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Injective W^*-algebras

By SIMON WASSERMANN
University of Glasgow

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Introduction. Let M be a W^*-algebra. If π is a non-degenerate faithful normal representation of M on a Hilbert space H, M is said to be *injective* [(8), § 5] if there is a projection of norm one from $L(H)$ onto $\pi(M)$. Tomiyama [(13), theorem 7·2] has shown that this definition is independent of the choice of representation π, and, moreover, that M is injective if and only if the commutant $\pi(M)'$ is injective for some such π. M is said to be *semidiscrete* [(8), § 3] if the identity map on M can be approximated in the topology of simple–weak* convergence by completely positive normal linear maps in $L(M)$ of finite rank which preserve the identity. If π is a non-degenerate faithful normal representation of M on a Hilbert space H, a *-homomorphism η of the algebraic tensor product $M \odot \pi(M)'$ into $L(H)$ is defined by

$$\eta(x \otimes x') = \pi(x)x' \quad (x \in M, x' \in \pi(M)').$$

If η has a bounded extension to $M \otimes \pi(M)'$ (the completion relative to the spatial C^*-cross norm) we shall say that M is *amenable*.

Effros and Lance (8) investigated the relationship between the above properties with a view to obtaining information about nuclear C^*-algebras. They were able to show that semidiscreteness is equivalent to amenability [(8), proposition 4·5] and that either property implies injectivity [(8), corollary 5·10]. They left open the converse problem.

In an important recent paper (5) Connes has succeeded in showing that every injective W^*-factor acting as a von Neumann algebra on a separable Hilbert space is semidiscrete (and, in fact, hyperfinite). Choi and Effros (2) have used this result, together with some messy direct integral theory, to show that every injective W^*-algebra which has a faithful normal representation on a separable Hilbert space is semidiscrete. From this they deduce that a separable C^*-algebra is nuclear (cf. section 3, remark 1) if and only if A^{**} is injective [(2), corollary 3·2]; they deduce the same result for general A from the separable case by further ingenious arguments [(3), theorem 3]. In this paper we give a direct proof that all injective W^*-algebras are amenable (hence semidiscrete), from which it follows easily that a C^*-algebra whose second dual is injective is nuclear.

Connes' programme in (5) was as follows. He first showed that an injective II_1 factor N on a separable Hilbert space satisfies a condition analogous to Følner's classical condition for the amenability of a discrete group (9) (cf. [(6), § 2]), from which it follows that N is amenable and so semidiscrete. Using deep and lengthy arguments on the automorphism group of a factor he then showed that N is necessarily the

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hyperfinite II_1 factor R of Murray and von Neumann. From this it follows easily that there is just one injective II_∞ factor on a separable infinite dimensional Hilbert space H, the factor $R_{0,1} = R \overline{\otimes} L(H)$. Using his earlier results concerning the decomposition of a type III factor as a crossed product of a semifinite W^*-algebra by a suitable group of automorphisms (cf. (4)), together with a result on the integral decomposition of injective W^*-algebras and deep results on the automorphism group of $R_{0,1}$, Connes showed, finally, that every injective type III factor is necessarily hyperfinite, and a fortiori semidiscrete.

In section 1 of this paper we use a straightforward generalization of Connes' method to show that a finite injective W^*-algebra satisfies an analogue of Følner's condition, and is therefore amenable. It is then easy to prove the same result in the general semifinite case. In section 2 we use the results of section 1 to show that any injective type III W^*-algebra is amenable. The tool which makes this possible is the Takesaki duality theorem [(12), §8], which asserts that every type III W^*-algebra is the crossed product of a semifinite W^*-algebra by a one-parameter group of automorphisms.

We shall frequently make use of the fact that a W^*-algebra M which has a direct product decomposition $M = \oplus_i M_i$ is injective (resp. semidiscrete) if and only if every M_i is injective (resp. semidiscrete) (cf. [(8), proposition 3·1]). We note also that any type I W^*-algebra is both injective and semidiscrete [(8), proposition 3·5]. A knowledge of the basic classification theory of W^*-algebras, as given in (7) and (11), is assumed.

1. *The semifinite case.* We consider first the case of finite injective W^*-algebras. If A is a C^*-algebra, the opposite of A is a C^*-algebra A^c *-antiisomorphic to A; A^c may be realized as the C^*-algebra with the same Banach space structure and involution as A, and product $x.y$ given by the element yx, for $x, y \in A^c$. If the map $x \to \tilde{x}$ is a linear *-antiisomorphism of A onto A^c (for example the identity map on A in the above identification), the map $x \to (x^*)^\sim$ is a conjugate linear *-isomorphism of A onto A^c. If M is a W^*-algebra, M^c is a W^*-algebra, and we assume that a particular conjugate linear *-isomorphism $x \to x^c$; $M \to M^c$ is chosen for each M.

The following proposition is a straightforward generalization of part of [(5), theorem 5·1]. To aid the reader a sketch of the proof is included; details are omitted except where the proof deviates from that in (5).

PROPOSITION 1. *Let M be a finite injective W^*-algebra with centre Z and centre-valued trace Φ. If $a_i, b_i \in M$ $(i = 1, \ldots, n)$ and $\epsilon > 0$, there is a normal state f on Z such that*

$$|f(\Phi(\Sigma a_i b_i^*))| \leqslant \|\Sigma a_i \otimes b_i^c\|_\alpha + \epsilon,$$

where α is the spatial C^*-cross norm on $M \odot M^c$.

Proof. Let M act as a von Neumann algebra on the Hilbert space H and let π be a projection of norm one from $L(H)$ onto M. If $a, b \in L(H)$ and $\phi \in L(H)_*$, denote by $b\phi a$ the normal linear functional $x \to \phi(axb)$ $(x \in L(H))$.

Let $(L(H)_*)^m$ denote the product of the predual of $L(H)$ with itself m times, with the norm $\|(\phi_1, \ldots, \phi_m)\| = \|\phi_1\| + \ldots + \|\phi_m\|$. If $u_1, \ldots, u_m$ are unitaries in M, the set

$$C = \{(\phi - u_1^* \phi u_1, \ldots, \phi - u_m^* \phi u_m) : \phi \quad \text{a normal state on} \quad L(H)\}$$

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is convex in $(L(H)_*)^m$. Let g be a normal state on Z; then $\psi = g \circ \Phi \circ \pi$ is a state on $L(H)$, and by the module property of projections of norm one [(13), theorem 3·1], $\psi(x) = \psi(uxu^*)$ for any unitary $u \in M$. As in [(5), beginning of § 5] the existence of such a ψ implies that the norm-closure of C contains 0. Thus for any $\epsilon > 0$ there is a normal state ϕ on $L(H)$ such that

$$\|u_j \phi - \phi u_j\| \leqslant \epsilon \quad (j = 1, \ldots, m).$$

By [(11), 1·15·3] there is a trace-class operator ρ in $L(H)$ such that $\rho \geqslant 0$, $\mathrm{Tr}\,(\rho) = 1$, and $\phi(x) = \mathrm{Tr}\,(\rho x)$ $(x \in L(H))$. Then $(u_j \phi u_j^*)(x) = \mathrm{Tr}\,((u_j \rho u_j^*)(x))$ and

$$|\mathrm{Tr}\,((u_j \rho u_j^* - \rho)(x))| \leqslant \epsilon \|x\| \quad (x \in M).$$

It follows that $\|u_j \rho u_j^* - \rho\|_1 \leqslant \epsilon$, where $\|\ \|_1$ is the trace-norm. Proceeding as in the proof of [(5), theorem 5·1, $7 \Rightarrow 6$; 6, § 2], one shows that there is a non-zero spectral projection e of $\rho^{\frac{1}{2}}$ such that e is of finite rank and $\|[e, u_j]\|_2 \leqslant \epsilon \|e\|_2$ ($\|\ \|_2$ denotes the Hilbert–Schmidt norm). It follows from this, using the fact that any element of M is a linear combination of eight unitaries, that M has the property

(F) For $x_1, \ldots, x_m \in M$ and $\epsilon > 0$ there is a non-zero projection $e \in L(H)$ of finite rank such that $\|[e, x_j]\|_2 \leqslant \epsilon \|e\|_2$.

Now if $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$, with $\|a_j\| \leqslant 1$ $(1 \leqslant j \leqslant n)$, and $\epsilon > 0$, there are unitaries $u_1, \ldots, u_k \in M$ and a positive $\delta < 1$ such that if $c = \Sigma a_j b_j^*$ and ϕ is a state on M

$$\|u_j \phi - \phi u_j\| \leqslant \delta \Rightarrow |\phi(c - \Phi(c))| \leqslant \epsilon',$$

where $\epsilon' = \epsilon/(n+1)$. (If not, there is an $\alpha > 0$ such that for any finite subset F of the unitary group of M and $n \in \mathbb{N}$, there is a state $\phi_{F,n}$ of M such that $\|u\phi_{F,n} - \phi_{F,n} u\| < 1/n$ $(u \in F)$ and $|\phi_{F,n}(c - \Phi(c))| > \alpha$. If ϕ is a weak* limit point of $\{\phi_{F,n}\}$ in M^*, ϕ is a trace state, so that, by [(7), p. 257, prop. 3], $\phi = \phi \circ \Phi$, which contradicts the last inequality.)

By property F there is a non-zero projection e in $L(H)$ of finite rank such that

$$\|[b_i, e]\|_2 \leqslant \epsilon' \|e\|_2, \quad \|[u_j, e]\|_2 \leqslant (\delta/3)\|e\|_2 \quad (i = 1, \ldots, n; j = 1, \ldots, k).$$

If ϕ is the state on M defined by $\phi(x) = \mathrm{Tr}\,(exe)/\mathrm{Tr}\,(e)$, it follows that

$$\|u_j \phi - \phi u_j\| \leqslant \delta \quad (j = 1, \ldots, k)$$

(cf. [(5), proof of $7 \Rightarrow 6$, p. 103]), which implies that $|\phi(\Phi(c) - c)| \leqslant \epsilon'$. It now follows, exactly as in the proof of [(5), theorem 5·1, $6 \Rightarrow 5$], that

$$|\Sigma(a_i b_i^* e|e)_2 - \Sigma(a_i e b_i^*|e)_2| \leqslant n\epsilon' \|e\|_2^2,$$

where $(x|y)_2 = \mathrm{Tr}\,(y^* x)$ for Hilbert–Schmidt operators x and y; and, using finite dimensional arguments, that

$$(1/\mathrm{Tr}\,(e))|\Sigma(a_i e b_i^*|e)_2| \leqslant \|\Sigma a_i \otimes b_i^c\|_\alpha.$$

The term on the left of this inequality is within $(n+1)\epsilon' = \epsilon$ of $\phi(\Phi(c))$, and taking $f = \phi|Z$, the proposition follows.

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COROLLARY 2. *With M as above,*

$$\|\Phi(\Sigma a_i b_i^*)\| \leqslant \|\Sigma a_i \otimes b_i^c\|_\alpha$$

for $a_i, b_i \in M$ $(i = 1, \ldots, n)$.

Proof. Let $a_i, b_i \in M$ $(i = 1, \ldots, n)$, $c = \Sigma a_i b_i^*$, and $\beta = \|\Sigma a_i \otimes b_i^c\|_\alpha$. Suppose that $\|\Phi(c)\| > \beta$. By the spectral theory of normal operators there is a non-zero projection e in Z and $\lambda \in \mathbb{C}$ such that $|\lambda| > \beta$ and $\|e\Phi(c) - \lambda e\| < \frac{1}{2}(|\lambda| - \beta)$. Let $\epsilon = \frac{1}{2}(|\lambda| - \beta)$. As M is injective, and $M = M_e \oplus M_{1-e}$, M_e is injective; by Proposition 1 there is a normal state f on M_e such that

$$\begin{aligned} |f(e\Phi(c))| &\leqslant \|\Sigma e a_i e \otimes (e b_i e)^c\|_\alpha + \epsilon \\ &= \|(e \otimes e^c)(\Sigma a_i \otimes b_i^c)(e \otimes e^c)\|_\alpha + \epsilon \\ &\leqslant \|\Sigma a_i \otimes b_i^c\|_\alpha + \epsilon \\ &= \tfrac{1}{2}(\beta + |\lambda|). \end{aligned}$$

However $|f(e\Phi(c)) - \lambda| \leqslant \|e\Phi(c) - \lambda e\| < \frac{1}{2}(|\lambda| - \beta) \Rightarrow |f(e\Phi(c))| > \frac{1}{2}(\beta + |\lambda|)$. This contradiction shows that $\|\Phi(c)\| < \beta$.

COROLLARY 3. *If M is finite and injective, it is semidiscrete.*

Proof. M has a direct product decomposition $M = \oplus_i M_i$, where, for each i, there is a faithful normal state ϕ_i on the centre of M_i. By the remarks of the introduction each M_i is injective, and it suffices to establish that each M_i is semidiscrete. Thus we may assume that there is a faithful normal state ϕ on the centre of M. Let $f = \phi \circ \Phi$; then f is a faithful normal trace state on M. By Corollary 2

$$|f(\Sigma a_i b_i^*)| \leqslant \|\Sigma a_i \otimes b_i^c\|_\alpha$$

for $a_i, b_i \in M$ $(i = 1, \ldots, n)$. The proof is now identical to the proof of [(5), theorem 5·1, 5 ⇒ 4]. We consider M represented standardly as a von Neumann algebra on the Hilbert space H with cyclic and separating trace vector ξ such that $f(x) = (x\xi|\xi)$ $(x \in M)$. Let J be the corresponding conjugate linear involution, so that $Jx\xi = x^*\xi$ $(x \in M)$. By [(7), I, §5] $JMJ = M'$, and the map $x \to Jx^*J$ $(x \in M)$ is a linear *-anti-isomorphism of M onto M'. Thus $M' \cong M^c$ and the above inequality becomes

$$|(\Sigma a_i J b_i J \xi|\xi)| \leqslant \|\Sigma a_i \otimes J b_i J\|_\alpha$$

or

$$|(\Sigma x_i y_i \xi|\xi)| \leqslant \|\Sigma x_i \otimes y_i\|_\alpha,$$

where $x_i \in M$, $y_i \in M'$. Let $\eta : M \odot M' \to L(H)$ be the *-homomorphism given by $\eta(\Sigma x_i \otimes y_i) = \Sigma x_i y_i$. The last inequality shows that the linear functional

$$w(x) = (\eta(x)\xi|\xi) \quad (x \in M \odot M')$$

is bounded relative to the spatial C^*-cross norm α on $M \odot M'$, so that w has a unique bounded extension to $M \otimes M'$; this extension is a state. Thus if $x, y \in M \odot M'$,

$$w(y^*x^*xy) \leqslant \|x\|_\alpha^2 w(y^*y).$$

Also $w(y^*y) = \|\eta(y)\xi\|^2$, so that

$$\|\eta(x)\eta(y)\xi\|^2 = w(y^*x^*xy) \leqslant \|x\|_\alpha^2 \|\eta(y)\xi\|^2.$$

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As $\eta(M \odot M')\xi$ is dense in H, it follows that $\|\eta(x)\| \leqslant \|x\|_\alpha$ $(x \in M \odot M')$; that is, M is amenable and hence semidiscrete.

COROLLARY 4. *Let M be a semifinite injective W*-algebra. Then M is semidiscrete.*

Proof. $M = M_1 \oplus M_2 \oplus M_3$, where M_1 is of type I, M_2 is of type II_1, and M_3 is of type II_∞. As any type I W^*-algebra is semidiscrete, and by Corollary 3, M_2 is semidiscrete, it suffices to assume that M is of type II_∞. Let e be a non-zero finite projection of M; by Zorn's lemma there is a maximal family $\{e_i\}_{i \in I}$ of mutually orthogonal projections each equivalent to e. Let $p = 1 - \Sigma_i e_i$. By the comparison theorem [(7), III, § 1, theorem 1] there is a central projection z such that

$$pz \prec ez, \quad (1-z)e \prec (1-z)p.$$

By the maximality of the family e_i, $z \neq 0$. As M is properly infinite, I has infinite cardinality. Thus $\Sigma_i ze_i \succ z = zp + \Sigma_i ze_i$, and $\Sigma_i ze_i \prec zp + \Sigma_i ze_i = z$. By [(7), II, § 1, prop. 1], $\Sigma_i ze_i \sim z$, and we can assume (replacing the ze_i by equivalent projections) that $\Sigma_i ze_i = z$. By [(7), I, § 2, prop. 5] $M_z \cong L(H) \bar{\otimes} M_0$, where $H = l^2(\mathrm{I})$ and $M_0 \cong M_{ze}$ is finite. M_z is injective, so that M_0 is also injective, by [(8), prop. 5·6], and hence semidiscrete, by Corollary 3. As $L(H)$ is semidiscrete, and a tensor product of semidiscrete W^*-algebras is again semidiscrete, by [(8), prop. 3·4], it follows that M_z is semidiscrete. Now by Zorn's lemma there is a maximal family of mutually orthogonal non-zero central projections $\{z_j\}_{j \in J}$ such that each M_{z_j} is semidiscrete. If $\Sigma_j z_j = q$ is not the identity, M_{1-q} is non-zero and injective, so that, by the above argument, there is a non-zero central projection $z \leqslant 1 - q$ such that M_z is semidiscrete. This contradicts the maximality of the family $\{z_j\}$. Thus $\Sigma z_j = 1$ and $M = \oplus_j M_{z_j}$, so that M is semidiscrete.

2. *The general case.* It is clear from the preceding section that in order to show that injectivity implies semidiscreteness we can restrict attention from now on to the case of type III W^*-algebras. Before doing this it is, however, necessary to recall the notation and certain results from (12), notably the duality theorem.

Let M be a W^*-algebra acting non-degenerately as a von Neumann algebra on the Hilbert space H, and let $t \to U(t)$ be a strongly continuous unitary representation of $\mathbb{R}$ on H such that $U(t) M U(-t) \subseteq M$ $(t \in \mathbb{R})$. Then $\mathbb{R}$ has a representation $t \to \alpha_t$ as a one parameter group of *-automorphisms of M, where $\alpha_t(x) = U(t) x U(-t)$ $(x \in M)$; the map $t \to \alpha_t$ is continuous in the simple-strong topology.

The space $K(H, \mathbb{R})$ of continuous H-valued functions on $\mathbb{R}$ of compact support is a prehilbert space with the norm

$$\|\xi\|_2 = \left\{ \int_{\mathbb{R}} \|\xi(t)\|^2 \, dt \right\}^{\frac{1}{2}} \quad (\xi \in K(H, \mathbb{R})).$$

Let $K(\mathbb{R})$ denote the space of complex-valued continuous functions on $\mathbb{R}$ of compact support. $K(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$. If $\xi_1, \ldots, \xi_n \in H, f_1, \ldots, f_n \in K(\mathbb{R})$, the function $\Sigma \xi_i f_i$ which takes the value $\Sigma \xi_i f_i(t)$ at $t \in \mathbb{R}$ is in $K(H, \mathbb{R})$, and, noting that

$$\Sigma \xi_i \otimes f_i \in H \otimes L^2(\mathbb{R}), \quad \|\Sigma \xi_i \otimes f_i\| = \|\Sigma \xi_i f_i\|_2.$$

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There is thus a unique linear isometric map Φ of $K(H,\mathbb{R})$ onto a dense subspace of $H\otimes L^2(\mathbb{R})$ satisfying

$$\Phi(\Sigma\xi_i f_i)=\Sigma\xi_i\otimes f_i,$$

and we shall identify $K(H,\mathbb{R})$ with its image under Φ.

If $\xi\in K(H,\mathbb{R})$, the function $s\to U(-s)\xi(s)$ is also in $K(H,\mathbb{R})$ and has the same norm. There is thus a unique unitary operator U on $H\otimes L^2(\mathbb{R})$ such that $(U\xi)(s)=U(-s)\xi(s)$ $(\xi\in K(H,\mathbb{R}))$. For $x\in M$ let $\pi_\alpha(x)=U(x\otimes 1)U^*$. If $\xi\in K(H,\mathbb{R})$,

$$(\pi_\alpha(x)\xi)(s)=U(-s)xU(s)\xi(s)=\alpha_{-s}(x)\xi(s).$$

It is clear that the map $x\to\pi_\alpha(x)$ is an amplification of M.

For $t\in\mathbb{R}$ let $\lambda(t)$ be the unitary operator on $H\otimes L^2(\mathbb{R})$ given by $(\lambda(t)\xi)(s)=\xi(s-t)$ $(\xi\in K(H,\mathbb{R}),\ s\in\mathbb{R})$. Let $R(M,\alpha)=\{\pi_\alpha(M),\ \lambda(\mathbb{R})\}''$; $R(M,\alpha)$ is the *crossed product* of M by $\mathbb{R}$ with respect to the action α.

For $y\in M'$ let $\pi'_\alpha(y)=y\otimes 1$, so that for $\xi\in K(H,\mathbb{R})$ $(\pi'_\alpha(y)\xi)(s)=y\xi(s)$. Then $\pi'_\alpha(M')\subseteq R(M,\alpha)'$.

For $t\in\mathbb{R}$ let $\mu(t)$ be the unitary operator on $H\otimes L^2(\mathbb{R})$ given by $(\mu(t)\xi)(s)=e^{-ist}\xi(s)$ $(\xi\in K(H,\mathbb{R}),\ s\in\mathbb{R})$. $t\to\mu(t)$ is a continuous unitary representation of $\mathbb{R}$, and it is easy to verify that

$$\mu(t)\pi_\alpha(x)\mu(-t)=\pi_\alpha(x)$$

$$\mu(t)\lambda(s)\mu(-t)=e^{-ist}\lambda(s)\quad(s,t\in\mathbb{R},x\in M).$$

Thus $\mu(t)R(M,\alpha)\mu(-t)=R(M,\alpha)$ and the map $x\to\mu(t)x\mu(-t)=\hat\alpha_t(x)$ $(x\in R(M,\alpha))$ is an automorphism of $R(M,\alpha)$. $\hat\alpha$ is the *dual action* of $\mathbb{R}$ on $R(M,\alpha)$.

The following lemma may be deduced from [(12), theorem 7·1]; the proof given here is, however, more direct.

LEMMA 5. *There is a projection of norm one from $R(M,\alpha)$ to $\pi_\alpha(M)$.*

Proof. (a) Let U be the unitary operator defined above, and for $t\in\mathbb{R}$ let $\lambda'(t)$ be the unitary operator on $H\otimes L^2(\mathbb{R})$ given by

$$(\lambda'(t)\xi)(s)=U(t)\xi(s+t)\quad(\xi\in K(H,\mathbb{R}),s\in\mathbb{R}).$$

Then

$$\lambda'(t)\in R(M,\alpha)'$$

and

$$\begin{aligned}(U^*\lambda'(t)U\xi)(s)&=U(s)U(t)U(-s-t)\xi(s+t)\\&=(\lambda(-t)\xi)(s)\quad(s,t\in\mathbb{R},\xi\in K(H,\mathbb{R})).\end{aligned}$$

Thus $U^*\lambda'(t)U=\lambda(-t)$ for $t\in\mathbb{R}$. Also $U^*\mu(t)U=\mu(t)$ $(t\in\mathbb{R})$. It is well known (cf. (10)) that $\{\lambda(\mathbb{R}),\mu(\mathbb{R})\}''=\mathbb{C}.1\otimes L(L^2(\mathbb{R}))$. Now suppose that $x\in R(M,\alpha)$ is fixed by every $\hat\alpha_t$. Then $[x,\mu(t)]=0$ for $t\in\mathbb{R}$; also, $[x,\lambda'(t)]=0$ for $t\in\mathbb{R}$, so that

$$U^*xU\in\{\lambda(\mathbb{R}),\mu(\mathbb{R})\}'=L(H)\otimes\mathbb{C}.1.$$

Thus $U^*xU=z\otimes 1$ for some $z\in L(H)$. If $y\in M'$ and $\xi\in K(H,\mathbb{R})$,

$$(U^*\pi'_\alpha(y)U\xi)(s)=U(s)yU(-s)\xi(s).$$

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As $x \in \pi_\alpha'(M')'$, $z \otimes 1 \in (U^* \pi_\alpha'(M')\, U)'$; that is, for $\xi \in K(H, \mathbb{R})$,

$$[z, U(s)\, yU(-s)]\, \xi(s) = 0 \quad (s \in \mathbb{R}).$$

Taking $\xi = \eta \otimes f$, with $\eta \in H$, $f \in K(\mathbb{R})$, it follows that $[z, U(s)\, yU(-s)] = 0$ for all $s \in \mathbb{R}$, $y \in M'$. Thus $z \in M$, and $x = U(z \otimes 1)\, U^* = \pi_\alpha(z)$. Thus $x \in \pi_\alpha(M)$. Conversely if $x \in \pi_\alpha(M)$ it is immediate that $[x, \mu(t)] = 0$ $(t \in \mathbb{R})$. So $\pi_\alpha(M)$ is the fixed point subalgebra of $R(M, \alpha)$ under $\hat{\alpha}$.

(*b*) It is now a standard result that there is a projection of norm one from $R(M, \alpha)$ to $\pi_\alpha(M)$. One way to see this is as follows. For $x \in R(M, \alpha)$ and $n \in \mathbb{N}$ let

$$\phi_n(x) = (1/2n) \int_{-n}^{n} \hat{\alpha}_t(x)\, dt.$$

This integral exists as a weak limit of finite linear combinations of the $\hat{\alpha}_t(x)$ as the map $t \to \hat{\alpha}_t(x)$ is strongly continuous. Moreover $\|\phi_n(x)\| \leqslant \|x\|$, and $\phi_n(\hat{\alpha}_s(x)) = \hat{\alpha}_s(\phi_n(x))$ $(s \in \mathbb{R}, n \in \mathbb{N})$. Also, $\|\phi_n(\hat{\alpha}_s(x)) - \phi_n(x)\| \to 0$ as $n \to \infty$ for each $s \in \mathbb{R}$. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$ and define

$$\phi(x) = \lim_{\mathscr{U}} \phi_n(x),$$

the limit being in the weak topology. Then $\|\phi(x)\| \leqslant \|x\|$, and

$$\phi(\hat{\alpha}_s(x)) = \hat{\alpha}_s(\phi(x)) = \phi(x),$$

for $x \in R(M, \alpha)$, $s \in \mathbb{R}$. Thus $\phi | \pi_\alpha(M)$ is the identity map, $\|\phi\| \leqslant 1$, and $\phi(x) \in \pi_\alpha(M)$ for each $x \in R(M, \alpha)$, by part (*a*) of the proof. So ϕ is the required projection of norm one.

One can construct the second crossed product $R(R(M, \alpha), \hat{\alpha})$, the crossed product of $R(M, \alpha)$ by $\mathbb{R}$ with respect to $\hat{\alpha}$. The duality theorem of Takesaki [(12), theorem 4·5] asserts that $R(R(M, \alpha), \hat{\alpha})$ is canonically isomorphic to the W^*-tensor product $M \overline{\otimes} L(L^2(\mathbb{R}))$, which is isomorphic to M if M is of type III. If moreover the one parameter automorphism group α is taken to be the modular automorphism group $t \to \sigma_t$ corresponding to a faithful normal semifinite weight on M, $R(M, \alpha)$ is semi-finite, by [(12), theorem 8·2].

We can now prove the main result of this paper.

THEOREM 6. *If M is an injective W^*-algebra, M is semidiscrete.*

Proof. M has a direct product decomposition $M = M_1 \oplus M_2$, where M_1 is semi-finite and M_2 is of type III. By earlier remarks it is sufficient to show that M_2 is semidiscrete. Thus we may assume, without loss of generality, that M is of type III. Let $t \to \sigma_t$ be the modular automorphism group of M corresponding to a faithful normal semi-finite weight on M, and let M be in the corresponding standard representation on H (cf. (1)), so that σ_t is implemented by a strongly continuous one parameter unitary group on H. By the remarks above $R(M, \sigma)$ is semifinite, and $R(R(M, \sigma), \hat{\sigma}) \cong M$. By Lemma 5 there is a norm one projection ϕ from $R(R(M, \sigma), \hat{\sigma})$ onto $\pi_{\hat{\sigma}}(R(M, \sigma))$; and, as M is injective, there is a norm one projection π from $L(H \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}))$ onto $R(R(M, \sigma), \hat{\sigma})$. Then $\phi \circ \pi$ is a projection of norm one from $L(H \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}))$ to $\pi_{\hat{\sigma}}(R(M, \sigma))$. Thus $R(M, \sigma)$ is injective, and, by Corollary 4, semidiscrete. By the

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remarks of the introduction it follows that $R(M,\sigma)$ is amenable. Thus if $a_i \in R(M,\sigma)$, $b_i \in R(M,\sigma)'$ $(i = 1,\dots,n)$,

$$\|\Sigma a_i b_i\| \leqslant \|\Sigma a_i \otimes b_i\|_\alpha.$$

In particular, if $x_i \in M$, $y_i \in M'$ $(i = 1,\dots,n)$,

$$\begin{aligned}\|\Sigma \pi_\sigma(x_i)\,\pi'_\sigma(y_i)\| &\leqslant \|\Sigma \pi_\sigma(x_i) \otimes \pi'_\sigma(y_i)\|_\alpha \\ &= \|\Sigma x_i \otimes y_i\|_\alpha,\end{aligned}$$

as π_σ and π'_σ are faithful representations of M and M', respectively. Let $\xi \in H, f \in K(\mathbb{R})$. Then

$$\begin{aligned}\|\Sigma \pi_\sigma(x_i)\,\pi'_\sigma(y_i)\,(\xi \otimes f)\|^2 &= \int_{\mathbb{R}} \|\Sigma \sigma_{-t}(x_i)\,y_i\,\xi\|^2\,|f(t)|^2\,dt \\ &\leqslant \|\Sigma \pi_\sigma(x_i)\,\pi'_\sigma(y_i)\|^2\,\|\xi\|^2\,\|f\|_2^2 \\ &\leqslant \|\Sigma x_i \otimes y_i\|_\alpha^2\,\|\xi\|^2\,\|f\|_2^2.\end{aligned}$$

As the H-valued function $t \to \Sigma \sigma_{-t}(x_i)\,y_i\,\xi$ is continuous and $f \in K(\mathbb{R})$ is arbitrary, it follows that

$$\|\Sigma \sigma_{-t}(x_i)\,y_i\,\xi\| \leqslant \|\Sigma x_i \otimes y_i\|_\alpha\,\|\xi\|$$

for $t \in \mathbb{R}$, $\xi \in H$; taking $t = 0$ we thus obtain $\|\Sigma x_i y_i\| \leqslant \|\Sigma x_i \otimes y_i\|_\alpha$. As the x_i and y_i are arbitrary, this shows that M is amenable, and so semidiscrete.

3. *Concluding remarks*

(1) Recall that a C^*-algebra A is *nuclear* if there is a unique C^*-norm on the algebraic tensor product $A \odot B$ of A with an arbitrary C^*-algebra B. Effros and Lance showed that if A is nuclear, A^{**} is injective; and if A^{**} is semidiscrete, A is nuclear [(8), theorem 6·4]. Choi and Effros (2, 3) showed that the conditions (i) A nuclear, (ii) A^{**} injective, (iii) A^{**} semidiscrete, are equivalent if A is separable, and from this deduced the equivalence of (i) and (ii) in general. An important consequence of this was an affirmative solution to the long-standing conjecture that the quotient of a nuclear C^*-algebra by a close two-sided ideal is again nuclear [(3), corollary 4]. By Theorem 6 one can assert the equivalence of the conditions (i), (ii), and (iii) in general. From this it follows immediately that a C^*-algebra A is nuclear if and only if A^{**} is semidiscrete. If B is the quotient of a nuclear C^*-algebra A by a closed two-sided ideal, $B^{**} = (A^{**})_e$, for some central projection e in A^{**}. Hence B^{**} is semidiscrete, which implies that B is nuclear.

(2) Let R be a von Neumann algebra on a Hilbert space H, and let A and B be weakly dense C^*-subalgebras of R and R', respectively. If η is the *-homomorphism $\Sigma a_i \otimes b_i \to \Sigma a_i b_i : A \odot B \to L(H)$, it is natural, in view of the previous remark, to ask whether R is amenable, whenever η is bounded relative to the spatial C^*-norm on $A \odot B$. The following example shows that this need not be the case.

Example 7. Let $M = \bigoplus_{n=1}^{\infty} M_n$, where M_n is the C^*-algebra of complex $n \times n$ matrices, let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$, and let $I_{\mathscr{U}}$ be the maximal two-sided ideal of M

$$I_{\mathscr{U}} = \{(x_n) : \lim_{\mathscr{U}} \mathrm{tr}_n(x_n^* x_n) = 0\}.$$

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Let $N = M/I_{\mathscr{U}}$ and let Φ be the quotient morphism $M \to N$. N is a II_1 factor, and is non-amenable, by [(14), proposition 1·7]. Moreover if N acts standardly on the Hilbert space K, N is isomorphic to N', and the corresponding identification of $N \odot N$ with $N \odot N'$, the latter regarded in the usual way as a *-subalgebra of $L(K)$, enables one to define a C^*-norm γ on $N \odot N$. γ is distinct from the spatial C^*-norm (as N is non-amenable), and there is a bounded *-homomorphism Ψ from $M \otimes M$ onto $N \otimes_\gamma N$ such that $\Psi | M \odot M = \Phi \odot \Phi$ (cf. (14), lemma 2·4). Let M act standardly on the Hilbert space H, so that M is antiisomorphic, and hence isomorphic, to M' (as M is antiisomorphic to itself). It follows that there is a *-homomorphism Φ' of M' onto N' and a bounded *-homomorphism of $M \otimes M'$ into $L(K)$ whose restriction to $M \odot M'$ is just $\Phi \odot \Phi'$. Let A and B be the C^*-algebras of operators on $H \oplus K$

$$A = \{(x, \Phi(x)) : x \in M\} \cong M, B = \{(x', \Phi'(x')) : x' \in M'\} \cong M'.$$

For $U \in \mathscr{U}$ let $e_U = (x_n)$, where $x_n = 1$ $(n \in U)$; $x_n = 0$ $(n \notin U)$. Then $e_U \to 0$ strongly in M, and $\Phi(e_U) = 1$, for each $U \in \mathscr{U}$. Thus $(e_U, \Phi(e_U)) \to (0, 1)$ strongly, and the strong closure of A is $M \oplus N$. Similarly the strong closure of B is $M' \oplus N' = A'$. If

$$a_i = (x_i, \Phi(x_i)) \in A, \quad b_i = (y_i, \Phi'(y_i)) \in B \quad (i = 1, \ldots, n),$$

$$\|\Sigma \Phi(x_i)\, \Phi'(y_i)\|_{L(K)} \leqslant \|\Sigma x_i \otimes y_i\|_\alpha$$

(by the result cited above); and M, being of type I, is amenable. Thus

$$\begin{aligned} \|\Sigma (x_i, \Phi(x_i))\, (y_i, \Phi'(y_i))\| &= \|\Sigma (x_i y_i, \Phi(x_i)\, \Phi'(y_i))\| \\ &\leqslant \|\Sigma x_i \otimes y_i\|_\alpha. \end{aligned}$$

However as N is not amenable, neither is $\bar{A} = M \oplus N$.

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