8

BPS solitons, supersymmetry, and duality

In addition to the direct prediction of physical objects, the study of solitons has also led to deeper insights into the structure of quantum field theories. We have already seen an early example of this, the discovery of the duality between the sine-Gordon and massive Thirring models [33]. A particularly fruitful direction was inspired by a special limit for the study of magnetic monopoles that was first proposed by Bogomolny [81] and by Prasad and Sommerfield [82]. It turns out that this BPS limit can be generalized and extended to many other systems, and that it has deep connections with supersymmetry and dualities.

Although there are a variety of solitons displaying BPS properties, in this chapter I will focus primarily on BPS magnetic monopoles. However, I will also use as examples a special case of the U(1) vortices that were studied in Chap. 3, as well as kink solutions in two spacetime dimensions.

The discussion in this chapter is necessarily limited. A review covering many of these topics in greater depth is [155].

8.1 The BPS limit as a limit of couplings

In Sec. 5.2 we considered magnetic monopoles in an SU(2) gauge theory with a triplet Higgs field governed by the potential

$$V(\boldsymbol{\phi}) = -\frac{\mu^2}{2}\boldsymbol{\phi}^2 + \frac{\lambda}{4}(\boldsymbol{\phi}^2)^2 + \frac{\lambda v^4}{4}. \tag{8.1}$$

The ansatz

$$\begin{aligned} \phi^a &= \hat{r}^a h(r), \\ A_i^a &= \epsilon^{aim}\hat{r}^m \left[\frac{1-u(r)}{er}\right], \\ A_0^a &= \hat{r}^a j(r), \end{aligned} \tag{8.2}$$

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led to the static field equations

$$\begin{aligned} 0 &= h'' + \frac{2}{r}\,h' - \frac{2u^2 h}{r^2} + \lambda(v^2 - h^2)h, \\ 0 &= u'' - \frac{u(u^2-1)}{r^2} - e^2 u(h^2 - j^2), \\ 0 &= j'' + \frac{2}{r}\,j' - \frac{2u^2 j}{r^2}. \end{aligned} \tag{8.3}$$

Three parameters enter these equations. Two of these, the Higgs field vacuum expectation value v and the gauge coupling e, can be eliminated from the equations by rescaling h and r. The resulting equations then depend only on the combination λ/e^2. Although these equations cannot be solved analytically for arbitrary values of λ/e^2, one might hope that they would be more tractable for some special values. In particular, Prasad and Sommerfield [82] proposed considering the case $\lambda/e^2 = 0$, obtained by taking a limit in which μ^2 and λ are taken to zero, but with the ratio $\mu^2/\lambda = v^2$ held fixed and the boundary condition $h(\infty) = v$ maintained. In this BPS limit the last term in the first of Eqs. (8.3) is absent, and by trial and error one can find the solution

$$\begin{aligned} h(r) &= v\coth(evr) - \frac{1}{er}, \\ u(r) &= \frac{evr}{\sinh(evr)}, \\ j(r) &= 0. \end{aligned} \tag{8.4}$$

This gives a monopole with $Q_M = 4\pi/e$ and mass

$$M = \frac{4\pi v}{e} = Q_M\, v. \tag{8.5}$$

This solution carries no electric charge. However, by noting that h and j obey identical equations when $\lambda = 0$, we see that dyon solutions can be obtained by a simple rescaling of the $Q_E = 0$ solution. For any real number γ there is a solution

$$\begin{aligned} h(r) &= v\coth(e\hat{v}r) - \frac{\cosh\gamma}{er}, \\ u(r) &= \frac{e\hat{v}r}{\sinh(e\hat{v}r)}\,, \\ j(r) &= -v\tanh\gamma\coth(e\hat{v}r) + \frac{\sinh\gamma}{er}, \end{aligned} \tag{8.6}$$

where $\hat{v} = v/\cosh\gamma$. The solution has magnetic charge $Q_M = 4\pi/e$, while from the gradient of the long-range tail of $j(r)$ we find that the electric charge is

$$Q_E = \frac{4\pi}{e}\,\sinh\gamma. \tag{8.7}$$

The mass of this dyon is

$$M = \frac{4\pi v}{e}\cosh\gamma = \sqrt{Q_M^2 + Q_E^2}\,v. \tag{8.8}$$

For these solutions $h(r)$ falls as $1/r$, in contrast to its usual exponential falloff. This is a consequence of the fact that the Higgs mass vanishes when μ^2 is taken

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to 0. Because this scalar field is now massless, it mediates a long-range force whose consequences will become evident presently.

8.2 Energy bounds

There are further striking features of this limit. With an eye to generalization to larger gauge groups later in this chapter, let us write the triplet Higgs field Φ as an element of the Lie algebra. The Lagrangian density for an arbitrary potential then takes the form

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\, F_{\mu\nu}^2 + \mathrm{tr}\,(D_\mu\Phi)^2 - V(\Phi), \tag{8.9}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu], \tag{8.10}$$

$$D_\mu\Phi = \partial_\mu\Phi - ie[A_\mu, \Phi]. \tag{8.11}$$

Let us assume that Φ is nonzero, with magnitude v at the minima of V, and that V is everywhere positive.

It is convenient to define

$$\begin{aligned}\mathcal{Q}_M &= 2\int d^2S^i\,\mathrm{tr}\,(\Phi B_i)\,,\\ \mathcal{Q}_E &= 2\int d^2S^i\,\mathrm{tr}\,(\Phi E_i),\end{aligned} \tag{8.12}$$

where $B_i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$, $E_i = F_{0i}$, and the integrals are over the surface at $r = \infty$. These quantities are, up to an overall rescaling, the magnetic and electric charges, with $\mathcal{Q}_M = vQ_M$ and $\mathcal{Q}_E = vQ_E$.

The energy functional can be written as

$$\begin{aligned}E &= \int d^3x\,\left[\mathrm{tr}\,E_i^2 + \mathrm{tr}\,B_i^2 + \mathrm{tr}\,(D_0\Phi)^2 + \mathrm{tr}\,(D_i\Phi)^2 + V(\Phi)\right]\\ &= \int d^3x\,\left[\mathrm{tr}\,(B_i \mp \cos\alpha D_i\Phi)^2 + \mathrm{tr}\,(E_i \mp \sin\alpha D_i\Phi)^2\right.\\ &\qquad \left. + \mathrm{tr}\,(D_0\Phi)^2 + V(\Phi)\right]\\ &\qquad \pm 2\int d^3x\,\left[\cos\alpha\ \mathrm{tr}\,(B_i\,D_i\Phi) + \sin\alpha\ \mathrm{tr}\,(E_i\,D_i\Phi)\right],\end{aligned} \tag{8.13}$$

where α is an arbitrary angle. The terms in the last integral can be rewritten by integrating by parts. With the aid of the Bianchi identity $D_iB_i = 0$ and Gauss's law,

$$D_iE_i = ie[\Phi, D_0\Phi], \tag{8.14}$$

this gives

$$\begin{aligned}2\int d^3x\,[\mathrm{tr}\,(B_i\,D_i\Phi)] &= 2\int d^3x\,[\partial_i\,\mathrm{tr}\,(B_i\Phi) - \mathrm{tr}\,(\Phi D_iB_i)] = \mathcal{Q}_M,\\ 2\int d^3x\,[\mathrm{tr}\,(E_i\,D_i\Phi)] &= 2\int d^3x\,[\partial_i\,\mathrm{tr}\,(E_i\Phi) - \mathrm{tr}\,(\Phi D_iE_i)]\\ &= \mathcal{Q}_E - 2ie\int d^3x\ \mathrm{tr}\,(\Phi[\Phi, D_0\Phi]) = \mathcal{Q}_E.\end{aligned} \tag{8.15}$$

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Substituting these results into Eq. (8.13), we obtain

$$\begin{aligned} E = \int d^3x \left[\mathrm{tr}\,(B_i \mp \cos\alpha D_i\Phi)^2 + \mathrm{tr}\,(E_i \mp \sin\alpha D_i\Phi)^2 \right. \\ \left. + \mathrm{tr}\,(D_0\Phi)^2 + V(\Phi) \right] \\ \pm \cos\alpha\, \mathcal{Q}_M \pm \sin\alpha\, \mathcal{Q}_E \end{aligned}$$

$$\geq \pm \cos\alpha\, \mathcal{Q}_M \pm \sin\alpha\, \mathcal{Q}_E. \tag{8.16}$$

This inequality holds for either choice of sign and for any value of α. For fixed values of the charges, the most stringent inequality,

$$E \geq \sqrt{\mathcal{Q}_M^2 + \mathcal{Q}_E^2}, \tag{8.17}$$

is obtained by setting $\alpha = \tan^{-1}(\mathcal{Q}_E/\mathcal{Q}_M)$ and choosing the upper or lower signs according to whether $\mathcal{Q}_M$ is positive or negative. Without loss of generality, let us assume that $\mathcal{Q}_M > 0$ and take the upper signs.

If $V(\Phi)$ is nonzero, the lower bound of the energy can never be attained by a nontrivial configuration. However, let us go to the BPS limit in which V is taken to zero but the asymptotic boundary conditions on Φ are maintained. With V removed, the remaining terms in the integrand in Eq. (8.16) are all perfect squares. They will all vanish, and the lower bound on the energy will be achieved, if [81, 156]

$$\begin{aligned} B_i &= \cos\alpha\, D_i\Phi, \\ E_i &= \sin\alpha\, D_i\Phi, \\ D_0\Phi &= 0. \end{aligned} \tag{8.18}$$

In particular, for vanishing electric charge $\alpha = 0$ and

$$B_i = D_i\Phi. \tag{8.19}$$

This is often referred to as the Bogomolny equation. We now use the fact that configurations that minimize the energy for fixed values of the charges are solutions of the full equations of motion, provided that they satisfy the Gauss's law constraint. Solutions of Eq. (8.18) do indeed satisfy this constraint, since for these

$$D_iE_i - ie[\Phi, D_0\Phi] = \tan\alpha\, D_iB_i - 0 = 0. \tag{8.20}$$

Hence, solutions of the first-order equations are in fact static solutions of the full field theory.[1] This can be verified explicitly by differentiating Eq. (8.18) and substituting the results into the second-order Euler–Lagrange equations.

[1] These are not the only static solutions of the theory. The existence of nontrivial solutions that do not obey Eq. (8.18) was demonstrated in [157, 158]. However, these correspond to saddle points of the energy, and are therefore not stable.

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For the particular case of unit magnetic charge in the SU(2) theory, substitution of the ansatz of Eq. (8.2) into Eq. (8.18) yields

$$\begin{aligned} h' &= \frac{1-u^2}{er^2}\sec\alpha, \\ u' &= -euh\cos\alpha, \\ j &= -h\sin\alpha, \end{aligned} \tag{8.21}$$

which are indeed satisfied by the solution in Eq. (8.6).

We have several remarkable results here. First, we have found a set of first-order equations whose solutions are also solutions of the full set of second-order field equations. Second, the energy of these solutions is determined completely by their charges. In particular, for solutions carrying only magnetic charges the energy is strictly proportional to that charge. Finally, the fact that these solutions minimize the energy for fixed charges guarantees their stability against decay to some other solution with the same magnetic and electric charges.

One might perhaps wonder whether these results are a consequence of the rather unusual limiting process that took the scalar field potential to zero but retained the boundary conditions implied by that potential. However, there are other systems with similar behavior that do not require any such problematic limiting procedure. An example is the (2+1)-dimensional U(1) gauge theory with a single complex field that was studied in Sec. 3.2. For static configurations with vanishing A_0, the energy functional implied by the Lagrangian density of Eq. (3.24) can be written as

$$E = \int d^2x \left[\frac{1}{2}|D_j\phi|^2 + \frac{1}{4}F_{jk}^2 + \frac{\lambda}{4}(|\phi|^2 - v^2)^2\right], \tag{8.22}$$

with $D_j\phi = \partial_j\phi + ieA_j\phi$ and $F_{jk} = \partial_j A_k - \partial_k A_j$. Now let us choose $\lambda = e^2/2$, which, it may be recalled, is the value that makes the vector and scalar masses equal and corresponds to the boundary between type I and type II superconductors.

The energy can then be written as [41, 81]

$$\begin{aligned} E &= \frac{1}{2}\int d^2x \left[|D_1\phi|^2 + |D_2\phi|^2 + F_{12}^2 + \frac{e^2}{4}(|\phi|^2 - v^2)^2\right] \\ &= \frac{1}{2}\int d^2x \left\{|D_1\phi \mp iD_2\phi|^2 + \left[F_{12} \pm \frac{e}{2}(|\phi|^2 - v^2)\right]^2 \right. \\ &\qquad \left. \pm i\left[(D_1\phi)^* D_2\phi - (D_2\phi)^* D_1\phi\right] \mp eF_{12}(|\phi|^2 - v^2)\right\} \\ &= \frac{1}{2}\int d^2x \left\{|D_1\phi \mp iD_2\phi|^2 + \left[F_{12} \pm \frac{e}{2}(|\phi|^2 - v^2)\right]^2\right\} \\ &\qquad \pm \frac{ev^2}{2}\int d^2x F_{12}, \end{aligned} \tag{8.23}$$

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where the third line is obtained by an integration by parts for which the surface term vanishes. Recalling Eq. (3.31), we recognize the last integral as being $2\pi/e$ times the vorticity n and see that we again have a lower bound on the energy in terms of the topological charge. For positive n we take the upper signs and have

$$E \geq \pi v^2 n, \tag{8.24}$$

with equality if

$$\begin{aligned} 0 &= D_1\phi - iD_2\phi, \\ 0 &= F_{12} + \frac{e}{2}(|\phi|^2 - v^2). \end{aligned} \tag{8.25}$$

By arguments similar to those for the BPS monopole, solutions of these first-order equations are also static solutions to the full set of second-order field equations. In contrast with the monopole case, these equations cannot be solved analytically. However, as we saw in Sec. 3.4, it is not hard to demonstrate the existence of axially symmetric solutions for arbitrary n, no matter what the value of λ/e^2.

A third example is the theory of a single scalar field in one space dimension governed by a potential $V(\phi)$, the case we studied in Sec. 2.1. If we assume that $V(\phi)$ is equal to zero at its global minima, we can find a function $W(\phi)$ such that

$$V = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2. \tag{8.26}$$

The energy of a static configuration can then be written as

$$\begin{aligned} E &= \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)\right] \\ &= \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx} \mp \sqrt{2V(\phi)}\right)^2 \pm \frac{d\phi}{dx}\frac{dW}{d\phi}\right] \\ &= \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx} \mp \sqrt{2V(\phi)}\right)^2\right] \pm T, \end{aligned} \tag{8.27}$$

where

$$T = W(\phi(\infty)) - W(\phi(-\infty)) \tag{8.28}$$

is a topological quantity. We have seen, in Sec. 2.1, that a solution to $d\phi/dx = \pm\sqrt{2V}$ is a solution to the full field equations. What is new here is the statement that its energy is equal to T.

Although T is topological, it does not always correspond to the topological charge

$$Q_{\text{top}} = \frac{1}{2v}\left[\phi(\infty) - \phi(-\infty)\right] \tag{8.29}$$

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that was defined in Sec. 2.1. For the ϕ^4 model, with $V = (\lambda/4)(\phi^2 - v^2)^2$,

$$W = \sqrt{\frac{\lambda}{2}}\left(v^2\phi - \frac{1}{3}\phi^3\right). \tag{8.30}$$

In this case, $W(\pm v) = \pm\sqrt{2\lambda}\, v^3/3$, and so T and $Q_{\rm top}$ agree up to a multiplicative constant. This is not so for the sine-Gordon model, where the potential

$$V = \frac{m^4}{\lambda}\left[1 - \cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right] \tag{8.31}$$

corresponds to

$$W = \frac{2m^3}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{2m}\phi\right). \tag{8.32}$$

The vacua are at $\phi = 2\pi N m/\sqrt{\lambda}$. The topological charge $Q_{\rm top}$ counts the total kink number; i.e., the difference ΔN between the value of N at $x = \infty$ and that at $x = -\infty$. In contrast, T only counts this modulo 2: it vanishes if ΔN is even, and is equal to $4m^3/\lambda$ if ΔN is odd.

A number of other field theories have been found with similar properties (e.g., [159–162]). All have an energy that can be written as a sum of perfect squares plus a topological term, implying a lower bound that is saturated when a set of first-order equations are satisfied. These properties can be understood as consequences of supersymmetry, as will now be explained.

8.3 Supersymmetry

In four spacetime dimensions the algebra of simple supersymmetry can be written as

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma^\mu)_{\alpha\beta}P_\mu, \tag{8.33}$$

where the supersymmetry generator Q is a four-component Majorana spinor with $\bar{Q} = Q^\dagger\gamma^0$, P_μ is the four-momentum, and subscripts from the beginning of the Greek alphabet denote Dirac indices. A more complex structure is possible in N-extended supersymmetry, with generators Q_r $(r = 1, \ldots, N)$. Here the most general form of the algebra of the supercharges can be written as

$$\{Q_{r\alpha}, \bar{Q}_{s\beta}\} = 2\delta_{rs}(\gamma^\mu)_{\alpha\beta}P_\mu + 2i\delta_{\alpha\beta}X_{rs} - 2(\gamma^5)_{\alpha\beta}Y_{rs}, \tag{8.34}$$

where $X_{rs} = -X_{sr}$ and $Y_{rs} = -Y_{sr}$ are central charges that commute with all of the supercharges and all of the generators of the Poincaré algebra.

Multiplying the left-hand side of Eq. (8.34) on the right by γ^0 gives the positive definite $4N \times 4N$ matrix $\mathcal{M}_{r\alpha,s\beta} \equiv \{Q_{r\alpha}, Q^\dagger_{s\beta}\}$. The positivity of its eigenvalues

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implies a lower bound on the mass. For example, taking $N = 2$ and working in the rest frame, with $P^\mu = \delta^{\mu 0} M$, we have

$$\mathcal{M} = 2 \begin{pmatrix} MI_4 & iX\gamma^0 - Y\gamma^5\gamma^0 \\ -iX\gamma^0 + Y\gamma^5\gamma^0 & MI_4 \end{pmatrix}, \tag{8.35}$$

where each entry represents a 4×4 block, with $X \equiv X_{12}$, $Y \equiv Y_{12}$, and I_4 being the 4×4 unit matrix. Requiring that the eigenvalues of $\mathcal{M}$ all be positive gives the inequality

$$M^2 - X^2 - Y^2 \geq 0. \tag{8.36}$$

For arbitrary N, this generalizes to

$$M \geq \sqrt{\frac{X_{rs}X_{rs} + Y_{rs}Y_{rs}}{2}}. \tag{8.37}$$

In order for a state to saturate this lower bound, $\mathcal{M}$ must have a zero eigenvalue, with means that there must be a linear combination of the Q_r and $Q_s^\dagger$ that vanishes when acting on the state. In other words, the state must be invariant under a subset of the supersymmetry transformations.

What does this have to do with the theories discussed in the previous sections? None of these include fermions, and so they clearly are not supersymmetric. However, they all have the property that they can be made supersymmetric by adding a few fields in a relatively simple fashion.

Let us start with the Yang–Mills theory with an adjoint representation scalar field Φ that was considered in the previous section. (Although we originally had the SU(2) theory in mind, the discussion below applies equally well for an arbitrary gauge group G.) To this we add a pseudoscalar field Θ and two Majorana fermion fields χ_r, all transforming under the adjoint representation, and take the Lagrangian to be [163]

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\mathrm{tr}\, F_{\mu\nu}^2 + \mathrm{tr}\,(D_\mu\Phi)^2 + \mathrm{tr}\,(D_\mu\Theta)^2 + i\ \mathrm{tr}\,\bar\chi_r\gamma^\mu D_\mu\chi_r \\ &+ie\ \mathrm{tr}\,\epsilon_{rs}\bar\chi_r[\chi_s,\Phi] + e\ \mathrm{tr}\,\epsilon_{rs}\bar\chi_r\gamma^5[\chi_s,\Theta] - V(\Phi,\Theta), \end{aligned} \tag{8.38}$$

where the scalar field potential is

$$V(\Phi,\Theta) = -e^2\mathrm{tr}\,[\Phi,\Theta]^2. \tag{8.39}$$

(This is positive, despite the minus sign, because the commutator is anti-Hermitian.) This potential vanishes whenever Φ and Θ commute, and thus has a continuous family of degenerate minima. We will focus primarily on the vacua for which[2] $\langle\Theta\rangle = 0$. With this choice of vacuum, we can look for solutions of

[2] This is actually the most general case in the SU(2) theory, where any mutually commuting Φ and Θ must be proportional to each other. By a transformation corresponding to an element of the R-symmetry, we can then set $\langle\Theta\rangle = 0$. This is not necessarily so for larger gauge groups.

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the classical field equations for which Θ vanishes at spatial infinity. It is easy to see that we can consistently set both Θ and the fermion fields χ_r to zero everywhere. The field equations then reduce to the same second-order equations as were obtained by taking the BPS limit of zero coupling. However, there is now no need to invoke a delicately tuned limit of parameters.

This Lagrangian is invariant under $N = 2$ supersymmetry transformations of the form

$$\begin{aligned}
\delta A_\mu &= i\bar{\zeta}_r \gamma_\mu \chi_r, \\
\delta \Phi &= \epsilon_{rs} \bar{\zeta}_r \chi_s, \\
\delta \Theta &= -i\,\epsilon_{rs} \bar{\zeta}_r \gamma^5 \chi_s, \\
\delta \chi_r &= -\frac{i}{2}\sigma^{\mu\nu} F_{\mu\nu} \zeta_r + i\epsilon_{rs}\gamma^\mu D_\mu (\Phi - i\gamma^5 \Theta)\zeta_s - e[\Phi, \Theta]\gamma^5 \zeta_r,
\end{aligned} \tag{8.40}$$

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu, \gamma^\nu]$ and the ζ_r $(r = 1, 2)$ are two independent Majorana spinor parameters.

The supercharges that generate these transformations can be written as spatial integrals of the time components of supercurrents S^μ_r. Calculating the anticommutators of the S^μ_r and then integrating to obtain the anticommutators of the supercharges shows that the latter are indeed of the form of Eq. (8.34), with the central charges,

$$\begin{aligned}
X_{12} &= 2\int d^2 S^i \,\mathrm{tr}\,(\Phi E_i + \Theta B_i), \\
Y_{12} &= 2\int d^2 S^i \,\mathrm{tr}\,(\Phi B_i + \Theta E_i),
\end{aligned} \tag{8.41}$$

being given by surface integrals at spatial infinity. If Φ, but not Θ, is nonzero as $r \to \infty$, we have

$$\begin{aligned}
X_{12} &= \mathcal{Q}_E, \\
Y_{12} &= \mathcal{Q}_M,
\end{aligned} \tag{8.42}$$

where $\mathcal{Q}_M$ and $\mathcal{Q}_E$ were defined in Eq. (8.12). The operator arguments given above then imply the lower bound

$$M \geq \sqrt{\mathcal{Q}_M^2 + \mathcal{Q}_E^2} \tag{8.43}$$

that we obtained in the previous section.

These arguments also tell us that any state that saturates this bound must be invariant under a portion of the supersymmetry. For magnetically charged states built upon a classical soliton solution, this means that the classical solution should be invariant under the transformations of Eq. (8.40) for at least some choices of the ζ_r. Because the classical solution is purely bosonic, with

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$\chi_1 = \chi_2 = 0$, the variations of the bosonic fields vanish automatically. The variations of the fermion fields, on the other hand, are in general nonzero. If $\Theta = 0$ everywhere, the condition for the vanishing of the $\delta\chi_r$ becomes

$$0 = -\frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu}\zeta_r + i\epsilon_{rs}\gamma^\mu D_\mu \Phi \zeta_s. \tag{8.44}$$

This is satisfied for arbitrary choices of the ζ_r only by a vacuum configuration with $F_{\mu\nu} = D_\mu \Phi = 0$. However, if the two supersymmetry parameters are related by

$$\zeta_1 = \gamma^5\gamma^0\zeta_2, \tag{8.45}$$

Eq. (8.44) becomes

$$0 = \left[E_k\gamma^k\gamma^5 - iB_k\gamma^k + i\gamma^\mu D_\mu\Phi\right]\epsilon_{rs}\zeta_s. \tag{8.46}$$

This is satisfied for all ζ_s if

$$\begin{aligned} B_i &= D_i\Phi, \\ E_i &= 0, \\ D_0\Phi &= 0. \end{aligned} \tag{8.47}$$

We have thus recovered the Bogomolny equation for a monopole carrying no electric charge. The corresponding equations for dyons, given in Eq. (8.18), can be obtained by taking

$$\zeta_1 = e^{i\alpha\gamma^5}\gamma^5\gamma^0\zeta_2. \tag{8.48}$$

In both cases, the BPS solutions are invariant under a one-parameter set of transformations, and thus preserve half of the supersymmetry.

Although it may seem that the supersymmetry algebra has simply provided an alternative derivation of the results of the previous section, we actually obtain more. Recall that supermultiplets can be obtained by acting on a base state with the various supercharges. If some subset of the supercharges annihilates the base state, the number of states obtained in this fashion is reduced and we have a "short" supermultiplet. For the Yang–Mills theory with $N = 2$ supersymmetry, multiplets that saturate the bound of Eq. (8.43) are composed of four states, while those with larger masses contain 16. This has implications for the effects of perturbative quantum corrections. *A priori*, one might expect that the classical relation that we have found between the mass and the topological charges of BPS monopoles would be modified by one-loop quantum effects. However, our algebraic results tell us that such a modification would also imply a change in the multiplet size, something that we would not expect to result from a perturbative correction. Hence, we conclude that for these states the relation between the mass and the central charges must be exact [164].

Although this discussion has focused on the BPS monopoles of the Yang–Mills theory, a similar analysis can be applied to the other examples where a soliton

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mass saturates a lower bound given by the topological charges. In all known cases, the bosonic theory can be simply extended to a supersymmetric one, and the condition that a state be invariant under a subset of the supersymmetry becomes, at the classical level, a set of first-order equations whose solutions necessarily satisfy the full set of field equations. Indeed, such partially supersymmetric states, even when not associated with magnetic monopoles, are now referred to as BPS states.[3]

In the case of the (2+1)-dimensional U(1) theory, we saw that the reduction of the vortex equations to first order was only possible when the vector and scalar masses are equal. This is precisely the case where the theory can be extended to one with $N = 2$ supersymmetry with the vector and scalar particles lying in the same $N = 2$ supermultiplet. For other values of the masses it is still possible to obtain a theory with extended supersymmetry by introducing a sufficient number of additional fields. However, requiring that a fraction of the supersymmetry be preserved gives classical equations that involve the additional fields in a nontrivial manner and so cannot be applied to the original theory.

The case of two spacetime dimensions is special, in that extended supersymmetry is not needed to obtain BPS states. A general supersymmetric scalar theory has a Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi - \frac{1}{2}[W'(\phi)]^2 - \frac{1}{2}W''(\phi)\bar{\psi}\psi, \tag{8.49}$$

where ψ is a Majorana fermion and the superpotential $W(\phi)$ is an arbitrary function of ϕ. By integrating the time component of the supercurrent and keeping careful track of surface terms, one obtains the supercharges, which satisfy

$$\{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma^\mu)_{\alpha\beta} P_\mu + 2i(\gamma^5)_{\alpha\beta} T, \tag{8.50}$$

where

$$T = W(\phi(\infty)) - W(\phi(-\infty)), \tag{8.51}$$

which we already encountered in Eq. (8.28), is a central charge. If we choose a basis where $\gamma^0 = \sigma_y$, $\gamma^1 = i\sigma_z$, and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_x$, and work in the rest frame, we find that

$$\begin{aligned} Q_1^2 &= M + T, \\ Q_2^2 &= M - T. \end{aligned} \tag{8.52}$$

A BPS state is obtained by requiring $Q_2 = 0$ or $Q_1 = 0$, according to whether T is positive or negative.

[3] In all of these cases, the squared quantities enter with positive coefficients. There are other examples, involving gravity, where the action can be written as a topological term plus a sum of squares with coefficients of varying sign. Setting each of the squared quantities to zero yields a stationary point of the action, and thus a classical solution, but these solutions do not necessarily preserve any supersymmetry [165].

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In this case, the vanishing of one of the Q_α does not imply a reduction in the size of the supermultiplet, so we cannot use the argument given above to show that perturbative effects cannot change the saturation of the mass bound. For quite some time there was dispute as to whether or not the saturation was maintained, with one-loop calculations using various regularization schemes purporting to show that the kink mass did or did not receive corrections. The issue was eventually resolved and clarified, and it was shown that if a kink obeys $M = T$ at the classical level, this relation is preserved by the leading quantum corrections [166–168]. There is a twist, however. At the quantum level there is an anomaly that contributes to T, so that the central charge becomes [168]

$$T = W(\phi(\infty)) + \frac{W''(\phi(\infty))}{4\pi} - W(\phi(-\infty)) - \frac{W''(\phi(-\infty))}{4\pi}. \tag{8.53}$$

The kink mass does receive perturbative corrections, but these are precisely equal to the corrections to T, provided that the anomaly terms are included.

In all of the above examples the soliton was invariant under the action of one half of the supersymmetry generators. This is not the only possibility. In $N = 4$ supersymmetric Yang–Mills theory there are four spinor parameters entering an arbitrary supersymmetry transformation, as compared to the two in the $N = 2$ theory. Imposing two relations between these leads to the BPS solutions that we have found, which are invariant under one half of the enlarged supersymmetry. However, there is now the possibility of imposing three constraints, leaving only one independent spinor parameter and giving a new class of solutions that preserve only one-fourth of the original supersymmetry [169–171]. Although this turns out not to be possible in the SU(2) theory, such 1/4-BPS solutions do exist with larger gauge groups. They obey a more complex set of first-order Bogomolny-type equations, and the relation between the mass and the electromagnetic charges is more complicated than in the 1/2-BPS case.

8.4 Multisoliton solutions

One would not ordinarily expect to find static solutions describing spatially separated solitons, because at any finite distance there would be nonzero intersoliton forces. A static multisoliton solution would require a remarkable cancellation between the attractive and the repulsive forces. However, the exact proportionality between the energy and the topological charge of BPS solutions suggests that just such a cancellation might take place. Indeed, we have seen that in the BPS monopole the Higgs field has a $1/r$ tail that corresponds to a long-range attractive force that has the potential to exactly cancel the magnetic repulsion. Similarly, the BPS U(1) vortices occur when the vector and scalar masses are equal, so that the exponential tails of the two fields fall at the same rate, again

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giving the possibility of cancellation between the attractive and the repulsive forces.

We can explore this further by counting the parameters needed to describe an arbitrary solution. In a solution describing noninteracting separated solitons there should be parameters reflecting independent degrees of freedom for each of the component solitons.

One might worry that this parameter counting could only be done after the putative multisoliton solution was actually found. In fact, this is not the case. Let us assume that we are given a BPS solution with some nonzero topological charge. This solution obeys a set of first-order Bogomolny-type equations. To count parameters, we need to know how many zero modes there are about this solution; i.e., how many linearly independent perturbations of the solution preserve the first-order equations. (In doing this counting, the zero modes corresponding to local gauge transformations must be excluded, e.g., by imposing a background gauge condition.) This question can be answered by index theory methods. Because these are most easily introduced in the context of instanton solutions, in Sec. 10.8, I will defer the details to Appendix B, and simply quote the results here.

For the (2+1)-dimensional U(1) theory, an index theorem shows that there are precisely $2n$ nongauge zero modes about any BPS solution with vorticity n, assuming that such a solution actually exists [41]. That existence is assured by the arguments of Sec. 3.4 that showed that rotationally invariant solutions exist for arbitrary n. We now see that these are simply particularly symmetric multivortex solutions in which the n component vortices happen to be located at the same point. The zero modes for $n = 1$ correspond to spatial translation of the vortex, so it is natural to assume that for $n > 1$ they correspond to independent translations of n noninteracting vortices and that the $2n$ parameters can be taken to be the two-dimensional positions of the n vortices. Indeed, as was already noted in Sec. 3.4, one can show that static multivortex solutions exist for any choice of the n vortex positions [42].

Similarly, one can show that there are $4n$ zero modes satisfying the background gauge condition about any SU(2) multimonopole solution carrying n units of magnetic charge [83]. Of these, $3n$ are naturally interpreted as specifying the positions of the component monopoles. The remaining n zero modes specify the U(1) phases for these monopoles. To understand these, recall that although the fourth zero mode about a single monopole corresponded simply to a global gauge rotation, a time-dependent excitation of this mode led to a dyonic solution with nonzero electric charge. For the case of n monopoles, one overall phase can be shifted by a global gauge transformation, but the $n - 1$ relative phases are gauge-invariant. Excitation of these n modes corresponds to endowing the component monopoles with independent electric charges. As with

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the vortices, the index theorem by itself does not establish the existence of multimonopole solutions. However, it can be shown [84] that for any set of n points, none of them too close together, there is an n-monopole BPS solution with Higgs field zeros at the specified points.[4]

Actually obtaining explicit multisoliton solutions is a nontrivial matter. We have seen that the solution describing n vortices superimposed at the origin is rotationally symmetric and can readily be obtained by numerical solution of a pair of ordinary differential equations. By contrast, in the case of monopoles there are no spherically symmetric configurations with $n > 1$, and so no such simple multimonopole solutions. A rather fruitful method for studying solutions is a generalization, due to Nahm [172–175], of the ADHM instanton construction that will be described in Sec. 10.10. This Nahm construction transforms the problem of finding BPS monopoles in three spatial dimensions to that of solving a set of nonlinear differential equations for three matrix functions of a single variable. Once this "Nahm equation" has been solved the spacetime gauge and Higgs fields can be obtained from the solutions of a linear ordinary differential equation.[5]

The Nahm equation can be solved completely for the two-monopole case, with the matrices given in terms of elliptic functions [177]. Some curious features of the resulting monopole solutions are worth noting. First, since a single monopole is spherically symmetric, one might have expected the solutions describing two spatially separated monopoles to be axially symmetric under rotations about the line separating them. This is never the case [178]. To understand this physically, imagine trying to patch together the solutions describing two widely separated monopoles with their centers on the z-axis at $z = \pm R$. In order to make the Higgs fields match up, the monopoles should be written in the string gauge form described in Sec. 5.3. There must then be a Dirac string emanating from each of the monopoles. Let us take these strings to run outward along the z-axis, with one running from $z = R$ to $z = \infty$ and the other from $z = -R$ to $z = -\infty$, so that we do not have to deal with overlapping strings. Considered in isolation, each of the monopoles has an axial symmetry corresponding to a combination of physical rotation about the z-axis and a global phase rotation in the unbroken U(1) that acts on the charged vector W field. Because their strings run in opposite directions, the compensating phase rotations for the two monopoles are in opposite directions. As a result, the combined configuration cannot be exactly axially symmetric. A gauge-invariant measure of the departure from axial symmetry is given by the scalar product

[4] There are solutions in which the monopole cores overlap, but in these the connection between zeros of the Higgs field and monopole positions is sometimes lost. As will be described below, there are solutions with more Higgs zeros than monopoles, with the difference accounted for by the presence of antizeros of the field.

[5] A detailed description of the Nahm construction for both SU(2) and larger gauge groups can be found in [155]. Both the Nahm construction and other approaches to multimonopole solutions are discussed in [176].

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$\mathbf{W}^*_{(1)} \cdot \mathbf{W}_{(2)}$, where $\mathbf{W}_{(a)}$ denotes the charged vector field of monopole a; this is $O(e^{-evR})$.

If the two monopoles are brought together so that the two zeros of the Higgs field coincide, the solution does become axially symmetric [179–182]. However, in contrast to what one might have expected from merging two spherical objects, the profiles of the energy density and Higgs fields are toroidal in shape.

Some unusual solutions with higher charge and extra symmetry have also been found. There is a solution with tetrahedral symmetry whose energy density contours resemble a tetrahedron with holes in its faces [183, 184]. Remarkably, this is a three-monopole solution. The Higgs field has four zeros, one at each vertex of the tetrahedron, but also a compensating antizero at the center. This demonstrates clearly that the zeros of the Higgs field need not coincide with the monopole positions when several monopoles are brought closely together [185]. There are also solutions corresponding to the other Platonic solids. There is a cubic $n = 4$ solution [183, 184] with a four-fold zero at the origin [186], an octahedral $n = 5$ solution [187] with zeros at the vertices and an antizero at the origin [186], a dodecahedral $n = 7$ solution [187] with a seven-fold zero at the center [186], and an icosahedral $n = 11$ solution [188].

8.5 The moduli space approximation

The multisoliton solutions of a given charge can be viewed as forming a manifold, known as the moduli space. A natural set of coordinates for the moduli space is given by the collective coordinates z_r corresponding to the zero modes about the multisoliton solutions.

This viewpoint leads to a useful method, the moduli space approximation [189], for understanding the dynamics in the low-energy limit. Let us denote the solutions on the moduli space by $\psi^{\rm cl}(\mathbf{x}; z)$, where ψ is to be understood as a multicomponent field comprising all of the fields in the theory. An arbitrary field configuration can then be written as

$$\psi(\mathbf{x}, t) = \psi^{\rm cl}(\mathbf{x}; z(t)) + \delta\psi(\mathbf{x}; z(t), t), \tag{8.54}$$

where $\delta\psi$ is orthogonal to motion on the moduli space. Thus, motion involving the zero modes is described by the time dependence of the collective coordinates, while motion involving the modes with nonzero eigenvalue is described by $\delta\psi$. If the total energy of the system is small compared to the lowest nonzero frequency, excitation of any mode except the zero modes is strongly suppressed and $\delta\psi$ can be ignored. In this approximation, the time dependence of the field comes entirely through the time dependence of the collective coordinates. With the motion thus restricted to the moduli space, the potential energy is constant and just equal to the static multisoliton energy $E_{\rm static}$. Assuming a kinetic energy of the standard form, the Lagrangian density can be integrated over the D-dimensional space to yield

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$$L = -E_{\text{static}} + \frac{1}{2} g_{rs}(z) \dot{z}_r \dot{z}_s + \cdots, \tag{8.55}$$

where the ellipsis represents terms that are ignored in this approximation and

$$g_{rs}(z) = \int d^D x \, \frac{\partial \psi^{\text{cl}}(\mathbf{x}; z)}{\partial z_r} \, \frac{\partial \psi^{\text{cl}}(\mathbf{x}; z)}{\partial z_s} \tag{8.56}$$

may be viewed as defining a metric on the moduli space.

We have thus reduced a field theory system with an infinite number of degrees of freedom to a system with a finite number of degrees of freedom. Furthermore, the classical trajectories of this reduced system are simply characterized. They are just geodesic motions defined with respect to the metric g_{rs}. To determine these exactly would require knowing the precise form of this metric. However, in some cases we can obtain considerable insight with only partial information about the metric.

As an example, consider the scattering of two BPS vortices. The moduli space is four dimensional. We could choose the z_r to be the x- and y-coordinates of the two vortices. However, because the two vortices are identical, interchanging their positions does not yield a new configuration, so we need to identify the corresponding points on the moduli space. A better choice of the collective coordinates is to have two of them specify the position of the center of mass and two to be the polar coordinates ρ and θ that describe the relative positions of the vortices. We then account for the identity of the two vortices by identifying θ and $\theta + \pi$.

Because the center-of-mass motion separates from the relative motion, the moduli space can be written as the product of a center of mass manifold and a relative space manifold. The former is simply the two-dimensional Euclidean plane, whose geodesics are straight lines describing the constant motion of the center of mass. The nontrivial dynamics is captured in the relative space manifold. Because the forces between the vortices fall exponentially at large distance, this manifold must be locally flat at large ρ. It would asymptotically approach R^2 were it not for the requirement that the points (ρ, θ) and $(\rho, \theta + \pi)$ be identified. With this identification, the asymptotic manifold becomes a portion of a cone. Without interactions, this would continue until the cone came to a point. Including the short-range interactions between the vortices rounds off the tip of the cone and removes this singularity [176].

Now consider two vortices moving directly at each other in the center-of-mass frame. At first thought, it seems almost obvious that the two will either pass right through each other or else be reflected back along their original paths. However, this is incorrect. The initial motion is described by a trajectory moving directly up the cone at some fixed θ_0. From the symmetry of the situation it is clear that this trajectory will continue up to the (rounded) tip of the cone and then straight down the opposite side. Unfolding the cone, we see that this corresponds to a shift from θ_0 to $\theta_0 + \pi/2$. Thus, two vortices that approach head-on scatter at right angles to their initial direction [190].

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Two complications arise when we try to apply the moduli space approximation to the scattering of monopoles. First, the presence of the massless electromagnetic field means that there is a continuum of normal frequencies running down to zero. No matter how slowly the monopoles are moving, excitation of some of these modes via electromagnetic radiation is always energetically allowed. This turns out not to be a problem [191, 192]. When the monopoles are separated by distances large compared to their core size, they can be treated as point magnetic charges and standard electromagnetic methods can be used to show that the total radiation falls as the fifth power of the monopole velocities. These methods fail when the monopole cores overlap, but the only modes affected by this overlap have wavelengths comparable to or smaller than the monopole core size and are therefore energetically inaccessible if the monopole velocity is sufficiently low.

The second complication arises from the fact that there is an unbroken gauge symmetry with a corresponding Gauss's law that must be satisfied. If we adopt a (4+1)-dimensional notation where $\Phi = A^4$, but with no dependence on x^4, so that $\partial_4 = 0$, Gauss's law can be written as

$$0 = D_a F^{a0}, \tag{8.57}$$

with a running from 1 to 4. If the time dependence of the fields is only through the collective coordinates, we then have

$$0 = D_a \left[D^a A^0 - \dot{z}_r \frac{\partial (A^{\rm cl})^a}{\partial z_r} \right]. \tag{8.58}$$

This requires that A_0 be nonzero and proportional to the collective coordinate velocities, and so of the form

$$A_0 = \dot{z}_r \epsilon_r. \tag{8.59}$$

Hence

$$F^{a0} = -\dot{z}_r \delta_r A^a, \tag{8.60}$$

where Gauss's law requires that

$$\delta_r A^a = \frac{\partial (A^{\rm cl})^a}{\partial z_r} - D_a \epsilon_r \tag{8.61}$$

obey the background gauge condition

$$0 = D_a \delta_r A^a. \tag{8.62}$$

Substituting these results into the field theory Lagrangian and then proceeding as before, we obtain a moduli space Lagrangian of the form in Eq. (8.55), but with the metric now given by

$$g_{rs}(z) = 2 \int d^3x \, {\rm tr} \, \delta_r A^a \delta_s A^a. \tag{8.63}$$

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Again, the exact form of the moduli space metric can only be found in some special cases. However, the asymptotic form in that part of moduli space describing n well-separated monopoles or dyons can be inferred by noting that in this limit the dynamics should be completely described by the purely Abelian long-range electric, magnetic, and scalar forces between point objects [193, 194]. Examination of this metric reveals that it develops a singularity if any of the intermonopole separations is taken to zero. This is symptomatic of the fact that new interactions, mediated by the massive non-Abelian gauge bosons, come into play at short distances.

The exact metric, including these corrections, for the specific case of two monopoles was found by Atiyah and Hitchin [195–197]. The full moduli space is eight dimensional, but three of the collective coordinates correspond to the center-of-mass position and a fourth to the overall U(1) phase, which is conjugate to the conserved total electric charge. The nontrivial dynamics is confined to the four-dimensional manifold spanned by the relative coordinates. This manifold must satisfy two important constraints. First, it must have an SO(3) isometry that reflects the three-dimensional rotational invariance of the dynamics. The second is more subtle. One consequence of the underlying supersymmetry is that a BPS monopole moduli space must be hyper-Kähler; i.e., it must have a natural quaternionic structure at each point.[6] These two requirements, together with the requirement that the metric have the correct asymptotic behavior at large monopole separation, reduce the problem of finding the metric to the solution of a set of ordinary differential equations for three metric functions that are easily analyzed [198].

With the Atiyah–Hitchin metric in hand, one can address the scattering of two BPS monopoles. In the particular case of head-on collisions, the scattering angle is $\pi/2$, with the monopoles emerging in the plane perpendicular to their initial line of approach [195, 196]. The particular direction in that plane is determined by the departure from axial symmetry in the initial two-monopole configuration.

8.6 BPS monopoles in larger gauge groups

Our discussion of BPS monopoles so far has focused on those arising in the SU(2) gauge theory. The close connection with extended supersymmetry suggests that in generalizing our results to a gauge group G with rank $r > 1$, we should consider only theories with adjoint representation Higgs fields. Such theories can be naturally extended to $N = 2$ ($N = 4$) supersymmetric Yang–Mills theory with two (six) adjoint representation scalar fields that can be viewed as elements of the Lie algebra. The scalar field potential is minimized (and set equal to zero) by any set of commuting scalar field vacuum expectation values.

[6] For a more precise definition and a discussion of the relation of the hyper-Kähler structure to the properties of the zero modes, see [155].

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Let us assume that G is simple, with gauge coupling e. Let us also assume that only one scalar field, Φ, has a nonzero vacuum expectation value Φ_0. Without any loss of generality, we can take Φ_0 to lie in the Cartan subalgebra, so that it can be written as a linear combination

$$\Phi_0 = \mathbf{h} \cdot \mathbf{H} \tag{8.64}$$

of the commuting generators H_j. The generators of the unbroken gauge group are those which commute with Φ_0. These are all of the H_j, together with any $E_{\boldsymbol{\alpha}}$ for which $\mathbf{h} \cdot \boldsymbol{\alpha} = 0$. We have two cases to consider. If $\mathbf{h}$ has nonzero inner products with all of the roots of G, then the unbroken gauge group is $\mathrm{U}(1)^r$; I will refer to this case as maximal symmetry breaking. The other possibility is that $\mathbf{h}$ is orthogonal to some of the roots. These roots are then the roots of a rank r' semisimple group K, and the unbroken group is $\mathrm{U}(1)^{r-r'} \times K$.

Let us start with the case of maximal symmetry breaking and consider a solution with total magnetic charge Q_M. At large distance the Higgs field and the magnetic charge must commute, so in a direction where the asymptotic Higgs field is Φ_0 we can require that the magnetic charge also lie in the Cartan subalgebra and write

$$Q_M = \frac{4\pi}{e} \mathbf{g} \cdot \mathbf{H}. \tag{8.65}$$

As was discussed in Sec. 6.1, the generalization of the Dirac quantization condition to larger gauge groups depends on the group representations that appear among the fields in the theory. The conditions on a nonsingular monopole solution are the most restrictive, because they must be consistent with all possible representations that could be added to the theory. The result is that we must require that

$$\mathbf{g} = \sum n_{\boldsymbol{\alpha}} \boldsymbol{\alpha}^*, \tag{8.66}$$

where

$$\boldsymbol{\alpha}^* = \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2} \tag{8.67}$$

is a root of the dual lattice.

However, since the simple roots $\boldsymbol{\beta}_a$ form a basis for the Lie algebra, we can re-express this as

$$\mathbf{g} = \sum_{a=1}^{r} n_a \boldsymbol{\beta}_a^*. \tag{8.68}$$

The integers n_a are the r topological charges corresponding to the r U(1) factors of the unbroken group.

There are many possible choices for the simple roots. A particularly natural choice is specified by requiring that they all obey

$$\mathbf{h} \cdot \boldsymbol{\beta}_a > 0. \tag{8.69}$$

With this choice, there is a unique set of r fundamental BPS monopole solutions, each carrying one unit of topological charge [199]. If $A_i^{(s)}(\mathbf{r}; v)$ and $\Phi^{(s)}(\mathbf{r}; v)$ are

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the fields of the SU(2) BPS monopole with Higgs vacuum expectation value v, the ath fundamental monopole solution is given explicitly by

$$A_i(\mathbf{r}) = \sum_{s=1}^{3} A_i^{(s)}(\mathbf{r}; \boldsymbol{\beta}_a \cdot \mathbf{h}) t_s(\boldsymbol{\beta}_a),$$
$$\Phi(\mathbf{r}) = \sum_{s=1}^{3} \Phi^{(s)}(\mathbf{r}; \boldsymbol{\beta}_a \cdot \mathbf{h}) t_s(\boldsymbol{\beta}_a) + [\mathbf{h} - (\mathbf{h} \cdot \boldsymbol{\beta}_a^*)\boldsymbol{\beta}_a] \cdot \mathbf{H}. \tag{8.70}$$

The $t_s(\boldsymbol{\beta}_a)$ are the generators of the SU(2) subgroup defined by $\boldsymbol{\beta}_a$, and are given by Eq. (A.7).

The topological charges of this monopole are

$$n_b = \delta_{ab}. \tag{8.71}$$

Its mass can be found by noting that the monopole mass calculation in Sec. 8.2 was independent of the gauge group. With vanishing electric charge, Eqs. (8.12), (8.64), and (8.65) tell us that the mass of the ath fundamental monopole is

$$m_a = \mathcal{Q}_M = \frac{4\pi}{e} \mathbf{h} \cdot \boldsymbol{\beta}_a^*. \tag{8.72}$$

Extending this calculation to arbitrary magnetic charge, we find that the energy of an arbitrary purely magnetic static solution is

$$M = \sum_{a=1}^{r} n_a m_a. \tag{8.73}$$

This suggests that any such solution should be viewed as being a multimonopole solution comprising appropriate numbers of fundamental monopoles of the various species. Further support for this interpretation comes from the index theory calculations of Appendix B. A straightforward extension of the calculation for the SU(2) monopole tells us that the number of zero modes about a BPS solution in a larger group is [199]

$$N = 4 \sum_{a=1}^{r} n_a. \tag{8.74}$$

This is just what should be expected for a collection of fundamental monopoles, each of which has three translational degrees of freedom and one global phase in the U(1) subgroup corresponding to its topological charge.

To make this discussion a bit more concrete, consider an SU(N) gauge theory and choose a basis where Φ_0, represented as an $N \times N$ Hermitian matrix, is diagonal with its eigenvalues decreasing along the diagonal. In a direction where the asymptotic Higgs field is Φ_0, the asymptotic magnetic field is

$$B_i = \frac{4\pi}{e} \frac{\hat{r}_i}{r^2} \operatorname{diag}(n_1, n_2 - n_1, \ldots, n_{r-1} - n_{r-2}, -n_{r-1}). \tag{8.75}$$

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The fundamental monopole solutions are obtained by inserting the appropriately rescaled SU(2) monopole solution into the 2×2 blocks that lie directly along the diagonal and then adding a suitable diagonal constant to Φ so as to obtain the correct asymptotic value.

Of course, we could also obtain a solution by embedding the SU(2) monopole in the SU(2) subgroup defined by a nonsimple root. For example, using the root $\boldsymbol{\alpha} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$ leads to a solution whose fields have nonzero values in their 11, 13, 31, and 33 matrix elements. This solution is spherically symmetric and at first sight does not appear to be essentially different from the fundamental monopole solutions. However, Eq. (8.74) tells us that it has eight zero modes, rather than four. This indicates that it is actually just a special case of a two-monopole solution. It can be continuously deformed in such a way that the two component monopoles, which are of different species, become spatially separated; this can be verified explicitly by using the Nahm construction to obtain the complete eight-parameter family of solutions [200].

In contrast with the case of two identical SU(2) monopoles, these solutions with two separated monopoles of different species are axially symmetric, reflecting the presence of two independent unbroken U(1) subgroups. We can also study their low-energy interactions using the moduli space approximation. The moduli space metric for this two-monopole case has been found [201–203]. Unlike the SU(2) case, the asymptotic form of the metric is exact.[7] This is a consequence of the presence of two conserved U(1) charges, which eliminates the possibility of interactions via the exchange of massive vector mesons.

Let us now turn to the case of nonmaximal symmetry breaking, where the unbroken gauge group has a non-Abelian factor K. In this case the vector $\mathbf{h}$ defined by the Higgs field vacuum expectation value is orthogonal to some roots, so Eq. (8.69) must be replaced by the weaker requirement that the simple roots obey $\boldsymbol{\beta}_a \cdot \mathbf{h} \geq 0$. This does not uniquely fix the set of simple roots, with the ambiguity in choosing the β_a corresponding to the action of the non-Abelian subgroup K. In addition, Eq. (8.72) seems to indicate that the fundamental monopoles corresponding to the simple roots that are orthogonal to $\mathbf{h}$ must be massless. However, it is easy to see that there can be no massless classical solitons. In fact, substitution of the appropriate $\boldsymbol{\beta}_a$ into Eq. (8.70) simply yields the vacuum solution. Nevertheless, we will see that there is a sense in which these massless monopoles exist.

To explore this further, consider the case of SU(3). With the symmetry broken to U(1)×U(1), there are two species of massive fundamental monopoles, as indicated in Fig. 8.1a. With the unbroken symmetry being SU(2)×U(1), there are two choices for the simple roots, shown in Fig. 8.1b. The two choices are related by a rotation in the SU(2) subgroup whose roots are $\boldsymbol{\beta}_2$ and $-\boldsymbol{\beta}_2 = \boldsymbol{\beta}_2'$.

[7] This result can be extended to the case of an arbitrary number of distinct fundamental monopoles [204–206].

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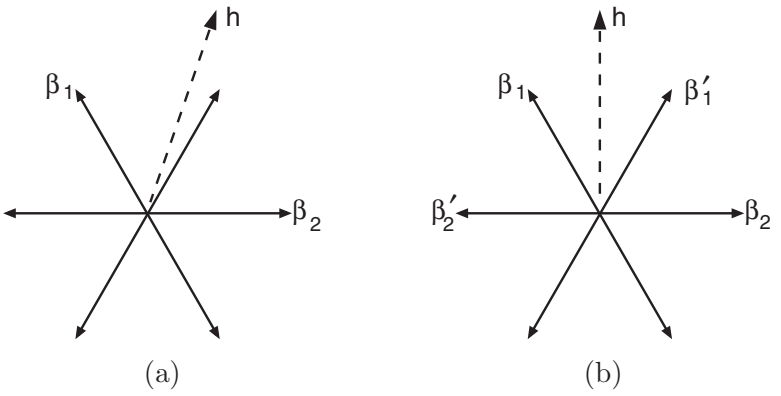


Fig. 8.1. The simple roots for (a) SU(3) maximally broken to U(1)×U(1) and (b) SU(3) broken to U(1)×SU(2). In the latter case there are two alternative pairs of simple roots.

With either choice one fundamental monopole, that corresponding to $\boldsymbol{\beta}_1$ or $\boldsymbol{\beta}'_1$, is massive, while the one corresponding to $\boldsymbol{\beta}_2$ or $\boldsymbol{\beta}'_2$ is massless. Note that $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}'_1$ lead to gauge-equivalent solutions.

The magnetic charge of these monopoles is given by Eq. (8.65). Because $\boldsymbol{\beta}_1$ has nonzero inner products with the roots of the unbroken SU(2), the magnetic charge of the massive monopoles has both Abelian U(1) and non-Abelian SU(2) components; that of the massless monopoles is purely non-Abelian. The case with nonmaximal symmetry breaking can be viewed as a limiting case of that with maximal symmetry breaking, corresponding to a rotation of the vector $\mathbf{h}$ that brings it to the vertical. This suggests that we examine the behavior of the solutions as this limit is approached. Taking this limit has no qualitative effect on the $\boldsymbol{\beta}_1$ monopole solution. On the other hand, the core region of the $\boldsymbol{\beta}_2$ monopole expands without bound, while at the same time the deviation of the fields from their vacuum values goes to zero. In the limit we are left with simply a vacuum solution.

We can also consider the behavior of a two-monopole solution containing one $\boldsymbol{\beta}_1$ monopole and one $\boldsymbol{\beta}_2$ monopole. Again, the core of the $\boldsymbol{\beta}_2$ monopole grows without bound [207]. In the limit we obtain a solution that is, up to a possible gauge transformation, just a single massive $\boldsymbol{\beta}_1$ monopole. This is to be expected, since $\boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^* = \boldsymbol{\beta}_1'^*$, which is related by an SU(2) rotation to $\boldsymbol{\beta}_1^*$

Now consider the case of two $\boldsymbol{\beta}_1$ monopoles and one $\boldsymbol{\beta}_2$ monopole. The new feature here is that the total magnetic charge, proportional to $2\boldsymbol{\beta}_1^* + \boldsymbol{\beta}_2^*$, lies entirely within the U(1) subgroup, and so remains purely Abelian in the limit of non-Abelian breaking. Starting with the monopoles at arbitrary locations and reducing the mass of the $\boldsymbol{\beta}_2$ monopole, we find that the core of the latter begins to expand, just as in the previous cases. However, when this core reaches the two massive monopoles, its expansion slows down and ends. In the limit, we have a solution containing two massive monopoles enclosed in what might be termed

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a cloud of non-Abelian field [207]. Inside this cloud the magnetic field is that which would be expected from the two massive monopoles, with both Abelian and non-Abelian components. Outside the cloud only the Abelian component survives.

This distinction between the case where the magnetic field is purely Abelian and that where it is not carries over to the zero modes and the moduli space. As was already noted in Sec. 6.4, when the magnetic field has a non-Abelian component there are non-normalizable zero modes corresponding to the action of the unbroken group on the long-range tail of the magnetic field. Furthermore, the counting of zero modes by the index theory methods described in Appendix B fails because of the poor behavior of the fields at large distance. Neither of these problems is present when the magnetic charge is purely Abelian. In this case, one finds that the number of zero modes is still given by Eq. (8.74), with the sum running over all species of fundamental monopoles, both massive and massless [208, 209]. Thus, even though the massless monopoles do not exist as isolated soliton solutions, their degrees of freedom survive as the collective coordinates describing the properties of the non-Abelian clouds. Furthermore, the moduli space metric behaves smoothly as one goes from the maximally broken case to the limit with nonmaximal breaking [209].

An example that shows this particularly clearly is provided by the case of SO(5) broken to SU(2)×U(1). With maximal symmetry breaking there are two species of massive monopoles, and the metric on the eight-dimensional moduli space of solutions containing one of each type is known exactly. With nonmaximal breaking there is a spherically symmetric solution with a massive monopole surrounded by a non-Abelian cloud whose form is known explicitly [210]. Using the explicit solution and the definition of the moduli space metric, Eq. (8.56), one can calculate the metric for the latter case and verify explicitly that it is the smooth limit of the metric for the former [209].

Examination of solutions with two massive monopoles and a single massless monopole [200, 211–214] reveals an ellipsoidal non-Abelian cloud. However, in solutions with multiple massless monopoles one finds that there is no longer a one-to-one correspondence between clouds and massless monopoles [215].

The moduli space approximation can be used to study the interactions of collections of massive monopoles and non-Abelian clouds [216–219]. Because of the presence of the additional massless fields corresponding to the gauge bosons of the unbroken non-Abelian group, there is no guarantee that the moduli space approximation will be valid. Indeed, application of this approximation leads to a prediction that the cloud radius eventually expands at greater than the speed of light, a clear sign that the approximation has broken down. However, comparison with numerical solution of the full field equations shows that there is a regime in which the moduli space approximation gives reliable predictions for the motion of the massive monopoles and for the exchange of energy between the massive and massless monopoles [220].

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Table 8.1 *The particle masses, charges, and spins in the BPS limit of the nonsupersymmetric SU(2) theory.*

	Mass	Q_E	Q_M	Spin
photon	0	0	0	1
ϕ	0	0	0	0
$W^{\pm}$	ev	$\pm e$	0	1
Monopole	$\frac{4\pi v}{e}$	0	$\pm\frac{4\pi}{e}$	0

8.7 Montonen–Olive duality

Montonen and Olive [221] pointed out an intriguing feature of the particle spectrum in the BPS limit of the SU(2) theory. Table 8.1 shows the masses and charges for the elementary particles implied by the Lagrangian of Eq. (5.45), together with those of the unit monopole and antimonopole. If one interchanges Q_M and Q_E and at the same time replaces e by $4\pi/e$, the entries for the gauge bosons and the monopoles are interchanged, but the overall spectrum of masses and charges remains the same.

The suggests that there might be a duality relating two equivalent formulations of the quantum field theory, similar to that between the sine-Gordon and massive Thirring models that was described in Sec. 2.7. In this case, however, the dual formulations of the theory would be identical in form except for the interchange of strong and weak coupling. In other words, the duality would actually be a self-duality.

Further support for this conjecture can be found by considering the forces between static objects. We have already seen that the energy of two static monopoles is just twice the monopole mass, regardless of their separation, so there is no force acting between them. Duality would require that the same be true for static W's. This can be tested by examining W-W scattering in the static limit. If the W's have the same sign of the electric charge, then in the limit where their spatial momenta are taken to zero the repulsive $1/r^2$ force corresponding to photon exchange is precisely canceled by the attractive $1/r^2$ force from Higgs exchange, just as required. (If the W's have opposite charges both forces are attractive, and there is a nonvanishing force for any value of the momenta.)

There is one very obvious difficulty. The gauge bosons have spin 1, whereas the supposedly dual quantum state built upon the spherically symmetric monopole solution is spinless. This contradiction is resolved by adding fermions and making the theory supersymmetric, something that we have seen is quite natural in the context of BPS solutions. In particular, recall from Sec. 5.7 that in the presence of a unit SU(2) monopole an adjoint representation Dirac fermion has two zero

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modes. With one such fermion (as one would have with $N = 2$ supersymmetry), there would be $2^2 = 4$ degenerate monopole states corresponding to the various possible occupation numbers of the zero modes. Two of these states would form a spin-1/2 multiplet, while the other two would be spinless, so there would still be no spin-1 dual to the W. But if we go to $N = 4$ supersymmetry there are two Dirac spinor fields, four fermionic zero modes, and thus $2^4 = 16$ degenerate states. These form one spin-1, four spin-1/2, and five spin-0 multiplets, and thus include not only the spin-1 counterpart of the W, but also the duals to the remaining members of the electrically charged $N = 4$ supermultiplet [222].

These are not the only particles in the theory. We have seen that there are dyons carrying unit magnetic charge and arbitrary integer electric charge. Duality requires that there also be one-particle states with unit electric charge and multiple magnetic charge. The existence of a dyonic state with two units of magnetic charge (in fact a supermultiplet of such states) was demonstrated by Sen, using arguments based on a supersymmetric extension of the moduli space dynamics discussed in Sec. 8.5 [223]. These states can be viewed as threshold bound states composed of two monopoles bound together through their interactions with the fermion fields. Extending this approach to states with higher magnetic charges poses technical difficulties, but there seems little doubt that the required states exist.

Now let us consider the extension of this conjecture to larger gauge groups, beginning with the case of maximal symmetry breaking, where the unbroken gauge group is a product of U(1) factors. In the elementary particle sector there is a massive gauge boson (with an accompanying supermultiplet) carrying electric-type U(1) charge (or charges) corresponding to each root $\boldsymbol{\alpha}$ of the original gauge group. Its mass is

$$M_{\boldsymbol{\alpha}} = e\mathbf{h} \cdot \boldsymbol{\alpha}. \tag{8.76}$$

If $\boldsymbol{\alpha}$ is a simple root $\boldsymbol{\beta}_a$, then the dual states will be obtained from the fundamental monopole corresponding to this root, which has a mass

$$m_a = \frac{4\pi}{e}\,\mathbf{h} \cdot \boldsymbol{\beta}_a^* = \frac{4\pi}{e}\,\mathbf{h} \cdot \frac{\boldsymbol{\beta}_a}{\boldsymbol{\beta}_a^2}. \tag{8.77}$$

If $\boldsymbol{\alpha}$ is not simple, a candidate for generating the dual states might be the solution, with energy

$$m_{\boldsymbol{\alpha}} = \frac{4\pi}{e}\,\mathbf{h} \cdot \boldsymbol{\alpha}^* = \frac{4\pi}{e}\,\mathbf{h} \cdot \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}, \tag{8.78}$$

obtained by embedding the SU(2) monopole in the SU(2) subgroup defined by $\boldsymbol{\alpha}$. However, we have seen that the counting of zero modes indicates that this solution is actually a particularly symmetric multimonopole solution, and so cannot give rise to the required one-particle magnetic dual states. Instead, the magnetic dual that we seek must be a threshold bound state of an appropriate collection of fundamental monopoles. The existence of such a bound state was first examined

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for the case of SU(3) broken to U(1)×U(1). In the notation of Fig. 8.1a, what is required is a bound state of a $\boldsymbol{\beta}_1$ monopole and a $\boldsymbol{\beta}_2$ monopole. By using the known moduli space metric for this two-monopole sector and applying Sen's method, one can verify the existence of the desired bound state [202, 203]. This calculation can be trivially extended to all other two-monopole cases. The result presumably extends to composite roots that are sums of arbitrary numbers of simple roots. However, this has not been verified, again because of the technical difficulty of the calculation.

Now note that the mass of the electrically charged particles, Eq. (8.77), involves $\boldsymbol{\alpha}$, while that of the magnetically charged particles, Eq. (8.78), involves $\boldsymbol{\alpha}^*$. For groups with a single root length μ (the "simply-laced" groups) the roots and their duals differ only by a multiplicative constant, and the only effect is that the Montonen–Olive substitution $e \leftrightarrow 4\pi/e$ must be replaced by $e\mu \leftrightarrow 4\pi/e\mu$. Indeed, the presence of the extra factor is not surprising if one realizes that the normalization of the gauge coupling depends on the convention for the root length.

The situation is more complex if there are two different root lengths. In this case, replacing each root by its dual is equivalent, up to an overall rescaling, to interchanging the long and the short roots. For some Lie algebras—F_4, G_2, and $\mathrm{SO}(5) = \mathrm{Sp}(4)$—this gives the original algebra, but with a relabeling of the roots. However, the interchange of long and short roots turns $\mathrm{SO}(2N+1)$ into $\mathrm{Sp}(2N)$, and vice versa.

Thus, the generalization of the Montonen–Olive conjecture is that theories based on simply laced gauge groups are self-dual under the interchange of electric and magnetic charges and weak and strong coupling. If the group is not simply laced, the duality maps the gauge theory onto one based on the dual Lie group.

New issues arise when we consider the case of nonmaximal symmetry breaking, where a non-Abelian subgroup remains unbroken. First, the elementary particle sector now includes some massless electrically charged particles, the "gluons" and their superpartners. The dual states are presumably related to the "massless monopoles". Although these do not exist as isolated classical solutions, they are in some sense manifested through the non-Abelian clouds and through the presence of their degrees of freedom in the moduli space metric. However, the exact relation between these and the gluon states needs to be clarified. Similarly, the precise mechanism by which the required non-Abelian multiplets of monopole states emerge from the classical solutions is yet to be fully worked out.

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