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Structural Properties and Edge Choosability of Planar Graphs without 6-Cycles

WANG WEIFAN[1] and KO-WEI LIH[2]

[1]Department of Mathematics, Liaoning University,
Shenyang 110036, P. R. China

[2]Institute of Mathematics, Academia Sinica,
Nankang, Taipei 115, Taiwan
(e-mail: makwlih@sinica.edu.tw)

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Let G be a planar graph without 6-cycles. We investigate structural properties of G and show that G is edge-$(\Delta(G)+1)$-choosable when its maximum degree $\Delta(G)$ is not 5. We also study the 3-degeneracy property of G.

1. Introduction

All graphs considered in this paper are finite, loopless and without multiple edges unless otherwise stated. Let G be a graph with the vertex set $V(G)$, the edge set $E(G)$, and the maximum degree $\Delta(G)$. An edge-k-colouring of G is a mapping ϕ from $E(G)$ to the set of colours $\{1,2,\ldots,k\}$ such that $\phi(x) \neq \phi(y)$ for any adjacent edges x and y of G. The graph G is edge-k-colourable if it has an edge-k-colouring. The *chromatic index* $\chi'(G)$ is the smallest integer k such that G is edge-k-colourable. The mapping L is said to be an *edge assignment* for the graph G if it assigns a list $L(e)$ of possible colours to each edge e of G. If G has some edge-k-colouring ϕ such that $\phi(e) \in L(e)$ for all edges e, then we say that G is edge-L-colourable or ϕ is an edge-L-colouring of G. We call G *edge-k-choosable* or *edge-k-list colourable* if it is edge-L-colourable for every edge assignment L satisfying $|L(e)| = k$ for all edges e. An edge-L-colouring ϕ of such an edge assignment L is also called an *edge-k-list colouring*. The *edge choice number* or *list chromatic index* $\chi'_l(G)$ of G is the smallest k such that G is edge-k-choosable.

The following seems to be the most well-known conjecture about edge choosability.

Conjecture 1.1. *If G is a multigraph, then $\chi'_l(G) = \chi'(G)$.*

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This conjecture has been proved for a few special cases, such as bipartite multigraphs [4], complete graphs of odd order [5], multicircuits [11], line-perfect multigraphs [9], and planar graphs G with $\Delta(G) \geqslant 12$ [2].

The following weaker conjecture on edge list colouring was proposed by Vizing (see [8]).

Conjecture 1.2. *Every graph G is edge-$(\Delta(G)+1)$-list colourable.*

An earlier result of Harris [6] shows that $\chi_l'(G) \leqslant 2\Delta(G)-2$ if G is a graph with $\Delta(G) \geqslant 3$. This implies Conjecture 1.2 for the case $\Delta(G) = 3$. Recently, Juvan, Mohar and Škrekovski [7] settled the case of $\Delta(G) = 4$. Conjecture 1.2 has also been confirmed for other special cases such as complete graphs [5], graphs with girth at least $8\Delta(G)(\ln\Delta(G)+1.1)$ [8], and planar graphs with $\Delta(G) \geqslant 9$ [1]. In this paper we will prove that Conjecture 1.2 holds for a planar graph G with no 6-cycles and $\Delta(G) \neq 5$.

2. Notation

A *plane* graph G is a particular drawing in the Euclidean plane of a certain planar graph. We denote its face set, order, and minimum degree by $F(G)$, $|G|$, and $\delta(G)$, respectively. Let $d_G(v)$ (or $d(v)$) denote the degree of v in G. Let $N_G(v)$ (or $N(v)$) denote the set of neighbours of the vertex v in G. For $f \in F(G)$, we use $b(f)$ to denote the boundary of f and write $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in the clockwise order. The set of boundary vertices of f is occasionally denoted by $V(f)$. A face f of G is called a *simple* face if $b(f)$ forms a cycle. Obviously, each k-face for $3 \leqslant k \leqslant 5$ is a simple face when $\delta(G) \geqslant 2$. Let $\lambda_G(f)$, or $\lambda(f)$ for short, denote the degree of a face f in G. A vertex (or a face) of degree k is called a *k-vertex* (or *k-face*). We say that two faces of a plane graph are *adjacent* if they share at least one common boundary edge. A vertex v is said to be *incident* to a face f, and *vice versa*, if v lies on the boundary of f. Let $F(v)$ denote the set of all faces that are incident to the vertex v. Furthermore, let $T(v), Q(v)$, and $P(v)$ denote, respectively, the set of 3-faces, the set of 4-faces, and the set of 5-faces that are incident to the vertex v. If $f \in F(G)$, let $T^*(f)$ denote the set of 3-faces that are adjacent to the face f. A *triangle* of a graph G is synonymous with a cycle of length 3. Two distinct triangles are said to be *adjacent* if they have one common edge. A *k-fan* F_k, $k \geqslant 3$, is a plane graph of order $k+1$ obtained from a path $P_k = x_1x_2\cdots x_k$ by adding a new vertex x_{k+1} and joining x_{k+1} to every x_i, $1 \leqslant i \leqslant k$. An edge xy of graph G is called *light* if the sum $d(x)+d(y)$ attains the smallest value among all the edges of G.

3. Light edges

Theorem 3.1. *If G is a plane graph without 6-cycles and $\delta(G) \geqslant 3$, then there is an edge $xy \in E(G)$ such that $d(x)+d(y) \leqslant 8$.*

Proof. To prove this by contradiction, we suppose that there is a connected plane graph G with the following properties:

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(a) G does not contain 6-cycles
(b) $\delta(G) \geqslant 3$
(c) $d(x) + d(y) \geqslant 9$ for every pair of adjacent vertices x and y of G.

The following is a list of some configurations that are excluded from G. We need this information to design our discharging procedures:

(P1) a simple 6-face
(P2) a 5-face adjacent to a 3-face
(P3) two adjacent 4-faces
(P4) a 4-face adjacent to two non-adjacent 3-faces
(P5) a 4-face having only one common edge with two adjacent 3-faces
(P6) a 3-face adjacent to three mutually non-adjacent 3-faces
(P7) a 5-fan.

Rewriting Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we have the following equivalent identity:

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (\lambda(f) - 6) = -12.$$

To define a weight function w on $V(G) \cup F(G)$, we let $w(v) = 2d(v) - 6$ if $v \in V(G)$ and $w(f) = \lambda(f) - 6$ if $f \in F(G)$. Thus $\sum\{w(x) |\ x \in V(G) \cup F(G)\} = -12$. Now we are going to define a discharging process that will distribute the weight $w(x)$ to its neighbouring elements while the sum of all weights is kept fixed.

Let $v \in V(G)$. If $d(v) = 3$, we do nothing. Otherwise, we discharge weights according to the following rules.

(R1) If $d(v) = 4$, then $w(v) = 2$. Since $0 \leqslant |T(v)| \leqslant 4$, we consider the following subcases.
(R1.1) If either $|T(v)| = 0$ or $|T(v)| = 4$, we transfer $1/2$ from v to every $f \in F(v)$.
(R1.2) If $|T(v)| = 1$, then $|Q(v)| \leqslant 2$ by (P3). We transfer 1 from v to the unique 3-face, $1/2$ from v to every $f \in Q(v)$ if $Q(v) \neq \emptyset$, and $(1 - \frac{1}{2}|Q(v)|)/|P(v)|$ from v to every $f \in P(v)$ if $P(v) \neq \emptyset$.
(R1.3) If $|T(v)| = 2$, then $|Q(v)| = |P(v)| = 0$ by (P2), (P4), and (P5). We transfer 1 from v to every $f \in T(v)$.
(R1.4) If $|T(v)| = 3$, then $|Q(v)| = |P(v)| = 0$. We transfer $2/3$ from v to every $f \in T(v)$.
(R2) If $d(v) = 5$, then $w(v) = 4$. Since $0 \leqslant |T(v)| \leqslant 3$ by (P7), there are four subcases to handle.
(R2.1) If $|T(v)| = 0$, we transfer $4/5$ from v to every $f \in F(v)$.
(R2.2) If $|T(v)| = 1$, we transfer $4/3$ from v to the unique 3-face and $2/3$ from v to every $f \in Q(v) \cup P(v)$ if $Q(v) \cup P(v) \neq \emptyset$.
(R2.3) If $|T(v)| = 2$, then it follows from (P2), (P3), (P4), and (P5) that both $|Q(v)|$ and $|P(v)|$ are $\leqslant 1$ and $|P(v)| = 1$ implies $|Q(v)| = 0$. We transfer $4/3$ from v to every $f \in T(v)$, $1/2$ from v to every $f \in Q(v)$ if $Q(v) \neq \emptyset$, and $(\frac{4}{3} - \frac{1}{2}|Q(v)|)/|P(v)|$ from v to every $f \in P(v)$ if $P(v) \neq \emptyset$.
(R2.4) If $|T(v)| = 3$, then $|Q(v)| = |P(v)| = 0$. We transfer $4/3$ from v to every $f \in T(v)$.

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(R3) If $d(v) \geqslant 6$, then $0 \leqslant |T(v)| \leqslant d(v)-2$. We transfer $3/2$ from v to every $f \in T(v)$ if $T(v) \neq \emptyset$, 1 from v to every $f \in Q(v)$ if $Q(v) \neq \emptyset$, and $1/2$ from v to every $f \in P(v)$ if $P(v) \neq \emptyset$.

(R4) For every face $f \in F(G)$ with $\lambda(f) \geqslant 6$, we transfer $w(f)/|T^*(f)|$ from f to every $f' \in T^*(f)$ if $T^*(f) \neq \emptyset$.

Let $Z(v,f)$ denote the amount transferred to the face f from a vertex v according to the rules (R1)–(R4). Let $Z(v)$ denote the total amount transferred out of v, that is, $Z(v) = \sum\{Z(v,f)|\ f \in F(v)\}$. We call a vertex v of G an *improper* vertex if $d(v) = 4$, $|T(v)| = 1$, $|Q(v)| = 2$, and $|P(v)| = 1$. If a 5-face f is incident to an improper vertex v, then $Z(v,f) = 0$ by (R1.2).

The following three claims summarize the consequences of the rules (R1)–(R4). The first two follow immediately upon inspection.

Claim 1. *For every vertex v with $d(v) \geqslant 4$, we have $Z(v,f) \geqslant 1/2$ for all $f \in Q(v)$.*

Claim 2. *If v is not an improper vertex and $d(v) \geqslant 4$, then $Z(v,f) \geqslant 1/2$ for every $f \in P(v)$.*

Claim 3. *For every $v \in V(G)$ with $d(v) \geqslant 3$, we have $w(v) \geqslant Z(v)$.*

Claim 3 follows immediately from the discharging rules if $3 \leqslant d(v) \leqslant 5$. If $d(v) = 6$, then $w(v) = 6$ and $|T(v)| \leqslant 4$. When $|T(v)| = 0$, $Z(v,f) \leqslant 1$ for every $f \in F(v)$, hence $Z(v) \leqslant 6$. When $|T(v)| = 1$, we have $|Q(v)| \leqslant 3$, $|P(v)| \leqslant 3$, and $|Q(v)|+|P(v)| \leqslant 5$. Thus $Z(v) \leqslant 11/2$. When $|T(v)| = 2$, we have $|Q(v)| + |P(v)| \leqslant 3$, hence $Z(v) \leqslant 6$. When $|T(v)| = 3$, we have $|Q(v)| + |P(v)| \leqslant 1$, hence $Z(v) \leqslant 11/2$. When $|T(v)| = 4$, obviously $|Q(v)| + |P(v)| = 0$, hence $Z(v) = 6$. If $d(v) = 7$, then $w(v) = 8$ and $|T(v)| \leqslant 5$. When $|T(v)| \leqslant 2$, then $Z(v) \leqslant 8$ by (R3). When $|T(v)| = i$ for $i = 3,4,5$, we have $|Q(v)| + |P(v)| \leqslant 5 - i$, hence $Z(v) \leqslant 5 + i/2$. If $d(v) = 8$, then $w(v) = 10$ and $|T(v)| \leqslant 6$. When $|T(v)| \leqslant 4$, we have $Z(v) \leqslant 10$. When $|T(v)| = 5$, we have $|Q(v)| + |P(v)| \leqslant 2$, hence $Z(v) \leqslant 19/2$. When $|T(v)| = 6$, we have $|Q(v)| + |P(v)| = 0$, hence $Z(v) \leqslant 9$. Finally we suppose $d(v) \geqslant 9$. Clearly, $|T(v)| \leqslant d(v) - 2$. If $|T(v)| \leqslant d(v) - 3$, then $Z(v) \leqslant (3/2)(d(v) - 3) + 3 = 2d(v) - 6 - (d(v) - 9)/2 \leqslant 2d(v) - 6 = w(v)$. If $|T(v)| = d(v) - 2$, we have $|Q(v)| = |P(v)| = 0$. So $Z(v) = 3(d(v) - 2)/2 = 2d(v) - 6 - (d(v) - 6)/2 \leqslant 2d(v) - 6 = w(v)$.

Let $w'(x)$ denote the final weight function when discharging is complete. Since the sum of all weights was kept fixed during discharging, if we can show $w'(x) \geqslant 0$ for every $x \in V(G) \cup F(G)$, then an obvious contradiction arises and the proof is complete. In fact, this is known to be true for all $x \in V(G)$ by Claim 3.

Now let $f \in F(G)$. If $\lambda(f) \geqslant 6$, it follows from (R4) that $w'(f) \geqslant 0$. If $\lambda(f) = 4$, then $w(f) = -2$. If the degree of every vertex in $b(f)$ is at least 4, then $w'(f) \geqslant 0$ by Claim 1. Otherwise, condition (c) implies that there are two non-adjacent boundary vertices having degree at least 6. Then $w'(f) \geqslant 0$ by (R3). If $\lambda(f) = 3$, then $w(f) = -3$. Let $f = [uvt]$ and suppose that $d(u) \geqslant d(v) \geqslant d(t)$. If $d(t) \geqslant 5$, then $w'(f) \geqslant 1$ by (R2). If $d(t) = 4$, then $d(u) \geqslant d(v) \geqslant 5$ by (c). It follows that $Z(t,f) \geqslant 1/2$, $Z(u,f) \geqslant 4/3$, and $Z(v,f) \geqslant 4/3$. Hence $w'(f) \geqslant 1/6$. If $d(t) = 3$, then $d(u) \geqslant d(v) \geqslant 6$ by (c). Thus (R3) guarantees $w'(f) \geqslant 0$. Finally suppose $\lambda(f) = 5$ and $f = [u_1u_2u_3u_4u_5]$. Hence $w(f) = -1$. If there is a boundary vertex whose degree is 3, then the two adjacent boundary vertices are of degree at least 6. Thus neither of them is an improper vertex. By Claim 2, $w'(f) \geqslant 0$. If all

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boundary vertices are of degree at least 4, then at most two of them are improper vertices, hence $w'(f) \geqslant 1/2$. Suppose that there were at least three improper boundary vertices. Then two of them, say u_1 and u_2, are adjacent: this yields an obvious contradiction to (c). □

Corollary 3.2. *If G is a planar graph without 6-cycles, then $\delta(G) \leqslant 4$.*

Proof. If $\delta(G) > 4$, then there is an edge xy such that $d(x) + d(y) \leqslant 8$ by Theorem 3.1. However, this is impossible. □

As an application of Theorem 3.1, we shall investigate the edge list colouring of a plane graph G without 6-cycles. It is obvious that every subgraph of a planar graph without 6-cycles is also a planar graph without 6-cycles. Moreover, every subgraph of an edge-k-list colourable graph is also edge-k-list colourable. These simple facts are essential for carrying out the induction in the following proof.

Theorem 3.3. *Every planar graph G without 6-cycles is edge-t-list colourable, where $t = \max\{7, \Delta(G) + 1\}$.*

Proof. The proof is carried out by induction on $|G| + |E(G)|$. The theorem holds trivially for $|G| + |E(G)| \leqslant 7$. Let G be a planar graph without 6-cycles such that $|G| + |E(G)| \geqslant 8$. Suppose that L is an edge assignment of G such that $|L(e)| = t$ for each $e \in E(G)$. If $\delta(G) \leqslant 2$, let v be a vertex of minimum degree in G. By the induction hypothesis, $G - v$ has an edge-L-colouring ϕ. If $d(v) \leqslant 1$, ϕ can be easily extended to an edge-L-colouring of G. If $d(v) = 2$, let x and y denote the two neighbours of v in G. We are able to extend ϕ to the edges vx and vy of G. This can be accomplished because there are at most $\Delta(G) - 1$ forbidden colours for vx and $\Delta(G)$ forbidden colours for vy, whereas the number of available colours is at least $\Delta(G) + 1$. If $\delta(G) \geqslant 3$, G contains an edge xy such that $d(x) + d(y) \leqslant 8$ by Theorem 3.1. The induction hypothesis implies that $G - xy$ has an edge-L-colouring ϕ. We can colour xy with some colour from $L(xy)$ that was not used by ϕ on the edges adjacent to xy. Since there exist at most 6 such edges and $|L(xy)| = t \geqslant 7$, the required colour is available. □

Combining Theorem 3.3 and the results of [6] and [7], we obtain the following.

Theorem 3.4. *If G is a planar graph without 6-cycles and $\Delta(G) \neq 5$, then $\chi'_l(G) \leqslant \Delta(G) + 1$.* □

4. 3-degeneracy

A graph is said to be *k-degenerate* if every induced subgraph contains a vertex of degree at most k. We are interested in knowing to what extent the lack of cycles of a fixed length implies 3-degeneracy. Clearly, a planar graph without 3-cycles is 3-degenerate since it always contains a vertex of degree at most 3. We proved in [10] that a planar graph

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without 5-cycles is 3-degenerate. Note that an icosidodecahedron, that is, the line graph of a dodecahedron, is a 4-regular planar graph without 4-cycles. Hence the lack of 4-cycles does not imply the 3-degeneracy of a planar graph. Choudum [3] constructed 4-regular 3-connected planar graphs without k-cycles for each $k \geqslant 7$. As far as we know, the case for $k = 6$ remains unsettled. Here we propose the following conjecture and establish some partial results.

Conjecture 4.1. *Every planar graph without* 6*-cycles is* 3*-degenerate.*

Theorem 4.2. *Let G be a plane graph without* 6*-cycles. If G does not contain two adjacent triangles, then G is* 3*-degenerate.*

Proof. Note that every subgraph of G is also a plane graph without 6-cycles that does not have two adjacent triangles. Thus we only need to prove $\delta(G) \leqslant 3$. First suppose that G is 2-connected. This implies that every face of G is a simple face.

To prove by contradiction, we assume that the theorem is false. Then the following properties hold:

(a′) $\delta(G) = 4$
(b′) G does not contain 6-cycles
(c′) any two 3-faces of G are non-adjacent.

The following is another equivalent form of Euler's formula:

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2\lambda(f) - 6) = -12.$$

Let w denote the weight function defined on $V(G) \cup F(G)$ by $w(v) = d(v) - 6$ if $v \in V(G)$ and $w(f) = 2\lambda(f) - 6$ if $f \in F(G)$. Thus $\sum\{w(x)|\ x \in V(G) \cup F(G)\} = -12$. For a face $f \in F(G)$, let $\mathrm{im}(f)$ denote the number of improper vertices in G which are incident to f. We define the following discharging rules.

(R′) If $\lambda(f) = 4$, we transfer $3/5$ from f to every improper 4-vertex of $V(f)$, and $(2 - \frac{3}{5}\mathrm{im}(f))/(4 - \mathrm{im}(f))$ to every other vertex of $V(f)$.
(R″) If $\lambda(f) \geqslant 5$, we transfer $w(f)/\lambda(f)$ from f to every vertex of $V(f)$.

The sum of all weights is kept fixed during discharging. Let the new weight function be w' after discharging is complete. In order to complete the proof by contradiction, it suffices to check $w'(x) \geqslant 0$ for every $x \in V(G) \cup F(G)$.

First suppose that $f \in F(G)$. If $\lambda(f) = 3$, then $w'(f) = w(f) = 0$. If $\lambda(f) \geqslant 5$, $w'(f) = 0$ by (R″). Now let $\lambda(f) = 4$. We claim that f is incident to at most two improper vertices. Thus $w'(f) = 0$ by (R′). If f is incident to at least three improper vertices, then f is adjacent to at least two 3-faces by (P3). This implies that G contains a 6-cycle, contradicting the property (b′). Next we consider $v \in V(G)$. We have $d(v) \geqslant 4$ by (a′). If $d(v) \geqslant 6$, then $w'(v) \geqslant w(v) \geqslant 0$. If $d(v) = 5$, v is incident to at least three faces of degree at least 4 by (c′). If the degree of a face f satisfies $\lambda(f) \geqslant 5$, then $w(f)/\lambda(f) = 2 - (6/\lambda(f)) \geqslant 4/5$. However, if $\lambda(f) = 4$, then the vertex v, which is not improper, again receives at least $2/5$ from f since f is incident to at most two improper vertices. We have $w'(v) \geqslant w(v) + 6/5 = 1/5$ by

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(R′) and (R″). Finally suppose that $d(v) = 4$. By (c′) and (b′), v is incident to at most two 3-faces and at most two 4-faces. If v is not incident to any 3-face, then v is not improper and $w'(v) \geqslant -2 + 4/5 + 8/5 = 2/5$. If v is incident to two 3-faces, then v is not incident to any 4-face or 5-face by (b′), that is, v is incident to exactly two faces of degree at least 7. Since $w(f)/\lambda(f) = 2 - 6/\lambda(f) \geqslant 8/7$ for every $f \in F(G)$ with $\lambda(f) \geqslant 7$, we have $w'(v) \geqslant -2 + 8/7 + 8/7 = 2/7$. Suppose that v is incident to exactly one 3-face f_0. When v is incident to at most one 4-face, v is not improper and $w'(v) \geqslant -2 + 2/5 + 8/5 = 0$. When v is incident to two 4-faces and one 5-face, that is, v is an improper vertex, we have $w'(v) \geqslant -2 + 6/5 + 4/5 = 0$. Finally, we assume that v is incident to two 4-faces, say f_1 and f_2, and one k-face, say f_3, for some $k \geqslant 7$. Thus f_1 is not adjacent to f_2 by (b′). This implies that both f_1 and f_2 are adjacent to f_0 and f_3. First note that v is not an improper vertex since $d(f_3) \geqslant 7$. If $\mathrm{im}(f_1) \geqslant 2$, it is easy to see that f_1 is adjacent to at least two 3-faces: this is impossible. Thus we have $\mathrm{im}(f_1) \leqslant 1$ and similarly $\mathrm{im}(f_2) \leqslant 1$. By (R′), v receives at least $7/15$ from f_i for $i = 1$ and 2. Consequently, $w'(v) \geqslant -2 + 7/15 + 7/15 + 8/7 = 8/105$. We have thus far proved the theorem for the 2-connected case.

Now suppose that G contains a cut vertex. Let B be a block of G such that B contains a unique cut vertex t of G. Thus $d_B(t) \geqslant 2$. We are going to show that B has a vertex v^* such that $d_B(v^*) = d_G(v^*) \leqslant 3$. If $d_B(t) \geqslant 4$, we may apply the preceding proof to the graph B to get a contradiction. So we assume $d_B(t) \leqslant 3$. Similarly, we define the weight function w on B by $w(v) = d_B(v) - 6$ for $v \in V(B)$ and $w(f) = 2\lambda_B(f) - 6$ for $f \in F(B)$. Note that $w(t) = d_B(t) - 6 \geqslant -4$. Applying Euler's formula to B, we obtain the following:

$$\sum \{w(x) |\ x \in (V(B) \cup F(B)) - \{t\}\} = -12 - w(t) \leqslant -8.$$

Let w' denote the new weight function when discharging is complete according to the rules (R′) and (R″), except that t receives nothing from its incident faces. We can argue similarly to the 2-connected case that $w'(x) \geqslant 0$ for all $x \in (V(B) \cup F(B)) - \{t\}$ when every vertex of $V(B) - \{t\}$ has degree at least 4. Since the sum of all weights was kept fixed during discharging, a contradiction is derived. □

Corollary 4.3. *Every planar graph without* 4*-cycles and* 6*-cycles is* 3*-degenerate.*

Proof. Since G does not contain any 4-cycle, G cannot contain two adjacent triangles. The result follows from Theorem 4.2. □

In the remaining part of this section, we assume that G is a plane graph without 6-cycles and 7-cycles, and $\delta(G) \geqslant 3$. In addition to (P1)–(P7), the following configurations are also excluded from G:

(P8) a simple 7-face
(P9) a 4-face adjacent to a 5-face
(P10) a 3-face adjacent to two 4-faces when $\delta(G) \geqslant 4$.

A 3-face f of G is said to be *fat* if $|T^*(f)| = 2$. A fat 3-face f is called 4-*fat* if $d(v) = 4$ for every $v \in V(f)$. Let $T_f(v)$ denote the set of all fat 3-faces that are incident to the vertex v. Let $T_f^*(g)$ and $ST_f^*(g)$ denote, respectively, the set of all fat 3-faces and the set of all

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4-fat 3-faces that are adjacent to the face g. A face f or a vertex v is said to be *major* if $\lambda(f) \geqslant 8$ or $d(v) \geqslant 5$.

Theorem 4.4. *If G is a plane graph without 6-cycles and 7-cycles, then G is 3-degenerate.*

Proof. It suffices to show $\delta(G) \leqslant 3$. First suppose that G is 2-connected. Thus each face of G is a simple face. To prove by contradiction, we assume that there is a 2-connected plane graph G that has the following properties:

(a″) G contains neither 6-cycles nor 7-cycles

(b″) $\delta(G) = 4$.

The following is another identity equivalent to Euler's formula:

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (\lambda(f) - 4) = -8.$$

Let w denote the weight function defined on $V(G) \cup F(G)$ by $w(v) = d(v) - 4$ if $v \in V(G)$ and $w(f) = \lambda(f) - 4$ if $f \in F(G)$. Thus $\sum\{w(x) |\ x \in V(G) \cup F(G)\} = -8$. We are going to use the following discharging rules. Again, the sum of all weights is kept fixed during discharging.

(r1) We transfer $1/2$ from a major vertex v to every face $g \in T_f(v)$ if $T_f(v) \neq \emptyset$.

(r2) We transfer 1 from a major face g to every face $g' \in ST_f^*(g)$ if $ST_f^*(g) \neq \emptyset$, and $1/2$ from a major face g to every face $g'' \in T^*(g) - ST_f^*(g)$ if $T^*(g) - ST_f^*(g) \neq \emptyset$.

Let w' denote the final weight function when discharging is complete. An obvious contradiction arises if we can show $w'(x) \geqslant 0$ for all $x \in V(G) \cup F(G)$. First consider $v \in V(G)$. By (b″), $d(v) \geqslant 4$. If $d(v) = 4$, then $w'(v) = 0$. If $d(v) = 5$, then $w(v) = 1$. It is easy to see from (P1)–(P10) that $|T_f(v)| \leqslant 2$. Thus we transfer at most 1 from v to its incident faces by (r1), so $w'(v) \geqslant 0$. If $d(v) \geqslant 6$, then $|T_f(v)| \leqslant |T(v)| \leqslant d(v) - 2$. Since we transfer at most $(d(v) - 2)/2 \leqslant d(v) - 4 = w(v)$ from v to its incident faces, we have $w'(v) \geqslant 0$. Now consider $f \in F(G)$. Obviously, $\lambda(f) \neq 6$ and 7. If $\lambda(f)$ is 4 or 5, then $w'(f) = w(f) \geqslant 0$. If $\lambda(f) = 3$, then $w(f) = -1$. If $|T^*(f)| = 0$, then f is adjacent to at least two major faces by (P1), (P2), (P8) and (P10). If $|T^*(f)| = 1$, then f is adjacent to exactly two major faces. Thus, for these two cases, we have $w'(f) \geqslant 0$ by (r2). If $|T^*(f)| = 2$, that is, f is fat, let f' denote the third face adjacent to f. Obviously, $\lambda(f') \geqslant 8$. When f is a 4-fat 3-face, f receives 1 from f' by (r2) and $w'(f) \geqslant 0$. When f is not 4-fat, $V(f)$ contains a major vertex u. Thus f receives $1/2$ from u by (r1) and $1/2$ from f' by (r2). Again, we have $w'(f) \geqslant 0$. If $\lambda(f) \geqslant 8$, let $f = [x_1 x_2 \cdots x_n]$. Let $f_{k,k+1}$ denote the face of G that shares the edge $x_k x_{k+1}$ with f, where the indices are taken modulo n.

Observation. *If $f_{i,i+1}$ is a 4-fat 3-face, then $f_{i-1,i}$ and $f_{i+1,i+2}$ are not 3-faces.*

Let $f_{i,i+1} = [ux_i x_{i+1}]$ be a 4-fat 3-face. There are distinct vertices u_1 and u_2 such that $[uu_1 x_i]$ and $[uu_2 x_{i+1}]$ are 3-faces adjacent to $[ux_i x_{i+1}]$. Suppose that $f_{i+1,i+2} = [vx_{i+1}x_{i+2}]$ is a 3-face. If $v = u_2$, then a 6-cycle $u_1 u u_2 x_{i+2} x_{i+1} x_i u_1$ arises, which contradicts (a″). It follows that $d(x_{i+1}) \geqslant |\{x_i, x_{i+2}, v, u, u_2\}| = 5$. However, this contradicts the fact that $f_{i,i+1}$ is 4-fat. Similarly, $f_{i-1,i}$ is not a 3-face.

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Now we may replace (r2) by the following two successive steps.

(r2.1) We transfer $1/2$ from f to all its adjacent faces $f_{i,i+1}$.

(r2.2) If $f_{i,i+1}$ is a 4-fat 3-face, then we transfer $1/2$ from $f_{i-1,i}$ to $f_{i,i+1}$.

Since $\lambda(f) \geqslant 8$ and we transfer at most $\lambda(f)/2$ from f to its adjacent faces by (r2.1), we have $\lambda(f)/2 \leqslant \lambda(f) - 4 = w(f)$; hence $w'(f) \geqslant 0$. By our Observation, (r2.1) and (r2.2) imply that every 4-fat 3-face $f_{i,i+1}$ receives 1 from f and any other 3-face receives $1/2$ from f. Hence the results of executing (r2.1) and (r2.2) are the same as (r2). We have thus proved the 2-connected case of the theorem.

Now suppose that G contains a cut vertex. Let B be a block of G such that B contains a unique cut vertex t of G. Thus $d_B(t) \geqslant 2$. We will prove that B has a vertex x_0 such that $d_B(x_0) = d_G(x_0) \leqslant 3$. This follows immediately from the preceding proof if $d_B(t) \geqslant 4$. We assume $d_B(t) \leqslant 3$. Define $M(t) = (d_B(t) - 4) + \sum\{\lambda_B(f) - 4 |\ f \in F(t)\}$. Then $M(t) \geqslant (d_B(t) - 4) - |F(t)| = -4$. Let w denote the weight function defined on B by $w(v) = d_B(v) - 4$ for $v \in V(B)$ and $w(f) = \lambda_B(f) - 4$ for $f \in F(B)$. Applying Euler's formula to B, we obtain the following:

$$\sum\{w(y) |\ y \in (V(B) - \{t\}) \cup (F(B) - F(t))\} = -8 - M(t) \leqslant -4.$$

Let w' denote the final weight function when discharging is complete according to the rules (r1) and (r2) except that

(i) we transfer nothing from t to its incident faces

(ii) each face in $F(t)$ receives nothing from its adjacent faces and incident vertices.

We can argue similarly to the 2-connected case that $w'(x) \geqslant 0$ for all $x \in (V(B) - \{t\}) \cup (F(B) - F(t))$ when every vertex of $V(B) - \{t\}$ has degree at least 4. Since the sum of all weights was kept fixed during discharging, a contradiction is derived. □

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