Ergod. Th. & Dynam. Sys. (2000), **20**, 483–499
Printed in the United Kingdom © *2000 Cambridge University Press*

Borel cocycles, approximation properties and relative property T

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(*Received* 10 *December* 1997 *and accepted in revised form* 29 *September* 1998)

Abstract. Let G and H be locally compact groups. Assume that G acts on a standard probability space (S, μ), μ being G-invariant. We prove that if there exists a Borel cocycle $\alpha : S \times G \longrightarrow H$ which is proper in an appropriate sense, then G inherits some approximation properties of H, for instance amenability or the so-called Haagerup Approximation Property. On the other hand, if G_0 is a closed subgroup of G, if the pair (G, G_0) has the relative property (T) of Margulis [**19**] and if either H has Haagerup Approximation Property, or if it is the unitary group of a finite von Neumann algebra with a similar property, then we give rigidity results analogous to that in [**23**] and [**1**].

1. *Introduction*

All locally compact groups considered here are assumed to be second countable, and von Neumann algebras are assumed to have separable predual. Let G be such a group. Assume that G acts on a standard probability space (S, μ), the measure μ being G-invariant. We are mainly interested here in questions of the following type:

> If α is a Borel cocycle from $S \times G$ to some second countable group H, under which conditions on α properties of H are inherited by G? Conversely, if G has some rigidity properties, what does this imply on H or on α?

Concerning the second question, Zimmer proved in Chapter 9 of [**24**] that if G has property T, if it acts ergodically on S and if H is a lcsc amenable group, then α is equivalent to a cocycle into a compact subgroup. Adams and Spatzier got analogous results when G is discrete and when H is a subgroup of the automorphism group of a tree (for instance a free group) or when it is a Coxeter group [**1**].

The aim of our article is mainly to give simpler and unified proofs of results taken from [**1, 12**] and [**23**].

§2 is devoted to approximation properties: we recall the definition of the Haagerup Approximation Property (HAP) (also known as a-T-menability), and we prove some

https://doi.org/10.1017/S0143385700000237 Published online by Cambridge University Press

stability results concerning that property: for instance if G contains a closed subgroup H possessing that property, and if $L^\infty(G/H)$ has a G-invariant state (for instance, if H is normal and if G/H is amenable or if H is a lattice) then G also has the same property. It turns out that this follows from a more general result concerning *proper Borel cocycles* (see §2 for a definition).

THEOREM A. *Let $\alpha : S \times G \longrightarrow H$ be such a cocycle and assume that H has property HAP and that the pair (S, G) is an amenable pair in the sense of* [**18**]. *Then G has the same property.*

Notice that μ need not be G-invariant. In the same section, we also extend Theorem 5.6 of [**12**] for proper Borel cocycles.

THEOREM B. *Let α, S and G be as in Theorem A. Assume furthermore that μ is G-invariant. Then the Cowling–Haagerup constants of G and H satisfy:*

$$\Lambda(G) \leq \Lambda(H).$$

In §3, we unify and slightly generalize results from [**1**] and from [**23**] in proving the following.

THEOREM C. *Suppose that G_0 is a closed subgroup of G, that the pair (G, G_0) has the relative property T, that G_0 acts ergodically on S and that H is a lcsc group with the HAP. Then for every arbitrary Borel cocycle α as above, $\alpha|S \times G_0$ is equivalent to a cocycle with values in a compact subgroup.*

Here we do not assume that α is proper any more. Finally, in §4 we recall the definition of the HAP for finite von Neumann algebras and we prove a result analogous to Theorem C for cocycles with values in the unitary group $U(M)$ of a finite von Neumann algebra M which has the above approximation property; this slightly extends Theorem C of [**21**].

THEOREM D. *Let $\alpha : S \times G \longrightarrow U(M)$ be a Borel cocycle, and assume that the pair (G, G_0) has the relative property T and that M is as above. For every Borel function $u : S \longrightarrow U(M)$, set*

$$\alpha_u(g) = \int_S u(s)\alpha(s, g)u(sg)^{-1}\, d\mu(s),$$

which is the mean value at g of the equivalent cocycle

$$\beta(s, g) = u(s)\alpha(s, g)u(sg)^{-1}.$$

Then the set $\{\alpha_u(g) : g \in G_0\}$ has compact closure for the topology induced by $\|\cdot\|_2$.

In particular, this implies that groups like $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ cannot have a faithful representation with discrete range in such a group $U(M)$.

2. *Proper Borel cocycles and approximation properties*

Let us recall the following definition (see [**5**] and [**3**]).

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Definition 2.1. A locally compact, second countable group H has the HAP if the abelian C^*-algebra $C_0(H)$ of all continuous functions on H tending to 0 at infinity has an approximate unit made with positive definite functions.

Notice that it is equivalent to the existence of a sequence $(\varphi_n)_{n\geq 1}$ of (continuous) positive definite functions on H such that:

(1) $\varphi_n(1) = 1$, for every $n \geq 1$;
(2) $\varphi_n \in C_0(H)$, for every $n \geq 1$;
(3) $\varphi_n \to 1$ uniformly on compact subsets of H.

These groups have played an important role recently: see [**2, 5, 15**] and Higson and Kasparov proved that if such a group is discrete, it satisfies the Connes–Kasparov conjecture (see also [**22**]). Moreover, they may be characterized in many ways. The following theorem gathers known conditions equivalent to the HAP.

THEOREM 2.2. *Let H be a lcsc group. Then the following properties are equivalent:*

(1) *H has the HAP;*
(2) *there is a continuous, conditionally negative definite function $\psi : H \longrightarrow \mathbb{R}_+$ which is* proper *in the following sense: for every $M > 0$, one can find a compact set $K \subset H$ with $\psi(x) > M$ for every $x \notin K$* [**2**]*;*
(3) *H is* a-T-menable [**15**]*: there exists an affine, isometric, continuous action α of H on some Hilbert space $\mathcal{H}$ such that for all bounded subsets B, C of $\mathcal{H}$, the set:*

$$\{x \in H : (\alpha(x)B) \cap C \neq \emptyset\}$$

is relatively compact.

The equivalence between (1) and (2) is due to Akemann and Walter [**2**], and the equivalence between (2) and (3) is taken from [**5**] where it is proved for countable groups, but the extension of the proof of Bekka *et al* [**5**] to the general case is straightforward.

Before showing (old and new) examples of such groups, we prove a fourth condition equivalent to the HAP that will be used later on; we recall first a definition taken from [**6**]: if $(\pi, \mathcal{H}_\pi)$ is a unitary representation of H, we say that π *is of class C_0* if, for all $\xi, \eta \in \mathcal{H}_\pi$, the coefficient function $h \mapsto \langle \pi(h)\xi, \eta\rangle$ belongs to $C_0(H)$.

PROPOSITION 2.3. *Let H be a lcsc group; then H has the HAP if and only if it has a unitary representation $(\pi, \mathcal{H}_\pi)$ such that:*

(a) *$\mathcal{H}_\pi$ is separable and $1_H \prec \pi$ (i.e. the trivial representation is weakly contained in π);*
(b) *π is of class C_0.*

Proof. The existence of such a representation clearly implies the existence of a sequence $(\varphi_n)_{n\geq 1}$ satisfying conditions (1)–(3) in Definition 2.1.

Conversely, if such a sequence exists, let $(\pi_n, \mathcal{H}_n, \xi_n)$ be the cyclic representation associated with φ_n. Then each $\mathcal{H}_n$ is separable, and $\pi = \bigoplus_{n\geq 1} \pi_n$ is a separable representation of H. Then $1_H \prec \pi$ since $\varphi_n \to 1$ uniformly on compact sets, and every coefficient of π belongs to $C_0(H)$ because all φ_n do. □

Examples 2.4. Each group in the families below has the HAP.

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(1) Free groups on finitely many generators: Haagerup proved in [**17**] that the word length function $|\cdot|$ on such a group is conditionally negative definite; this motivates our terminology because then the sequence $(e^{-\frac{1}{n}|\cdot|})_{n\geq 1}$ satisfies required properties of Definition 2.1.
(2) Amenable lcsc groups: indeed, the left regular representation fulfills conditions (a) and (b) of Proposition 2.3; see also [**5**] for the case of discrete amenable groups, where an affine proper action is constructed.
(3) Finitely generated Coxeter groups: the natural word length function is conditionally negative definite [**8**].
(4) The rank one Lie groups $SO(1,n)$ and $SU(1,n)$: this follows from [**14**]; see also [**13**], and the recent nice proof for $SO(1,n)$ in [**20**].
(5) All groups that act properly and isometrically on some cube complex CAT(0) [**9**].
(6) Direct products and free products of finitely many groups with the HAP: see [**2**] and Proposition 2.5(3) below.
(7) Every closed subgroup of a group with the HAP: this follows trivially from Proposition 2.3.

We are going to prove the following result which seems to be new; its proof is postponed after Theorem 2.7 since the proof of statements (1) and (2) is based on it.

PROPOSITION 2.5. *If a lcsc group G satisfies one of the conditions below, then it has the HAP.*

(1) *G contains a closed subgroup H having the HAP and the homogeneous space G/H is amenable in the sense of Eymard (see below).*
(2) *G is the middle term of a short exact sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1,$$

where K is compact and H has the HAP.
(3) *$G = \Gamma_1 *_A \Gamma_2$ is an amalgamated product of two countable groups with the HAP containing a common finite subgroup A.*

Remark. For example, if H has the HAP and if A is an amenable group acting continuously on H, then the semidirect product group $A \ltimes H$ has the same property.

Notice also that compactness of the normal subgroup K above cannot be removed: indeed, $SL_2(\mathbb{R})$ has the HAP, and it is a quotient group by $\mathbb{R}^2$ of $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ which does not have the HAP.

It turns out that the two above conditions are particular cases of a more general situation that we describe now: let G and H be lcsc groups. Assume that G acts on a standard probability space (S, μ) in a measure class preserving way, but we do not assume that μ is G-invariant. Let $\chi(s,g) = d\mu(sg)/d\mu(s)$ be the corresponding Radon–Nikodym cocycle. Let $\alpha : S \times G \longrightarrow H$ be a Borel cocycle: for all $g, h \in G$, one has

$$\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$$

for μ-almost every $s \in S$. Recall also that two cocycles α and β are *equivalent* if there exists a Borel function $\varphi : S \longrightarrow H$ such that, for every $g \in G$,

$$\beta(s, g) = \varphi(s)\alpha(s, g)\varphi(sg)^{-1}$$

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for a.e. $s \in S$ (see [**24**]). Finally, let $(\pi, \mathcal{H})$ be a unitary representation of H. One associates a representation π_α of G on $L^2(S, \mu) \otimes \mathcal{H}$ by

$$(\pi_\alpha(g)\xi)(s) = \pi(\alpha(s, g))\xi(sg)\sqrt{\chi(s, g)}$$

for every $\xi \in L^2(S, \mu) \otimes \mathcal{H} = L^2(S, \mu, \mathcal{H})$.

For instance, the representation $\pi_S : G \longrightarrow U(L^2(S, \mu))$ given by

$$(\pi_S(g)\xi)(s) = \xi(sg)\sqrt{\chi(s, g)}$$

and associated with the action of G on S is the special case corresponding to the trivial representation of H.

Moreover, if H is a closed subgroup of G, if $\alpha(s, g) = \gamma(s)g\gamma(sg)^{-1}$, where $\gamma : G/H \longrightarrow H$ is a Borel cross-section for the canonical projection, then π_α is equivalent to the induced representation $\mathrm{Ind}_H^G(\pi)$.

Thus the representation π_α is an induced representation of π from H to G through α.

Definition 2.6. Let $\alpha : S \times G \longrightarrow H$ be as above. Then α is a *proper Borel cocycle* if there exists a generating family $\mathcal{A}$ of Borel subsets of S such that:

(1) S is a countable union of elements of $\mathcal{A}$;

(2) for every $A \in \mathcal{A}$ and for every compact subset K of G, $\alpha(A \times K)$ is relatively compact;

(3) for every $A \in \mathcal{A}$ and for every compact subset L of H, the set $K(A, L)$ of elements $g \in G$ such that $\mu(\{s \in A \cap Ag^{-1} : \alpha(s, g) \in L\}) > 0$ is relatively compact.

Remarks.

(1) We chose the adjective *proper* because condition (3) above can be interpreted as follows: for $A \in \mathcal{A}$ and $L \subset H$ compact, set

$$K'(A, L) = \{g \in G : \exists s \in A \cap Ag^{-1} \text{ with } \alpha(s, g) \in L\}$$

and define $\beta : A \times G \longrightarrow S \times H$ by

$$\beta(s, g) = (sg, \alpha(s, g)).$$

Then $K'(A, L)$ contains $K(A, L)$ and $K'(A, L) = pr_2(\beta^{-1}(A \times L))$.

(2) We will see in the proof of Proposition 2.5 how to define proper Borel cocycles in the presence of a group G and a closed subgroup H of G.

(3) One also gets examples of Borel cocycles coming from geometry (this is inspired by [**12**]). Suppose that Γ is a countable group that acts (on the right) on a compact manifold X, preserving a connection and a G-structure, where G is a suitable Lie group. Denote by P the associated frame bundle. Using a bounded measurable section of the projection $P \longrightarrow X$, we get a trivialization of $P \cong X \times G$, under which the corresponding action of Γ on $X \times G$ is given by

$$(x, g)\gamma = (x\gamma, g\alpha(x, \gamma)),$$

where $\alpha : X \times \Gamma \longrightarrow G$ is a Borel cocycle. Then, if the action of Γ on P is proper, α is a proper cocycle (see p. 232 of [**12**]). We will return to such a situation in Theorem 2.10.

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Recall from [**18**] that (S, G) is an *amenable pair* if $1_G \prec \pi_S$; this is equivalent to the existence of a G-invariant state on $L^\infty(S, \mu)$, and when S is a homogeneous space G/H, it corresponds to Eymard's amenability for homogeneous spaces.

THEOREM 2.7. *Let $\alpha : S \times G \longrightarrow H$ be a proper Borel cocycle and assume that (S, G) is an amenable pair.*

(1) *If H is amenable, then G is also amenable.*

(2) *If H has the HAP, then so does G.*

We need the following proposition for the proof of statement (1).

PROPOSITION 2.8. *Let (S, μ) be a G-space and let $\alpha : S \times G \longrightarrow H$ be a proper Borel cocycle. Denote by λ the left regular representation of H, and by λ_α the representation of G corresponding to λ and α. Then every coefficient of λ_α is a coefficient of the regular representation λ_G of G. In particular, λ_α is quasi-equivalent to a subrepresentation of λ_G.*

Before giving the proof of Proposition 2.8, we need to recall definitions of Fourier–Stieltjes and Fourier algebras of locally compact groups (see for instance [**11**]): let G be as above; $B(G)$ denotes the algebra of all coefficients of unitary representations of G, and $A(G)$ denotes the subspace (in fact, the closed ideal) consisting of all coefficients of the regular representation of G. More precisely, $A(G)$ is the norm closure in $B(G)$ of the subspace $C_c(G) \cap B(G)$, where $C_c(G)$ is the space of all continuous and compactly supported functions on G.

Proof of Proposition 2.8. For $\xi, \eta \in \mathcal{H} = L^2(S) \otimes L^2(H)$ and $g \in G$, set

$$\varphi_{\xi,\eta}(g) = \langle \lambda_\alpha(g)\xi, \eta \rangle.$$

It is sufficient to prove that $\varphi_{\xi,\eta} \in A(G)$ for ξ and η in a total subset $\mathcal{T}$ of $\mathcal{H}$. In order to choose it, observe that conditions (2) and (3) of Definition 2.6 are still satisfied by every set A belonging to the algebra generated by $\mathcal{A}$. This implies that the family $\mathcal{T} = \{\chi_A \otimes \xi : A \in \mathcal{A} \text{ and } \xi \in C_c(G)\}$ is a total subset of $\mathcal{H}$. Fix $A, B \in \mathcal{A}$ and $\xi, \eta \in C_c(G)$. Then set

$$\varphi(h) = \langle \lambda(h)\xi, \eta \rangle \quad \forall h \in H,$$

and

$$\psi(g) = \langle \lambda_\alpha(g)\chi_A \otimes \xi, \chi_B \otimes \eta \rangle \quad \forall g \in G.$$

We claim that ψ has compact support; indeed, φ has compact support, and one checks easily that

$$\psi(g) = \int_{B \cap Ag^{-1}} \varphi(\alpha(s, g))\sqrt{\chi(s, g)}\, d\mu(s) \quad \forall g \in G.$$

If $L = \text{supp}(\varphi)$, then it is straightforward to check that

$$\text{supp}(\psi) \subset \overline{K(A \cup B, L)}.$$

This proves that $\psi \in C_c(G) \cap B(G)$. □

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Proof of Theorem 2.7. We prove statement (2) first because arguments are needed in the proof of the first part.

Let $(\pi, \mathcal{H})$ be a representation of H such that $1_H \prec \pi$ and π is of class C_0. We are going to prove that the corresponding representation π_α of G satisfies these conditions.

(1) π_α weakly contains 1_G. Since (S, G) is an amenable pair, one has $1_G \prec \pi_S$. Thus we prove that $\pi_S \prec \pi_\alpha$. It suffices to show that if $\varphi \in L^2(S, \mu)$ is a unit vector which is a simple function with support in a set $A \in \mathcal{A}$, then for every $\varepsilon > 0$ and every compact subset $K \subset G$, there exists a unit vector $\xi \in L^2(S, \mu, \mathcal{H})$ such that

$$\sup_{g \in K} |\langle \pi_S(g)\varphi, \varphi\rangle - \langle \pi_\alpha(g)\xi, \xi\rangle| \leq \varepsilon.$$

Set $L = \overline{\alpha(A \times K)}$, which is a compact subset of H.

There exists a unit vector $\eta \in \mathcal{H}$ such that

$$\sup_{h \in L} |\langle \pi(h)\eta, \eta\rangle - 1| \leq \varepsilon.$$

Then set $\xi = \varphi \otimes \eta$, which is a unit vector of $L^2(S, \mu, \mathcal{H})$. We get for $g \in K$

$$\begin{aligned}
|\langle \pi_S(g)\varphi, \varphi\rangle - \langle \pi_\alpha(g)\xi, \xi\rangle| &= \left| \int_A (1 - \langle \pi(\alpha(s, g))\eta, \eta\rangle)\varphi(sg)\overline{\varphi(s)}\sqrt{\chi(s, g)}\, d\mu(s) \right| \\
&\leq \int_A |1 - \langle \pi(\alpha(s, g))\eta, \eta\rangle||\varphi(sg)||\varphi(s)|\sqrt{\chi(s, g)}\, d\mu(s) \\
&\leq \varepsilon \int_S |\varphi(sg)|\, |\varphi(s)|\sqrt{\chi(s, g)}\, d\mu(s) \\
&\leq \varepsilon \|\pi_S(g)|\varphi|\|_2 \|\varphi\|_2 \\
&= \varepsilon.
\end{aligned}$$

This proves (1).

(2) The representation π_α is of class C_0. Indeed, as in the proof of Proposition 2.8, we assume that $\mathcal{A}$ is an algebra of Borel sets. This implies that the family $\mathcal{T} = \{\chi_A \otimes \xi : A \in \mathcal{A}, \xi \in \mathcal{H}\}$ is a total subset of $L^2(S, \mu, \mathcal{H})$. So let us prove that for fixed $A, B \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}$, the coefficient function

$$\varphi : g \mapsto \langle \pi_\alpha(g)\chi_A \otimes \xi, \chi_B \otimes \eta\rangle$$

belongs to $C_0(G)$.

If $\varepsilon > 0$ is given, there exists a compact subset $L \subset H$ such that

$$|\langle \pi(h)\xi, \eta\rangle| \leq \varepsilon \quad \forall h \notin L.$$

Let us show that for $g \notin K = \overline{K(A \cup B, L)}$, one has

$$|\varphi(g)| \leq \varepsilon.$$

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Indeed, if $g \notin K$, then $\mu(X_g) = 0$, where $X_g = \{s \in B \cap Ag^{-1} : \alpha(s, g) \in L\}$; thus

$$\begin{aligned}|\varphi(g)| &= \left| \int_S \langle \pi(\alpha(s, g))\xi, \eta\rangle \chi_A(sg)\chi_B(s)\sqrt{\chi(s, g)}\, d\mu(s) \right| \\ &\leq \int_{B\cap Ag^{-1}\cap X_g^c} |\langle \pi(\alpha(s, g))\xi, \eta\rangle| \sqrt{\chi(s, g)}\, d\mu(s) \\ &\leq \varepsilon\mu(S)^{1/2} \left(\int_S \frac{d\mu(sg)}{d\mu(s)}\, d\mu(s) \right)^{1/2} \\ &= \varepsilon.\end{aligned}$$

To prove statement (1), observe that if H is amenable then the regular representation λ of H has the same properties as π above. In particular, $1_G \prec \lambda_\alpha$ and by Proposition 2.8, λ_α is quasi-equivalent to the regular representation of G. This proves that G is amenable. □

Proof of Proposition 2.5. (1) Take $S = G/H$ gifted with a quasi-invariant probability measure μ. Let $p : G \longrightarrow S$ be the canonical projection $g \mapsto Hg$ and let $\gamma : S \longrightarrow G$ be a regular Borel cross-section for p: $p(\gamma(s)) = s\ \forall s$, and $\gamma(S) \cap p^{-1}(p(K))$ is relatively compact for every compact set $K \subset G$; moreover, for every compact set $C \subset S$, there exists a compact set $C_1 \subset G$ such that $C = p(C_1)$.

Let $\mathcal{A}$ be the family of all compact subsets of S. Using regularity of γ, it is easy to see that $\overline{\gamma(C)}$ is compact for every $C \in \mathcal{A}$.

Define $\alpha : S \times G \longrightarrow H$ by

$$\alpha(s, g) = \gamma(s)g\gamma(sg)^{-1}.$$

Let us check that α satisfies conditions of Definition 2.6: condition (1) is obvious; if $A \in \mathcal{A}$ and if $K \subset G$ is compact, then $\alpha(A \times K) \subset \gamma(A)K\gamma(AK)^{-1}$ which is relatively compact because AK is compact; thus condition (2) is satisfied.

Finally, if $A \in \mathcal{A}$ and if $L \subset H$ is compact, then it is easy to see that $K(A, L) \subset \gamma(A)^{-1}L\gamma(A)$ which is relatively compact.

(2) Take $S = K$ with its normalized Haar measure μ; G acts on K by conjugation and μ is G-invariant. Take for $\mathcal{A}$ all Borel subsets of K and define $\alpha : K \times G \longrightarrow H$ by

$$\alpha(k, g) = p(g),$$

where $p : G \longrightarrow H$ is the canonical projection.

Then conditions (1) and (2) of Definition 2.6 are obviously fulfilled. Let now L be a compact subset of H; since p is a continuous homomorphism with compact kernel, it is proper. Hence $\alpha^{-1}(L) = K \times p^{-1}(L)$ is compact. This implies condition (3) of Definition 2.6.

(3) Choose for $i = 1, 2$ a representing set $S_i \ni 1$ for classes gA in the group Γ_i. Recall that every $g \in G$ admits a unique normal form

$$g = s_1(g) \dots s_k(g)a(g),$$

with $a(g) \in A$, $s_j(g) \in S_{i_j} \setminus \{1\}$, and, if $k \geq 2$, $i_j \neq i_{j+1}$ for $1 \leq j \leq k - 1$. Then we claim that for $i = 1, 2$, we can choose a proper, real-valued, conditionally negative definite function ψ_i on Γ_i satisfying:

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(a) $\psi_i|A = 0$;
(b) $\psi_i|\Gamma_i \setminus A \geq 1$.

Indeed, as A is a finite subgroup of Γ_i and as Γ_i has the HAP, using suitable averages on A, there exists a sequence $(\varphi_{i,n})_{n\geq 1} \subset C_0(\Gamma_i)$ such that:

(i) $\varphi_{i,n}|A = 1\ \forall n$;
(ii) $\varphi_{i,n}$ is positive definite;
(iii) $\varphi_{i,n}(\gamma) \to 1$ as $n \to \infty\ \forall \gamma \in \Gamma_i$.

Then the proper, conditionally negative definite function ψ_i constructed by Akemann and Walter in Theorem 10 of [**2**] satisfies condition (a) above. If it does not satisfy condition (b), simply replace it by $\psi_i + (1 - \chi_A)$.

Now define $\psi : G \longrightarrow \mathbb{R}^+$ by

$$\psi(s_1 \dots s_k a) = \psi_{i_1}(s_1) + \cdots + \psi_{i_k}(s_k),$$

for every normal form $s_1 \dots s_k a \in G$. Then ψ is obviously proper, and for every $t > 0$,

$$\exp(-t\psi) = \exp(-t\psi_1) * \exp(-t\psi_2)$$

in the sense of [**7**]. Hence $\exp(-t\psi)$ is positive definite for all $t > 0$, which implies that ψ is conditionally negative definite. □

We are going to exhibit another property of H inherited by G in the presence of a proper Borel cocycle. To do this, we need a definition taken from [**11**]. Denote by $M_0A(G)$ the space of *completely bounded multipliers* of $A(G)$; it admits various definitions, but we quote the one to be used here: a function ψ on G belongs to $M_0A(G)$ if and only if there exist a Hilbert space $\mathcal{H}$ and continuous functions $\xi, \eta : G \longrightarrow \mathcal{H}$ such that

$$\psi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle \quad \forall x, y \in G.$$

It is a little tedious, but not difficult, to see that ξ and η above need only be measurable to ensure that ψ is an element of $M_0A(G)$. Moreover, $\|\psi\|_{M_0A} = \min\{\|\xi\|_\infty, \|\eta\|_\infty\}$, where the minimum is taken over all pairs as above.

Definition 2.9. The *Cowling–Haagerup constant* $\Lambda(G)$ is the infimum of all numbers $L \geq 1$ for which there exists a net $(\psi_i)_{i\in I} \subset A(G)$ satisfying: $\|\psi_i\|_{M_0A} \leq L\ \forall i$ and $\psi_i \to 1$ uniformly on compact sets. Moreover, G is said to be *weakly amenable* if $\Lambda(G) < \infty$.

For instance the main theorem of [**11**] states that if $G = SO(1,n)$ or $SU(1,n)$ then $\Lambda(G) = 1$, and if $G = Sp(1,n)$ then $\Lambda(G) = 2n - 1$.

In Theorem 5.6 of [**12**], Cowling and Zimmer proved that if Γ is a (countable) discrete group acting on a measure space (X, μ), where μ is finite and Γ-invariant, and if $\alpha : X \times \Gamma \longrightarrow G$ is a Borel cocycle satisfying some technical conditions (that imply properness in our sense), then $\Lambda(\Gamma) \leq \Lambda(G)$. An example of such a situation is discussed in Remark (3) after Definition 2.6.

We present here a more general result with a simpler proof.

THEOREM 2.10. *Let G, H be lcsc groups, and let (S, μ) be a standard G-space with a G-invariant probability measure. If there exists a proper Borel cocycle $\alpha : S \times G \longrightarrow H$ then $\Lambda(G) \leq \Lambda(H)$.*

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This result should also be compared with Examples 4.10 of [**3**].

The following lemma is inspired by [**16**].

LEMMA 2.11. *With the above assumptions, let $\varphi \in M_0A(H)$. Define $\widehat{\varphi} : G \longrightarrow \mathbb{C}$ by*

$$\widehat{\varphi}(g) = \int_S \varphi(\alpha(s, g))\, d\mu(s).$$

Then $\widehat{\varphi} \in M_0A(G)$, and $\|\widehat{\varphi}\|_{M_0A} \leq \|\varphi\|_{M_0A}$.

Proof. Let $\mathcal{H}$ be a Hilbert space and $\xi, \eta : H \longrightarrow \mathcal{H}$ be bounded continuous functions such that

$$\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle \quad \forall x, y \in H.$$

As H is second countable, we assume that $\mathcal{H}$ is separable, so that we have no problem of measurability below. Then set $\mathcal{K} = L^2(S, \mathcal{H})$ and define $\widehat{\xi}$ and $\widehat{\eta} : G \longrightarrow \mathcal{K}$ by

$$\widehat{\xi}(g)(s) = \xi(\alpha(s, g))$$

and

$$\widehat{\eta}(g)(s) = \eta(\alpha(s, g)).$$

Then, using for fixed $g, h \in G$, the identity

$$\alpha(s, g) = \alpha(s, h)\alpha(sh, h^{-1}g) \quad \mu\text{-a.e.},$$

we get:

$$\begin{aligned}
\langle \widehat{\xi}(g), \widehat{\eta}(h) \rangle &= \int_S \langle \xi(\alpha(s, g)), \eta(\alpha(s, h)) \rangle\, d\mu(s) \\
&= \int_S \varphi(\alpha(s, h)^{-1}\alpha(s, g))\, d\mu(s) \\
&= \int_S \varphi(\alpha(sh, h^{-1}g))\, d\mu(s) \\
&= \int_S \varphi(\alpha(s, h^{-1}g))\, d\mu(s) \\
&= \widehat{\varphi}(h^{-1}g),
\end{aligned}$$

since μ is G-invariant. □

Proof of Theorem 2.10. As in the proof of Theorem 2.7, we assume that $\mathcal{A}$ is an algebra of subsets of S. Fix a compact set $K \subset G$, a number $C > \Lambda(H)$ and $\varepsilon > 0$. Let $A \in \mathcal{A}$ be such that $\mu(S \setminus A) \leq \varepsilon/2(C + 1)$. Then $L = \overline{\alpha(A \times K)}$ is a compact subset of H, and there exists $\varphi \in A(H)$ such that

$$\|\varphi\|_{M_0A} \leq C$$

and

$$\|\varphi|L - 1\|_\infty \leq \varepsilon/2.$$

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For $g \in K$ we get

$$\begin{aligned}|\widehat{\varphi}(g) - 1| &= \left| \int_S \varphi(\alpha(s, g))\, d\mu(s) - 1 \right| \\ &\leq \int_A |\varphi(\alpha(s, g)) - 1|\, d\mu(s) + \int_{S \setminus A} |\varphi(\alpha(s, g)) - 1|\, d\mu(s) \\ &\leq \varepsilon.\end{aligned}$$

Observe finally that $\widehat{\varphi} \in A(G)$ since, as in the proof of Proposition 2.8, if $\varphi(h) = \langle \lambda(h)\xi, \eta \rangle$, then $\widehat{\varphi}(g) = \langle \lambda_\alpha(g)\xi', \eta' \rangle$, where $\xi'(s) = \xi$ and $\eta'(s) = \eta\ \forall s \in S$. □

COROLLARY 2.12. *If G is locally isomorphic to $Sp(1, n)$, if H is locally isomorphic to $Sp(1, m)$, if (S, μ) is a G-space with invariant measure and if there is a proper cocycle $\alpha : S \times G \longrightarrow H$, then $n \leq m$.*

(Compare with Theorem 1.1 of [**12**].)

3. *Cocycles and relative property T*

As in the previous sections, G denotes a lcsc group; let G_0 be a closed subgroup of G. Recall from Theorem 1.1 of [**4**] that the following conditions on the pair (G, G_0) are equivalent:

(1) (G, G_0) has the relative property T: if $(\pi, \mathcal{H}_\pi)$ is a unitary representation of G such that $1_G \prec \pi$, then $\mathcal{H}_\pi$ contains a non-zero H-invariant vector;

(2) for every sequence (φ_n) of continuous, positive definite functions on G such that $\varphi_n \to 1$ uniformly on compact sets, one has

$$\|\varphi_n | G_0 - 1\|_\infty \to 0;$$

(3) every continuous, conditionally negative definite function on G is bounded on G_0.

We will use this result in the next section, as well as the following proposition which can be compared to Theorem 1.4 of [**4**].

PROPOSITION 3.1. *Let G be a lcsc group and let G_0 and G_1 be closed subgroups of G. Set $G_2 = G_0 \cap G_1$, and assume that G_1 is of finite covolume in G, that G_0 is normal in G and that the pair (G, G_0) has the relative property T. Then so does the pair (G_1, G_2).*

Proof. Let $(\pi, \mathcal{H})$ be a unitary representation of G_1 such that $1_{G_1} \prec \pi$. Denote by ρ the induced representation of π. Then $1_G \prec \rho$ by continuity of inducing, and there exists a non-zero G_0-invariant vector in $\mathcal{H}_\rho$. Realizing ρ as in Chapter 3, pp. 31–32 of [**13**], there exists a non-zero Borel function $\xi : G \longrightarrow \mathcal{H}$ such that:

(1) $\forall g_1 \in G_1, \xi(gg_1) = \pi(g_1^{-1})\xi(g), dg$-a.e.;

(2) $\forall g_0 \in G_0, \xi(g_0 g) = \xi(g), dg$-a.e.

By Lemma 2.2.16 of [**24**], there exists a left G_0-invariant conul subset S_1 of G and a Borel function $\xi_1 : G \longrightarrow \mathcal{H}$ such that:

(3) $\xi_1 | S_1 = \xi | S_1$ and $\xi_1(g_0 g) = \xi_1(g)\ \forall g_0 \in G_0, \forall g \in G$.
Observe that S_1 is also right G_2-invariant: indeed, if $\gamma \in G_2$, if $s \in S_1$, then $s\gamma = (s\gamma s^{-1})s$ belongs to S_1 because $s\gamma s^{-1} \in G_0$.

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Now we claim that for every $\gamma \in G_2$, one has

$$\pi(\gamma)\xi_1(g) = \xi_1(g) \quad dg\text{-a.e.}$$

Indeed, fix $\gamma \in G_2$ and set

$$S(\gamma) = S_1 \cap S_1\gamma \cap \{g : \xi(g\gamma^{-1}) = \pi(\gamma)\xi(g)\},$$

which is a conul subset of G. If $g \in S(\gamma)$, we get

$$\begin{aligned}\pi(\gamma)\xi_1(g) &= \pi(\gamma)\xi(g)\\ &= \xi(g\gamma^{-1})\\ &= \xi_1(g\gamma^{-1}) \quad \text{since } g\gamma^{-1} \in S_1\\ &= \xi_1((g\gamma^{-1}g^{-1})g)\\ &= \xi_1(g) \quad \text{since } g\gamma^{-1}g^{-1} \in G_0.\end{aligned}$$

Thus, using again Lemma 2.2.16 of [**24**], there exists a Borel function $\xi_2 : G \longrightarrow \mathcal{H}$ such that:

(4) $\xi_2(g) = \pi(\gamma)\xi_2(g)\ \forall\gamma \in G_2, \forall g \in G$ and $\xi_2 = \xi_1$ a.e. As $\xi_1 \neq 0$, we can find a $g \in G$ such that $\xi_2(g) \neq 0$, and we set $\eta = \xi_2(g)$, which is a non-zero G_2-invariant vector of $\mathcal{H}$. □

We end the present section with a generalization of Theorem 9.1.1 of [**24**] and of the main results of [**1**].

THEOREM 3.2. *Let (G, G_0) be a pair with the relative property T. Assume that G acts on (S, μ) in such a way that μ is G-invariant and that G_0 acts ergodically on S. Let H be a lcsc group with the HAP and let $\alpha : S \times G \longrightarrow H$ be a Borel cocycle. Then the restriction α_0 of α to $S \times G_0$ is equivalent to a cocycle with range in a compact subgroup of H.*

Proof. Let $(\pi, \mathcal{H})$ be a representation of H satisfying conditions (a) and (b) of Proposition 2.3. Since π is of class C_0, the orbit $\pi(H)\xi$ is closed in $\mathcal{H}$ for every ξ; moreover, if $\xi \neq 0$, its stabilizer in H is a compact subgroup of H.

Now, arguing exactly as in the proof of Lemma 9.1.2 of [**24**], we see that α_0 is equivalent to a cocycle with range in a compact subgroup of H if and only if there is a function $\varphi : S \longrightarrow \mathcal{H}_1$, the norm one vectors in $\mathcal{H}$, such that for every $g \in G_0$,

$$\pi(\alpha(s, g))\varphi(sg) = \varphi(s) \quad \mu\text{-a.e.}$$

Next, as in the proof of Theorem 9.1.1 of [**24**], we show that for every $\varepsilon > 0$ and every compact set $K \subset G$ there is a unit vector $\xi \in L^2(S, \mu, \mathcal{H})$ such that

$$\|\pi_\alpha(g)\xi - \xi\| \leq \varepsilon \quad \forall g \in K.$$

(We use the fact that $1_H \prec \pi$.)

Since (G, G_0) has the relative property T, it follows that there exists a $\pi \circ \alpha_0$-invariant function φ as required above. □

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4. *Cocycles with values in finite von Neumann algebras*

We begin this section by recalling the definition of the HAP for finite von Neumann algebras (with separable predual).

Let M be such an algebra and let tr be a fixed faithful, normal, normalized trace on M; the associated norm is

$$\|x\|_2 = \sqrt{\mathrm{tr}(x^*x)} \quad \forall x \in M.$$

Completing $(M, \|\cdot\|_2)$, we obtain the Hilbert space $L^2(M)$ on which M acts naturally on the left and on the right.

If A, B are C^*-algebras, a linear map $\Phi : A \longrightarrow B$ is *completely positive* (in short, cp) if for every $n \geq 1$, the map $\Phi \otimes i_n : A \otimes M_n(\mathbb{C}) \longrightarrow B \otimes M_n(\mathbb{C})$ is positive. Observe that if Φ is a cp map on M such that $\mathrm{tr} \circ \Phi \leq \mathrm{tr}$, then it extends to a linear bounded operator on $L^2(M)$, still denoted by Φ, because

$$\Phi(x)^*\Phi(x) \leq \|\Phi\|\Phi(x^*x) \quad \forall x \in M.$$

The following definition is taken from [**10**] and [**3**].

Definition 4.1. Let M be as above. We say that it possesses the HAP if there exists a sequence $(\Phi_n)_{n\geq 1}$ of cp, normal, unital maps on M with the following properties:

(1) $\mathrm{tr} \circ \Phi_n \leq \mathrm{tr}$ $\forall n \geq 1$;
(2) each Φ_n extends to a compact operator on $L^2(M)$;
(3) for every $x \in M$, $\|\Phi_n(x) - x\|_2 \to 0$ as $n \to \infty$.

Examples 4.2.

(1) Let Γ be a countable group and let $L(\Gamma)$ be the von Neumann algebra generated by the regular representation of Γ. Then $L(\Gamma)$ has the HAP if and only if Γ does [**10**].
(2) If M is (finite and) injective, then it has property HAP, and each Φ_n can be taken of finite rank.
(3) If M has the HAP and if N is a von Neumann subalgebra of M, then it has the same property: use the natural normal conditional expectation from M onto N.
(4) It is straightforward to check that if M and N have the HAP, then so does their tensor product von Neumann algebra $M \otimes N$.

Let (G, G_0) and (S, μ) be as above, but we do not assume that G_0 acts ergodically on S anymore. Let also M be a finite von Neumann algebra with separable predual and gifted with a faithful, normal, normalized trace tr.

Let us recall Theorem C of [**21**] (which was stated for property T only, but the extension to the relative property T is straightforward).

THEOREM 4.3. *Let (G, G_0) be a pair with the relative property T, let M be a finite von Neumann algebra with the HAP and let $\sigma : G \longrightarrow U(M)$ be a (continuous) homomorphism. Then the set $\sigma(G_0)$ is relatively compact with respect to the topology induced by $\|\cdot\|_2$.*

For instance, this proves that $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ cannot have a faithful representation with discrete range into the unitary group of such an algebra.

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Taking into account Theorem 3.2, one could expect that if σ above is replaced by a cocycle $\alpha : S \times G \longrightarrow U(M)$, then $\alpha | S \times G_0$ has relatively compact range. Unfortunately, we do not know if this is true in full generality. Nevertheless, we have the following result concerning transitive actions.

COROLLARY 4.4. *Let G,G_0 and G_1 be as in Proposition 3.1, with $G_0 \subset G_1$, and set $S = G/G_1$. Let M be a finite von Neumann algebra with the HAP. Then every Borel cocycle $\alpha : S \times G \longrightarrow U(M)$ is equivalent to a cocycle $\beta : S \times G \longrightarrow U(M)$ such that $\beta(S \times G_0)$ is relatively compact.*

Proof. By Proposition 4.2.13 of [**24**], α equivalent to a strict cocycle $\beta : S \times G \longrightarrow U(M)$ of the form: $\beta(s, g) = \sigma(\gamma(s)g\gamma(sg)^{-1})$, where $\gamma : S \longrightarrow G$ is a Borel cross-section and $\sigma : G_1 \longrightarrow U(M)$ is a continuous homomorphism. Then $\sigma(G_0)$ is relatively compact and $\beta(S \times G_0)$ is contained in $\sigma(G_0)$ since $\gamma(s)g\gamma(sg)^{-1} = \gamma(s)g\gamma(s)^{-1} \in G_0$ for every $s \in S$ and every $g \in G_0$, using the fact that G_0 is normal and contained in G_1. □

In the general case, we have the following.

THEOREM 4.5. *Let (G, G_0) be a pair with the relative property T, let (S, μ) be a standard G-space with G-invariant probability measure and let M be as in Corollary 4.4. For every Borel function $u : S \longrightarrow U(M)$ and every Borel cocycle $\alpha : S \times G \longrightarrow U(M)$, set*

$$\alpha_u(g) = \int_S u(s)\alpha(s, g)u(sg)^{-1}\, d\mu(s),$$

for $g \in G$.

Then the set $\{\alpha_u(g) : g \in G_0\}$ is relatively compact in M.

LEMMA 4.6. *Let G, M and α be as above. Let Φ be a normal cp map on M such that $\Phi(1) = 1$. Set*

$$\varphi_\Phi(g) = \int_S \mathrm{tr}(\Phi(\alpha(s, g))\alpha(s, g)^{-1})\, d\mu(s) \quad \forall g \in G.$$

Then φ_Φ is a unital, continuous, positive definite function on G.

Proof. As $\alpha(s, 1) = 1$ for a.e. $s \in S$, we get that $\varphi_\Phi(1) = 1$.

Let us prove that φ_Φ is a coefficient of a unitary representation $(\pi, \mathcal{H})$ of G: let $\mathcal{H}_\Phi$ be the correspondence from M to M associated with Φ (see [**21**, p. 550]). $\mathcal{H}_\Phi$ is a normal M-bimodule with a unit vector ξ_Φ which is cyclic in the following sense: the set of vectors $x\xi_\Phi y$, with $x, y \in M$ is total in $\mathcal{H}_\Phi$. Moreover, the scalar product is determined by

$$\langle x_1\xi_\Phi y_1, x_2\xi_\Phi y_2\rangle = \mathrm{tr}(\Phi(x_2^* x_1)y_1 y_2^*) \quad \forall x_i, y_i \in M.$$

Now set $\mathcal{H} = L^2(S, \mu, \mathcal{H}_\Phi)$ and define $\pi : G \longrightarrow U(\mathcal{H})$ by

$$\pi(g)\xi(s) = \alpha(s, g)\xi(sg)\alpha(s, g)^{-1} \quad \forall \xi \in \mathcal{H}.$$

Then π is a Borel homomorphism, and since $\mathcal{H}$ is separable, π is continuous. Define

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$\xi \in \mathcal{H}$ by $\xi(s) = \xi_\Phi \ \forall s \in S$. It is a unit vector and we have

$$\begin{aligned}\langle \pi(g)\xi, \xi\rangle &= \int_S \langle(\pi(g)\xi)(s), \xi(s)\rangle \, d\mu(s)\\ &= \int_S \langle \alpha(s,g)\xi_\Phi \alpha(s,g)^{-1}, \xi_\Phi\rangle \, d\mu(s)\\ &= \int_S \operatorname{tr}(\Phi(\alpha(s,g))\alpha(s,g)^{-1}) \, d\mu(s)\\ &= \varphi_\Phi(g),\end{aligned}$$

which proves the lemma. □

Proof of Theorem 4.5. Set $\beta(s,g) = u(s)\alpha(s,g)u(sg)^{-1}$, for $(s,g) \in S \times G$, which is a Borel cocycle. Let $(\Phi_n)_{n\geq 1}$ be as in Definition 4.1 and set $\varphi_n = \varphi_{\Phi_n}$ as in Lemma 4.6 above, with α replaced by β.

For fixed $(s,g) \in S \times G$, the sequence $\operatorname{tr}(\Phi_n(\beta(s,g))\beta(s,g)^{-1})$ tends to 1 as $n \to \infty$, hence $\varphi_n(g) \to 1$ for every $g \in G$. Arguing as in the proof of Proposition 2.2 of [**21**], we see that $\varphi_n \to 1$ uniformly on compact subsets of G. As (G, G_0) has the relative property T, we get

$$\|\varphi_n|G_0 - 1\|_\infty \to 0$$

as $n \to \infty$.

In order to prove that $\{\alpha_u(g) : g \in G_0\}$ is relatively compact, it suffices to check that for every $\varepsilon > 0$, there exist $x_1, \ldots, x_m \in M$ such that

$$\{\alpha_u(g) : g \in G_0\} \subset \bigcup_{j=1}^m B(x_j, \varepsilon),$$

where $B(x,\varepsilon) = \{y \in M : \|x - y\|_2 \leq \varepsilon\}$.

Then fix $\varepsilon > 0$ and choose n large enough so that $\|\varphi_n|G_0 - 1\|_\infty \leq \varepsilon^2/8$.

By compactness of Φ_n, there exist $x_1, \ldots, x_m \in M$ such that

$$\Phi_n(M_1) \subset \bigcup_{j=1}^m B(x_j, \varepsilon/2),$$

where M_1 is the unit ball of M for the operator norm.

Let $g \in G_0$ be arbitrary; as $\alpha_u(g) \in M_1$, there exists $1 \leq j \leq m$ such that $\Phi_n(\alpha_u(g)) \in B(x_j, \varepsilon/2)$. But one has

$$\begin{aligned}\|\alpha_u(g) - \Phi_n(\alpha_u(g))\|_2^2 &\leq \left(\int_S \|\beta(s,g) - \Phi_n(\beta(s,g))\|_2 \, d\mu(s)\right)^2\\ &= \left(\int_S \|1 - \Phi_n(\beta(s,g))\beta(s,g)^{-1}\|_2 \, d\mu(s)\right)^2\\ &\leq \int_S \|1 - \Phi_n(\beta(s,g))\beta(s,g)^{-1}\|_2^2 \, d\mu(s)\\ &= \int_S [1 - 2\Re \operatorname{tr}(\Phi_n(\beta(s,g))\beta(s,g)^{-1})\\ &\quad + \operatorname{tr}(\Phi_n(\beta(s,g))^*\Phi_n(\beta(s,g)))] \, d\mu(s)\end{aligned}$$

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$$\leq 2\Re(1-\varphi_n(g))$$
$$\leq 2|1-\varphi_n(g)|$$
$$\leq \varepsilon^2/4.$$

This shows that $\alpha_u(g) \in B(x_j, \varepsilon)$. □

Remark. Retain hypotheses of Theorem 4.5. If α is equivalent to some homomorphism $\pi : G \longrightarrow U(M)$ then $\alpha|S \times G_0$ is equivalent to a cocycle with relatively compact range, by Robertson's theorem.

The following example shows the existence of a cocycle $\alpha : S \times G \longrightarrow U(M)$ which is not equivalent to a homomorphism: Set $G = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$, $\Gamma = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and $S = G/\Gamma$. Set also $M = \lambda(SL_2(\mathbb{Z}))''$, which is a finite von Neumann algebra with property HAP. Now define a homomorphism $\sigma : \Gamma \longrightarrow U(M)$ by

$$\sigma(\gamma, v) = \lambda(\gamma), \quad \text{for } \gamma \in SL_2(\mathbb{Z}) \text{ and } v \in \mathbb{Z}^2.$$

Let also $c : S \longrightarrow G$ be a regular Borel cross-section for the canonical projection $p : G \longrightarrow S$, so that $\alpha : S \times G \longrightarrow U(M)$, defined by

$$\alpha(s, g) = \sigma(c(s)gc(sg)^{-1}),$$

is a Borel cocycle. If α were equivalent to some homomorphism $\pi : G \longrightarrow U(M)$ then by Proposition 4.2.16 of [**24**], σ would extend to a homomorphism from G to $U(M)$ which would be non-trivial, but this is impossible.

References

[1] S. Adams and R. Spatzier. Kazhdan groups, cocycles and trees. *Amer. J. Math.* **112** (1990), 271–287.

[2] C. Akemann and M. Walter. Unbounded negative definite functions. *Can. J. Math.* **33** (1981), 862–871.

[3] C. Anantharaman. Amenable correspondences and approximation properties for von Neumann algebras. *Pac. J. Math.* **171** (1995), 309–341.

[4] T. Bates and A. G. Robertson. Positive definite functions and relative property (T) for subgroups of discrete groups. *Bull. Austral. Math. Soc.* **52** (1995), 31–39.

[5] M. Bekka, P. A. Cherix and A. Valette. Proper affine isometric actions of amenable groups. *Novikov Conjecture, Index Theorems and Rigidity (London Math. Soc. Lecture Notes Series, 227).* Cambridge University Press, Cambridge, 1995.

[6] M. Bekka and P. de la Harpe. Représentations d'un groupe faiblement équivalentes à la représentation régulière. *Bull. Soc. Math. France* **122** (1994), 333–342.

[7] F. Boca. Free products of completely positive maps and spectral sets. *J. Funct. Anal.* **97** (1991), 251–263.

[8] M. Bozejko, T. Januszkiewicz and R. Spatzier. Infinite Coxeter groups do not have Kazhdan's property. *J. Operator Theory* **19** (1988), 63–67.

[9] M. Bridson. Remarks on a talk of Higson. Unpublished notes.

[10] M. Choda. Group factors of the Haagerup type. *Proc. Japan Acad.* **59** (1983), 174–209.

[11] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.* **96** (1989), 507–549.

[12] M. Cowling and R. Zimmer. Actions of lattices in $Sp(1, n)$. *Ergod. Th. & Dynam. Sys.* **9** (1989), 221–237.

[13] P. de la Harpe and A. Valette. La propriété (T) pour les groupes localement compacts. *Astérisque, Soc. Math. France* **175** (1989).

[14] J. Faraut and K. Harzallah. Distances hilbertiennes invariantes sur un espace homogène. *Ann. Inst. Fourier* **24** (1974), 171–217.

https://doi.org/10.1017/S0143385700000237 Published online by Cambridge University Press

[15] M. Gromov. *Asymptotic Invariants of Infinite Groups (London Math. Soc. Lect. Notes Ser., 182)*. London Math. Soc., 1993.
[16] U. Haagerup. Group C^*-algebras without the completely bounded approximation property. Unpublished.
[17] U. Haagerup. An example of a non-nuclear C^*-algebra which has the metric approximation property. *Invent. Math.* **50** (1979), 279–293.
[18] P. Jolissaint. Invariant states and a conditional fixed point property for affine actions. *Math. Ann.* **304** (1996), 561–579.
[19] G. Margulis. Finitely-additive invariant measures on Euclidean spaces. *Ergod. Th. & Dynam. Sys.* **2** (1982), 383–396.
[20] A. G. Robertson. Crofton formulae and geodesic distance in hyperbolic spaces. *J. Lie Theory* **8** (1998), 163–172.
[21] A. G. Robertson. Property (T) for II_1 factors and unitary representations of Kazhdan's groups. *Math. Ann.* **296** (1993), 547–555.
[22] J. L. Tu. La conjecture de Baum–Connes pour les feuilletages moyennables. *Preprint*, 1997.
[23] R. Zimmer. On the cohomology of ergodic actions of semisimple Lie groups and discrete subgroups. *Amer. J. Math.* **103** (1981), 937–950.
[24] R. Zimmer. *Ergodic Theory and Semisimple Groups*. Birkhauser, Boston, 1984.

https://doi.org/10.1017/S0143385700000237 Published online by Cambridge University Press