HAMMOCKS AND THE NAZAROVA–ROITER ALGORITHM

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1. *Introduction*

Hammocks have been considered by Brenner [**1**], who gave a numerical criterion for a finite translation quiver to be the Auslander–Reiten quiver of some representation-finite algebra. Ringel and Vossieck [**11**] gave a combinatorial definition of left hammocks which generalised the concept of hammocks in the sense of Brenner, as a translation quiver H and an additive function h on H (called the hammock function) satisfying some conditions. They showed that a thin left hammock with finitely many projective vertices is just the preprojective component of the Auslander–Reiten quiver of the category of $\mathscr{S}$-spaces, where $\mathscr{S}$ is a finite partially ordered set (abbreviated as 'poset'). An important role in the representation theory of posets is played by two differentiation algorithms. One of the algorithms was developed by Nazarova and Roiter [**8**], and it reduces a poset $\mathscr{S}$ with a maximal element a to a new poset $\mathscr{S}' = {}_a\partial\mathscr{S}$. The second algorithm was developed by Zavadskij [**13**], and it reduces a poset $\mathscr{S}$ with a suitable pair (a, b) of elements a, b to a new poset $\mathscr{S}' = \partial_{(a,b)}\mathscr{S}$. The main purpose of this paper is to construct new left hammocks from a given one, and to show the relationship between these new left hammocks and the Nazarova–Roiter algorithm. In a later paper [**5**], we discuss the relationship between hammocks and the Zavadskij algorithm.

In §2 of this paper, we recall some basic definitions and facts. Let H be a left hammock with the hammock function h_H, and $p(a)$ be a thin projective vertex of H that is different from the source. In §3, we construct a new left hammock ${}_aH$ and determine its hammock function $h_{({}_aH)}$. We prove that the subquiver denoted by $H/{}_aH$ consisting of all vertices x satisfying $h_H(x) - h_{({}_aH)}(x) \neq 0$ is also a left hammock. We show that, if H is infinite, then ${}_aH$ is also infinite. In the case in which H is a hammock, the relationship between the hammock induced by a projective vertex of H and the Nazarova–Roiter algorithm is stated in §6. The case in which $p(a)$ is a non-thin projective vertex of H is discussed in §4. In §5, we consider the hammock induced by some injective vertex of H. The corresponding results relating to $\ell(\mathscr{S})$, the category of $\mathscr{S}$-spaces, are also described.

Throughout this paper, all algebras are assumed to be finite-dimensional (associative) basic algebras with a unit over an algebraically closed field. All modules are finitely generated right modules. We denote by A-mod the category of A-modules. The composition of two morphisms $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_3$ is denoted by fg. All posets are assumed to be finite. We denote by $\mathbb{N}$, $\mathbb{N}_1$, $\mathbb{Z}$ the set of natural numbers, positive integers and integers, respectively. For all unexplained notation, refer to [**10**, **11**].

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2. *Preliminaries*

2.1. *Left hammocks and hammocks*

Let $H = (H_0, H_1, \tau)$ be a proper translation quiver. We define inductively the full subquivers ${}^dH$ of H. First of all, ${}^{-1}H$ is the empty quiver, and z belongs to ${}^dH$ if and only if $z^- \subseteq {}^{d-1}H$. Also, ${}^\infty H = \bigcup_{d\in N} {}^dH$. Thus, for all $d \in N \cup \{\infty\}$, we see that ${}^dH$ is a predecessor closed subquiver, and we may consider it as a translation quiver, using the restriction of τ. Suppose that H has a unique source ω, and $H = {}^\infty H$. Then we define $h_H: H_0 \to \mathbb{Z}$ inductively as follows. By abuse of notation, let $h_H(\tau x) = 0$ for x projective (note that, in this case, τx is not defined). Now, let $h_H(\omega) = 1$, and, for $x \neq \omega$, with h_H already defined on all proper predecessors of x, let $h_H(x) = \sum_{y\to x} h_H(y) - h_H(\tau x)$, where the sum is taken over all arrows ending at x. With these preparations, we are able to recall the main definition: the translation quiver H is said to be a *left hammock* provided that the following are true:

(1) $H = {}^\infty H$.
(2) H has a unique source ω and $h_H(\omega) = 1$.
(3) h_H takes values in the set N_1 of positive integers.
(4) If q is an injective vertex, then $h_H(q) \geqslant \sum_{q\to y} h_H(y)$.

When H is a left hammock, the function h_H is said to be its *hammock function*. A vertex x of H is called *thin* if $h_H(x) = 1$. A left hammock H is said to be *thin* provided that $h_H(p) = 1$ for any projective vertex p of H. A left hammock H is called a *hammock* if $|H_0| < \infty$. A hammock is always thin and has a unique sink, say ω'.

2.2. $\mathcal{S}$*-spaces*

Fix some field k. Given a poset $\mathcal{S}$, an $\mathcal{S}$*-space* $V = (V_\omega, V_s)_{s\in\mathcal{S}}$ is given by a vector space V_ω over k and subspaces V_s of V_ω, for $s \in \mathcal{S}$, such that $V_s \subseteq V_t$ for $s \leqslant t$. We call V_ω the *total space* of V, and define its k-dimension by $\underline{\dim}_\omega V = \dim_k V_\omega$. Given two $\mathcal{S}$-spaces V, W, a map $f: V \to W$ is given by a k-linear map $f_\omega: V_\omega \to W_\omega$ satisfying $f_\omega(V_s) \subseteq W_s$ for all $s \in \mathcal{S}$; the induced map $V_s \to W_s$ is denoted by f_s. The posets that we consider are always assumed to be finite. We denote the category of $\mathcal{S}$-spaces V with $\dim_k V_\omega < \infty$ by $\ell(\mathcal{S})$. For convenience, we denote $\mathrm{Hom}_{\ell(\mathcal{S})}(V, W)$ for two $\mathcal{S}$-spaces V and W by $\mathrm{Hom}_{\mathcal{S}}(V, W)$. We denote by $\mathcal{S}^+$ the poset obtained from $\mathcal{S}$ by adding an element ω with $s < \omega$ for all $s \in \mathcal{S}$. For $s \in \mathcal{S}^+$, *projective objects* in $\ell(\mathcal{S})$, denoted by $P_{\mathcal{S}}(s)$ for $s \in \mathcal{S}^+$, are defined by putting $P_{\mathcal{S}}(s)_t = k$ for $t \geqslant s$ and $P_{\mathcal{S}}(s)_t = 0$ for $t \not\geqslant s$, where $t \in \mathcal{S}^+$. In a similar way, we denote by $\mathcal{S}^-$ the poset obtained from $\mathcal{S}$ by adding an element ω' with $s > \omega'$ for all $s \in \mathcal{S}$. The *injective objects* in $\ell(\mathcal{S})$, denoted by $Q_{\mathcal{S}}(s)$ with $s \in \mathcal{S}^-$, are defined by putting $Q_{\mathcal{S}}(s)_t = k$ for $t \not\leqslant s$, and $Q_{\mathcal{S}}(s)_t = 0$ for $t \leqslant s$, where $t \in \mathcal{S}^+$. For $t \in \mathcal{S}$ and $V \in \ell(\mathcal{S})$, we have $\dim_k \mathrm{Hom}_{\mathcal{S}}(P_{\mathcal{S}}(t), V) = \dim_k V_t$ and $\dim_k \mathrm{Hom}_{\mathcal{S}}(V, Q_{\mathcal{S}}(t)) = \dim_k V_\omega - \dim_k V_t$. An $\mathcal{S}$-space V is *thin* if its total space V_ω is one-dimensional. We denote by $\tau_{\mathcal{S}}$ the *Auslander–Reiten translation* in $\ell(\mathcal{S})$. It is well known that the Auslander–Reiten quiver of $\ell(\mathcal{S})$ always has a unique preprojective component, denoted by $\mathcal{P}_{\mathcal{S}}$, which is standard.

Given a Krull–Schmidt category Λ, let us define the full subcategories ${}_a\Lambda$. First of all, ${}_{-1}\Lambda$ contains only the zero object. Second, an indecomposable object X of Λ belongs to ${}_a\Lambda$ if and only if any indecomposable object Y of Λ with $\mathrm{rad}(Y, X) \neq 0$

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belongs to ${}_{d-1}\Lambda$. Finally, ${}_{\infty}\Lambda = \bigcup_{d\in\mathbb{N}} {}_{d}\Lambda$. Let $\mathscr{S}$ be a poset. We observe that ${}_{\infty}\ell(\mathscr{S})$ is just the full subcategory of $\ell(\mathscr{S})$ whose indecomposable objects occur in $\mathscr{P}_{\mathscr{S}}$. Therefore ${}_{\infty}\ell(\mathscr{S}) \cong \operatorname{add} k(\mathscr{P}_{\mathscr{S}})$, where $k(\mathscr{P}_{\mathscr{S}})$ denotes the mesh category for $\mathscr{P}_{\mathscr{S}}$.

There is a strong relationship between thin left hammocks and the representation theory of posets, which is described in the following theorem developed by Ringel and Vossieck [**11**].

THEOREM 2.1. *Let $\mathscr{S}$ be a finite poset, and k be a field. Then the preprojective component $\mathscr{P}_{\mathscr{S}}$ of the Auslander–Reiten quiver of $\ell(\mathscr{S})$ is a thin left hammock with finitely many projective vertices. The hammock function on $\mathscr{P}_{\mathscr{S}}$ is $\underline{\dim}_{\omega}$. Conversely, given a thin left hammock H with n projective vertices, there exists a unique poset $\mathscr{S} := \mathscr{S}(H)$ with $n-1$ elements such that $\operatorname{add} k(H) \cong {}_{\infty}\ell(\mathscr{S})$ as categories, and $H \cong \mathscr{P}_{\mathscr{S}}$ as translation quivers.*

From now on, we will take any thin left hammock H as the preprojective component $\mathscr{P}_{\mathscr{S}}$ for $\mathscr{S} = \mathscr{S}(H)$. Accordingly, we have a bijective map $p: \mathscr{S}^+ \to \{\text{projective vertices of } H\}$ where $p(s)$ is the vertex corresponding to $P_{\mathscr{S}}(s)$. Let $(\mathscr{S}^-)^0$ be the subset of $\mathscr{S}^-$ such that the injective object $Q_{\mathscr{S}}(s)$ occurs in ${}_{\infty}\ell(\mathscr{S})$ for $s \in (\mathscr{S}^-)^0$. Then we have a bijective map $q: (\mathscr{S}^-)^0 \to \{\text{injective vertices of } H\}$, where $q(s)$ is the vertex corresponding to $Q_{\mathscr{S}}(s)$. In particular, we find that $\operatorname{Hom}_{k(H)}(p(s), p(t)) \neq 0$ if and only if $s \geqslant t$ in $\mathscr{S}^+$ and $\operatorname{Hom}_{k(H)}(q(s), q(t)) \neq 0$ if and only if $s \geqslant t$ in $\mathscr{S}^-$.

2.3. *Incidence algebras and socle-projective modules*

Let k be a field. Given a Krull–Schmidt category Λ, a Λ-*module* M is a finitely presented functor $\Lambda^{\mathrm{op}} \to k$-mod. We denote by Λ-mod the category of all Λ-modules, and by Λ-spmod we denote the full subcategory of Λ-mod generated by all modules $M \in \Lambda$-mod which have a projective socle. Clearly, Λ-spmod is closed under direct sums, direct summands, submodules and extensions. A module M in Λ-spmod is said to be *thin* if M has a simple socle. We will use the following easy result.

LEMMA 2.1. *Let Λ be a Krull–Schmidt category, where $M, N \in \Lambda$-spmod.*

(1) *Assume that $0 \neq \psi \in \operatorname{Hom}_{\Lambda}(M, N)$ and M is thin. Then ψ is a monomorphism.*

(2) *Assume that $0 \neq \quad \in \operatorname{Hom}_{\Lambda}(M, N)$, where $0 \neq \phi \in \operatorname{Hom}_{\Lambda}(N, L)$ and M, N are thin. Then $\quad \phi \neq 0$.*

Proof. Suppose that ψ is not a monomorphism. Then $\operatorname{soc}(\ker(\psi)) = \operatorname{soc} M$ since M is thin. As a consequence, $\operatorname{soc}(\operatorname{Im}(\psi)) \cong \operatorname{soc}(M/\ker(\psi))$ is not projective. This is a contradiction to the fact that Λ-spmod is closed under submodules. Thus assumption 1 holds and assumption 2 follows at once.

Let A be an algebra over an algebraically closed field k with projective left socle W. It is clear that the projective A-modules are socle-projective. Therefore a relative projective module in A-spmod is just a projective module in A-mod. A module E is a relative injective module in A-spmod if and only if $E \in \operatorname{add}(\tau_A B \oplus I_A(W))$, where $I_A(W)$ is the injective hull of the socle of A and $B := A/W$ (see [**3**]).

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LEMMA 2.2. *Let A be an algebra over an algebraically closed field with projective socle, and let $P_A(\omega)$ be the unique simple projective module.*

(1) *Let $P_A(a)$, where $a \neq \omega$, be an indecomposable projective module and $I_A(a) = D\mathrm{Hom}_A(P_A(a), A)$. Then $\mathrm{Hom}_A(I_A(a), M) = 0$ for any module $M \in A$-spmod.*

(2) *We have $\mathrm{Hom}_A(X, Q_A(\omega')) \neq 0$ for each $X \in A$-spmod.*

Proof. The socle of M is generated by the simple projective module $P_A(\omega)$, but $I_A(a)$ has no composition factor which is isomorphic to $P_A(\omega)$. This proves part 1 of the lemma. For part 2, $X \in A$-spmod means that $\mathrm{Hom}_A(P_A(\omega), X) \neq 0$. Thus $\mathrm{Hom}_A(X, Q_A(\omega')) \neq 0$.

Let $\mathscr{S}$ be a poset, and let k be an algebraically closed field. By $A(\mathscr{S}) := k\mathscr{S}^+$, we mean the *$k$-incidence algebra* of the enlarged poset $\mathscr{S}^+$. Note that $P_A(\omega)$ is the unique simple projective $A(\mathscr{S})$-module. The following theorem was developed by Ringel and Vossieck [**11**].

THEOREM 2.2. *Let H be a left hammock with source ω, and let k be a field. Let $\mathscr{P}(H,k)$ be the full additive subcategory of $k(H)$ the indecomposable objects of which are just the projective vertices of H. Define the functor $\mathbf{M}: k(H) \to \mathscr{P}(H,k)$-mod by $\mathbf{M}(x) = \mathrm{Hom}_{k(H)}(-, x)|\mathscr{P}(H,k)$. Then the following are true:*

(1) *There is a unique simple projective object in $\mathscr{P}(H,k)$-mod, namely $\mathbf{M}(\omega)$. An object X of $\mathscr{P}(H,k)$-mod belongs to $\mathscr{P}(H,k)$-spmod if and only if its socle is generated by $\mathbf{M}(\omega)$.*

(2) *${}_\infty(\mathscr{P}(H,k)\text{-spmod})$ has Auslander–Reiten sequences.*

(3) *$\mathbf{M}$ induces the equivalence $k(H) \cong {}_\infty(\mathscr{P}(H,k)\text{-spmod})$ (as categories).*

(4) *$H \cong \Gamma_{\infty(\mathscr{P}(H,k)\text{-spmod})}$ (as translation quivers), where $\Gamma_{\infty(\mathscr{P}(H,k)\text{-spmod})}$ is the Auslander–Reiten quiver of ${}_\infty(\mathscr{P}(H,k)\text{-spmod})$.*

If the left hammock H has only finitely many projective vertices, $\mathscr{P}(H,k)$ is a finite category; therefore $\mathscr{P}(H,k)$-mod $\cong A(H)$-mod for some finite dimensional algebra $A(H)$, and $k(H) \cong {}_\infty(A\text{-spmod})$, $H \cong \Gamma_{\infty(A\text{-spmod})}$, where $A = A(H)$. We call $A(H)$ the *finite dimensional algebra corresponding to H*. Note that, if H is a thin left hammock with finitely many projective vertices, then $A(H)$ is just the incidence algebra of the poset $\mathscr{S}(H)$.

3. *Hammocks induced by a thin projective object*

Let Λ be a Krull–Schmidt category, and let Ξ be a class of objects of Λ. For $x, y \in \Lambda$, we denote by $\mathrm{Hom}_\Lambda(x,y)_\Xi$ the subspace of all the maps in $\mathrm{Hom}_\Lambda(x,y)$ which factor through some object of Ξ.

THEOREM 3.1. *Let k be a field. Let H be a left hammock with source ω, and let h_H be the hammock function of H. Assume that $p(a) \neq p(\omega)$ is a thin projective vertex of H. Then the following are true:*

(1) *${}_aH = \{x \in H | \mathrm{Hom}_{k(H)}(p(a), x) \neq 0\}$ is a left hammock with source $p(a)$. The hammock function on ${}_aH$ is*

$$h_{({}_aH)} = \dim_k \mathrm{Hom}_{k(H)}(p(a), -) = \dim_k \mathrm{Hom}_{k(H)}(p(\omega), -)_{\{p(a)\}}$$

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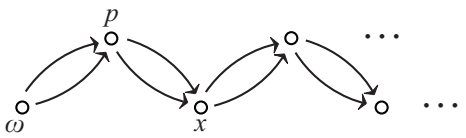


FIG. 1. Left hammock.

(2) $H/_{a}H = \{x \in H \mid h_H(x) - h_{(aH)}(x) \neq 0\}$ *is a left hammock with source* ω. *The hammock function on* $H/_{a}H$ *is*

$$h_{(H/aH)} = h_H - h_{(aH)} = \dim_k \mathrm{Hom}_{k(H)}(p(\omega), -) - \dim_k \mathrm{Hom}_{k(H)}(p(\omega), -)_{\{p(a)\}}$$

In order to prove Theorem 3.1, we need the following lemma.

LEMMA 3.1. *Let* H *be a left hammock with source* ω, *and let* h_H *be the hammock function of* H. *Let* k *be a field. Assume that* $p(a) \neq p(\omega)$ *is a thin projective vertex of* H. *Then* $\dim_k \mathrm{Hom}_{k(H)}(p(a), x) = \dim_k \mathrm{Hom}_{k(H)}(p(\omega), x)_{\{p(a)\}}$ *for* $x \in k(H)$.

Proof. It is clear that

$$\mathrm{Hom}_{k(H)}(p(\omega), x)_{\{p(a)\}} = \mathrm{Hom}_{k(H)}(p(\omega), p(a)) \mathrm{Hom}_{k(H)}(p(a), x)$$

Therefore,

$$\dim_k \mathrm{Hom}_{k(H)}(p(\omega), x)_{\{p(a)\}} \leqslant \dim_k \mathrm{Hom}_{k(H)}(p(a), x)$$

since $\dim_k \mathrm{Hom}_{k(H)}(p(\omega), p(a)) = 1$. On the other hand, we choose and then fix some $0 \neq \phi \in \mathrm{Hom}_{k(H)}(p(\omega), p(a))$. Assume that $\dim_k \mathrm{Hom}_{k(H)}(p(a), x) = r$ and that $\quad_1, \ldots, \quad_r$ is a basis of $\mathrm{Hom}_{k(H)}(p(a), x)$ as a k-vector space. Then Lemma 2.1 implies that the $\phi\quad_1, \ldots, \phi\quad_r$ are linearly independent. Therefore, we have

$$\dim_k \mathrm{Hom}_{k(H)}(p(a), x) \leqslant \dim_k \mathrm{Hom}_{k(H)}(p(\omega), x)_{\{p(a)\}}$$

REMARK 3.1. The assumption that $p(a)$ is thin in Lemma 3.1 cannot be omitted. In the left hammock shown in Figure 1, we have $h_H(p) = 2$. It is easy to see that $\dim_k \mathrm{Hom}_{k(H)}(p, x) = 2$, but $\dim_k \mathrm{Hom}_{k(H)}(\omega, x)_{\{p\}} = 3$.

Proof of Theorem 3.1. In the case in which $x \in {}_{a}H$ is not a projective vertex of H, let $0 \to \mathbf{M}(\tau x) \to \oplus_{y \to x} \mathbf{M}(y) \to \mathbf{M}(x) \to 0$ be an Auslander–Reiten sequence in $\mathscr{P}(H, k)$-spmod, where $\mathbf{M}$ is the equivalence functor $\mathbf{M}: k(H) \cong {}_{\infty}(\mathscr{P}(H, k)\text{-spmod})$. Applying $\mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), -)$ to the sequence above, we obtain

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(\tau x)) &\longrightarrow \sum_{y \to x} \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(y)) \\ \longrightarrow \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(x)) &\longrightarrow 0 \end{aligned}$$

This sequence is exact, since x is not a projective vertex. Therefore

$$\begin{aligned} \dim_k \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(\tau x)) + \dim_k \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(x)) \\ = \sum_{y \to x} \dim_k \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(y)) \end{aligned}$$

and therefore $h_{(aH)}(x) + h_{(aH)}(\tau x) = \sum_{y \to x} h_{(aH)}(y)$. In the case in which $x \in {}_{a}H$ is a projective vertex of H that is different from $p(a)$, then $\oplus_{y \to x} \mathbf{M}(y) = \mathrm{rad}\, \mathbf{M}(x)$. It is obvious that

$$\dim_k \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \bigoplus_{y \to x} \mathbf{M}(y)) = \dim_k \mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(x))$$

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Therefore $h_{({}_aH)}(x) = \sum_{y\to x} h_{({}_aH)}(y)$. Thus $h_{({}_aH)}$ is an additive function on ${}_aH$. (In fact, $h_{({}_aH)}$ is an additive function on H if we put $h_{({}_aH)}(x) = 0$ for $x \notin {}_aH$.) It is clear that ${}_aH$ has the unique source $p(a)$.

Now let q be an injective vertex of ${}_aH$. If q is also an injective vertex of H and not a sink, then $|q^+| = 1$ and $h_H(q) = h_H(y) = 1$, where $q^+ = \{y\}$. From Lemma 3.1, we know that $h_H(y) \geqslant h_{({}_aH)}(y)$. Therefore $h_{({}_aH)}(q) = h_H(q) = h_H(y) \geqslant h_{({}_aH)}(y)$. If q is not an injective vertex in H, then there exists the Auslander–Reiten sequence

$$0 \longrightarrow \mathbf{M}(q) \longrightarrow \sum_{q\to y} \mathbf{M}(y) \longrightarrow \mathbf{M}(\tau^- q) \longrightarrow 0$$

Applying the functor $\mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), -)$ to this exact sequence, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{P}(H,k)}(M(p(a)), M(q)) \longrightarrow \sum_{q\to y} \mathrm{Hom}_{\mathscr{P}(H,k)}(M(p(a)), M(y)) \longrightarrow 0$$

Therefore $h_{({}_aH)}(q) \geqslant \sum_{q\to y} h_{({}_aH)}(y)$. Thus we have the proof of part 1 of Theorem 3.1.

In order to prove part 2 of the theorem, assume that x is not the source. Then we have

$$h_H(x) + h_H(\tau x) = \sum_{y\to x} h_H(y)$$

and

$$h_{({}_aH)}(x) + h_{({}_aH)}(\tau x) = \sum_{y\to x} h_{({}_aH)}(y)$$

for $x \in H$ (we put $h_{({}_aH)}(x) = 0$ for $x \notin {}_aH$). Therefore

$$h_{({}_aH/{}_aH)}(x) + h_{({}_aH/{}_aH)}(\tau x) = \sum_{y\to x} h_{({}_aH/{}_aH)}(y)$$

Thus, $H/{}_aH$ is a proper translation quiver with source ω, and there is an additive function $h_{({}_aH/{}_aH)}$ on it such that $h_{({}_aH/{}_aH)}(\omega) = 1$ and $h_{({}_aH/{}_aH)}(p) = \sum_{y\to p} h_{({}_aH/{}_aH)}(y)$ for any projective vertex $p \neq \omega$ of $H/{}_aH$. We claim that ω is the unique source in $H/{}_aH$. If it is not, assume that ω_0 is another source. Then we have

$$h_{({}_aH/{}_aH)}(\omega_0) = \sum_{y\to\omega_0} h_{({}_aH/{}_aH)}(y) - h_{({}_aH/{}_aH)}(\tau\omega_0) = 0$$

This is a contradiction. Now, let q be an injective vertex in $H/{}_aH$. If q is also an injective vertex in H, then $|q^+| = 1$ and $h_H(q) = h_H(y) = 1$ for $q^+ = \{y\}$. This implies that $h_{({}_aH/{}_aH)}(q) = 1$ and $h_{({}_aH/{}_aH)}(y) \leqslant 1$. As a result, $h_{({}_aH/{}_aH)}(q) \geqslant h_{({}_aH/{}_aH)}(y)$. If q is not an injective vertex in H, it is clear that $h_{({}_aH/{}_aH)}(q) \geqslant \sum_{q\to y} h_{({}_aH/{}_aH)}(y)$. This completes the proof.

REMARK 3.2. Let H be a thin left hammock, and let $p(a)$ be a projective vertex. Then it is possible that the left hammocks ${}_aH$ and $H/{}_aH$ are not thin.

EXAMPLE 3.1. Consider the thin left hammock H shown in Figure 2. Then ${}_aH$ is as shown in Figure 3, and $H/{}_aH$ is as shown in Figure 4. We see that ${}_aH$ and $H/{}_aH$ are not thin.

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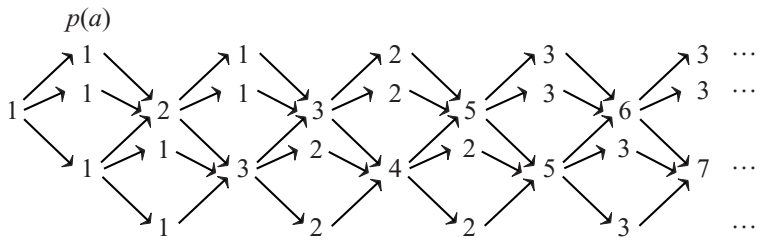


FIG. 2. Thin left hammock H.

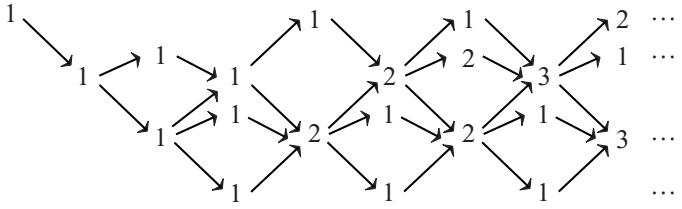


FIG. 3. ${}_aH$.

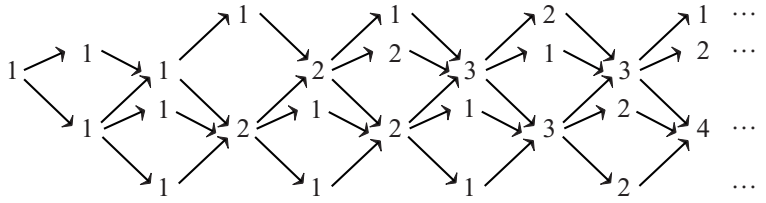


FIG. 4. $H/{}_aH$.

For a poset $\mathscr{S}$ and a field k, we know that $\mathscr{P}_{\mathscr{S}}$, the preprojective component of the Auslander–Reiten quiver of $\ell(\mathscr{S})$, is a thin left hammock with finitely many projective vertices, with the hammock function $h_H(X) = \dim_k X_\omega$. Thus, we have the following.

COROLLARY 3.1. *Let $\mathscr{S}$ be a poset, let k be a field, and let h_H be the hammock function of $\mathscr{P}_{\mathscr{S}}$. Then, for $a \in \mathscr{S}$, we have the following:*

(1)
$$ {}_aH = \{X \in \mathscr{P}_{\mathscr{S}} \mid \mathrm{Hom}_{\mathscr{S}}(P_{\mathscr{S}}(a), X) \neq 0\} = \{X \in \mathscr{P}_{\mathscr{S}} \mid X_a \neq 0\} $$
is a left hammock with source $P_{\mathscr{S}}(a)$. The hammock function on ${}_aH$ is $h_{({}_aH)}(X) = \dim_k X_a$.

(2)
$$ H/{}_aH = \{X \in \mathscr{P}_{\mathscr{S}} \mid h_H(X) - h_{({}_aH)}(X) \neq 0\} = \{X \in \mathscr{P}_{\mathscr{S}} \mid \dim_k X_\omega - \dim_k X_a \neq 0\} $$
is a left hammock with source ω. The hammock function on $H/{}_aH$ is
$$ h_{(H/{}_aH)}(X) = h_H(X) - h_{({}_aH)}(X) = \dim_k X_\omega - \dim_k X_a. $$

PROPOSITION 3.1. *Let k be a field. Let H be a left hammock, and let $p(a)$ and $p(b)$ be thin projective vertices of H that are different from $p(\omega)$. Assume that $\mathrm{Hom}_{k(H)}(p(a), p(b)) \neq 0$. Then we have ${}_aH \supseteq {}_bH$ and ${}_b({}_aH) = {}_bH$, where ${}_aH \supseteq {}_bH$ means that ${}_bH$ is a translation subquiver of ${}_aH$.*

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Proof. Since $\mathrm{Hom}_{k(H)}(p(a), p(b)) \neq 0$, we can choose some $0 \neq \quad \in \mathrm{Hom}_{k(H)}(p(a), p(b))$. Then $\quad$ is a monomorphism, since $p(a)$ is thin by Lemma 2.1. Now, for any $x \in {}_bH$, we take $0 \neq \phi \in \mathrm{Hom}_{k(H)}(p(b), x)$. Then ϕ is a monomorphism and $0 \neq \quad \phi \in \mathrm{Hom}_{k(H)}(p(a), x)$, again by Lemma 2.1. As a consequence, ${}_aH \supseteq {}_bH$.

For a given $p(a)$, we denote by ${}_{a^\wedge}\mathscr{M}$ the object class in $k(H)$ the objects x of which satisfy $\mathrm{Hom}_{k(H)}(p(a), x) = 0$. It is easy to see that $k({}_aH) = k(H)/{}_{a^\wedge}\mathscr{M}$. Note that $\mathrm{Hom}_{k(H)}(p(b), x)_{{}_{a^\wedge}\mathscr{M}} = 0$ for any vertex $x \in H$, since $k(H)$ is directing and $p(a)$ is a predecessor of $p(b)$. Thus we have

$$\begin{aligned}
{}_b({}_aH) &= \{x \in {}_aH \mid \mathrm{Hom}_{k({}_aH)}(p(b), x) \neq 0\} \\
&= \{x \in H \mid \mathrm{Hom}_{k(H)/{}_{a^\wedge}\mathscr{M}}(p(b), x) \neq 0\} \\
&= \{x \in H \mid \mathrm{Hom}_{k(H)}(p(b), x)/\mathrm{Hom}_{k(H)}(p(b), x)_{{}_{a^\wedge}\mathscr{M}} \neq 0\} \\
&= \{x \in H \mid \mathrm{Hom}_{k(H)}(p(b), x) \neq 0\} \\
&= {}_bH
\end{aligned}$$

For a representation-finite poset $\mathscr{S}$ and $t \in \mathscr{S}$, we denote by ${}_tH$ the left hammock induced by $P_{\mathscr{S}}(t)$ as in Corollary 3.1. Now, we define a poset $\bar{\mathscr{S}} = \{{}_tH \mid t \in \mathscr{S}\}$, the poset formed by left hammocks ${}_tH, t \in \mathscr{S}$, ordered ${}_sH \geqslant {}_tH$ if and only if ${}_sH \supseteq {}_tH$. Then we have the following corollary.

COROLLARY 3.2. *There is an isomorphism of posets $\theta\colon \mathscr{S} \to \bar{\mathscr{S}}$ which is created by sending t to ${}_tH$.*

Proof. Clearly θ is a bijective map. We need only to prove that $s \geqslant t$ if and only if ${}_sH \supseteq {}_tH$. Suppose that $s \geqslant t$ in $\mathscr{S}$. This means that $\mathrm{Hom}_{k(H)}(P_{\mathscr{S}}(s), P_{\mathscr{S}}(t)) \neq 0$. Therefore, Proposition 3.1 implies that ${}_sH \supseteq {}_tH$. On the other hand, if ${}_sH \supseteq {}_tH$, then $P_{\mathscr{S}}(t) \in {}_sH$. Therefore $\mathrm{Hom}_{k(H)}(P_{\mathscr{S}}(s), P_{\mathscr{S}}(t)) \neq 0$, and so $s \geqslant t$.

Theorem 3.2 extends the result shown in part 1 of Theorem 3.1.

THEOREM 3.2. *Let k be a field. Let H be a thin left hammock, and let z be a thin vertex of H. Then ${}_sH = \{x \in H \mid \mathrm{Hom}_{k(H)}(z, x) \neq 0\}$ is a left hammock with source z. The hammock function is $h_{({}_zH)} = \dim_k \mathrm{Hom}_{k(H)}(z, -)$.*

Proof. Since z is thin, $\dim_k \mathrm{Hom}_{k(H)}(\omega, z) = 1$. Therefore we can choose a non-zero path in $k(H)$ of the form $x_0 = \omega \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \to x_{n-1} \xrightarrow{\alpha_n} x_n = z$, where x_i, $1 \leqslant i \leqslant n$, are vertices of H, and α_i, $1 \leqslant i \leqslant n$, are arrows in H. Now, x_1 is a direct successor of the source of H. Since H is thin, it follows that x_1 is a thin projective vertex in H. By Theorem 3.1, we obtain a left hammock ${}_{x_1}H$ with source x_1. Clearly, $x_1 \xrightarrow{\alpha_2} \cdots \longrightarrow x_{n-1} \xrightarrow{\alpha_n} x_n = z$ is a non-zero path in $k({}_{x_1}H)$.

Since H is thin, we also know that there are no multiple arrows in $k(H)$. Therefore, there are no multiple arrows in $k({}_{x_1}H)$. Thus x_2, as a direct successor of the source x_1 in ${}_{x_1}H$, is a thin projective vertex of ${}_{x_1}H$. By Theorem 3.1 again, we obtain a left hammock ${}_{x_2}({}_{x_1}H)$, and ${}_{x_2}({}_{x_1}H)$ is just ${}_{x_2}H = \{x \in H \mid \mathrm{Hom}_{k(H)}(x_2, x) \neq 0\}$. This follows from the proof of Proposition 3.1, and the hammock function is $h_{({}_{x_2}H)} = \dim_k \mathrm{Hom}_{k(H)}(x_2, -)$. Again, z is a thin vertex in ${}_{x_2}H$ and $x_2 \xrightarrow{\alpha_3} \cdots \longrightarrow x_{n-1} \xrightarrow{\alpha_n} x_n = z$ is a non-zero path in $k({}_{x_2}H)$.

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By induction on the length of the path from ω to z, we obtain a left hammock ${}_zH$ with source z. The hammock function is $h_{({}_zH)} = \dim_k \mathrm{Hom}_{k(H)}(z, -)$.

REMARK 3.3. Let H be a thin left hammock, and let z be a thin vertex of H. Assume that z is not a projective vertex. Then clearly $\{x \in H \mid h_H(x) - h_{({}_zH)}(x) \neq 0\}$ is not a left hammock, since $(h_H - h_{({}_zH)})(\tau z) + (h_H - h_{({}_zH)})(z) < \sum_{y \to z}(h_H - h_{({}_zH)})(y)$.

The following result is analogous to Proposition 3.1.

PROPOSITION 3.2. *Let H be a thin left hammock, and let y and z be a thin vertex of H that is different from ω. Assume that* $\mathrm{Hom}_{k(H)}(y, z) \neq 0$. *Then we have* ${}_yH \supseteq {}_zH$ *and* ${}_z({}_yH) = {}_zH$.

THEOREM 3.3. *Let H be an infinite left hammock. Assume that $p(a)$ is a thin projective vertex of H. Then the left hammock ${}_aH$ is also infinite.*

Proof. Let $\mathscr{P}(d)$ be the full subcategory of $\mathscr{P}(H, k)$ given by all projective vertices p with $p \in {}^dH$. We can identify $\mathscr{P}(d)$-mod with the full subcategory of $\mathscr{P}(H,k)$-mod given by all modules X with $X(p') = 0$ for all projective vertices $p' \notin {}^dH$. Note that we have the inclusions $\mathscr{P}(0)\text{-mod} \subseteq \mathscr{P}(1)\text{-mod} \subseteq \ldots \subseteq \mathscr{P}(d)\text{-mod} \subseteq \ldots$ and we have ${}_\infty\mathscr{P}(H, k)\text{-mod} = \bigcup_{d \in \mathbb{N}} \mathscr{P}(d)\text{-mod}$, since any object in $\mathscr{P}(H, k)$-mod is of finite length. Note that $\mathbf{M}(\omega)$ belongs to $\mathscr{P}(0)$-mod, and thus to all $\mathscr{P}(d)$-mod, where $\mathbf{M}$ is the equivalence functor $\mathbf{M}\colon k(H) \cong {}_\infty(\mathscr{P}(H, k)\text{-spmod})$. We denote by $\mathscr{F}(d)$ the set of all objects in $\mathscr{P}(d)$-mod with a socle generated by $\mathbf{M}(\omega)$; thus we also have $\mathscr{F}(d) = (\mathscr{P}(H, k)\text{-spmod}) \cap (\mathscr{P}(d)\text{-mod})$. Let $Q_\omega(d)$ be the injective hull of $\mathbf{M}(\omega)$ in $\mathscr{P}(d)$-mod, and then we have $Q_\omega(d) \in \mathscr{F}(d)$-mod and the inclusions $Q_\omega(0) \subseteq Q_\omega(1) \subseteq \ldots \subseteq Q_\omega(d) \subseteq \ldots$ in $\mathscr{P}(H,k)$-spmod.

Consider a projective vertex $p(a) \neq p(\omega)$, and assume that $p(a) \in {}^{d_0}H \setminus {}^{d_0-1}H$. In order to prove the result, it is sufficient to prove the following assertion: for every d such that $d_0 < d \in \mathbb{N}$, there is an $x_d \in {}^dH \setminus {}^{d-1}H$, and $\psi_d \in \mathrm{Hom}_{\mathscr{F}(d)}(\mathbf{M}(p(a)), \mathbf{M}(x_d))$ and $\phi_d \in \mathrm{Hom}_{\mathscr{F}}(\mathbf{M}(x_d), Q_\omega(d))$ such that $\psi_d \phi_d \neq 0$.

We can show this by induction on d. In the case in which $d = d_0$, we only need to put $x_{d_0} = p(a)$. Now, assume that the assertion holds for d. That is, there are $x_d \in ({}^dH) \setminus ({}^{d-1}H)$ and $\psi_d \in \mathrm{Hom}_{\mathscr{F}(d)}(\mathbf{M}(p(a)), \mathbf{M}(x_d))$ and $\phi_d \in \mathrm{Hom}_{\mathscr{F}}(\mathbf{M}(x_d), Q_\omega d))$ such that $\psi_d \phi_d \neq 0$. We denote by ι the embedding map $\iota\colon Q_\omega(d) \to Q_\omega(d+1)$ in $\mathscr{P}(d+1)$-mod. Let $\phi'_d = \phi_d \iota$; then $\psi_d \phi'_d \neq 0$. Let (f_{y_d}) be the maps $(f_{y_d})\colon x_d \to \oplus_{y_d \in x_d^+} y_d$ in $k(H)$. Then, by [**11**], $\mathbf{M}(f_{y_d})\colon \mathbf{M}(x_d) \to \oplus_{y_d \in x_d^+} \mathbf{M}(y_d)$ in $\mathscr{P}(H,k)$-spmod, in fact in $\mathscr{F}(d+1)$, is a source map. Thus ϕ'_d can be lifted. That is, there are $(g_{y_d}) \in \mathrm{Hom}_{\mathscr{F}}(\oplus_{y_d \in x_d^+} \mathbf{M}(y_d), Q_\omega(d+1))$ such that $(f_{y_d})(g_{y_d}) = \phi'_d$. Thus $\psi_d(f_{x_d})(g_{x_d}) \neq 0$. As a result, there is some indecomposable summand of $\oplus_{y_d \in x_d^+} \mathbf{M}(y_d^+)$, say $\mathbf{M}(y_d^0)$, and a corresponding map, say $f_{y_d^0}$, such that $\psi f_{y_d^0} g_{y_d^0} \neq 0$. We put $x_{d+1} := y_d^0$ and $\psi_{d+1} := \psi_d f_{y_d^0}$ and $\phi_{d+1} := g_{y_d^0}$. Then it is clear that $x_{d+1} \in {}^{d+1}H \setminus {}^dH$ and $\psi_{d+1} \in \mathrm{Hom}_{\mathscr{F}(d+1)}(\mathbf{M}(p(a)), \mathbf{M}(x_{d+1}))$ and $\phi_{d+1} \in \mathrm{Hom}_{\mathscr{F}}(\mathbf{M}(x_{d+1}), Q_\omega(d+1))$. Thus the assertion is proved.

REMARK 3.4. Let H be an infinite thin left hammock and let $p(a)$ be a projective vertex of H. Then $H/{}_aH$ may be finite.

EXAMPLE 3.2. In the infinite thin left hammock H shown in Figure 5, we can see that $H/{}_aH$ is finite and $H/{}_bH$ is infinite.

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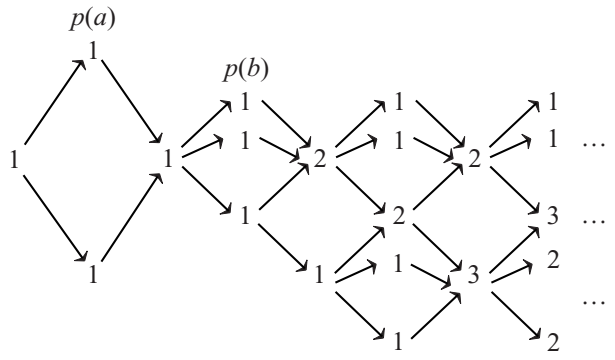


FIG. 5. Infinite thin left hammock H.

4. *The case in which a projective object is not thin*

Let k be an algebraical closed field. Let H be a left hammock with finitely many projective vertices, and let A be the finite dimensional algebra (over k) corresponding to H. Let $\{e_\omega, e_a, e_b, \ldots\}$ be the set of primitive idempotents of A. Note that any indecomposable relative injective module in A-spmod is of the form $Q_A(a) = \tau_A B_A(a)$ or $Q_A(\omega') = I_A(\omega)$, where $B_A(a) = P_A(a)/\mathrm{soc}(P_A(a))$ (see [**3**]). An indecomposable projective A-module $P_A(a)$ is called a *last projective module* if $\mathrm{Hom}_A(P_A(a), P_A(b)) = 0$ for $b \neq a$.

LEMMA 4.1. *Let H be a left hammock with finitely many projective vertices, and let A be the finite dimensional algebra corresponding to H. Assume that $P_A(a)$ is a last projective A-module. Then there is an irreducible map $Q_A(a) \xrightarrow{\chi} Q_A(\omega')$.*

Proof. Suppose that the result is false. Let us consider the Auslander–Reiten sequence $0 \to \tau_A Q_A(\omega') \to \oplus_{i\in I} M_i \to Q_A(\omega') \to 0$ in A-spmod, where M_i, $i \in I$, are indecomposable modules. Obviously, M_i, $i \in I$, are relative injective modules by Lemma 2.2. Now, we know that $\mathrm{Hom}_A(Q_A(a), Q_A(\omega')) \neq 0$ by Lemma 2.2 again. Let $0 \neq \psi \in \mathrm{Hom}_A(Q_A(a), Q_A(\omega'))$. Since ψ is a non-zero non-split epimorphism, ψ can be lifted to $\oplus_{i\in I} M_i$. This means that there is some relative injective module $Q_A(b)$ with $a \neq b$ such that $\mathrm{Hom}_A(Q_A(a), Q_A(b)) \neq 0$. Therefore, $\mathrm{Hom}_A(B_A(a), B_A(b)) \neq 0$. We choose $0 \neq \zeta \in \mathrm{Hom}_A(B_A(a), B_A(b))$. Let $P_A(a) \xrightarrow{\eta_a} B_A(a)$ and $P_A(b) \xrightarrow{\eta_b} B_A(b)$ be projective covers in A-mod. Then $\eta_a \zeta$ is not zero, and it can be lifted to $P_A(b)$. This implies that $\mathrm{Hom}_A(P_A(a), P_A(b)) \neq 0$. Thus the assumption that there is no irreducible map $Q_A(a) \to Q_A(\omega')$ leads to a contradiction, and the lemma is proved.

PROPOSITION 4.1. *Let H be a left hammock with finitely many projective vertices, and let A be the finite dimensional algebra corresponding to H. Assume that $P_A(a)$ is a non-thin projective A-module. Then $Q_A(a)$ does not occur in the preprojective component $\mathscr{P}_{A\text{-spmod}}$.*

Proof. First we consider the case in which $P_A(a)$ is a last projective module. If $Q_A(a) \in \mathscr{P}_{A\text{-spmod}}$, then $Q_A(\omega') \in \mathscr{P}_{A\text{-spmod}}$ follows from Lemma 4.1. Since $k(H) \cong \mathscr{P}_{A\text{-spmod}}$, we see that $\mathscr{P}_{A\text{-spmod}}$ is finite and $P_A(a)$ is thin. This is a contradiction.

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Now we consider the case in which $P_A(a)$ is not a last projective module. Let $e = \sum_{b\in J} e_b$, where $J = \{b \mid \mathrm{Hom}_A(P_A(a), P_A(b)) \neq 0\}$. We put $D = A/AeA$. Then D-mod is naturally a full subcategory of A-mod and D-spmod is a full subcategory of A-spmod. Note that $P_A(a)$ is a non-thin last projective D-module and $B_A(a)$ is also a D-module. By induction, we see that $\tau_D B_A(a)$ does not occur in $\mathscr{P}_{D\text{-spmod}}$. Let $0 \to \tau_D B_A(a) \xrightarrow{f_1} N \xrightarrow{g_1} B_A(a) \to 0$ be the Auslander–Reiten sequence in D-mod, and let $0 \to \tau_A B_A(a) \xrightarrow{f} M \xrightarrow{g} B_A(a) \to 0$ be the Auslander–Reiten sequence in A-mod. Then g_1 can be lifted to M. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_D B_A(a) & \xrightarrow{f_1} & N & \xrightarrow{g_1} & B_A(a) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \xi} & & \| & & \\ 0 & \longrightarrow & \tau_A B_A(a) & \xrightarrow{f} & M & \xrightarrow{g} & B_A(a) & \longrightarrow & 0 \end{array}$$

It is easy to see that $\psi \neq 0$. This shows that $Q_A(a) = \tau_A B_A(a)$ does not occur in $\mathscr{P}_{A\text{-spmod}}$, since, otherwise, there are only finitely many predecessors of $\tau_D B_A(a)$ in $\mathscr{P}_{A\text{-spmod}}$ and hence in $\mathscr{P}_{D\text{-spmod}}$.

COROLLARY 4.1. *Let H be a left hammock with finitely many projective vertices, and let A be the finite dimensional algebra corresponding to H. Assume that $P_A(a)$ is a non-thin projective A-module and that* $\mathrm{Hom}_A(P_A(a), P_A(b)) \neq 0$. *Then $Q_A(b)$ does not occur in the preprojective component $\mathscr{P}_{A\text{-spmod}}$.*

Proof. Since $\mathrm{Hom}_A(P_A(a), P_A(b)) \neq 0$, it follows that

$$\mathrm{Hom}_A(P_A(a)/\mathrm{soc}\, P_A(a), P_A(b)/\mathrm{soc}\, P_A(b)) \neq 0$$

since $\mathrm{soc}\, P_A(a)$ and $\mathrm{soc}\, P_A(b)$ are generated by the simple projective module $P_A(\omega)$. In other words, $\mathrm{Hom}_A(B_A(a), B_A(b)) \neq 0$. Thus $\mathrm{Hom}_A(Q_A(a), Q_A(b)) \neq 0$ since $Q_A(a) = \tau_A B_A(a)$ and $Q_A(b) = \tau_A B_A(b)$. This shows that $Q_A(b)$ does not occur in $\mathscr{P}_{A\text{-spmod}}$.

THEOREM 4.1. *Let H be a left hammock with source ω, with the hammock function h_H. Assume that $p(a)$ is a non-thin projective vertex of H. Then* ${}_AH = \{x \in H \mid \mathrm{Hom}_{k(H)}(p(a), x) \neq 0\}$ *is a left hammock with source $p(a)$. The hammock function on ${}_aH$ is $h_{({}_aH)} = \dim_k \mathrm{Hom}_{k(H)}(p(a), -)$.*

Proof. (1) It is clear that ${}_aH$ has a unique source $p(a)$.

(2) Let $0 \to \mathbf{M}(\tau x) \to \oplus_{y\to x} \mathbf{M}(y) \to \mathbf{M}(x) \to 0$ be an Auslander–Reiten sequence in $\mathscr{P}(H,k)$-spmod, where $\mathbf{M}$ is the equivalence functor $\mathbf{M}\colon k(H) \cong {}_\infty(\mathscr{P}(H,k)\text{-spmod})$. Then, applying $\mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(P(a)), -)$ to the sequence above, we obtain $h_{({}_aH)}(x) + h_{({}_aH)}(\tau x) = \sum_{y\to x} h_{({}_aH)}(y)$.

(3) Now let q be an injective vertex of ${}_aH$. If q is not an injective of H, then there exists an exact sequence $0 \to \mathbf{M}(q) \to \sum_{q\to y} \mathbf{M}(y) \to \mathbf{M}(\tau^- q) \to 0$. Apply the functor $\mathrm{Hom}_{\mathscr{P}(H,k)}(\mathbf{M}(p(a)), -)$ to this exact sequence. Note that q is an injective vertex of ${}_aH$. Thus we have the exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{P}(H,k)}(M(p(a)), M(q)) \to \textstyle\sum_{q\to y} \mathrm{Hom}\, \mathscr{P}(H,k)\,(M(p(a)), M(y)) \to 0$$

Thus $h_{({}_aH)}(q) \geqslant \sum_{q\to y} h_{({}_aH)}(y)$.

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Assume that q is also an injective vertex of H, and not a sink, and let $q \in {}^{d}H$. We consider the full subcategory $\mathscr{P}(d+2)$ of $\mathscr{P}(H,k)$ given by all projective vertices p with $p \in {}^{d+2}H$. We can consider $\mathscr{P}(d+2)$-modules as $\mathscr{P}(H,k)$-modules. Since $\mathscr{P}(d+2)$ is a finite category, there is a finite dimensional algebra A with A-mod $\cong \mathscr{P}(d+2)$-mod, and $k({}^{d+2}H) \cong {}_{d+2}(\mathscr{P}(H,k)\text{-spmod}) \cong {}_{d+2}(A\text{-spmod})$. For this equivalence, we denote by $\mathbf{M}$ the corresponding equivalence functor. We can write $\mathbf{M}(p(a)) = P_A(a)$, $\mathbf{M}(q(b)) = Q_A(b)$ with $a \neq b$ by Proposition 4.1. Using Corollary 4.1, it follows that $\mathrm{Hom}_A(P_A(a),\ P_A(b)) = 0$. Let $0 \to Q_A(b) \to X \to B_A(b) \to 0$ be the Auslander–Reiten sequence in A-mod and $Q_A(b) \to \oplus_{q \to y} \mathbf{M}(y)$ be the source map. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_A(b) & \longrightarrow & X & \longrightarrow & B_A(b) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \chi} & & & & \\ & & Q_A(b) & \longrightarrow & \oplus_{y \in q^+} \mathbf{M}(y) & & & & \end{array}$$

Observe that $\mathrm{Hom}_A(P_A(a),\ B_A(b)) = 0$, since $\mathrm{Hom}_A(P_A(a),\ P_A(b)) = 0$, and χ is an epimorphism. Thus we have

$$\dim_k \mathrm{Hom}_A(P_A(a), Q_A(b)) = \dim_k \mathrm{Hom}_A(P_A(a), X) \geqslant \dim_k \mathrm{Hom}_A(P_A(a), \oplus_{q \to y} \mathbf{M}(y))$$

In other words, $h_{({}_aH)}(q) \geqslant \sum_{q \to y} h_{({}_aH)}(y)$. Thus Theorem 4.1 is proved.

REMARK 4.1. Let H be a left hammock with a hammock function of H. Let $p(a)$ be a projective vertex of H, and let $H/_aH$ be the full translation subquiver $\{x \in H \mid h_H(x) - h_{({}_aH)}(x) \neq 0\}$. Then $H/_aH$ is a left hammock if and only if $p(a)$ is thin. In fact, the sufficiency follows from Theorem 3.1. For necessity, assume that $p(a)$ is non-thin. It is easy to see that $p(a)$ occurs in $H/_aH$ as a projective vertex, but

$$h_H(p(a)) - h_{({}_aH)}(p(a)) < \sum_{y \in p(a)^-} (h_H(y) - h_{({}_aH)}(y))$$

5. *Hammocks induced by an injective object*

LEMMA 5.1. *Let k be a field. Let H be a left hammock with source ω, and h_H be the hammock function of H. Assume that $q(a) \neq q(\omega')$ is an injective vertex of H. Then we have* $\dim_k \mathrm{Hom}_{k(H)}(p(\omega), x) \geqslant \dim_k \mathrm{Hom}_{k(H)}(x, q(a))$ *for all* $x \in H$.

Proof. Assume that $q(a) \in {}^{d}H$. We consider the full subcategory $\mathscr{P}(d+2)$ of $\mathscr{P}(H, k)$ given by all projective vertices p with $p \in {}^{d+2}H$. We can consider $\mathscr{P}(d+2)$-modules as $\mathscr{P}(H,k)$-modules. Since $\mathscr{P}(d+2)$ is a finite category, there is a finite dimensional algebra A with A-mod $\cong \mathscr{P}(d+2)$-mod, and $k({}^{d+2}H) \cong {}_{d+2}(\mathscr{P}(H,k)\text{-spmod}) \cong {}_{d+2}(A\text{-spmod})$. For this equivalence, we denote by $\mathbf{M}$ the corresponding equivalence functor. We can write $\mathbf{M}(p(a)) = P_A(a)$ and $\mathbf{M}(q(a)) = Q_A(a)$ and $B_A(a) = P_A(a)/\mathrm{soc}\, P_A(a)$. Since $Q_A(a)$ is an injective vertex of H, $P_A(a)$ is thin according to Proposition 4.1. Let $0 \to P_A(\omega) \to P_A(a) \to B_A(a) \to 0$ be a minimal projective resolution. We have the following exact sequence: $0 \to Q_A(a) \to I_A(\omega) \to I_A(a) \to 0$. Thus, for $X \in {}_{d+2}(A\text{-spmod})$, we have $0 \to \mathrm{Hom}_A(X, Q_A(a)) \to \mathrm{Hom}_A(X, I_A(\omega)) \to \mathrm{Hom}_A(X, I_A(a)) \to 0$. Therefore,

$$\dim_k \mathrm{Hom}_A(P_A(\omega), X) = \dim_k \mathrm{Hom}_A(X, I_A(\omega)) \geqslant \dim_k \mathrm{Hom}_A(X, Q_A(a))$$

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In other words,

$$\dim_k \mathrm{Hom}_{k(H)}(p(\omega), x) \geqslant \dim_k \mathrm{Hom}_{k(H)}(x, q(a))$$

for $x \in {}^{d+2}H$. The inequality holds naturally for $x \in H \setminus {}^{d+2}H$, since $\mathrm{Hom}_{k(H)}(x, q(a)) = 0$.

THEOREM 5.1. *Let k be a field. Let H be a left hammock with the hammock function h_H. Assume that $q(a) \neq q(\omega')$ is an injective vertex of H. Then the following are true:*

(1) *$H_a = \{x \in H \mid \mathrm{Hom}_{k(H)}(x, q(a)) \neq 0\}$ is a hammock with source ω and sink $q(a)$. The hammock function on H_a is $h_{(H_a)} = \dim_k \mathrm{Hom}_{k(H)}(-, q(a))$.*

(2) *$H/H_a = \{x \in H \mid h_H(x) - h_{(H_a)}(x) \neq 0\}$ is a left hammock with source $p(a)$. The hammock function on H/H_a is $h_{(H/H_a)} = h_H - h_{(H_a)}$.*

(3) *$H_a = H/{}_aH$ and ${}_aH = H/H_a$.*

Proof. Assume that $q \in {}^dH$. We consider the full subcategory $\mathscr{P}(d+2)$ of $\mathscr{P}(H, k)$ given by all projective vertices p with $p \in {}^{d+2}H$. Naturally, $\mathscr{P}(d+2)$-modules are $\mathscr{P}(H, k)$-modules. We have $\mathscr{P}(d+2)$-mod $\cong A$-mod for some finite dimensional algebra A. Since $\mathscr{P}(d+2)$ is a finite category, we have $k({}^{d+2}H) \cong {}_{d+2}(\mathscr{P}(H, k)\text{-spmod}) \cong {}_{d+2}(A\text{-spmod})$. For this equivalence, we denote by $\mathbf{M}$ the corresponding equivalence functor. We can write $\mathbf{M}(p(a)) = P_A(a)$ and $\mathbf{M}(q(a)) = Q_A(a)$ and $B_A(a) = P_A(a)/\mathrm{soc}(P_A(a))$. Since $Q_A(a)$ is an injective vertex of H, $P_A(a)$ is thin, by Proposition 4.1. Let $0 \to P_A(\omega) \to P_A(a) \to B_A(a) \to 0$ be the minimal projective resolution. Thus we have the following exact sequence for $X \in A$-spmod: $0 \to \mathrm{Hom}_A(B_A(a), X) \to \mathrm{Hom}_A(P_A(a), X) \to \mathrm{Hom}_A(P_A(\omega), X) \to \mathrm{Ext}^1_A(B_A(a), X) \to 0$. Note that $\mathrm{soc}\, X$ is generated by the simple projective module $P_A(\omega)$ and $B_A(a)$ has no composition factor which is isomorphic to $P_A(\omega)$, and so we see that $\mathrm{Hom}_A(B_A(a), X) = 0$. The Auslander–Reiten formula and Lemma 2.2 imply that

$$\mathrm{D}\,\overline{\mathrm{Hom}}_A(X, \tau_A B_A(a)) = \mathrm{D}\,\mathrm{Hom}_A(X, \tau_A B_A(a))$$

Hence

$$\dim_k \mathrm{Hom}_A(P_A(\omega), X) = \dim_k \mathrm{Hom}_A(P_A(a), X) + \dim_k \mathrm{Hom}_A(X, \tau_A B_A(a))$$

In other words,

$$\dim_k \mathrm{Hom}_{k(H)}(p(\omega), x) = \dim_k \mathrm{Hom}_{k(H)}(p(a), x) + \dim_k \mathrm{Hom}_{k(H)}(x, q(a))$$

This shows that $H_a = H/{}_aH$ where $h_{(H_a)} = h_{(H/{}_aH)}$ and ${}_aH = H/H_a$ where $h_{({}_aH)} = h_{(H/H_a)}$. We have proved part 3 of the theorem. Then part 3 implies part 1 and part 2.

COROLLARY 5.1. *Let $\mathscr{S}$ be a poset, let k be a field, and let h_H be the hammock function of $P_{\mathscr{S}}$. Then, for $Q_{\mathscr{S}}(a) \in P_{\mathscr{S}}$, we have the following:*

(1)

$$H_a = \{X \in \mathscr{P}_{\mathscr{S}} \mid \mathrm{Hom}_{\mathscr{S}}(X_\omega, Q_{\mathscr{S}}(a)) \neq 0\} = \{X \in \mathscr{P}_{\mathscr{S}} \mid X_\omega / X_A \neq 0\}$$

is a hammock with source $P_{\mathscr{S}}(\omega)$ and sink $Q_{\mathscr{S}}(a)$. The hammock function on H_a is

$$h_{(H_a)}(X) = \dim_k \mathrm{Hom}_{\mathscr{S}}(X, Q_{\mathscr{S}}(a)) = \dim_k X_\omega - \dim_k X_a$$

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(2)
$$H/H_a = \{X \in \mathscr{P}_{\mathscr{S}} \mid h_H(X) - h_{(H_a)}(X) \neq 0\} = \{X \in \mathscr{P}_{\mathscr{S}} \mid X_a \neq 0\}$$

is a left hammock with source $P(a)$. *The hammock function on* H/H_a *is*

$$h_{(H/H_a)}(X) = h_H(X) - h_{(H_a)}(X) = \dim_k X_a$$

(3) $H_a = H/_aH$ *and* ${}_aH = H/H_a$.

PROPOSITION 5.1. *Let* H *be a left hammock, and let* $q(a)$ *and* $q(b)$ *be injective vertices of* H *that are different from* $q(\omega')$. *Assume that* $\mathrm{Hom}_{k(H)}(q(a), q(b)) \neq 0$. *Then we have* $H_a \subseteq H_b$ *and* $(H_b)_a = H_a$.

Proof. The proof of this proposition is analogous to the proof of Proposition 3.1.

For a given representation-finite poset $\mathscr{S}$ and $t \in \mathscr{S}$, we denote by H_t the hammock induced by $Q_{\mathscr{S}}(t)$, as in Corollary 5.1. We define the poset $\tilde{\mathscr{S}} = \{H_t \mid t \in \mathscr{S}\}$ formed by the left hammocks H_t, $t \in \mathscr{S}$, with $H_s \geqslant H_t$ if and only if $H_s \subseteq H_t$. Then the following corollary is analogous to Corollary 3.2.

COROLLARY 5.2. *There is an isomorphism of posets* $\theta: \mathscr{S} \to \tilde{\mathscr{S}}$ *which is created by sending* t *to* H_t.

6. *Hammocks and the Nazarova–Roiter algorithm*

Let $\mathscr{S}$ be a poset. Let a be a maximal element of $\mathscr{S}$. Following Nazarova and Roiter, one can construct a new poset ${}_a\partial\mathscr{S}$ the vertices of which consist of (i) all the elements of $\mathscr{S}$ except a, and (ii) new elements $s+t$, for any pair s, t such that s, t and a are mutually incomparable. The order relationships for ${}_a\partial\mathscr{S}$ are as follows: (i) for $s, t \in \mathscr{S}$, the order follows from the order on $\mathscr{S}$, (ii) if $x \leqslant s$ or $x \leqslant t$, then $x \leqslant s+t$, (iii) if $x \geqslant s$ and $x \geqslant t$, then $x \geqslant s+t$, and (iv) $s+t \leqslant u+v$ if and only if, first, $s \leqslant u$ or $s \leqslant v$, and, second, $t \leqslant u$ or $t \leqslant v$. We say that ${}_a\partial\mathscr{S}$ is obtained from $\mathscr{S}$ by *differentiation with respect to the maximal element* a.

THEOREM 6.1. *Let* H *be a hammock, and let* $\mathscr{S}(H)$ *be the poset corresponding to* H. *Assume that* $p(a)$ *is a projective vertex of* H *that is different from* ω. *Then* $\mathscr{S}({}_aH)$ *is obtained from* $\mathscr{S}(H)$ *as follows: there is a finite sequence of vertices* $x_1, x_2, \ldots, x_l = p(a)$ *and a finite sequence of posets* $\mathscr{S}_1 = \mathscr{S}(H), \mathscr{S}_2, \ldots, \mathscr{S}_l$ *such that the following are true:*

(1) x_i *is a maximal element of* $\mathscr{S}_i$.

(2) $\mathscr{S}_i = {}_{x_{i-1}}\partial\mathscr{S}_{i-1}$ *for* $i = 2, \ldots, l$, *that is,* $\mathscr{S}_i$ *is obtained from* $\mathscr{S}_{i-1}$ *by differentiation with respect to* x_{i-1} *in the sense of Nazarova and Roiter.*

(3) $\mathscr{S}_l \cong \mathscr{S}({}_aH)$.

For a given poset $\mathscr{S}$, width $(\mathscr{S})$ is the maximal number of pairwise incomparable elements. For $a \in \mathscr{S}$, we set $a_\wedge = \{x \in \mathscr{S} \mid x \leqslant a\}$. Let $\mathscr{S}$ be a poset with a maximal

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element a, and let ${}_a\partial\mathscr{S}$ be the poset obtained from $\mathscr{S}$ by differentiation with respect to a. We recall the *differentiation functor* ${}_a\partial\colon \ell(\mathscr{S})\to\ell({}_a\partial\mathscr{S})$, which is defined by $({}_a\partial(V))_\omega = V_a$ and $({}_a\partial(V))_s = V_a\cap V_s$ and $({}_a\partial(V))_{s+t} = V_a\cap(V_s+V_t)$ and ${}_a\partial(\phi)=\phi_a$. Theorem 6.2 was developed by Gabriel [**2**].

THEOREM 6.2. *Let $\mathscr{S}$ be a poset, and let $a\in\mathscr{S}$ be a maximal element such that* width $(\mathscr{S}\setminus a_\wedge)\leqslant 2$. *Assume that ${}_a\partial\mathscr{S}$ is obtained from $\mathscr{S}$ by differentiation with respect to a. Then ${}_a\partial$ induces an equivalence $\ell(\mathscr{S})/{}_a\mathscr{N}\cong\ell({}_a\partial\mathscr{S})$, where ${}_a\mathscr{N}$ is the object class of $\ell(\mathscr{S})$ consisting of all $\mathscr{S}$-spaces V with $V_a=0$.*

Proof of Theorem 6.1. Since H is a hammock, we have width $(\mathscr{S}(H))\leqslant 3$. Therefore, width $(\mathscr{S}(H)\setminus a_\wedge)\leqslant 2$ for each maximal element a.

In the case in which the vertex a is a maximal element of $\mathscr{S}(H)$, we have

$$\ell(\mathscr{S}({}_aH))\cong k({}_aH)\cong k(H)/{}_a\mathscr{M}\cong\ell(\mathscr{S})/{}_a\mathscr{N}\cong\ell({}_a\partial\mathscr{S})$$

The hammock function is uniquely determined, and so $\mathscr{S}({}_aH)\cong{}_a\partial\mathscr{S}$.

In the case in which the vertex a is not a maximal element of $\mathscr{S}$, we can choose some vertex x_1 in H where (i) x_1 is a maximal element of $\mathscr{S}(H)$, and (ii) $\mathrm{Hom}_{k(H)}(x_1, p(a))\neq 0$. Then we consider ${}_{x_1}H$. Of course, $p(a)$ is a projective vertex of ${}_{x_1}H$. If a is not a maximal element in $\mathscr{S}({}_{x_1}H)$, we can choose some vertex x_2 in ${}_{x_1}H$ where (i) x_2 is a maximal element of $\mathscr{S}({}_{x_1}H)$ and (ii) $\mathrm{Hom}_{k({}_{x_1}H)}(x_2, p(a))\neq 0$.

By Proposition 3.1, ${}_{x_2}({}_{x_1}H) = {}_{x_2}H$. We consider the hammock ${}_{x_2}H$. In this way, we obtain a sequence of vertices $x_1, x_2, \ldots, x_l = p(a)$ and a sequence of posets $\mathscr{S}_1=\mathscr{S}(H)$, $\mathscr{S}_i=\mathscr{S}({}_{x_i}H)$, $i=2,\ldots,l$, such that (i) x_i is a maximal element of $\mathscr{S}_i$, for $i=1,\ldots,l$, and (ii) $\mathrm{Hom}_{k({}_{x_{i-1}}H)}(x_i, p(a))\neq 0$ for $i=2,\ldots,l$. In fact, we obtain a non-zero path in $k(H)\omega\xrightarrow{\alpha_1}x_1\xrightarrow{\alpha_2}\cdots\xrightarrow{\alpha_{l-1}}x_{l-1}\xrightarrow{\alpha_l}x_l=p(a)$ where $p(a)\in{}^lH\setminus{}^{l-1}H$. Thus ${}_{x_i}H = {}_{x_i}({}_{x_{i-1}}H)$ and $\mathscr{S}_i = {}_{x_{i-1}}\partial\mathscr{S}_{i-1}$, for $i=2,\ldots,l$. In particular, ${}_aH = {}_a({}_{x_{l-1}}(\ldots{}_{x_2}({}_{x_1}H)\ldots))$ and $\mathscr{S}({}_aH) = {}_a\partial(\mathscr{S}({}_{x_{l-1}}H)) = ({}_a\partial)({}_{x_{l-1}}\partial)\mathscr{S}({}_{x_{i-2}}H) = \ldots = ({}_a\partial)({}_{x_{l-1}}\partial)\ldots({}_{x_2}\partial)({}_{x_1}\partial)\mathscr{S}(H)$. This completes the proof.

The differentiation algorithm with respect to minimal elements is analogous to the differentiation algorithm with respect to maximal elements. (See in [**7**] the remark following Lemma 2, and also see [**12**]). We have the following result which is analogous to Theorem 6.1.

THEOREM 6.3. *Let H be a hammock, and let $\mathscr{S}(H)$ be the poset corresponding to H. Assume that $q(a)$ is an injective vertex of H that is different from ω'. Then $\mathscr{S}(H_a)$ is obtained from $\mathscr{S}(H)$ as follows. There is a finite sequence of vertices $x_1, x_2, \ldots, x_l = q(a)$ and a finite sequence of posets $\mathscr{S}_1=\mathscr{S}(H), \mathscr{S}_2, \ldots, \mathscr{S}_l$ such that the following are true:*

(1) *x_i is a minimal element of $\mathscr{S}_i$.*

(2) *$\mathscr{S}_i = \partial_{x_{i-1}}\mathscr{S}_{i-1}$ for $i=2,\ldots,l$, that is, $\mathscr{S}_i$ is obtained from $\mathscr{S}_{i-1}$ by a differentiation with respect to the minimal element x_{i-1}.*

(3) *$\mathscr{S}_l\cong\mathscr{S}(H_a)$.*

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References

1. R. Brenner, 'A combinatorial characterization of finite Auslander–Reiten quivers', *Representation theory I. Finite dimensional algebras*, Lecture Notes in Mathematics 1177 (Springer, Berlin, 1986) 13–49.

2. P. Gabriel, 'Représentations indécomposable des ensembles ordonnés', *Sém. P. Dubreil. Paris* (1972–73) 301–304.

3. O. Kerner, 'On socle-projective categories and tilting modules', *Comm. Algebra* 20 (1992) 2459–2470.

4. M. M. Klejner, 'Partially ordered sets of finite type', *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov*. 28 (1972) 32–41 (*J. Soviet Math*. 3 (1975) 607–615).

5. Y. Lin, 'Hammocks and the algorithm of Zavadskij', *J. Algebras* 23 (1996) 694–720.

6. Y. Lin, 'Hammocks and the algorithms of Nazarova–Roiter and of Zavadskij', PhD Thesis, Bielefeld, 1994.

7. L. A. Nazarova, 'Partially ordered sets of infinite type', *Izv. Akad. Nauk SSSR Ser. Mat*. 39 (1975) 963–991 (*Math. USSR-Izv*. 9 (1975) 917–938).

8. L. A. Nazarova and A. V. Roiter, 'Representations of partially ordered sets', *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov*. 28 (1972) 5–31 (*J. Soviet Math*. 3 (1975) 585–606).

9. C. M. Ringel, *Representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Notes 116 (London Mathematical Society, London, 1986) 7–79.

10. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics 1099 (Springer, Berlin, 1984).

11. C. M. Ringel and D. Vossieck, 'Hammocks', *Proc. London Math. Soc*. (3) 54 (1987) 216–246.

12. D. Simson, *Linear representations of partially ordered sets and vector space categories* (Gordon and Breach, 1992).

13. A. G. Zavadskij, 'Differentiation with respect to a pair of points', *Matrix problems* (Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1977) 115–121 (in Russian).

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