number of half-systems on F of the system $|\beta|$†.

5. We return to the hypothetical double surface F^{2n*} for which the prime sections $|C^*|$ form part or whole of the canonical system $|K|$. The case where F^n is rational does not need further consideration, so that we can assume this surface to have sectional genus $p > 0$.

We attempt to construct the bicanonical model F^{8n*} of F^{2n*}, that is, the surface whose prime sections map the curves of the complete system $|2K|$. Among the curves of $|2K|$ are curves $|2C^*|$, double curves superimposed on the curves of $|2C|$, the double of the system of prime sections of F^n. We assume first that these do not make up all the curves $|2K|$, that is, that F^{8n*} is not a double surface, the coincident point-pairs of F^{2n*} being separated on this model. We consider the three-dimensional locus V_3 of lines joining these point-pairs of F^{8n*}. There must exist a linear space ϖ meeting all the lines of V_3, but not meeting F^{8n*}, from which V_3 projects down into a surface F^{4n}, whose prime sections map the complete system $|2C|$ on F^n; F^{8n*} projects doubly into F^{4n} from ϖ.

The images on F^{8n*} of the curves C^* are ∞^r curves H, say, of order $4n$ and genus $2n+1$, on which the primes of space cut out sets of the canonical series (the complete canonical series if F^{2n*} is regular). The lines joining the point-pairs on the curves H form scrolls R of order $2n+2p-2$ and genus p. From the containing space of one of these scrolls R the surface F^{8n*} projects into the canonical model of F^{2n*}, that is, into F^{2n*} itself if $|C^*|$ and $|K|$ are identical.

Let us fix our attention on one of these scrolls R and its bisecant curve H of order $4n$ and genus $2n+1$. The generators of R all meet ϖ in points of a curve $\mathfrak{d}$, lying perhaps in a sub-space ϖ' of ϖ. From ϖ' R projects down into a curve C^{2n} of order $2n$ and genus p, whose prime sections map the linear series cut out on the corresponding curve C of F^n by the complete system $|2C|$. Thus the directrix curve $\mathfrak{d}$ of R is of order $2p-2$. If H is normal, that is, if F^{2n*} is regular, the whole figure lies in $[2n]$, and the directrix curves $\mathfrak{d}$ and C^{2n} are both normal. If C^{2n} is not special it will lie in a $[2n-p]$ and consequently ϖ' will be of dimension $[p-1]$ and $\mathfrak{d}$ a canonical curve of ϖ'. This will certainly be the case if $n > p-1$ and it will also be true if $n = p-1$ provided that the original simple surface F^n is

† Dr Du Val has given a topological proof of this suggestion (*Proc. Cambridge Phil. Soc.* 30 (1934), 309). I am also greatly indebted to him for some general discussion and enlightenment on these matters.

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not itself canonical. If H is not normal the primes of ϖ' will still cut out some of the sets of the canonical series on $\mathfrak{S}$, provided that F^n is not canonical, and primes through C^{2n} will cut the scroll R in a canonical set of its generators. In the case $n = p - 1$, and when F^n is canonical, the curve C^{2n} will be a canonical curve (or the projection of one) but $\mathfrak{S}$ will not: the primes through C^{2n} will not in this case cut out canonical sets of the generators of R but contact sets of the bicanonical series of generators, that is, the sets of a half-series of the bicanonical series which is different from the canonical series. As an example of this we may take the case of a canonical octavic curve Γ of genus five in [4] which can be projected from a line l doubly into a plane quartic C of genus three (there being no branch points). The scroll R_8 of order eight and genus three, generated by the joins of the pairs of the involution on Γ, has C as simple directrix and l as a quadruple directrix. The generators of R set up a (1, 4) correspondence between the points of l and the points of C: to the ∞^1 points of l correspond the ∞^1 sets of a g_4^1 on C, namely the contact points of one of the Steiner systems of quadritangent conics.

We leave for further consideration the case when F^n is canonical.

In general V_3 meets ϖ in the points of a surface ϕ† which is in (1, 1) correspondence with the surface F^{4n}, the correspondence being set up by the lines of V_3. There is an associated correspondence between ϕ and F^n. In this correspondence the prime sections C of F^n are mapped on ϕ by the curves, of order $2p - 2$ and genus p, in which ϕ is met by the scrolls R. It follows that the prime sections of ϕ, which must be normal since F^{8n*} is normal, are mapped on F^n by the complete system $|C'|$ adjoint to $|C|$.

We thus deduce a geometrical construction for the locus V_3. In space of suitably high dimension we take the image F^{4n} of F^n given by $|2C|$, lying in a space π, and the image ϕ of F^n given by the system $|C'|$, lying in a space ϖ: V_3 is the locus of joins of corresponding points. A quadric primal through F^{8n*}, which does not contain V_3 entirely, meets it residually in a scroll of its lines, determined by having as its directrix curve the quadric section of ϕ. A quadric primal through such a scroll meets V_3 residually in a surface F^{8n*} of the required type, that is a surface whose canonical system is half the system of prime sections and for which a linear sub-system, at least, of the canonical system consists of curves which belong to the involution cut out by the lines of V. It does not follow that all such surfaces F^{8n*} can be obtained as partial quadric sections of V_3.

† ϕ may be a multiple surface, and it may also in particular close up into a curve or a point.

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If the system $|C|$ of F^n is contained in $|C'|$, that is, if the geometric genus p_g of F^n is greater than zero, then the directrix curve in ϖ of a scroll R lies in a sub-space of ϖ, and hence, on projection from the space of R, the surface F^{8n*} will not shut up into F^{2n*}, and consequently F^{2n*} is not normal, the system $|C^*|$ being only a sub-system of $|K|$. Thus if we have a surface F^{8*} in [3] consisting of two sheets superimposed on a quartic surface F^4 and branching along a quadric section, the planes do not cut out the complete canonical system on F^{8*}: the canonical model actually lies in [4], and is the quadric section of a quartic point-cone.

F^{8n*} is itself a double surface if $|C'|$ does not exist. In this case F^n is ruled, and F^{2n*} belongs to the class of surfaces previously mentioned which contain a pencil of hyperelliptic curves.

Hence, provided that F^n is not itself canonical, we have the essential condition that $p_g = 0$ if F^{2n*} is to be canonical.

It is not difficult to see that the case where F^n is itself canonical cannot lead to a double canonical surface F^{2n*}. If F^n is canonical the directrix surface ϕ of V_3 must be the image of F^n given by a linear system of curves $|D|$ on F^n, which cut out on a general curve C sets of a non-special g^{p-2}_{2p-2}, whose double is the bicanonical series on C. Hence we must have $|2D| = |2C'| = |4C|$. Further, if the primes of $[r]$ are to cut out the complete canonical system on F^{2n*}, it follows as before that $|C|$ must not be contained in the linear system $|D|$. This implies that the arithmetic genus p_a of F^n is negative. For the system $|D|$ is, like $|2C|$, of genus $3p-2$ and grade $4p-4$: therefore, since $|D|$ certainly cannot be contained in the canonical system $|C|$ (because a curve D is of order $2p-2$ and a curve C of order $p-1$), its freedom is, by the Riemann-Roch theorem, $r_D \geqslant p-1+p_a$. The postulation of a curve C to $|D|$ is at most $p-1$, and therefore, if p_a is zero or positive, the system $|D-C|$ will certainly exist. The only surfaces with negative p_a which have $p_g \geqslant 2$, and whose canonical models F^n might therefore be expected to lead to canonical double surfaces F^{2n*}, are the so-called elliptic surfaces, which contain a pencil of genus p_g of elliptic curves†. The canonical curves on a surface of this type are reducible, consisting of canonical sets of $2p_g - 2$ curves of the pencil; the canonical model F^n degenerates into a canonical curve of genus p_g. If we have such a surface and double it without a branch curve, the double surface itself contains a pencil of (double) elliptic curves and its complete canonical system consists of the double curves superimposed on the canonical curves of the simple surface.

6. We can at once find the arithmetic genus p_a^* of the canon-

† Castelnuovo, "Sulle superficie aventi il genere aritmetico negativo", *Rend. di Palermo*, 20 (1905), 55–60. For these surfaces $p_a = -1$.

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ical double surface F^{2n*} in $[r]$, if this exists. The bigenus $P_2{}^*$ of F^{2n*} is, as we have seen, equal to $\rho_{2C}+\rho_{C'}+2$, where ρ_D denotes the freedom of a complete linear system $|D|$ on the simple surface F^n. The linear genus $p^{(1)*}$ of F^{2n*} is $2n+1$, whence, by the formula $P_2=p^{(1)}+p_a$, we have

$$p_a{}^* = \rho_{2C}+\rho_{C'}-2n+1,$$

and the irregularity of F^{2n*} is $r+2n-\rho_{2C}-\rho_{C'}$.

If F^n has geometric genus zero and arithmetic genus p_a ($\leqslant 0$), then, if $|2C|$ is a regular system, we have

$$\rho_{2C}=4n-(2p+n-1)+1+p_a=3n-2p+2+p_a.$$

Since $\rho_{C'}=p-1+p_a$, it follows that the irregularity of F^{2n*} is equal to $r-n+p-1-2p_a$.

If the system $|C|$ is also regular we have $r=n-p+1+p_a$, whence the irregularity of F^{2n*} is equal to $-p_a$, which is the same as that of F^n.

7. As an example of a surface whose canonical model is a double ruled surface we may take the surface F^{12} in [4] which is the intersection of a quartic primal Φ^4, having a double plane π, with an elliptic cubic line-cone E^3, vertex l. F^{12} contains a quartic of genus two in each plane of E^3; it has a double elliptic cubic Γ in π and four triple points A_1, A_2, A_3, A_4 on l. The canonical system of F^{12} is cut out by quadric primals passing through Γ, A_1, A_2, A_3, A_4, that is, through π and l. This ∞^5 system of quadrics is compounded of the congruence of lines λ which meet π and l. If we put the system of quadrics in linear correspondence with the primes of [5], the space [4] is transformed into a Segre $V_3{}^3$ of general type, each point of $V_3{}^3$ arising from a line λ. The ∞^1 generating planes of $V_3{}^3$ correspond to the ∞^1 solids through π (each such solid contains a sheaf of lines λ), and the ∞^2 lines of the conjugate generation correspond to the ∞^2 planes through l (each of which planes contains a pencil of lines λ). The planes of E^3 transform into lines of $V_3{}^3$ generating an elliptic sextic scroll R^6 which has an elliptic cubic directrix in each plane of $V_3{}^3$. Each point of R^6 corresponds to two points of F^{12}. Thus the canonical model of F^{12} is a surface R^{12*} consisting of two sheets superimposed on R^6. The branch curve β of R^{12*} meets each generator of R^6 in six points: β is of order twenty-four and is linearly equivalent on R^6 to $2C-2K$, where C is a prime section of R^6 and K is a curve of the canonical system. The curves K have of course only a virtual existence, being equivalent on R^6 to -2γ, where γ is one of the ∞^1 plane cubics of R^6. The curve $\beta\equiv 2C+4\gamma$ can be cut out on R^6 by a sextic primal passing through twelve generators of R^6 which meet each curve γ in its complete intersection with a quartic lying in its plane.

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For the double scroll R^{12*} we have $p_g^* = 6$: p_a^* is equal to the difference between the number of linearly independent curves of the system $|2C|$ on R^6 and the linear genus of R^{12*}, which is thirteen. $|2C|$ is cut out on R^6 by quadrics, which are ∞^{20} in number, and of which ∞^2 pass through R^6, namely those which contain V_3^3: hence $p_a^* = 5$, and the double scroll has irregularity one (cf. § 6). The same result follows by considering the surface F^{12} in [4]. The conditions of adjointness, that of passing through Γ, A_1, A_2, A_3, A_4, for a primal of sufficiently high (> 2) order n are in number $3n + 4$, or ten in the case $n = 2$. Actually the postulation to quadrics is only nine, the difference between these two numbers being the irregularity of F^{12}.

8. A much more interesting example of a double canonical surface is that of a surface consisting of two sheets superimposed on the Enriques sextic surface F^6 in [3]. F^6 has for double lines the six edges of a tetrahedron and has triple points at the vertices: it has characteristics $p_g = p_a = 0$, $P_2 = 1$. The ∞^3 plane sections $|C|$ of F^6 are sextics of genus four and the adjoint system $|C'|$ is also ∞^3 and consists of sextics of genus four cut out by cubic surfaces with nodes at the vertices of the tetrahedron. The systems $|C|$, $|C'|$ are distinct but the complete systems $|2C|$, $|2C'|$ are identical: the system $|2C|$ or $|2C'|$ is ∞^{12} and consists of curves of order twelve and genus thirteen cut out by sextic surfaces of the same type as F^6. There is one surface cutting out the null-system $|2C' - 2C|$, namely the four faces of the tetrahedron.

Now take the point-cone V_3^6 in [4], with vertex O, whose cross-section is the Enriques surface F^6, and let F^{12} be its section by a general quadric primal Ω. F^{12} has four pairs of triple points and through any two pairs there passes a double conic. The canonical system $|K|$ is cut out by cubic primals through these conics. It imposes twenty-six conditions on a cubic primal to make it contain the conics (eight conditions to go through the eight triple points and then three more for each conic), and there are five linearly independent cubic primals through F^{12}, these degenerating into the quadric Ω together with a prime. Hence there are

$$\binom{7}{4} - 26 - 5 = 4$$

independent curves K, and the surface has $p_g = p_a = 4$. The system $|K|$ must be cut out by the ∞^3 cones with vertex O whose cross-sections are cubic surfaces with nodes at the vertices of the corresponding tetrahedron.

Projecting from O we thus get a double Enriques sextic F^6, branching along a quadric section, on which $|K|$ is cut out by the four-nodal cubic surfaces: transforming by these cubics we obtain

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a double canonical surface F^{12*} consisting of a repeated Enriques sextic F^6 branching along the section β by a sextic surface which passes doubly through the edges of the tetrahedron and has quadruple points at the vertices. The canonical curves K of F^{12*} are the plane sections, double curves of genus $p^{(1)*} = 13$ and grade $p^{(2)*} = 12$.

An apparent paradox occurs when the original double Enriques surface branches along its section by a quadric Q through the vertices of the tetrahedron. If we regard this double surface as the projection of the quadric section of a cone in [4], that is as a surface lying on the double threefold whose branch surface is Q, then just as before the canonical system is cut out by the four-nodal cubic surfaces. On transforming by these cubics we get as the canonical model a double Enriques surface again branching along its intersection with a quadric Q' through the vertices of the corresponding tetrahedron; whereas we should expect the canonical system of this surface to be cut out by the four-nodal cubics. The explanation has already been indicated in § 4. The double surface is not uniquely defined by its branch curve. To fix it completely we can regard it as lying on a double threefold whose branch surface is assigned: a simple model of the surface can then at once be found†. In our case the transformation by four-nodal cubics turns the double threefold whose branch surface is Q into a double threefold whose branch surface consists of Q' together with the four faces of a tetrahedron. Thus if we have a double Enriques surface whose branch curve is its intersection with a quadric Q through the vertices of the tetrahedron, its complete canonical system is cut out by four-nodal cubics or planes according as the branch surface of the double threefold on which the double surface may be supposed to lie is taken to be Q or Q together with the four faces of the tetrahedron

If the branch curve of the double surface F^{12*} is taken to be the intersection of F^6 with another surface G^6 of the same type, that is, a general member of $|2C|$, it is not so easy by constructing a simple model to see what the possible canonical systems are; but there seems no reason to doubt that $|K|$ is cut out either by the planes or by the four-nodal cubics, according as the double threefold on which F^{12*} lies branches along G^6 or along G^6 together with the four faces of the tetrahedron.

Taking the double canonical F^{12*} we denote as before by $|C|$, $|C'|$ the system of plane sections of the simple surface F^6 and the system adjoint to this, and by $|C^*|$ $(= |K|)$, $|C'^*|$ the corresponding systems of double curves on F^{12*}. The property of the simple

† If the branch surface of the double threefold is, in non-homogeneous coordinates, $f(x, y, z) = 0$, the double threefold may be regarded as the projection of the primal $t^2 = f(x, y, z)$ in [4]: and any double surface lying on the double threefold is transformable into the section of this primal by a cone.

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surface that the complete systems $|2C|$, $|2C'|$ are identical is carried over to the double surface, and we have the result that the bicanonical system $|2K|$ of F^{12*} has two distinct half-systems, $|K|$ and $|C'^*|$. On a curve K the bicanonical system $|2K|$ cuts the complete canonical series g_{24}^{12}, while on a curve C'^* which like K is of genus thirteen, it cuts a complete non-special series g_{24}^{11}. This is brought out quite clearly on the bicanonical model F^{48*} of F^{12*}. The bigenus P_2^* of F^{12*} is given by

$$P_2^* = p^{(1)*} + p_a^* = 13 + 4 = 17,$$

and therefore F^{48*} lies in [16]. The associated locus of lines V_3 is constructed as follows. In [16] we take a [3] ϖ and a [12] π, in general position. In ϖ we take the (Enriques sextic) surface G^6 whose plane sections are mapped on F^6 by $|C'|$, and in π we take the surface M^{24} whose prime sections are mapped on F^6 by the ∞^{12} system $|2C| \equiv |2C'|$: G^6 and M^{24} are in $(1, 1)$ correspondence and the locus of joins of corresponding points is the required locus V_3. The order of V is forty-two: for a prime through M^{24} meets G^6 in a curve of order six to which corresponds a curve of order twelve on M^{24}, and hence the intersection of the prime with V_3 consists of M^{24} together with a scroll of order eighteen. Denote the images of C, C' on the two surfaces by C_g, C_g', and C_m, C_m'. C_g is a canonical sextic of genus four, C_g' is a plane sextic of genus four, and C_m, C_m' are both curves of order twelve and genus four, lying in [8]'s. The locus of joins of corresponding points of a C_g and its associated C_m is a scroll R of order eighteen and genus four lying in a [12]. The locus of joins of corresponding points of a C_g' and its associated C_m' is a scroll R' of order eighteen and genus four lying in an [11]. V_3 contains ∞^4 surfaces of the type M^{24}, cut out by the ∞^4 primes through a scroll R', and there are correspondingly ∞^4 [12]'s like π.

The bicanonical surface F^{48*} may be taken as cut out from V_3 by a quadric primal Φ through two of the scrolls R' (or through a scroll of lines of V_3 which meets G^6 in a quadric section)†. Φ, as containing twelve generators of an R, meets it residually in a curve H of order twenty-four and genus thirteen, a canonical curve in [12]: the ∞^3 curves H correspond to the prime sections $|K|$ of F^{12*} and form the canonical system of F^{48*}. Φ meets a scroll R' residually in a curve H' of order twenty-four and genus thirteen, lying in an [11]: the ∞^3 curves H' are the images of the curves C'^* of F^{12}. Two curves H make up the complete section of F^{48*} by a prime through ϖ. Two curves H' also make up a prime section of F^{48*}; for it is easily seen that their containing [11]'s lie in a unique prime, which cannot meet F^{48*} other than in the curves H'. Thus

† Although it has not been established that F^{48*} can always be obtained as a partial quadric section of V_3 it is sufficient for our purpose that it behaves as though it were thus obtained.

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the ∞^3 systems $|H|$, $|H'|$ are half-systems of the system of prime sections, which latter system cuts the complete canonical g_{24}^{12} on each curve H and a complete non-special g_{24}^{11} on each curve H'.

9. It should be possible to construct other examples of canonical surfaces of two sheets superimposed on a normal surface F^n, which has zero geometric genus and is neither rational nor referable to a scroll. We need not of course restrict ourselves to the case when F^n is simple.

The problem of finding triple canonical surfaces is more difficult except in one case. Castelnuovo† has pointed out that if the general curve of the canonical system of a surface contains a g_3^1 then the characteristic series on a curve of that system is compounded with the g_3^1, whence the canonical system belongs to an involution of the third order: in this case the canonical surface is triple. It may be questioned whether this property of the canonical system can occur other than in the case of a surface whose canonical model is the triple plane mentioned above‡. It is very easy to show that it cannot. In fact the required canonical model has to be a surface $F^{3n}{}^*$ of three sheets superimposed on a normal surface F^n with rational linear sections. F^n is therefore a plane, a rational normal scroll, or a Veronese surface. In the second case $F^{3n}{}^*$ contains a pencil of triple lines: since a curve of a pencil (without base points) has no characteristic set it follows that a prime, which cuts out a curve of the canonical system of $F^{3n}{}^*$, must meet each triple line in a set of its canonical series, which is absurd because a canonical set must consist of an even number of points. We see similarly that F^n cannot be a Veronese surface F^4 by considering the net of triple conics on $F^{12}{}^*$. A prime meets such a triple conic in six points, which must be equivalent on the triple conic to a canonical set minus a characteristic set; but a characteristic set consists of an odd number, three, of points.

In the case where F^n is a plane we can discover the nature of the branch curve by constructing the bicanonical model $F^{12}{}^*$ of the triple plane, on which the sets of three superimposed points of the canonical surface are transformed into sets of three collinear points. $F^{12}{}^*$ can be shewn without difficulty to be the section by a cubic primal of the Veronese point-cone V_3^4 in [6]§.

If the surface F^n is a normal surface with sectional genus $p\,(>0)$ we might still attempt to discover the corresponding canonical triple surface F^{3n} if this exists, by consideration of the bicanonical model $F^{12n}{}^*$. In connection with this we should consider the locus

† Castelnuovo, "Osservazioni intorno alla geometria sopra una superficie, Nota II", *Rend. Ist. Lomb.* (2), 24 (1891), 307–318.

‡ Cf. Enriques, *loc. cit.* p. 347.

§ Du Val, *Journal Lond. Math. Soc.* 8 (1933), 11–18.

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V_4 of planes joining the images of the ∞^2 sets of three superimposed points of F^{3n}: in particular, if F^{3n} was irregular, the planes of V_4 might degenerate into lines or even points, the last case arising if the bicanonical model is itself triple. So far I have been unable to obtain any new examples of triple canonical surfaces or to prove their non-existence.

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