HOMOGENIZATION THEORY APPLIED TO UNSATURATED SOLID-LIQUID MIXTURE

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ABSTRACT

In this study, we present theoretical derivation of seepage flow in unsaturated and static soil using Homogenization theory. The derivation started in the microscopic scale in the soil. The representative elementary volume (REV) in the soil is set to be one order larger than the scale of characteristic length of pore. Solids in the REV are assumed to be rigid and cohesionless. The liquid velocity in the pore is slow. By no-slip boundary condition on the solid boundary in REV, we could obtain the microscopic flow conditions. Using spatial ensemble average under the microscopic scale, we obtain the relation between water content, pressure head and velocities in macroscopic scale. This macroscopic averaged equation is validated to be equal to Richards' equation.

Keywords: Homogenization theory, Unsaturated, Solid-liquid mixture, Richards' equation.

1. INTRODUCTION

It is well accepted concept that landslide together with enough water can form debris flows. But the mechanism for landslides and occurrence of debris flows are different. Landslide is a bulky motion of a soil where particle displacement is important. However, debris flow is a flowing process where strain rate is important. This means that as the motion changes from landslides to debris flows, the modeling of these physics must be changed. Furthermore, if we consider these phenomena from scales of particles, the major difference would be the interactions between solids and solids as well as solids and fluid in the pore scale.

From the continuum point of view, there are many theories used to interpret the flowing properties of debris flow. An extensive review was given by Ancey [1]. Many theories are validated useful and practical in certain domains. However, most of these theories can be used either when bulk material is almost stationary (such as soil) or has large movement (such as debris flows and avalanches). In the aspect of landslide modeling, Iverson *et al*. [2] gave a review for models involving the effect of pore pressures and granular temperature in the mobilization of debris-flow. In the same paper, they assessed the relationship between Coulomb failure and liquefaction, and considered the role of granular temperature and soil volume change in an infinite-slope formulation. Iverson [3] used a multiple time-scale approach together with Richards' equation to develop a mathematical model to evaluate effects of rainfall infiltration on landslide occurrence and acceleration. The model provided a tool to assess the possibility of landslide triggered by rainfall and post-failure motion. But this approach still used the continuum concept to model sliding process macroscopically. From the authors' knowledge, no theory has accounted for the transition from a pile of stationary mass to debris flows. To tackle this problem, we shall start from a theory that is accurate at a condition close to initiation of solid motion.

As the continuum motion is actually the result from small scale motion, there should be methods to examine the small scale motion and then transfer small scale motion to that of continuum. In such small scale, one should be able to visualize how particles start from dry and stationary state to fully saturated but stationary state, and then start to move.

Homogenization theory [4], which based on the multiple-scale perturbation method together with spatial average technique, is adapted in our problem. It has been applied to porous medium flow and used to derive the averaged seepage flow in pores under saturated and static soil [4] successfully. As the first attempt to study the initiation mechanism of debris-flow, we shall apply homogenization theory on unsaturated soil. Therefore, in this study, we only consider the case of solids in static and start by deriving the unsteady flow motion of liquid in the unsaturated pore as the first step towards our goal.

Under the assumption of static solids, we shall ignore the effect of solid-air interaction. Without any

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DOI : 10.1017/jmech.2012.36
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https://doi.org/10.1017/jmech.2012.36 Published online by Cambridge University Press

assumption of constitutive law for the water-soil mixture, we begin to derive seepage flow condition in the representative element volume (REV from here on) under the microscopic length scale - the scale of pores. Then we use ensemble average to obtain the averaged flow condition in the macroscopic scale - the scale of total soil-water mixture. Using our formulation, we can obtain the Richards' equation [5] by theoretical derivation. Since the original Richards' equation was derived based on the semi-empirical Darcy's law, this can be a support for our formulation.

2. INTRODUCTION TO HOMOGENIZATION THEORY

Homogenization theory is a method to obtain motion equations by using multiple-scale perturbation method together with ensemble average in smaller scales. There are two different characteristic length scales in our problem. The characteristic length scale of a typical pore in soil is defined as the microscopic characteristic length l. On the other hand, the characteristic length scale of bulk soil is the macroscopic length L. Using these two scales, we define the small parameter ε as

$$\varepsilon = \frac{l}{L} \ll 1 , \tag{1}$$

where l and L are micro- and macro-scopic characteristic length scales respectively. With this small parameter ε defined in Eq. (1), the multiple-scale functional dependence for spatial and temporary independent variables as

$$x_i = (x_{0i}, x_{1i}, x_{2i}, ...) \tag{2}$$

and

$$t = (t_0, t_1, t_2, ...) , \tag{3}$$

where the subscript index i indicates three major directions in Cartesian coordinates, $i \in \{x, y, z\}$ and $x_{1i} = \varepsilon x_{0i}$, $x_{2i} = \varepsilon^2 x_{0i}$, ... and $t_1 = \varepsilon t_0$, $t_2 = \varepsilon^2 t_0$, ..., *etc*. Physical dependant variables are also expanded in ε as

$$\Phi(x_i, t) = \Phi_0(x_i, t) + \varepsilon\Phi_1(x_i, t) + \varepsilon^2\Phi_2(x_i, t) + ... , \tag{4}$$

where dependent variable $\Phi(x_i,t)$ can represent velocity, pressure or other perturbed physical variables. Substituting Eqs. (2), (3) and (4) into the governing equations of the problem of interest, we can solve the microscopic solution with the boundary conditions. Then we use the spatial ensemble average over a REV to obtain the averaged physical variable representing the macroscopic property. The ensemble average is defined as

$$\langle\Phi\rangle_{(k)} = \frac{1}{|\Omega_{(k)}|} \iiint_{\Omega_{(k)}} \Phi \, d\,\Omega_{(k)} , \tag{5}$$

where $|\Omega_{(k)}|$ is the total volume of k^{th}-order REV. $\langle\Phi\rangle_{(k)}$ is the volume average of Φ in the k^{th}-order REV. $\langle\Phi\rangle_{(k)}$ represents the $(k+1)^{\text{th}}$-order property and is the function of macroscopic independent variables $x_{(k+1)i}$, $x_{(k+2)i}$, ..., *etc*. The volume $\Omega_{(k)}$ can also be a function of time in an unsaturated soil. Then the physical properties of the solid-liquid mixture can be found using these averaged results with boundary conditions under macroscopic scale.

3. GOVERNING EQUATIONS

In this study, our goal is to derive the seepage flow condition in unsaturated soil under static solid structure. We consider the pores in soil are large enough for water to form a free surface (Fig. 1). Besides, solids itself in soil is assumed to be rigid, and water is considered incompressible. If solid structure is stationary, we only need the governing equations for water, which are the continuity and Navier-Stokes equations.

$$\frac{\partial u_i}{\partial x_i} = 0 , \quad i, j = x, y, z \tag{6}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} , \quad i, j = x, y, z \tag{7}$$

where $p = p' - \rho g z$. ρ and μ are the density and dynamic viscosity of water, which are all constants. g is the gravitational acceleration. p is dynamic pressure where p' is total pressure.

For boundary conditions, we need them for different scales. At microscopic scale, there are kinematic boundary conditions (KBC) at free surface and solid-water interface, and a dynamic boundary condition (DBC) at free surface.

$$u_i = 0 \ \text{ on } \ \Gamma , \ i, j = x, y, z \tag{8}$$

$$\frac{D\mathrm{H}}{Dt} = 0 \quad \text{at free surface}$$
$$\mathrm{H}(x, y, z, t) = z - h(x, y, t) = 0 , \tag{9}$$

$$\left[(-p - \rho g z + \delta_s \kappa)\,\delta_{ij} + 2\mu e_{ij}\right] \cdot \frac{\partial \mathrm{H}}{\partial x_j} = 0 \ \text{ at free surface}$$
$$\mathrm{H} = 0 , \ i, j = x, y, z \tag{10}$$

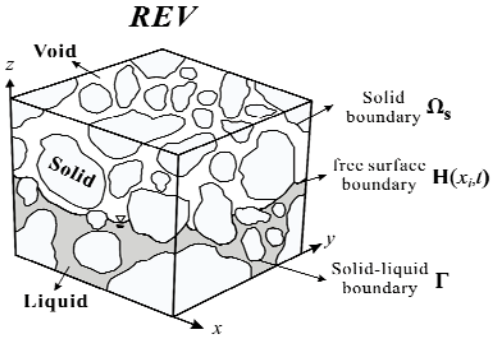


Fig. 1 Definition of free surface of water, solid and solid-liquid boundary in the REV. All boundaries and free surface are three dimensional

https://doi.org/10.1017/jmech.2012.36 Published online by Cambridge University Press

where $e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$. Γ is the solid-water boundary in REV (Fig. 1). Free water surface boundary is expressed by the function $\mathrm{H}(x, y, z, t) = z - h(x, y, t) = 0$ where $h(x, y, t)$ defines the vertical elevation of free water surface in unsaturated REV. δ_s is the surface-tension coefficient ranging from 0.0757N/m at 0°C to 0.0589N/m at 100°C [6]; and κ is the curvature of the free surface. δ_{ij} and e_{ij} are Kronecker delta function and the strain-rate tensor respectively. Equation (8) is the no-slip condition on the solid surface due to static solid structure assumption. By the definition of dynamic pressure p, the free surface DBC yields Eq. (10) with the substitution $p' = p - \rho g z$. Boundary conditions Eqs. (9) and (10) are used only for unsaturated microscopic REV.

From the definition of spatial average Eq. (5), we could obtain the volume averaged seepage flow condition in 0th-order REV by substituting $u_{(k)i}$ into Eq. (5) to give

$$\left\langle u_{(k)i}\right\rangle_0 = \frac{1}{|\Omega_0|}\iiint_{\Omega_{l0}} u_{(k)i}\, d\Omega_0 , \tag{11}$$

where $|\Omega_0|$ and Ω_{l0} are the total REV volume and liquid volume of 0th-order respectively. We also define the porosity η and water content θ in kth-order as

$$\eta_{(k)} = 1-\frac{\Omega_{s(k)}}{|\Omega_{(k)}|}, \tag{12}$$

$$\theta_{(k)} = \frac{\Omega_{l(k)}}{|\Omega_{(k)}|}, \tag{13}$$

where $\Omega_{s(k)} = \iiint_{\Omega_{s(k)}} dV$ and $\Omega_{l(k)} = \iiint_{\Omega_{l(k)}} dV$.

$\Omega_{s(k)}$ is the total solid volume in kth-order REV, and $\eta_{(k)}$ and $\theta_{(k)}$ are functions of $x_{(k+1)i}$, $x_{(k+2)i}$, $\cdots$, t_0, $t_1\cdots$. If the kth-order REV is saturated, we could have the relation that $|\Omega_{(k)}| = \Omega_{l(k)} + \Omega_{s(k)}$. But in unsaturated REV, $\Omega_{l(k)}$ can vary in time. In most soil, water content θ ranges from 0 to η, and the porosity η ranges from 0.25 to 0.75 [7].

4. NORMALIZATION

In our problem, the outer physical excitation to the microscopic REV is the macroscopic pressure gradient. The order of magnitude of pressure is defined from external input from macroscopic scale and can not be defined microscopically alone by the physical system. Therefore, the order of p cannot be expressed explicitly on microscopic scale. We shall simply denote the pressure scale as P. Due to the viscous effects dominate in the flow in pores in REV, the viscous term in micro-scale, $O(\mu U / l^2)$, must be as important as the macro-scale pressure gradient, $O(P / L)$. This gives the characteristic scale of velocity $O(Pl^2 / \mu L)$. So all scales are defined as

$$(x_i, p, u_i, t, \mathrm{H}) = \left(l, P, \frac{Pl^2}{\mu L}, \frac{\mu L}{Pl}, l\right)(\tilde{x}_i, \tilde{p}, \tilde{u}_i, \tilde{t}, \tilde{\mathrm{H}}). \tag{14}$$

where $\tilde{}$ indicates the normalized dimensionless variables. Also, we define the Reynolds number of seepage flow in pores as $\mathrm{Re} = \rho l^2 P / \mu^2$, where in porous media $\mathrm{Re} = O(1)$. Substituting all dimensionless variables in Eq. (14) into Eqs. (6) to (10), and omitting all the tildes, we obtain the dimensionless equations as

$$\frac{\partial u_i}{\partial x_i} = 0 , \tag{15}$$

$$\varepsilon^2\,\mathrm{Re}\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i}+\varepsilon\frac{\partial^2 u_i}{\partial x_j\,\partial x_j} . \tag{16}$$

With normalized boundary conditions,

$$u_i = 0 \quad \text{on} \quad \Gamma , \tag{17}$$

$$\frac{D\mathrm{H}}{Dt} = 0 \ \text{ at } \ \mathrm{H} = 0 , \tag{18}$$

$$\left[(-p-\rho g z+\beta\kappa\delta_{ij}+2e_{ij})\right]\cdot\frac{\partial \mathrm{H}}{\partial x_j} = 0 \ \text{ at } \ \mathrm{H} = 0 , \tag{19}$$

where $\beta = \delta_s Pl^2 / \mu L$ and is the ratio of surface tension to shear stress at free surface in the pores.

Velocity, water pressure and free surface are expanded in terms of ε,

$$u_i = u_{0i} + \varepsilon u_{1i} + \ldots , \tag{20}$$

$$p = p_0 + \varepsilon p_1 + \ldots , \tag{21}$$

$$\mathrm{H} = \mathrm{H}_0 + \varepsilon \mathrm{H}_1 + \ldots . \tag{22}$$

Substituting Eqs. (20) to (22) into Eqs. (15) to (19) and collecting terms of the same order, we obtain

$O(\varepsilon^0)$:

Continuity: $$\frac{\partial u_{0i}}{\partial x_{0i}} = 0 . \tag{23}$$

Momentum: $$0 = -\frac{\partial p_0}{\partial x_{0i}} . \tag{24}$$

$O(\varepsilon^1)$:

Continuity: $$\frac{\partial u_{1i}}{\partial x_{0i}}+\frac{\partial u_{0i}}{\partial x_{1i}} = 0 . \tag{25}$$

Momentum: $$0 = -\left(\frac{\partial p_1}{\partial x_{0i}}+\frac{\partial p_0}{\partial x_{1i}}\right)+\frac{\partial^2 u_{0i}}{\partial x_{0j}\,\partial x_{0j}} . \tag{26}$$

The KBC of kth-order in ε on wetted solid-liquid interface in saturated soil are

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$$u_{0i} = u_{1i} = ... = 0 \quad \text{for} \quad x_i \in \Gamma . \tag{27}$$

In unsaturated soil, KBC of different orders at free surface are

$O(\varepsilon^0)$:

$$\frac{\partial \mathrm{H}_0}{\partial t_0} + u_{0j}\frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 , \quad \mathrm{H}_0 = z - h_0 = 0 . \tag{28}$$

$O(\varepsilon^1)$:

$$\frac{\partial \mathrm{H}_1}{\partial t_0} + u_{0i}\left(\frac{\partial \mathrm{H}_1}{\partial x_{0i}} + \frac{\partial \mathrm{H}_0}{\partial x_{1i}}\right) + \frac{\partial \mathrm{H}_0}{\partial t_1} + \left(u_{1i} - \mathrm{H}_1\frac{\partial u_{0i}}{\partial z_0}\right)\frac{\partial \mathrm{H}_0}{\partial x_{0i}} = 0 ,$$
$$\mathrm{H}_0 = z - h_0 = 0 . \tag{29}$$

Dynamic boundary conditions at free surface are

$O(\varepsilon^0)$:

$$\left(-p_0 - \rho g z_0 + \beta\frac{\partial^2 \mathrm{H}_0}{\partial x_{0k}\,\partial x_{0k}} + 2(e_{ij})_{0,0}\right)\frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 ,$$
$$\mathrm{H}_0 = z - h_0 = 0 . \tag{30}$$

$O(\varepsilon^1)$:

$$\left[\left(-p_0 - \rho g z_0 + \beta\frac{\partial^2 \mathrm{H}_0}{\partial x_{0k}\,\partial x_{0k}}\right)\delta_{ij} + 2(e_{ij})_{0,0}\right]\left(\frac{\partial \mathrm{H}_1}{\partial x_{0j}} + \frac{\partial \mathrm{H}_0}{\partial x_{1j}}\right)$$
$$+\left\{\left[-p_1 - \rho g z_1 + \beta\left(\frac{\partial^2 \mathrm{H}_1}{\partial x_{0k}\,\partial x_{0k}} + \frac{\partial^2 \mathrm{H}_0}{\partial x_{1k}\,\partial x_{0k}}\right)\right]\right.$$
$$\left.\delta_{ij} + 2(e_{ij})_{1,0} + 2(e_{ij})_{0,1}\right\}\frac{\partial \mathrm{H}_0}{\partial x_{0j}}$$
$$+\left\{\mathrm{H}_1\frac{\partial p_0}{\partial z_0} + 2\frac{\partial}{\partial z_0}\left(\frac{\partial h_1 u_{0i}}{\partial x_{0j}} + \frac{\partial h_1 u_{0j}}{\partial x_{0i}}\right)\right\}\frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 ,$$
$$\mathrm{H}_0 = z - h_0 = 0 , \tag{31}$$

where $(e_{ij})_{m,n} = \left(\frac{\partial u_{mi}}{\partial x_{nj}} + \frac{\partial u_{ni}}{\partial x_{mj}}\right)$. Equations (23) to (31) are all the equations and boundary conditions needed in the present problem.

5. UNSTEADY FLOW IN UNSATURATED SOIL

In the whole solid-water mixture, there must be saturated and unsaturated REV. We first derive the flow condition for saturated REV, and then continue to unsaturated REV. From Eq. (24), we obtain

$$p_0 = p_0(x_{1i}, x_{2i}, ..., t) . \tag{32}$$

This implies 0^{th}-order pressure depends on macroscopic pressure effect for 0^{th}-order REV. To solve u_{0i}, we combine Eq. (32) with Eq. (26) to obtain

$$\frac{\partial p_0}{\partial x_{1i}}(x_{1i}, x_{2i}, ...) = \left(-\frac{\partial p_1}{\partial x_{0i}} + \frac{\partial^2 u_{0i}}{\partial x_{0j}\partial x_{0j}}\right)(x_{0i}, x_{1i}, x_{2i}, ...) , \tag{33}$$

where the two terms on RHS of Eq. (33) depends on microscopic variable x_{oi} and higher order macroscopic variables x_{1i}. But the LHS of Eq. (33) only depends on macroscopic variables. Therefore, LHS simply equals a constant under the variable x_{0i}. So Eq. (33) is a linear equation in x_{0i}, and the two terms on RHS are linearly related to the LHS term $\partial p_0 / \partial x_{1i}$. It is also proved from Auriault [4] and Mei and Auriault [8] that the solution form of u_{0i} and p_1 from Eq. (33) can be formally represented by

$$u_{0i} = K_{ij}\frac{\partial p_0}{\partial x_{1j}} \quad \text{and} \quad p_1 = A_j\frac{\partial p_0}{\partial x_{1j}} + \overline{p}_1 , \tag{34a,b}$$

due to the linearity of Eq. (33). K_{ij} and A_j are the 2^{nd} and 1^{st}-order tensors representing the geometrical properties in the 0^{th}-order REV, and they depend on microscopic variables, x_{0i}. $\overline{p}_1$ is the function of x_{1i}, x_{2i}, …, t, and is a constant representing macroscopic pressure effect for 0^{th}-order pressure. Physically K_{ij} can be regarded as hydraulic conductivity. Substituting Eqs. (34a) and (34b) into Eqs. (23) and (26), we obtain

$$\frac{\partial K_{ij}}{\partial x_{0i}} = 0 , \tag{35}$$

$$-\delta_{ij} = -\frac{\partial A_j}{\partial x_{0i}} + \frac{\partial^2 K_{ij}}{\partial x_{0k}\,\partial x_{0k}} , \tag{36}$$

and with the assumption that macroscopic pressure gra dient is not zero, $\partial p_0 / \partial x_{1i} \neq 0$; otherwise, the solution would be trivial. So that boundary conditions for water in saturated REV become

$$K_{ij} = 0 \quad \text{for} \quad x_i \in \Gamma . \tag{37}$$

For unsaturated REVs, by substituting Eqs. (34a) and (34b) into Eqs. (28) and (29), the free surface boundary conditions become

$$\frac{\partial \mathrm{H}_0}{\partial t_0} + K_{jk}\frac{\partial p_0}{\partial x_{1k}}\frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 , \tag{38}$$

$$\left[\left(-p_0 - \rho g z_0 + \beta\frac{\partial^2 \mathrm{H}_0}{\partial x_{0k}{}^2}\right) + \left(\frac{\partial K_{jk}}{\partial x_{0i}} + \frac{\partial K_{ik}}{\partial x_{0j}}\right)\frac{\partial p_0}{\partial x_{1k}}\right]\frac{\partial \mathrm{H}_0}{\partial x_{0j}} = 0 . \tag{39}$$

Equations (35) to (37) are a boundary-value-problem for solving K_{ij} and A_j in saturated REV. For specific cases such as sample formed with many identical glass balls, it is possible to define Γ and then solve the whole set of equations. However, generally it is difficult to define the solid boundary Γ in the microscopic REV without any knowledge for the solid structure. Due to the complex composition of heterogeneous materials in nature, one often obtain K_{ij} and A_j through experimental or computational methods [9].

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In saturated REV, substituting Eq. (34a) into Eq. (25) and applying ensemble average in a saturated REV, we find

$$\frac{1}{|\Omega_0|}\iiint_{\Omega_{l0}}\frac{\partial u_{1i}}{\partial x_{0i}}d\Omega_0+\frac{\partial}{\partial x_{1i}}\left(\left\langle K_{ij}\right\rangle_0\frac{\partial p_0}{\partial x_{1j}}\right)=0\,. \tag{40}$$

Using Gauss theorem to the first term on LHS of Eq. (40), we obtain

$$Q_{REV}=\frac{1}{|\Omega_0|}\iint_{S_{REV}}u_{1i}\cdot n_i\,dS=-\frac{\partial}{\partial x_{1i}}\left(\left\langle K_{ij}\right\rangle_0\frac{\partial p_0}{\partial x_{1j}}\right), \tag{41}$$

where Q_{REV} is averaged net volume flux for a REV; S_{REV} represents all the wet surfaces of REV and depends on microscopic variables x_{0i}; $u_{1i}\cdot n_i$ is the normal volume flux through the saturated REV surfaces; $\left\langle K_{ij}\right\rangle_0$ is the averaged hydraulic conductivity in microscopic REV, and it can be regarded as the representative hydraulic conductivity in the macro-scale. Equation (41) only depends on macroscopic variables.

Since solid matrix is assumed to be rigid and fixed spatially, the net total volume flux for a saturated REV through all control surfaces must be zero, so

$$\iint_{S_{REV}}u_i\cdot n_i\,dS=0\,. \tag{42}$$

By substituting the perturbed velocity, Eq. (20), into Eq. (42), we find

$$\iint_{S_{REV}}u_{0i}\cdot n_i\,dS+\varepsilon\iint_{S_{REV}}u_{1i}\cdot n_i\,dS+O(\varepsilon^2)=0\,. \tag{43}$$

From the principle of perturbation theory, terms of different orders are independent. So integrations at different orders in Eq. (43) must all be zero. This implies that Q_{REV} in Eq. (41) in saturated REV is also zero. Furthermore, if the soil is isotropic and homogeneous on the macro-scale, we could simplify $\left\langle K_{ij}\right\rangle_0$ to $K\delta_{ij}$, where K is a constant. With these conditions, Eq. (41) becomes

$$\frac{\partial^2 p_0}{\partial x_{1i}\,\partial x_{1i}}=0\,. \tag{44}$$

Equation (44) is the macroscopic averaged 1st-order continuity in saturated, isotropic and homogeneous soil. The result is the same as in Mei and Auriault [8]. If we have macroscopic boundary conditions of some specified problem, we could apply these boundary conditions to Eq. (44) to obtain the pressure distribution p_0. Then substituting this solved p_0 back to Eq. (34a), we could obtain microscopic velocity of seepage condition of the specified problem.

For unsaturated REV, we apply ensemble average to Eq. (23) to yield

$$\left\langle\frac{\partial u_{0i}}{\partial x_{0i}}\right\rangle_0=\frac{1}{|\Omega_0|}\iiint_{\Omega_{l0}}\frac{\partial u_{0i}}{\partial x_{0i}}d\Omega_0=0\,. \tag{45}$$

With Divergence theorem, Eq. (45) becomes

$$\left\langle\frac{\partial u_{0i}}{\partial x_{0i}}\right\rangle_0=\frac{1}{|\Omega_0|}\iint_S u_{0i}\cdot n_i\,dS_0=0\,, \tag{46}$$

where S is the boundary surface of water in 0th-order REV. There are three different kinds of boundaries. They are water-solid interface S_Γ, water-air interface S_H and water area on boundary of a REV S_{REV}, respectively. Separating RHS of Eq. (46) for different kinds of interfaces, we obtain

$$\begin{aligned}\iint_S u_{0i}\cdot n_i\,dS_0&=\iint_{S_H}u_{0i}\cdot n_i\,dS_0+\iint_{S_{REV}}u_{0i}\cdot n_i\,dS_0\\&\quad+\iint_{S_\Gamma}u_{0i}\cdot n_i\,dS_0=0\,,\end{aligned} \tag{47}$$

where n_i is the unit normal vector of each surface. With no-slip condition, the last term on RHS of Eq. (47) is zero. Before discussing the first term in RHS of Eq. (47), we divide the free surface kinematic boundary condition, Eq. (38), by $|\nabla_0 H_0|$ to yield

$$\frac{1}{|\nabla_0 H_0|}\frac{\partial H_0}{\partial t_0}+K_{jk}\frac{\partial p_0}{\partial x_{1k}}\cdot n_j=0\,, \tag{48}$$

where $n_j=\dfrac{1}{|\nabla_0 H_0|}\dfrac{\partial H_0}{\partial x_{0j}}$ is unit normal vector of free surface. The subscript zero of the operator ∇_0 means taking gradient with respect to 0th-order spatial independent variables x_0, y_0 and z_0. From Eq. (34a), the second term in Eq. (48) is $u_{0i}\cdot n_i$ and physically is the normal flux at the free surface. So it can be expressed as

$$u_{0i}\cdot n_i=-\frac{1}{|\nabla_0 H_0|}\frac{\partial H_0}{\partial t_0},\ \text{on}\ S_H\,. \tag{49}$$

Finally, substituting Eq. (49) into Eq. (47), we obtain

$$-\iint_{S_H}\frac{1}{|\nabla_0 H_0|}\frac{\partial H_0}{\partial t_0}dS_0+\iint_{S_{REV}}u_{0i}\cdot n_i\,dS_0=0\,. \tag{50}$$

Equation (50) is the averaged 0th-order continuity. In Fig. 2, We find that the inner product of the surface outward unit normal vector n_i for an infinitesimal free surface dS_0, with the unit vector in z-direction, n_{zi} = (0, 0, 1), is

$$\cos\theta=\frac{\nabla_0 H_0\cdot n_{zi}}{|\nabla_0 H_0|}=\frac{1}{|\nabla_0 H_0|}, \tag{51}$$

where θ is angle between unit normal vector of free surface and z-direction. Therefore, the integrand of the first integral in Eq. (50) becomes

$$\frac{dS_0}{|\nabla_0 H_0|}=dS_0\cos\theta=d\hat{S}_0\,, \tag{52}$$

where $d\hat{S}_0$ is the vertical projection area of the infinitesimal free surface dS_0 on z-plane. With Eq. (52) and putting $H_0=z-h_0$ into Eq. (50) to yield

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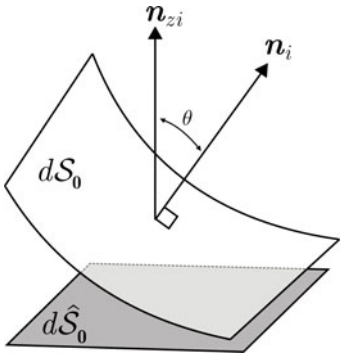


Fig. 2 Infinitesimal surface dS and its projection area $d\hat{S}$. (n_i and n_{zi} are unit normal vector of water free surface and unit vector in z-direction respectively)

$$-\iint_{\hat{S}_H} \frac{\partial h_0}{\partial t_0} d\hat{S}_0 = \iint_{S_{REV}} u_{0i} \cdot n_i \, dS_0 \,. \tag{53}$$

Note that if there is discontinuous distribution of water in pores, the direction of the surface will add or subtract water from the integration. From Eq. (53), we can find that microscopic surface integral of temporal variation of free surface elevation in 0^{th}-order REV must balance the net volume outward flux on REV surfaces.

6. VERIFICATION WITH RICHARDS' EQUATION

Physically, temporal variation of water content in a REV should equal total volume flux rate through REV boundaries. Equation (53) gives this information. In what follows, we shall prove rigorously that Eq. (53), a relation between volume flux and free surface elevation in 0^{th}-order REV, is actually the Richards' equation.

First, we need to define water content in an unsaturated REV. The water volume Ω_{l0} in REV can be regarded as surface integral of height of water

$$\Omega_{l0} = \iint_{\hat{S}_H} h_0 \, d\hat{S}_0 \,, \tag{54}$$

where Ω_{l0}, h_0 and $\hat{S}_H$ are functions of time variable; Ω_{l0} depends on macroscopic spatial variables x_{1i}, x_{2i}, …. Now we expand Eq. (54) with small time interval Δt_0

$$\begin{aligned}\Omega_{l0}(t_0+\Delta t_0) &= \iint_{\hat{S}_H(t+\Delta t)} h_0(t_0+\Delta t_0)\, d\hat{S}_0 \\ &= \iint_{\hat{S}_H(t)} \left(h_0(t_0) + \Delta t_0 \frac{\partial h_0}{\partial t_0} \ldots \right) d\hat{S}_0 \\ &\quad + \oint_{\Gamma_H} \Delta t_0 \, U_{\hat{S}_H} \left(h_0(t_0) + \Delta t_0 \frac{\partial h_0}{\partial t_0} \ldots \right) dl \,, \end{aligned} \tag{55}$$

where $U_{\hat{S}_H}$ is the outward normal velocity on the boundary of surface $\hat{S}_H$; Γ_H defines the contour line of vertically projected free surface boundary. The line integral in Eq. (55) represents outward flux on surface $\hat{S}_H$. However, the boundary of $\hat{S}_H$ is also the solid-liquid interface. From KBC on this solid-liquid boundary, Eq. (17), the velocity is zero. The vertical projection of zero velocity is still zero, so $U_{\hat{S}_H} = 0$, and Eq. (55) becomes

$$\Omega_{l0}(t_0+\Delta t_0) = \iint_{\hat{S}_H(t)} \left(h_0(t_0) + \Delta t_0 \frac{\partial h_0}{\partial t_0} \right) d\hat{S}_0 + O(\Delta t_0^{\,2}) \,. \tag{56}$$

Subtracting Eq. (54) from Eq. (56) and taking the limit $\Delta t_0 \to 0$, we obtain

$$\frac{\partial \Omega_{l0}}{\partial t_0} = \lim_{\Delta t_0 \to 0} \frac{\Omega_{l0}(t_0+\Delta t_0) - \Omega_{l0}(t_0)}{\Delta t_0} = \iint_{\hat{S}_H} \frac{\partial h_0}{\partial t_0} d\hat{S}_0 \,. \tag{57}$$

Equation (57) states that temporal variation of water volume in 0^{th}-order REV can be calculated with the temporal variation of free surface elevation. By definition of water content in Eq. (13),

$$\frac{\partial \theta_0}{\partial t_0} = \frac{1}{|\Omega_0|} \frac{\partial \Omega_{l0}}{\partial t_0} = \frac{1}{|\Omega_0|} \iint_{\hat{S}_H} \frac{\partial h_0}{\partial t_0} d\hat{S}_0 \,. \tag{58}$$

Equations (58) and (53) can be combined to be

$$\frac{\partial \theta_0}{\partial t_0} = -\frac{1}{|\Omega_0|} \iint_{S_{REV}} u_{0i} \cdot n_i \, dS_0 \,. \tag{59}$$

Equation (59) means temporal variation of water content equals average net volume flux through all REV boundary surfaces. We shall prove Eq. (59) is just the Richards' equation. The velocity concept used in Richards' equation is seepage velocity, which is an averaged velocity and not the real water velocity in the pores. In order to verify our result, Eq. (59), with Richards' equation, we have to use the same definition for velocity. Since in macro-scale, micro-scopic volume is considered very small, so a REV in micro-scale can be considered as an infinitesimal small volume under macro-scale. Therefore, we consider the REV as a small rectangle as in Fig. 1 with Δx_1, Δy_1, Δz_1 the length in x, y and z-direction, respectively. For this small volume, $|\Omega_0| = \Delta x_1 \Delta y_1 \Delta z_1$, we can evaluate the surface integral in Eq. (59) from each surface. Defining the surface averaged flux in i direction by U_{0i}.

$$U_{0i} = \frac{1}{\Delta S_i} \iint_{S_{0i}} u_{0i} \cdot n_i \, dS_0 \,, \quad i = x, y, z \,, \tag{60}$$

where $\Delta S_x = \Delta y_1 \Delta z_1$, $\Delta S_y = \Delta x_1 \Delta z_1$ and $\Delta S_z = \Delta x_1 \Delta y_1$. Let's examine the integral in x-direction first, we have

$$U_{0x}^{+}(x_{1i}, \ldots) \cdot \Delta y_1 \Delta z_1 = \iint_{S_{0x}^{+}} u_{0i} \cdot n_x \, dS_0 = \iint_{S_{0x}^{+}} u_{0i} \, dS_0 \tag{61}$$

on positive yz-plane S_{0x}^{+}, and

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$$U_{0x}^{-}(x_{1i},\ldots)\cdot\Delta y_1\,\Delta z_1=\iint_{S_{0x}^{-}}u_{0i}\cdot n_x\,dS_0=-\iint_{S_{0x}^{-}}u_{0i}\,dS_0 \qquad (62)$$

on negative yz-plane S_{0x}^{-}. So the net flux in x-direction is

$$\begin{aligned}\iint_{S_{0x}^{+}+S_{0x}^{-}}u_{0i}\cdot n_x\,dS_0&=U_{0x}^{+}(x_{1i},\ldots)\cdot\Delta y_1\,\Delta z_1-U_{0x}^{-}(x_{1i},\ldots)\cdot\Delta y_1\,\Delta z_1\\&=U_{0x}(x_1+\Delta x_1,\,y_1,\,z_1,\,x_{2i},\ldots)-U_{0x}(x_{1i},\ldots)\\&=\Delta x_1\,\Delta y_1\,\Delta z_1\left(\frac{\partial U_{0x}}{\partial x_1}+O(\Delta x_1)\right).\end{aligned}$$

Then

$$\begin{aligned}\iint_{S_{0x}}u_{0i}\cdot n_x\,dS_0&=\Delta x_1\,\Delta y_1\Delta z_1\left(\frac{\partial U_{0x}}{\partial x_1}+O(\Delta x_1)\right)\\&=|\,\Omega_0\,|\left(\frac{\partial U_{0x}}{\partial x_1}+O(\Delta x_1)\right),\end{aligned} \qquad (63)$$

where $|\,\Omega_0\,|=\Delta x_1\Delta y_1\Delta z_1$. Similarly, in y and z-direction, we could also obtain

$$\iint_{S_{0y}}u_{0i}\cdot n_y\,dS_0=|\Omega_0|\left(\frac{\partial U_{0y}}{\partial y_1}+O(\Delta y_1)\right), \qquad (64)$$

and

$$\iint_{S_{0z}}u_{0i}\cdot n_z\,dS_0=|\Omega_0|\left(\frac{\partial U_{0z}}{\partial z_1}+O(\Delta z_1)\right). \qquad (65)$$

With Eqs. (63) to (65) and omitting small terms, Eq. (59) becomes

$$\frac{\partial\theta_0}{\partial t_0}=-\frac{\partial U_{0i}}{\partial x_{1i}},\quad i=x,\,y,\,z\,. \qquad (66)$$

Finally, the velocity solution, Eq. (34a), is used in Eq. (66) to reach

$$\frac{\partial\theta_0}{\partial t_0}=-\frac{\partial}{\partial x_{1i}}\left[\frac{1}{\Delta S_i}\iint_{S_{0i}}\left(K_{ij}\frac{\partial p_0}{\partial x_{1j}}\cdot n_i\right)dS_0\right],\quad i,\,j=x,\,y,\,z\,. \qquad (67)$$

Defining

$$\hat{K}_i=\frac{1}{\Delta S_i}\iint_{S_{0j}}K_{ji}\,dS_0$$

as the surface averaged micro hydraulic conductivity K_{ij} in three directions under macro-scale. Equation (67) becomes

$$\frac{\partial\theta_0}{\partial t_0}=-\frac{\partial}{\partial x_{1i}}\left(\hat{K}_i\frac{\partial p_0}{\partial x_{1i}}\right). \qquad (68)$$

Equation (68) interprets the temporal variation of water content in a REV is equivalent to negative divergence of macroscopic pressure gradient multiplying by hydraulic conductivity. Equation (68) is exactly the Richards' equation [5].

7. CONCLUSION AND DISCUSSION

Without any constitutive assumption for unsaturated solid-liquid mixture, we successfully use homogenization theory to obtain the equation governs water content variation. The derived result is proved to be the Richards' equation. From this result, we can conclude that homogenization theory can be used to describe water motion in the problem of unsteady and unsaturated solid-liquid mixtures. Therefore, this result can be a sound foundation for studying the initiation of solids under unsteady and unsaturated solid-liquid mixture.

ACKNOWLEDGEMENTS

We gratefully appreciate National Science Council (Grant No. NSC 96-2625-Z-002-006-MY3) in Taiwan for supporting this research. Valuable and useful comments and suggestions from reviewers are also acknowledged.

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(Manuscript received January 20, 2011, accepted for publication October 13, 2011.)

https://doi.org/10.1017/jmech.2012.36 Published online by Cambridge University Press