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Degree Powers in Graphs: The Erdős–Stone Theorem

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Dedicated to the memory of our dear friend Richard Schelp

Let $1 \leqslant p \leqslant r+1$, with $r \geqslant 2$ an integer, and let G be a graph of order n. Let $d(v)$ denote the degree of a vertex $v \in V(G)$. We show that if

$$\sum_{v \in V(G)} d^p(v) > (1-1/r)^p n^{p+1},$$

then G has more than

$$\frac{1}{2^{6r(r+1)} r^r} n^{r-1}$$

$(r+1)$-cliques sharing a common edge. From this we deduce that if

$$\sum_{v \in V(G)} d^p(v) > (1-1/r)^p n^{p+1} + C,$$

then G contains more than

$$\frac{C}{p 2^{6r(r+1)+1} r^r} n^{r-p}$$

cliques of order $r+1$.

In turn, this statement is used to strengthen the Erdős–Stone theorem by using $\sum_{v \in V(G)} d^p(v)$ instead of the number of edges.

1. Introduction and main results

Given a graph G and a vertex $u \in V(G)$, we write $d(u)$ for the degree of u. The sum

$$f_p(G) = \sum_{u \in V(G)} d^p(u),$$

is a much studied parameter in graph theory, especially for $p = 2$. In [4], Caro and Yuster

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raised a Turán-type problem for $f_p(G)$: Given $p \geqslant 1$, how large can $f_p(G)$ be if G has no subgraph of a particular type? It turns out that such problems are usually more difficult for $p > 1$ than for $p = 1$, and often lead to involved analytical questions. Problems of this type have been subsequently studied by several authors: see, *e.g.*, [2], [5] , [8], [11] and [12]. In particular, in [2] it was shown that for every real p, $1 \leqslant p < r$, if G is a graph of sufficiently large order n and has no clique of order $r+1$, then

$$f_p(G) \leqslant f_p(T_r(n)),$$

where $T_r(n)$ denotes the r-partite Turán graph of order n.

In this note we continue this line of investigation and strengthen the Erdős–Stone theorem using $f_p(G)$ instead of the number of edges, that is to say, $f_1(G)/2$. This result is deduced from a number of other results about $f_p(G)$, which seem to be of some interest on their own.

We start with a general theorem about hereditary properties. Recall that a *graph property* is a class of graphs closed under graph isomorphisms. A graph property $\mathcal{Q}$ is called *hereditary* if $G \in \mathcal{Q}$ implies that every induced subgraph of G also belongs to $\mathcal{Q}$. As usual, we set $\mathcal{Q}_n = \{G \in \mathcal{Q} \text{ and } V(G) = [n]\}$. We shall use both $f_p(G)$ and $\sum_{u \in V(G)} d^p(u)$ to improve readability.

Theorem 1.1. *Let $p \geqslant 1$ and let $\mathcal{Q}$ be a hereditary property. Then the limit*

$$\lim_{n\to\infty} \frac{1}{n^{p+1}} \max\left\{ \sum_{v \in V(G)} d^p(v) : G \in \mathcal{Q}_n \right\}$$

exists.

In particular, for the Turán graph $T_r(n)$ one can show that

$$f_p(T_r(n)) \geqslant (1-1/r)^p n^{p+1} + O(n^{p-1}).$$

Thus, writing K_r for the complete graph of order r, for the hereditary property of being K_{r+1}-free the limit in Theorem 1.1 is at least $(1-1/r)^p$. It turns out that for $p \leqslant r+1$ there is also a matching upper bound.

Theorem 1.2. *Let $r \geqslant 2$ and $1 \leqslant p \leqslant r+1$. If G is a K_{r+1}-free graph of order n, then*

$$\sum_{v \in V(G)} d^p(v) \leqslant (1-1/r)^p n^{p+1}. \tag{1.1}$$

We shall deduce Theorem 1.2 from the following analytical inequality.

If $x_1, \dots, x_r$ are non-negative numbers such that $x_1 + \cdots + x_r = 1$, then

$$\sum_{i=1}^{r} x_i (1-x_i)^{r+1} < (1-1/r)^{r+1}$$

unless $x_1 = \cdots = x_r = 1/r$.

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What Theorem 1.2 tells us is that if $r \geqslant 2$, $1 \leqslant p \leqslant r+1$, and G is a graph of order n with

$$\sum_{v \in V(G)} d^p(v) > (1-1/r)^p n^{p+1}, \tag{1.2}$$

then $K_{r+1} \in G$. In fact, a stronger and subtler result holds. In order to state it, we need the following definitions from [2].

An r-joint of size t is a union of t distinct r-cliques sharing an edge. The maximum size of an r-joint in a graph G is called the r-jointsize of G and is denoted by $\mathrm{js}_r(G)$.

The following theorem shows that condition (1.2) implies not only the existence of a single K_{r+1}, but that of large $(r+1)$-joints as well.

Theorem 1.3. *Let $r \geqslant 2$, $1 \leqslant p \leqslant r+1$, and let G be a graph of order n. If*

$$\sum_{v \in V(G)} d^p(v) > (1-1/r)^p n^{p+1}$$

then

$$\mathrm{js}_{r+1}(G) > \frac{1}{2^{6r(r+1)} r^r} n^{r-1}.$$

In the above theorem, the coefficient $2^{-6r(r-1)} r^{-r}$ is far from being optimal, but it makes the statement valid for all n; this coefficient can be increased when n is sufficiently large.

Joints play a crucial role in extremal and spectral graph theory (see, *e.g.*, [9] and [10]). A typical use of them is in proving lower bounds on the number $k_{r+1}(G)$ of $(r+1)$-cliques, as in the following theorem.

Theorem 1.4. *Let $r \geqslant 2$, $1 \leqslant p \leqslant r+1$ and $C > 0$. If G is a graph of order n and*

$$\sum_{u \in V(G)} d^p(u) > (1-1/r)^p n^{p+1} + C,$$

then

$$k_{r+1}(G) > \frac{C}{p 2^{6r(r+1)+1} r^r} n^{r-p}.$$

Note that in this theorem C may depend on n; in particular, we shall be especially interested in the case when it is in the range $\Omega(n^p)$ to $o(n^{p+1})$. Denote by $K_r(s_1, s_2, \ldots, s_r)$ the complete r-partite graph with class sizes $s_1, s_2, \ldots, s_r$. Combining Theorem 1.4 with a result in [7], we shall immediately obtain the following strengthening of the classical Erdős–Stone theorem.

Theorem 1.5. *Let r, p, c and n satisfy*

$$r \geqslant 2, \quad 1 \leqslant p \leqslant r+1, \quad \frac{c}{2^{6r(r+1)+1} r^r (r+1)} > (\log n)^{-1/(r+1)}.$$

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If G is a graph of order n and

$$\sum_{u\in V(G)} d^p(u) > (1-1/r)^p n^{p+1} + cn^{p+1},$$

then G contains a $K_{r+1}(s,\dots,s,t)$, where

$$s = \left\lfloor \frac{c^{r+1}}{2^{6r^3+12r^2+7r+1} r^{r(r+1)} (r+1)^{(r+1)}} \log n \right\rfloor \quad \text{and} \quad t = \lceil n^{1-c^r} \rceil. \tag{1.3}$$

As in Theorem 1.4, the parameter c above may depend on n, *e.g.*, letting $c = 1/\log\log n$, the conclusion is meaningful for sufficiently large n. Observe also the following peculiarity of the graphs $K_{r+1}(s,\dots,s,t)$ in the conclusion of the theorem: if the statement holds for some c, then it holds also for all positive $c' < c$ provided n is large enough. That is to say, when n increases, in addition to the graph $K_{r+1}(s,\dots,s,t)$ guaranteed by the theorem, we can find other, larger and more lopsided graphs $K_{r+1}(s',\dots,s',t')$ with $s' < s$ and $t' > t$.

It is possible that a version of Theorem 1.5 can be deduced from some results of Pikhurko and Taraz [12] as well, but such a deduction does not seem too easy.

2. Proofs

Most of our notation is taken from [1]. Given a graph G, we write:

- $|G|$ for the number of vertices of G;
- $\Gamma_G(u)$ for the set of neighbours of a vertex u, and $d_G(u)$ for $|\Gamma_G(u)|$;
- $\delta(G)$ for the minimum degree of G;
- $G-u$ for the graph obtained by removing the vertex $u \in V(G)$;
- $G-uv$ for the graph obtained by removing the edge $uv \in E(G)$;
- $K_s(G)$ for the set of s-cliques of G, and $k_s(G)$ for $|K_s(G)|$;
- K_r for the complete graph of order r.

We shall need Lemma 5 of [3] and Theorem 1 of [7]: for ease of use we restate them here.

Lemma A. *Let $r \geqslant 2$. If G is a graph of order n, containing a K_{r+1}, and with*

$$\delta(G) > \left(1 - \frac{1}{r} - \frac{1}{r^2(r^2-1)}\right) n,$$

then

$$\mathrm{js}_{r+1}(G) > \frac{n^{r-1}}{r^{r+3}}.$$

Theorem B. *Let $r \geqslant 3$, $(\log n)^{-1/r} \leqslant \alpha \leqslant 1/2$, and let G be a graph of order n. If $k_r(G) > \alpha n^r$, then G contains a $K_r(s,\dots,s,t)$ with $s = \lfloor \alpha^r \log n \rfloor$ and $t > n^{1-\alpha^{r-1}}$.*

In our proofs we shall use several versions of the inequality

$$(1+x)^p > 1 + px,$$

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which is valid for $x \geqslant -1$ and $p \geqslant 1$. In particular, one can easily deduce that if $k > p-1$ and $p \geqslant 1$, then

$$\left(1+\frac{1}{k}\right)^p < 1+\frac{p}{k+1-p}. \tag{2.1}$$

Proof of Theorem 1.1. Although our proof goes along familiar lines, we give it for the sake of completeness.

Set

$$f_p(\mathcal{Q},n) = \max\{f_p(G) : G \in \mathcal{Q}_n\},$$

and let $G \in \mathcal{Q}_n$ be a graph such that $f_p(\mathcal{Q},n) = f_p(G)$.

For every vertex $u \in V(G)$ we have

$$\begin{aligned} f_p(\mathcal{Q},n-1) &\geqslant \sum_{v\in V(G-u)} d^p_{G-u}(v) = \sum_{v\in V(G-u)\setminus\Gamma(u)} d^p_G(v) + \sum_{v\in\Gamma(u)} (d_G(v)-1)^p \\ &\geqslant f_p(\mathcal{Q},n) - d^p_G(u) - p\sum_{v\in\Gamma(u)} d^{p-1}_G(v). \end{aligned}$$

Summing this inequality for all $u \in V(G)$, we obtain

$$nf_p(\mathcal{Q},n-1) \geqslant nf_p(\mathcal{Q},n) - \sum_{u\in V(G)} d^p_G(u) - p\sum_{u\in V(G)}\sum_{v\in\Gamma(u)} d^{p-1}_G(v) = (n-p-1)f_p(\mathcal{Q}_n). \tag{2.2}$$

When $n > p+1$ we see that

$$\frac{f_p(\mathcal{Q},n)}{f_p(\mathcal{Q},n-1)} \leqslant \frac{n}{n-p-1} = 1+\frac{p+1}{n-p-1} < \left(1+\frac{1}{n-p-1}\right)^{p+1} \leqslant \frac{(n-p)^{p+1}}{(n-p-1)^{p+1}}.$$

Thus, for $n > p$ the sequence

$$\frac{f_p(\mathcal{Q},n)}{(n-p)^{p+1}}$$

is non-increasing, and therefore it is converging; clearly, so is $f_p(\mathcal{Q},n)/n^{p+1}$ as well, completing the proof. □

Proof of Theorem 1.2. For $1 \leqslant p \leqslant r+1$, Jensen's inequality implies that

$$\frac{1}{n}\sum_{u\in V(G)} d^p(u) \leqslant \left(\frac{1}{n}\sum_{u\in V(G)} d^{r+1}(u)\right)^{p/(r+1)},$$

so it suffices to prove inequality (1.1) for $p = r+1$.

In [6] Erdős showed that if G is a K_{r+1}-free graph, then there exists an r-partite graph H with $V(H) = V(G)$ such that $d_G(u) \leqslant d_H(u)$ for every $u \in V(G)$. Clearly this implies that the maximum of $f_{r+1}(G)$ among the K_{r+1}-free graphs G of order n is attained on some complete r-partite graph, say the complete r-partite graph $K_r(n_1,\ldots,n_r)$ of order n with

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part sizes $n_1,\dots,n_r$. In this case we have

$$f_{r+1}(K_r(n_1,\dots,n_r)) = \sum_{i=1}^{r} n_i(n-n_i)^{r+1},$$

and so setting

$$F(x_1,\dots,x_r) = \sum_{i=1}^{r} x_i(1-x_i)^{r+1},$$

we see that

$$\frac{f_p(G)}{n^{r+2}} \leqslant \max\{F(x_1,\dots,x_r) : x_1+\cdots+x_r = 1 \text{ and } x_i \geqslant 0 \text{ for } i=1,\dots,r\}. \tag{2.3}$$

We shall prove that

$$F(x_1,\dots,x_r) \leqslant (1-1/r)^{r+1},$$

with equality possible only if $x_1 = \cdots = x_r$. Let $(x_i)_1^r$ be a vector for which the maximum in (2.3) is attained and assume that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_r$.

Assume first that $r=2$, and choose x so that $x_1 = (1-x)/2$, $x_2 = (1+x)/2$, where $0 \leqslant x \leqslant 1$. Then

$$\begin{aligned} F(x_1,x_2) &= \frac{(1-x)}{2}\left(1-\frac{(1-x)}{2}\right)^3 + \frac{(1+x)}{2}\left(1-\frac{(1+x)}{2}\right)^3 \\ &= \frac{1}{16}(1-x^2)((1-x)^2+(1+x)^2) = \frac{1}{8}(1-x^4). \end{aligned}$$

Obviously $F(x_1,x_2)$ is maximum when $x=0$, and so $x_1 = x_2$.

Let us now assume that $r \geqslant 3$. Routine calculations show that the function $x(1-x)^{r+1}$ increases for $0 \leqslant x \leqslant 1/(r+2)$ and decreases for $1/(r+2) \leqslant x \leqslant 1$; also, it is concave for $0 \leqslant x \leqslant 2/(r+2)$ and convex for $2/(r+2) \leqslant x \leqslant 1$. If $x_r \leqslant 2/(r+2)$, the concavity of $x(1-x)^{r+1}$ implies that $x_r = x_1$ and the proof is completed, so we shall assume that $x_r > 2/(r+2)$. Note that in fact $x_r < 1$, for otherwise $F(x_1,\dots,x_r) = 0$. Thus, the convexity of $x(n-x)^{r+1}$ in the interval $(2/(r+2),1)$ implies that only x_r belongs to this interval. Again, the concavity of $x(1-x)^{r+1}$ in the interval $[0,2/(r+2)]$ implies that $x_1 = \cdots = x_{r-1}$. Hence, setting $x = x_1$, we see that $x_r = 1-(r-1)x$.

Using Lagrange multipliers, we find that

$$\frac{\partial F(x_1,\dots,x_r)}{\partial x_1} = \frac{\partial F(x_1,\dots,x_r)}{\partial x_r},$$

which in our case implies that

$$((r+2)x-1)(1-x)^r = (r+1-(r+2)(r-1)x)(r-1)^r x^r. \tag{2.4}$$

We shall show that equality (2.4) is not possible. Assume for a contradiction that (2.4) holds for some x. Clearly we have $x > 1/(r+2)$, and from $1-(r-1)x > 2/(r+2)$ we see that

$$\frac{1}{r+2} < x < \frac{r}{(r+2)(r-1)}.$$

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Write I for the open interval $(1/(r+2), r/((r+2)(r-1))]$ and note first that (2.4) is equivalent to

$$-r+1+\frac{2}{(r+2)x-1}=\frac{1}{(r-1)^r}\left(\frac{1}{x}-1\right)^r. \tag{2.5}$$

Let

$$g(x)=-r+1+\frac{2}{(r+2)x-1},$$
$$h(x)=\frac{1}{(r-1)^r}\left(\frac{1}{x}-1\right)^r,$$

and let $L(x)$ be the linear function

$$\begin{aligned}L(x)&=r-1-2(r+2)(r-1)^2\left(x-\frac{r}{(r+2)(r-1)}\right)\\&=g\left(\frac{r}{(r+2)(r-1)}\right)+\left(x-\frac{r}{(r+2)(r-1)}\right)g'\left(\frac{r}{(r+2)(r-1)}\right).\end{aligned}$$

Note that both $g(x)$ and $h(x)$ are decreasing convex functions in I. Hence, by Taylor's expansion, $g(x)\geqslant L(x)$ for all $x\in I$. We shall show that if $r\geqslant 4$ and $x\in I$, then $h(x)<L(x)$, which contradicts (2.5) and completes the proof for $r>3$. Indeed, if $r\geqslant 4$, we find that

$$h\left(\frac{1}{r+2}\right)=\frac{(r+1)^r}{(r-1)^r}<3r-3=L\left(\frac{1}{r+2}\right),$$

and for $r\geqslant 3$, we find that

$$h\left(\frac{r}{(r+2)(r-1)}\right)=\frac{(r^2-2)^r}{r^r(r-1)^r}<r-1=L\left(\frac{r}{(r+2)(r-1)}\right).$$

For $r=3$ we can apply the same method with a slight modification. Note that we have

$$h(x)<h\left(\frac{1}{5}\right)=8,$$

while for $x>6/25$,

$$g(x)>g(6/25)=8.$$

This means that if x is a solution to (2.5), then $x>6/25$. To finish the proof note that

$$h(6/25)=\frac{6859}{1728}<\frac{22}{5}=14-40\cdot\frac{6}{25}=L\left(\frac{6}{25}\right).$$

Therefore equation (2.5) has no solution in I, and the proof is completed. □

Proof of Theorem 1.4. We shall deduce Theorem 1.4 from Theorem 1.3. Since $f_p(G)>(1-1/r)p n^{p+1}$, Theorem 1.3 implies that $\mathrm{js}_{r+1}(G)>n^{r-1}/(2^{6r(r+1)}r^r)$, which means that some edge of G is contained in at least $n^{r-1}/(2^{6r(r+1)}r^r)$ cliques of order $r+1$. Set $k=0$ and $G_0=G$, and perform the following procedure:

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repeat

find an edge uv *in* G_k *that is contained in at least* $n^{r-1}/(2^{6r(r+1)}r^r)$ *cliques of order* $r+1$;

let $G_{k+1} = G_k - uv$;

add 1 *to* k;

until $f_p(G_k) \leqslant (1-1/r)^p n^{p+1}$.

Note that at the beginning of each iteration we have $f_p(G_k) > (1-1/r)^p n^{p+1}$, and by Theorem 1.3, $\mathrm{js}_{r+1}(G_k) > n^{r-1}/(2^{6r(r+1)}r^r)$, *i.e.*, there is an edge uv contained in at least $n^{r-1}/(2^{6r(r+1)}r^r)$ cliques of order $r+1$. To estimate how the removal of uv affects $f_p(G_k)$, we find that

$$\sum_{w\in V(G)} d^p_{G_{k+1}}(w) = \sum_{w\in V(G)\setminus\{u,v\}} d^p_{G_k}(w) + (d_{G_k}(v)-1)^p + (d_{G_k}(u)-1)^p$$
$$\geqslant \sum_{w\in V(G)} d^p_{G_k}(w) - pd^{p-1}_{G_k}(v) - pd^{p-1}_{G_k}(u) > \sum_{w\in V(G)} d^p_{G_k}(w) - 2pn^{p-1}.$$

Therefore, upon exiting the procedure, we have

$$(1-1/r)^p n^{p+1} \geqslant f_p(G_k) = \sum_{w\in V(G)} d^p_{G_k}(w) > \sum_{w\in V(G)} d^p_G(w) - 2kpn^{p-1}$$
$$> (1-1/r)^p n^{p+1} + C - 2kpn^{p-1},$$

and so

$$k > \frac{C}{2pn^{p-1}}.$$

Hence, by removing edges the procedure destroys at least

$$k\frac{1}{2^{6r(r+1)}r^r}n^{r-1} > \frac{C}{2p2^{6r(r+1)}r^r}n^{r-p}$$

$(r+1)$-cliques, implying the assertion. □

To simplify the proof of Theorem 1.3, we give two preliminary lemmas. Although the second one seems just a general version of the first, a number of different details require two separate lemmas.

Lemma 2.1. *Let* G *be a graph of order* n, *containing a* K_3. *If* $\mathrm{js}_3(G) \leqslant n/1500$ *and* u *is a vertex with* $d_G(u) = \delta(G)$, *then*

$$\sum_{v\in V(G-u)} d^3_{G-u}(v) > \sum_{v\in V(G)} d^3_G(v) - \frac{499}{1000}n^3.$$

Proof. Since $\mathrm{js}_3(G) \leqslant n/1500 < n/32$, Lemma A implies that $\delta(G) \leqslant 5n/12$. Let u be a vertex such that $d(u) = \delta(G)$, and note that

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$$\sum_{v\in V(G-u)} d^3_{G-u}(v) = \sum_{v\in V(G-u)\setminus\Gamma_G(u)} d^3_G(v) + \sum_{v\in\Gamma_G(u)} (d_G(v)-1)^3$$
$$> \sum_{v\in V(G)} d^3_G(v) - \delta^3 - 3\sum_{v\in\Gamma_G(u)} d^2_G(v).$$

In particular, if $v\in\Gamma_G(u)$, then $|\Gamma_G(v)\cap\Gamma_G(u)|\leqslant \mathrm{js}_3(G)\leqslant n/1500$, and therefore

$$d_G(v)+\delta\leqslant n+n/1500.$$

Hence,

$$-3\sum_{v\in\Gamma_G(u)} d^2_G(v)\geqslant -3\sum_{v\in\Gamma_G(u)}(n+n/1500-\delta)^2 = -3\delta(n+n/1500-\delta)^2,$$

implying that

$$\sum_{v\in V(G-u)} d^3_{G-u}(v) > \sum_{v\in V(G)} d^3_G(v) - 3\delta(n+n/1500-\delta)^2-\delta^3.$$

Simple calculations show that $-3\delta(n+n/1500-\delta)^2-\delta^3$ is non-increasing in δ, and so

$$\sum_{v\in V(G-u)} d^3_{G-u}(v) > \sum_{v\in V(G)} d^3_G(v) - 3(5/12)(1+1/1500-5/12)^2n^3-(5/12)^3n^3$$
$$> \sum_{v\in V(G)} d^3_G(v) - \left(\frac{1}{2}-\frac{1}{1000}\right)n^3,$$

completing the proof. □

We find that proving Lemma 2.2 before the proof of Theorem 1.3 greatly simplifies the latter. However, the logical position of Lemma 2.2 in our argument needs clarification as it may seem to be circular reasoning. We use Lemma 2.2 to prove Theorem 1.3 by induction on r, but to prove the lemma we apply Theorem 1.4 for $r'<r$, which in turn is deduced from Theorem 1.3 for $r'<r$. Therefore, the induction assumption makes our argument logically correct.

Lemma 2.2. *Let $r\geqslant 3$ and let G be a graph of order n, containing a K_{r+1}. If*

$$\mathrm{js}_{r+1}(G)\leqslant \frac{1}{2^{6r^2}r^r}n^{r-1}, \tag{2.6}$$

and u is a vertex with $d_G(u)=\delta(G)$, then

$$\sum_{v\in V(G-u)} d^{r+1}_{G-u}(v) > \sum_{v\in V(G)} d^{r+1}_G(v) - \left(r+2-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}n^{r+1}.$$

Proof. Note that Lemma A, together with (2.6), implies that

$$\delta(G)\leqslant\left(1-\frac{1}{r}-\frac{1}{r^2(r^2-1)}\right)n, \tag{2.7}$$

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Set $\delta = \delta(G)$ and let u be a vertex with $d(u) = \delta$; note that

$$\sum_{v\in V(G-u)} d_{G-u}^{r+1}(v) = \sum_{v\in V(G-u)\setminus\Gamma_G(u)} d_G^{r+1}(v) + \sum_{v\in\Gamma_G(u)} (d_G(v)-1)^{r+1} \geqslant \sum_{v\in V(G)} d_G^{r+1}(v) - \delta^{r+1} - (r+1)\sum_{v\in\Gamma(u)} d_G^r(v). \tag{2.8}$$

It is easy to see that the assertion holds if $\delta \leqslant n/8$. Indeed, in view of (2.8), all we need to show in this case is that

$$\delta^{r+1} + (r+1)\sum_{v\in\Gamma(u)} d_G^r(v) < \left(r+2-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}n^{r+1}.$$

This inequality does hold, since by (2.7) and in view of $(1-1/r)^{r+1} \geqslant 1/8$, we find that

$$\begin{aligned}\delta^{r+1} + (r+1)\sum_{v\in\Gamma(u)} d_G^r(v) &< \frac{n^{r+1}}{8^{r+1}} + (r+1)\delta n^r \\ &\leqslant \frac{1}{8^r}(1-1/r)^{r+1}n^{r+1} + (r+1)\frac{n^{r+1}}{8} \\ &< \left(1-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}n^{r+1} + (r+1)(1-1/r)^{r+1}n^{r+1}.\end{aligned}$$

Therefore, to the end of the proof we may and shall assume that $\delta > n/8$.

Set $H = G[\Gamma_G(u)]$. Note first that

$$k_r(H) \leqslant \frac{1}{2^{6r^2}r^{r+1}}n^r, \tag{2.9}$$

for otherwise there would be a vertex in $\Gamma(u)$ that is common to at least

$$\frac{rk_r(H)}{\delta} > \frac{n^{r-1}}{2^{6r^2}r^r}$$

cliques in $K_r(H)$, and so

$$\mathrm{js}_{r+1}(G) > \frac{1}{2^{6r^2}r^r}n^{r-1},$$

contradicting (2.6).

We shall use inequality (2.9) to prove that for every $k = 1,\dots,r$,

$$\sum_{v\in\Gamma(u)} d_H^k(v) \leqslant \left(1-\frac{1}{r-1}\right)^k \delta^{k+1} + \frac{2(r-1)^{r-1}}{8^{r+1}r^r}\delta n^k. \tag{2.10}$$

Indeed, if this inequality fails for some $k \in [r]$, then applying Theorem 1.4 to the graph H with $p = k$ and

$$C = \frac{2(r-1)^{r-1}}{8^{r+1}r^r}\delta n^k,$$

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we find that

$$k_r(H) > \frac{1}{k2^{6r(r-1)+1}(r-1)^{r-1}} \cdot \frac{2(r-1)^{r-1}}{8^{r+1}r^r}\delta n^k \delta^{r-1-k}$$
$$> \frac{1}{2^{6r^2-3r+3}r^{r+1}} \cdot \frac{n^r}{8^{r-k}} \geqslant \frac{1}{2^{6r^2}r^{r+1}}n^r,$$

contradicting (2.9).

Now, using inequality (2.10) and the inequality $d_G(v) \leqslant n - \delta + d_H(v)$ whenever $v \in \Gamma_G(u)$, we obtain

$$\sum_{v\in\Gamma(u)} d_G^r(v) \leqslant \sum_{v\in\Gamma(u)} (n-\delta+d_H(v))^r = \sum_{i=0}^{r}\binom{r}{i}(n-\delta)^{r-i}\sum_{v\in\Gamma(u)} d_H^i(v)$$
$$\leqslant \delta(n-\delta)^r + \delta\sum_{i=1}^{r}\binom{r}{i}(n-\delta)^{r-i}\left(\left(1-\frac{1}{r-1}\right)^i\delta^i + \frac{2(r-1)^{r-1}}{8^{r+1}r^r}n^i\right)$$
$$= \delta\sum_{i=0}^{r}\binom{r}{i}(n-\delta)^{r-i}\left(1-\frac{1}{r-1}\right)^i\delta^i + \frac{2(r-1)^{r-1}}{8^{r+1}r^r}\delta\sum_{i=1}^{r}\binom{r}{i}(n-\delta)^{r-i}n^i$$
$$< \delta\left(n-\frac{\delta}{r-1}\right)^r + \frac{2(r-1)^{r-1}}{8^{r+1}r^r}\delta(2n-\delta)^r$$
$$< \delta\left(n-\frac{\delta}{r-1}\right)^r + \frac{2(r-1)^{r-1}}{8^{r+1}r^r}2^r\left(1-\frac{1}{r}\right)n^{r+1}$$
$$= \delta\left(n-\frac{\delta}{r-1}\right)^r + \frac{1}{4^{r+1}}\frac{(r-1)^r}{r^{r+1}}n^{r+1}.$$

A simple proof by induction shows that

$$\frac{1}{4^{r+1}}\frac{(r-1)^r}{r^{r+1}} < \frac{1}{2r^2(r+1)^2}\left(1-\frac{1}{r}\right)^{r+1}$$

for $r \geqslant 3$, and so, in view of (2.8), we get

$$\sum_{v\in V(G-u)} d_{G-u}^{r+1}(v) \tag{2.11}$$
$$> \sum_{v\in V(G)} d_G^{r+1}(v) - \delta^{r+1} - (r+1)\delta\left(n-\frac{\delta}{r-1}\right)^r - \frac{1}{2r^2(r+1)}\left(1-\frac{1}{r}\right)^{r+1}n^{r+1}.$$

On the other hand, using calculus, one can show that the function

$$\delta^{r+1} + (r+1)\delta\left(n-\frac{\delta}{r-1}\right)^r$$

is increasing in δ, and so, recalling (2.7), we find that

$$\delta^{r+1} + (r+1)\delta\left(n-\frac{\delta}{r-1}\right)^r \leqslant \left(1-\frac{1}{r}-\frac{1}{r^2(r^2-1)}\right)^{r+1}$$
$$+ (r+1)\left(1-\frac{1}{r}-\frac{1}{r^2(r^2-1)}\right)\left(1-\frac{1}{r-1}\left(1-\frac{1}{r}-\frac{1}{r^2(r^2-1)}\right)\right)^r$$

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$$= \left(1 - \frac{1}{r} - \frac{1}{r^2(r^2-1)}\right)^{r+1} + (r+1)\left(1 - \frac{1}{r} - \frac{1}{r^2(r^2-1)}\right)\left(1 - \frac{1}{r} + \frac{1}{r^2(r-1)^2(r+1)}\right)^r.$$

To estimate the right-hand side of this inequality, let us divide both sides by $(1-1/r)^{r+1}$, thus obtaining

$$\left(\delta^{r+1} + (r+1)\delta\left(n - \frac{\delta}{r-1}\right)^r\right)(1-1/r)^{-r-1} \tag{2.12}$$
$$\leqslant \left(1 - \frac{1}{r(r-1)^2(r+1)}\right)^{r+1} + (r+1)\left(1 - \frac{1}{r(r-1)^2(r+1)}\right)\left(1 + \frac{1}{r(r-1)^3(r+1)}\right)^r.$$

We shall estimate the two terms of the right-hand side above separately. Using inequality (2.1) and some algebra, we see that

$$\left(1 - \frac{1}{r(r-1)^2(r+1)}\right)^{r+1} < 1 - \frac{r+1}{r(r-1)^2(r+1)+r+1} = 1 - \frac{1}{r(r-1)^2+1},$$

and also

$$\left(1 + \frac{1}{r(r-1)^3(r+1)}\right)^r < 1 + \frac{r}{r(r-1)^3(r+1)-r} = 1 + \frac{1}{(r-1)^3(r+1)-1}.$$

Furthermore, from the last inequality we obtain

$$\begin{aligned}
&(r+1)\left(1 - \frac{1}{r(r-1)^2(r+1)}\right)\left(1 + \frac{1}{r(r-1)^3(r+1)}\right)^r \\
&< (r+1)\left(1 - \frac{1}{r(r-1)^2(r+1)}\right)\left(1 + \frac{1}{(r-1)^3(r+1)-1}\right) \\
&= (r+1)\left(1 + \frac{1}{(r-1)^3(r+1)-1}\right) - \frac{(r-1)(r+1)}{r((r-1)^3(r+1)-1)} \\
&\leqslant r+1 + \frac{r+1}{r((r-1)^3(r+1)-1)} \\
&\leqslant r+1 + \frac{1}{r(r-1)^3-1}.
\end{aligned}$$

Now, using the bounds for the two terms on the right-hand side of (2.12), we see that

$$\begin{aligned}
&\left(\delta^{r+1} + (r+1)\delta\left(n - \frac{\delta}{r-1}\right)^r\right)(1-1/r)^{-r-1} \\
&\quad < \left(r+2 - \frac{1}{r(r-1)^2+1} + \frac{1}{r(r-1)^3-1}\right)n^{r+1} \\
&\quad < \left(r+2 - \frac{1}{r^2(r+1)}\right)n^{r+1},
\end{aligned}$$

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which gives

$$\delta^{r+1} + (r+1)\delta\left(n - \frac{\delta}{r-1}\right)^r < \left(r + 2 - \frac{1}{r^2(r+1)}\right)(1-1/r)^{r+1}n^{r+1}.$$

Hence, from (2.11) we obtain

$$\sum_{v\in V(G-u)} d_{G-u}^{r+1}(v) > \sum_{v\in V(G)} d_G^{r+1}(v) - \left(r + 2 - \frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}n^{r+1},$$

completing the proof. □

Proof of Theorem 1.3. We shall apply induction on r, making use of the fact that if Theorem 1.3 holds for some r, then Theorem 1.4 and Lemma 2.2 hold for the same r as well.

Let us first prove the assertion for $r = 2$. We claim that in this case

$$\mathrm{js}_3(G) > n/16000 > \frac{1}{2^{38}}n,$$

as required. Assume for a contradiction that $\mathrm{js}_3(G) \leqslant n/16000$. Theorem 1.2 implies that $K_3 \subset G$, and so

$$n \geqslant 16000\mathrm{js}_3(G) \geqslant 16000.$$

Set $i = 0$ and $G_0 = G$, and let us perform the following procedure:

while $\mathrm{js}_3(G_i) \leqslant |G_i|/1500$ **do begin**
select a vertex in G_i *such that* $d_{G_i}(u) = \delta(G_i)$;
let $G_{i+1} = G_i - u$;
add 1 *to* i;
end

Note that

$$\frac{1}{8}k^4 - \frac{499}{1000}k^3 > \frac{1}{8}(k-1)^4,$$

for $k > 1500$. Hence, we see that if $|G_i| \geqslant n/10 > 1500$ and $f_3(G_i) > |G_i|^4/8$ at the beginning of the **while** loop, then G_i contains a K_3, and since $\mathrm{js}_3(G_i) \leqslant |G_i|/1500$, Lemma 2.1 implies that G_{i+1} also satisfies $f_3(G_{i+1}) > |G_{i+1}|^4/8$.

We claim that at the beginning of the **while** loop we always have $|G_i| > n/10$. Indeed, otherwise at some iteration we would have $|G_i| = \lfloor n/10 \rfloor$, and the following inequalities would hold:

$$\lfloor n/10 \rfloor^4 = |G_i|^4 > f_3(G_i) > f_3(G_1) - \frac{499}{1000}\sum_{i=1}^{n} i^3 > \frac{1}{8}n^4 - \frac{499}{1000}\frac{(n+1)^4}{4}.$$

Hence,

$$\frac{n^4}{10^4} > \frac{1}{8}n^4 - \frac{499}{4000}(n+1)^4,$$

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and so

$$1\mu(G) - \mu(G)\frac{3}{2498} = \frac{\frac{499}{4000}}{\left(\frac{1}{8} - \frac{1}{10^4}\right)} > \frac{n^4}{(n+1)^4} > 1 - \frac{4}{n+1} > 1 - \frac{4}{16000}.$$

This contradiction shows that, when the procedure stops, we have $|G_i| > n/10$. Now, since

$$\mathrm{js}_3(G_i) > \frac{1}{1500}|G_i|,$$

we obtain

$$\mathrm{js}_3(G) \geqslant \mathrm{js}_3(G_i) > \frac{1}{1500}|G_i| > \frac{1}{16000}n,$$

as claimed.

Assume now that $r \geqslant 3$ and that the assertion holds for all integers between 2 and $r-1$. Assume for a contradiction that

$$\mathrm{js}_{r+1}(G) \leqslant \frac{1}{2^{6r(r+1)}r^r}n^{r-1}.$$

Since Theorem 1.2 implies that $K_{r+1} \subset G$, we see that

$$n \geqslant (2^{6r(r+1)}r^r \mathrm{js}_{r+1}(G))^{1/(r-1)} \geqslant (2^{6r(r+1)}r^r)^{1/(r-1)} > 8^{r+2}r.$$

Furthermore, as in the case $r = 2$, we set $i = 0$, $G_0 = G$, and perform the following procedure:

while $\mathrm{js}_{r+1}(G) \leqslant 2^{-6r^2}r^{-r}|G_i|^{r-1}$ **do begin**
select a vertex in G_i *such that* $d_{G_i}(u) = \delta(G_i)$;
let $G_{i+1} = G_i - u$;
add 1 *to* i;
end

Note that if a number k satisfies $k > 2r^2(r+1)(r+2)^2$, then

$$(1-1/r)^{r+1}k^{r+2} - \left(r+2-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}k^{r+1} > (1-1/r)^{r+1}(k-1)^{r+2}.$$

Hence, if $|G_i| \geqslant n/8$ and $f_{r+1}(G_i) > (1-1/r)^{r+1}|G_i|^{r+2}$ at the beginning of the **while** loop, then G_i contains a K_{r+1} and since $\mathrm{js}_{r+1}(G) \leqslant 2^{-6r^2}r^{-r}|G_i|^{r-1}$, Lemma 2.2 implies that G_{i+1} also satisfies $f_{r+1}(G_{i+1}) > (1-1/r)^{r+1}|G_{i+1}|^{r+2}$, in view of

$$|G_i| \geqslant n/8 > 8^{r+1}r > 2r^2(r+1)(r+2)^2.$$

We claim that at the beginning of the **while** loop we always have $|G_i| > n/8$. Indeed, otherwise at some iteration we would have $|G_i| = \lfloor n/8 \rfloor$, and the following inequalities would hold:

$$\begin{aligned}\left(\frac{n}{8}\right)^{r+2} &> f_{r+1}(G_i) > f_{r+1}(G_1) - \left(r+2-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}\sum_{i=1}^{n} i^{r+1} \\ &> (1-1/r)^{r+1}n^{r+2} - \left(r+2-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}\frac{(n+1)^{r+2}}{r+2},\end{aligned}$$

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implying that

$$\left(r+2-\frac{1}{2r^2(r+1)}\right)(1-1/r)^{r+1}\frac{(n+1)^{r+2}}{r+2} > ((1-1/r)^{r+1}-1/8^{r+2})n^{r+2},$$

and so,

$$\begin{aligned}1-\frac{1}{2r^2(r+1)(r+2)} &> \left(1-\frac{1}{8^{r+2}(1-1/r)^{r+1}}\right)\frac{n^{r+2}}{(n+1)^{r+2}}\\ &> \left(1-\frac{1}{8^{r+2}(1-1/r)^{r+1}}\right)\left(1-\frac{r+2}{n+1}\right)\\ &> 1-\frac{1}{8^{r+2}(1-1/r)^{r+1}}-\frac{r+2}{n+1}\\ &> 1-\frac{1}{8^{r+1}}-\frac{r+2}{8^{r+2}r}.\end{aligned}$$

It is easy to see that for $r \geqslant 3$ we have

$$\frac{1}{2r^2(r+1)(r+2)} > \frac{1}{8^{r+1}}+\frac{r+2}{8^{r+2}r},$$

and this contradiction shows that, when the procedure stops, we have $|G_i| > n/8$.

Therefore,

$$\mathrm{js}_{r+1}(G) \geqslant \mathrm{js}_{r+1}(G_i) > \frac{1}{2^{6r^2}r^r}\frac{n^{r-1}}{8^{r-1}} > \frac{1}{2^{6r(r+1)}r^r}n^{r-1}.$$

This completes the induction step and the proof of the theorem. □

Proof of Theorem 1.5. The proof is an immediate consequence of Theorem 1.4 and Theorem B. Indeed, setting

$$C = cn^{p+1},$$

Theorem 1.4 implies that

$$k_{r+1}(G) > \frac{C}{p2^{6r(r+1)+1}r^r}n^{r-p} > \frac{c}{2^{6r(r+1)+1}r^r(r+1)}n^{r+1}.$$

Now, letting

$$\alpha = \frac{c}{2^{6r(r+1)+1}r^r(r+1)},$$

we see that $\alpha > (\log n)^{-1/(r+1)}$ and so Theorem B implies that G contains a $K_{r+1}(s',\ldots,s',t')$, where

$$s' = \lfloor \alpha^{r+1}\log n \rfloor \quad \text{and} \quad t' > n^{1-\alpha^r}.$$

Note that here s' is equal to s in (1.3) and t' is greater than or equal to t in (1.3), so Theorem 1.5 is proved. □

3. Concluding remarks

Here we formulate several open questions related to the general topic of this paper.

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Given an integer $r \geqslant 2$, what is the maximum $p = p_r$ such that if G is a K_{r+1}-free graph, then $f_p(G) \leqslant f_p(T_r(|G|))$ for every graph G?

Another, formally similar, but otherwise rather different, question was raised in [2] and is reiterated here.

Given an integer $r \geqslant 2$, what is the maximum $p = p'_r$ such that if G is a K_{r+1}-free graph, then $f_p(G) \leqslant f_p(T_r(|G|))$ provided $|G|$ is sufficiently large?

In [2], we have determined that $p'_2 = 3$ and in general $p'_r < r + \sqrt{2r}$. Given the result of Erdős [6] used in the proof of Theorem 1.2, this is an essentially analytical and number-theoretical problem. However, contrary to the authors of [12], we believe that finding p'_r is important, as it may have practical implications.

In fact, Pikhurko and Taraz [12] discuss even more general problems, namely, letting φ be a non-negative increasing real function and setting

$$f_\varphi(G) = \sum_{u \in V(G)} \varphi(d_G(u)),$$

they raise Turán-type and Erdős–Stone-type extremal questions using $f_\varphi(G)$ instead of the number of edges. Under certain restrictions on φ, they give several results in a rather general set-up. Nonetheless, we believe that for possible applications it is important to study concrete functions φ, for which the results of [12] say nothing. We end up with one such example.

Given an integer $r \geqslant 2$, what is the maximum integer k such that if G is a K_{r+1}-free graph of order n, then

$$\sum_{u \in V(G)} \binom{d_G(u)}{k} \leqslant \sum_{u \in V(T_r(n))} \binom{d_{T_r(n)}(u)}{k}?$$

The combinatorial implications of this question are obvious, as

$$\sum_{u \in V(G)} \binom{d_G(u)}{k}$$

counts the $(k+1)$-vertex subgraphs of G with a dominating vertex.

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