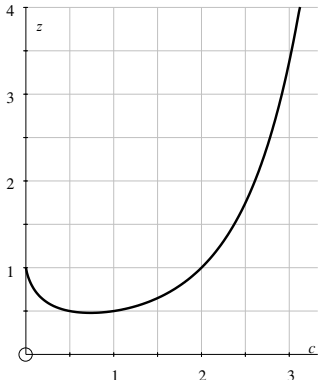



FIGURE 4: Graph of $z = (\frac{1}{2}c)^c$

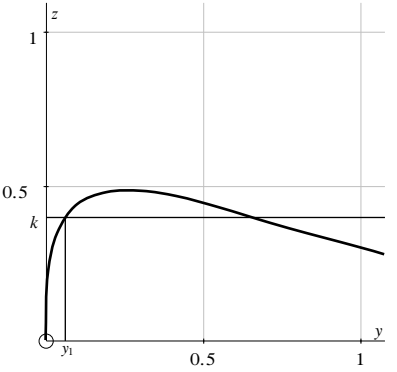



FIGURE 5: Graph showing solutions to $0.3^y y^{0.3} = 0.4$

The reason that graph plotters struggle near O is that the graph approximates a right-angled bend there. This is because, for fixed (small) values of $x$, $z = x^y y^x$ has the shape of graph typified by Figure 5 with a maximum when $\dfrac{dz}{dy} = x^y y^{x-1}(x + y \ln x) = 0$ or $y = -\dfrac{x}{\ln x}$. It follows that $y_1$, the smaller of the two $y$-values with $x^y y^x = k$, satisfies $0 < y_1 < -\dfrac{x}{\ln x}$. Thus $0 < \dfrac{y_1}{x} < -\dfrac{1}{\ln x}$ which tends to 0 as $x \downarrow 0$.

Finally, we note that one of the drawbacks of implicit representations of curves is that natural questions (such as, in Figure 3(b), what is the area of the bounded component?) usually do not have neat analytical solutions.

*Reference*

1. G. Pólya and G. Szegö, *Problems and theorems in analysis I*, Springer (1972) p. 38, Problem 170.

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## Two trajectories: a dynamic monkey tale

The "monkey drop" experiment is an engaging and instructive demonstration of basic kinematic principles. A toy stuffed monkey is suspended at height $b$ above the horizontal floor and at horizontal distance $a$ from a toy cannon on the floor which is loaded with a golf ball projectile. The cannon is aimed directly at the monkey, that is at the angle $\alpha = \arctan(b/a)$ relative to the floor. A trigger mechanism simultaneously fires the cannon and releases the monkey. To the amusement, not to say astonishment, of onlookers the projectile arches gracefully through the air and strikes the monkey before it reaches the floor [1]. A trial in which the launch speed of the projectile is increased, but the firing angle is unchanged,

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also results in a strike [2]. This behaviour is explained physically (air resistance being ignored) by three fundamental kinematic principles: Galileo's law of fall; the principle of inertia; and the synthesis of planar motion from its vertical and horizontal components. A dynamical analysis of the experiment via Newton's differential equations of motion is also well within the grasp of students in an introductory calculus course.

We treat a slightly modified model in which the monkey is *thrown* downward rather than dropped from rest. This modified model requires deeper analysis than the dropped monkey model but remains instructive and accessible to calculus students. Analysis of the modified model leads to conclusions that significantly differ from those of the standard model and provides a platform for discussions of several mathematical topics including the intermediate value theorem, implicitly defined functions, assorted calculus techniques and numerical computation.

*The dropped monkey*

Suppose the projectile is fired from the origin of the standard rectangular coordinate system at a firing angle $\theta \in (0, \pi/2)$ relative to the positive $x$-axis and with initial speed $v > 0$. Integrating Newton's equations of motion gives a parametric representation, in terms of the time $t$ since firing, of the projectile's trajectory:

$$x(t) = (v \cos\theta)t, \qquad y(t) = (v \sin\theta)t - \tfrac{1}{2}gt^2, \tag{1}$$

where $g$ is the constant acceleration of gravity. The projectile trajectory intersects the line $x = a$ at the time $T(\theta)$ given by

$$T(\theta) = \frac{a}{v \cos\theta}. \tag{2}$$

We need consider only projectile trajectories which have positive altitude at $x = a$, that is, $v \sin\theta > \frac{1}{2}gT(\theta)$. In view of (2), this is equivalent to

$$\sin 2\theta > \frac{ag}{v^2},$$

and hence we require

$$v^2 > ag \tag{3}$$

and $\theta \in (\theta_0, \theta_1)$ where:

$$\theta_0 = \frac{1}{2}\arcsin\left(\frac{ag}{v^2}\right), \qquad \theta_1 = \frac{\pi}{2} - \theta_0.$$

Note that $\pi/4 \geqslant \theta_0 \searrow 0$ and $\pi/4 \leqslant \theta_1 \nearrow \pi/2$ as $v \to \infty$. Consequently each angle in $(0, \pi/2)$ is eventually bracketed by the interval $(\theta_0, \theta_1)$ as $v$ increases indefinitely.

The trajectory of the dropped monkey is described by

$$x_m(t) = a, \qquad y_m(t) = b - \tfrac{1}{2}gt^2.$$

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When the projectile reaches the line $x = a$ the monkey is at altitude

$$y_m\big(T(\theta)\big) = b - \tfrac{1}{2}g\,T(\theta)^2.$$

Therefore, the projectile strikes the monkey if, and only if, $y\big(T(\theta)\big) = y_m\big(T(\theta)\big)$, that is

$$v \sin\theta\, T(\theta) = b,$$

or equivalently, $\theta = \arctan(b/a) = \alpha$. We also require that the projectile has positive altitude when it strikes the monkey, that is we require that $\theta_0 < \alpha < \theta_1$, which is equivalent to

$$\frac{ag}{v^2} < 2\sin\alpha\cos\alpha = \frac{2ab}{a^2 + b^2}. \tag{4}$$

Note that this implies that $\dfrac{ag}{v^2} < 1$ as required by condition (3).

*To sum up*:

In the dropped monkey experiment a strike occurs at positive altitude uniquely at the fixed firing angle $\alpha = \arctan(b/a)$ for any launch speed $v$ satisfying (4).

*The thrown monkey*

Consider now the scenario in which the monkey is not dropped from rest but is projected downward with initial velocity $\dot{y}_m(0) = -v_m < 0$. The altitude of the monkey at time $t$ is

$$y_m(t) = b - v_m t - \tfrac{1}{2}gt^2.$$

Our goal is to show that for any sufficiently large launch speed $v$ there is a unique firing angle $\theta^*(v)$ which results in a strike on the monkey before it hits the floor. A strike on the monkey occurs for a firing angle $\theta \in (\theta_0, \theta_1)$ which is a root of the function

$$\begin{aligned} f(\theta) &= y\big(T(\theta)\big) - y_m\big(T(\theta)\big) = v\sin\theta\, T(\theta) - b + v_m T(\theta) \\ &= a\tan\theta - b + a\frac{v_m}{v}\sec\theta. \end{aligned} \tag{5}$$

This function is strictly increasing since

$$f'(\theta) = a\sec^2\theta + a\frac{v_m}{v}\sec\theta\tan\theta > 0.$$

Also, for any firing angle $\theta \geqslant \alpha = \arctan(b/a)$,

$$f(\theta) = a\tan\theta - b + v_m T(\theta) \geqslant a\tan\alpha - b + v_m T(\theta) = v_m T(\theta) > 0,$$

and hence no firing angle $\theta \geqslant \alpha$ achieves a strike on the monkey. We seek a sufficient condition on $v$ that leads to an intercept angle irrespective of the relative sizes of $a$ and $b$. For this we require that $v > v_m$, for otherwise any firing angle $\theta$ leading to a strike results in

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$$0 = f(\theta) = a\tan\theta + a\left(\frac{v_m}{v}\right)\sec\theta - b > a\sec\theta - b$$

which is possible only if $b > a$.

We have assumed $v$ is large enough to satisfy condition (4), which is equivalent to $\alpha \in (\theta_0, \theta_1)$. Therefore we restrict our search to firing angles $\theta \in (\theta_0, \alpha)$, as all such firing angles satisfy $y\big(T(\theta)\big) > 0$.

Since $\theta_0 \to 0$ and $T(\theta_0) \to 0$ as $v \to \infty$ we have

$$f(\theta_0) = a\tan\theta_0 + v_m T(\theta_0) - b < 0 \tag{6}$$

for sufficiently large $v$. Also, as already noted, $f(\alpha) > 0$. Since $f$ is strictly increasing the intermediate value theorem guarantees the *existence* and *uniqueness* of a firing angle that achieves a strike on the monkey at a positive altitude.

For each sufficiently large launch speed $v$ there is a unique firing angle $\theta^*(v) \in (\theta_0, \alpha)$ which results in a strike on the monkey.

Furthermore, the implicit function theorem [3, p. 482] ensures that the function $v \to \theta^*(v)$ is continuously differentiable.

This result contrasts sharply with the dropped monkey model of the previous section. There the *same* firing angle, namely $\alpha$, serves for *all* launch speeds $v$ satisfying (4). Also in the dropped monkey model the firing angle $\alpha$ is given explicitly in terms of geometric parameters only; in the thrown monkey model the implicitly defined *intercept angle* $\theta^*(v)$ depends on the launch speed and also on the dynamic parameters $v_m$ and $g$.

*The intercept angle*

For a sufficiently large launch speed $v$ the intercept angle $\theta^*(v)$ may be reliably computed by the most basic root finder – the bisection method [4, p. 42]. Before providing an illustrative example it is instructive to establish a *specific* sufficient condition on the launch speed $v$ that guarantees

$$f(\theta_0) = a\tan\theta_0 + \frac{av_m}{v}\sec\theta_0 - b < 0,$$

that is

$$\tan\left(\frac{\arcsin c}{2}\right) + \frac{v_m}{v}\sec\left(\frac{\arcsin c}{2}\right) < \frac{b}{a},$$

where for notational convenience we have set $c = \dfrac{ag}{v^2} < 1$. On applying trigonometric half-angle formulas this is seen to be equivalent to

$$\frac{c}{1+\sqrt{1-c^2}} + \frac{\sqrt{2}\,v_m}{v}\frac{1}{\sqrt{1+\sqrt{1-c^2}}} < \frac{b}{a}.$$

Since $\dfrac{c}{1+\sqrt{1-c^2}} < c$ and $\dfrac{1}{\sqrt{1+\sqrt{1-c^2}}} < 1$, the condition

$$\frac{ag}{v^2} + \frac{\sqrt{2}\,v_m}{v} < \frac{b}{a} \tag{7}$$

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along with (4) is sufficient to ensure a unique root $\theta^*(v) \in (\theta_0(v), \alpha)$ of the function $f(\theta)$.

For example, suppose $a = b = 10$, $g = 32$, $v_m = 1$ and $v = 20$. Then $\alpha = \frac{1}{4}\pi$, and $\theta_0 = 0.5 \arcsin \frac{4}{5}$. In this case (4) and (7) are satisfied and a dozen iterations of the bisection method with initial interval $[\theta_0, \alpha]$, implemented in a simple *MATLAB* program, traps $\theta^*(v)$ in the interval [0.75001, 0.75004], and hence $\theta^*(v) \approx 0.7500$ radians, rounded to 4 significant digits. Results of further numerical computations with increasingly large $v$ indicate that $\theta^*(v)$ is an increasing function of $v$. This seems natural as physical intuition suggests that an increase in firing speed, without an increase in the firing angle, would seem to result in the projectile passing below the monkey. A mathematical proof that $\theta^*(v)$ is an increasing function is easy to construct by use of implicit differentiation. In fact, since

$$0 = f\left(\theta^*(v)\right) = a \tan \theta^*(v) + v_m T\left(\theta^*(v)\right) - b,$$

we find after multiplying by $\cos \theta^*(v)/a$ and a little rearranging,

$$0 = \sin \theta^*(v) + v_m v^{-1} - \frac{b}{a} \cos \theta^*(v).$$

Differentiating with respect to $v$ and solving for $\frac{v_m}{v^2}$ gives

$$\frac{v_m}{v^2} = \left(\cos \theta^*(v) + \frac{b}{a} \sin \theta^*(v)\right) \frac{d\theta^*}{dv}.$$

We then have

$$0 < \frac{v_m}{v^2} \leqslant \frac{v_m}{v^2} \sec \theta^*(v) = \left(1 + \frac{b}{a} \tan \theta^*(v)\right) \frac{d\theta^*}{dv},$$

and so $\frac{d\theta^*}{dv} > 0$, proving that the strike angle is an increasing function of the launch speed. Furthermore, since $\theta^*(v) < \alpha$,

$$\frac{v_m}{v^2} < \left(1 + \frac{b}{a} \tan \alpha\right) \frac{d\theta^*}{dv} = \left(1 + \left(\frac{b}{a}\right)^2\right) \frac{d\theta^*}{dv},$$

and hence we can say something about the rate of increase of the launch angle:

$$\frac{d\theta^*}{dv} > \frac{v_m}{v^2} \frac{a^2}{a^2 + b^2}.$$

*An explicit solution*

There is another method for computing the intercept angle, that is, the root $\theta^*(v) \in (\theta_0, \alpha)$ of the function

$$f(\theta) = a \tan \theta + \frac{a v_m}{v} \sec \theta - b.$$

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Setting $z = \tan\theta$, we seek a root $z^* \in (\tan\theta_0, \tan\alpha)$ of

$$z + \frac{v_m}{v}\sqrt{1 + z^2} - \frac{b}{a} = 0. \tag{8}$$

Any such root $z^*$ of this equation satisfies

$$z^* < z^* + \frac{v_m}{v}\sqrt{1 + (z^*)^2} = \frac{b}{a}.$$

Also, under conditions (4) and (7), (8) has a unique root $z^* \in \left(\tan\theta_0, \frac{b}{a}\right)$, as shown earlier. Each root of (8) is also a root of the quadratic equation

$$\left[\left(\frac{v_m}{v}\right)^2 - 1\right]z^2 + 2\frac{b}{a}z + \left[\left(\frac{v_m}{v}\right)^2 - \left(\frac{b}{a}\right)^2\right] = 0.$$

The discriminant

$$D = 4\left(\frac{v_m}{v}\right)^2\left[1 + \left(\frac{b}{a}\right)^2 - \left(\frac{v_m}{v}\right)^2\right]$$

of this quadratic is positive, guaranteeing two real roots. The larger root of the quadratic,

$$\frac{\frac{b}{a} + \sqrt{\frac{D}{4}}}{1 - \left(\frac{v_m}{v}\right)^2},$$

is therefore greater than $\frac{b}{a} = \tan\alpha$ and hence is extraneous. The smaller root then gives an explicit formula for the intercept angle,

$$\theta^*(v) = \arctan\left(\frac{\frac{b}{a} - \sqrt{\frac{D}{4}}}{1 - \left(\frac{v_m}{v}\right)^2}\right).$$

For the test case treated above, all sufficient conditions are satisfied and this explicit formula produces

$$\theta^* = \arctan\left(\frac{400 - \sqrt{799}}{399}\right) \approx 0.7500 \text{ radians}$$

rounded to four decimal places, as was computed previously by the bisection method.

*Conclusion*

Various aspects of the foregoing analysis have provided interesting and engaging enrichment material in several classes, including first term calculus, basic analysis and mathematical modelling. The model may be modified in various ways to construct other individual or group project activities. A particularly illuminating project investigates a model in which

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the monkey is projected *upward* with an initial speed $v_m$. Students may be asked to formulate the model, perform the appropriate mathematical analysis, implement computational procedures, and compare and contrast the conclusions derived with those obtained in the analysis of the model presented in this Note.

*References*

1. http://www.youtube.com/watch?v=MkKEwrshYWs
2. http://www.youtube.com/watch?v=0jGZnMf3rPo
3. R. Courant, *Differential and integral calculus*, Vol. 1, translated by E. J. McShane, Blackie and Son, Glasgow (1957).
4. J. Thomas King, *Introduction to numerical computation*, McGraw Hill, New York (1984).

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