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ARTICLE

Avoiding long Berge cycles: the missing cases $k = r + 1$ and $k = r + 2$

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Abstract

The maximum size of an r-uniform hypergraph without a Berge cycle of length at least k has been determined for all $k \geqslant r + 3$ by Füredi, Kostochka and Luo and for $k < r$ (and $k = r$, asymptotically) by Kostochka and Luo. In this paper we settle the remaining cases: $k = r + 1$ and $k = r + 2$, proving a conjecture of Füredi, Kostochka and Luo.

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1. Introduction

Given a hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ and $E(\mathcal{H})$ denote the set of vertices and hyperedges of $\mathcal{H}$, respectively, and let $e(\mathcal{H}) = |E(\mathcal{H})|$. A hypergraph is called r-uniform if all of its hyperedges have size r. For convenience, we refer to an r-uniform hypergraph as an r-graph. Berge introduced the following definitions of a cycle and a path in a hypergraph.

Definition. A *Berge cycle* of length l in a hypergraph is a set of l distinct vertices $\{v_1, \ldots, v_l\}$ and l distinct hyperedges $\{e_1, \ldots, e_l\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo l.

A *Berge path* of length l in a hypergraph is a set of $l + 1$ distinct vertices $v_1, \ldots, v_{l+1}$ and l distinct hyperedges $e_1, \ldots, e_l$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leqslant i \leqslant l$. We say that such a Berge path is between v_1 and v_{l+1}. The vertices $v_1, \ldots, v_{l+1}$ are called the *defining vertices* of the path.

Győri, Katona and Lemons extended the well-known Erdős–Gallai theorem to hypergraphs by showing the following.

Theorem 1.1 (Győri, Katona and Lemons [9]). *Let $\mathcal{H}$ be an r-uniform hypergraph with no Berge path of length k. If $k > r + 1 > 3$, we have*

$$e(\mathcal{H}) \leqslant \frac{n}{k}\binom{k}{r}.$$

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If $r \geqslant k > 2$, we have

$$e(\mathcal{H}) \leqslant \frac{n(k-1)}{r+1}.$$

For the case $k = r+1$, Győri, Katona and Lemons conjectured that the upper bound should have the same form as the $k > r+1$ case. This was settled by Davoodi, Győri, Methuku and Tompkins [1], who showed the following.

Theorem 1.2 (Davoodi, Győri, Methuku and Tompkins [1]). *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k. Then*

$$e(\mathcal{H}) \leqslant \frac{n}{k}\binom{k}{r} = n.$$

The bounds in the above two theorems are sharp for each k and r for infinitely many n. Győri, Methuku, Salia, Tompkins and Vizer [10] proved a significantly smaller upper bound on the maximum number of hyperedges in an n-vertex r-graph with no Berge path of length k under the assumption that it is connected. Their bound is asymptotically exact when r is fixed and k and n are sufficiently large. The notion of Berge cycles and Berge paths was generalized to arbitrary Berge graphs F by Gerbner and Palmer in [7], and the (3-uniform) Turán number of Berge-$K_{2,t}$ was determined asymptotically in [5]. The general behaviour of the Turán number of Berge-F, as the uniformity increases, was studied in [8].

Recently, Füredi, Kostochka and Luo [3] proved exact bounds similar to Theorem 1.1 for hypergraphs avoiding long Berge cycles. To state their results, we need the following notation.

Notation. Let $\mathcal{H}$ be a hypergraph. Then its 2-*shadow*, $\partial_2(\mathcal{H})$, is the collection of pairs that lie in some hyperedge of $\mathcal{H}$. Given a set $S \subseteq V(\mathcal{H})$, the subhypergraph of $\mathcal{H}$ induced by S is denoted by $\mathcal{H}[S]$.

We say $\mathcal{H}$ is *connected* if $\partial_2(\mathcal{H})$ is a connected graph. A hyperedge $h \in E(\mathcal{H})$ is called a *cut-hyperedge* of $\mathcal{H}$ if $\mathcal{H} \setminus \{h\} := (V(\mathcal{H}), E(\mathcal{H}) \setminus \{h\})$ is not connected.

When we say D is a block of $\partial_2(\mathcal{H})$, we may either mean D is the vertex-set of the block, or D is the edge-set of the block depending on the context.

Theorem 1.3 (Füredi, Kostochka and Luo [3]). *Let $r \geqslant 3$ and $k \geqslant r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then*

$$e(\mathcal{H}) \leqslant \frac{n-1}{k-2}\binom{k-1}{r}.$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected, and for every block D of $\partial_2(\mathcal{H})$, $D = K_{k-1}$ and $\mathcal{H}[D] = K^r_{k-1}$.

Moreover, Kostochka and Luo [11] found exact bounds for $k \leqslant r-1$ and asymptotic bounds for $k = r$. Let us remark that their asymptotic bound in the case $k = r$ also follows from Theorem 1.5 stated below. (More recently, extending [3], Füredi, Kostochka and Luo [4] proved exact bounds and determined the extremal examples for all n when $k \geqslant r+4$.)

The two cases $k = r+2$ and $k = r+1$ remained open. For the case $k = r+2$, Füredi, Kostochka and Luo [3] conjectured that a statement similar to that of Theorem 1.3 holds, and mentioned that the answer is unknown for the case $k = r+1$. In this paper, we prove this conjecture.

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Theorem 1.4. *Let $r \geqslant 3$ and $n \geqslant 1$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length $r+2$ or longer. Then*

$$e(\mathcal{H}) \leqslant \frac{r+1}{r}(n-1).$$

Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected, and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D] = K^r_{r+1}$.

In the case $k = r+1$, we prove the following exact result and characterize the extremal examples.

Theorem 1.5. *Let $r \geqslant 3$ and $n \geqslant 1$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length $r+1$ or longer. Then $e(\mathcal{H}) \leqslant n-1$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected, and for every block D of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of r hyperedges.*

Let us note that our proof method is very different from the proof of Füredi, Kostochka and Luo [3] and that of Kostochka and Luo [11]. One of our main ideas in proving the above two theorems is an unusual application of Hall's theorem to vertices of the hypergraph (instead of applying it to edges in the shadow of the hypergraph). This allows us to assign a distinct hyperedge to each vertex, which is then very helpful in finding Berge cycles. The notion of connectivity (*i.e.* the notion of 'cut-hyperedges') in Berge hypergraphs has also been very important. These ideas may have further applications (see *e.g.* [6]).

Note that Theorem 1.5 easily implies Theorem 1.2. In fact, it gives the following stronger form. Here we quickly prove this implication.

Theorem 1.6. *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length k. Then*

$$e(\mathcal{H}) \leqslant \frac{n}{k}\binom{k}{r} = n.$$

Moreover, equality holds if and only if each connected component D of $\partial_2(\mathcal{H})$ is K_{r+1}, and $\mathcal{H}[D] = K^r_{r+1}$.

Proof. We proceed by induction on n. The base cases $n \leqslant r+1$ are easy to check. Let $\mathcal{H}$ be an r-uniform hypergraph containing no Berge path of length $k = r+1$ such that $e(\mathcal{H}) \geqslant n$. Then by Theorem 1.5, $\mathcal{H}$ contains a Berge cycle $\mathcal{C}$ of length $r+1$ or longer. $\mathcal{C}$ must be of length exactly $r+1$, otherwise it would contain a Berge path of length $r+1$. Let $v_1, \ldots, v_{r+1}$ and $e_1, \ldots, e_{r+1}$ be the vertices and edges of $\mathcal{C}$ where $\{v_i, v_{i+1}\} \subseteq e_i$ (indices are taken modulo $r+1$). For any i with $1 \leqslant i \leqslant r+1$, if e_i contains a vertex $v \notin \{v_1, \ldots, v_{r+1}\}$, then $v_{i+1}e_{i+1}v_{i+2}e_{i+2}\cdots e_{i-1}v_ie_iv$ forms a Berge path of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, all of the edges e_i (for $1 \leqslant i \leqslant r+1$) are contained in the set $S := \{v_1, \ldots, v_{r+1}\}$. That is, $\mathcal{H}[S] = K^r_{r+1}$. It is easy to see that S forms a connected component in $\partial_2(\mathcal{H})$ because if any hyperedge h of $\mathcal{H}$ (with $h \notin \mathcal{C}$) contains a vertex of $\mathcal{C}$, then $\mathcal{C}$ can be extended to form a Berge path of length $r+1$.

Let $S_1, S_2, \ldots, S_t$ be the vertex sets of connected components of $\partial_2(\mathcal{H})$. As noted before, one of them, say S_1, is equal to S. We delete the vertices of S_1 from $\mathcal{H}$ to form a new hypergraph $\mathcal{H}'$; note that

$$|V(\mathcal{H}')| = |V(\mathcal{H})| - (r+1) \quad \text{and} \quad |E(\mathcal{H}')| = |E(\mathcal{H})| - (r+1)$$

and the connected components of $\partial_2(\mathcal{H}')$ are $S_2, \ldots, S_t$. By induction $|E(\mathcal{H}')| \leqslant |V(\mathcal{H}')|$. Thus

$$|E(\mathcal{H})| = |E(\mathcal{H}')| + (r+1) \leqslant |V(\mathcal{H}')| + (r+1) = |V(\mathcal{H})|.$$

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Moreover, if $|E(\mathcal{H})| = |V(\mathcal{H})|$, then $|E(\mathcal{H}')| = |V(\mathcal{H}')|$, so by the induction hypothesis each connected component S_i ($i \geqslant 2$) of $\partial_2(\mathcal{H}')$ is K_{r+1}, and $\mathcal{H}'[S_i] = K^r_{r+1}$, proving the theorem. □

Structure of the paper. In Section 2 we prove some basic lemmas which are used in our proofs. In Section 3 we prove Theorem 1.4, and in Section 4 we prove Theorem 1.5.

2. Basic lemmas

We will use the following two lemmas.

Lemma 2.1. *For any $r \geqslant 3$, if a set S of size $r+1$ contains r hyperedges of size r, then between any two vertices $u, v \in S$, there is a Berge path of length r consisting of these hyperedges.*

Proof. Let $\mathcal{H}$ be the hypergraph consisting of r hyperedges on $r+1$ vertices. First note that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. (Indeed, there is at most one hyperedge that does not contain x and at most one hyperedge that does not contain y.) This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least three hyperedges contained in S. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2, \ldots, x_{r+1}$ of length r in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_{r+1}$. We want to show that there are distinct hyperedges containing the pairs x_ix_{i+1} for each $1 \leqslant i \leqslant r$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \ldots, x_rx_{r+1}\}$ in one class and the r hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We will show that Hall's condition holds. As noted before, every pair is contained in a hyperedge. Given any two distinct pairs x_ix_{i+1} and x_jx_{j+1}, there is at most one hyperedge that does not contain either of them; that is, at least $r-1$ hyperedges contain one of them. Thus we need $2 \leqslant r-1$ for Hall's condition to hold, but this is true as we assumed $r \geqslant 3$. Moreover, if we take any $3 \leqslant j \leqslant r$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \leqslant r$, but this is true by assumption. This finishes the proof of the lemma. □

Lemma 2.2. *For any $r \geqslant 4$, if a set S of size $r+1$ contains $r-1$ hyperedges of size r, then between any two vertices $u, v \in S$, there is a Berge path of length $r-1$ consisting of these hyperedges.*

Proof. The proof is similar to that of Lemma 2.1. Let $\mathcal{H}$ be the hypergraph consisting of $r-1$ hyperedges on $r+1$ vertices. First note that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least $r-1 \geqslant 3$ hyperedges contained in S. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2 \cdots x_r$ of length $r-1$ in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_r$. We want to show that there are distinct hyperedges containing the pairs x_ix_{i+1} for each $1 \leqslant i \leqslant r-1$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \ldots, x_{r-1}x_r\}$ in one class and the $r-1$ hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We show that Hall's condition holds. As noted before, every pair is contained in a hyperedge. Given any two distinct pairs x_ix_{i+1} and x_jx_{j+1}, there is at most one hyperedge that does not contain either of them; that is, at least $r-2$ hyperedges contain one of them. Thus we need $2 \leqslant r-2$ for Hall's condition to hold, but this is true as we assumed $r \geqslant 4$. Moreover, if we take any $3 \leqslant j \leqslant r-1$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \leqslant r-1$ for Hall's condition to hold, and this is true by assumption. This finishes the proof of the lemma. □

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3. Proof of Theorem 1.4 ($k = r + 2$)

We will prove the theorem by induction on n. For the base cases, note that if $1 \leqslant n \leqslant r$ then the statement of the theorem is trivially true. If $n = r + 1$, the statement is true since there are at most $r + 1$ hyperedges of size r on $r + 1$ vertices. Moreover, equality holds if and only if $\mathcal{H} = K^r_{r+1}$.

We will show that the statement is true for $n \geqslant r + 2$ assuming it is true for all smaller values. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices having no Berge cycle of length $r + 2$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S \subseteq V(\mathcal{H})$ of vertices, the number of hyperedges of $\mathcal{H}$ incident to the vertices of S is at least $|S|$. Indeed, suppose there is a set $S \subseteq V(\mathcal{H})$ with fewer than $|S|$ hyperedges incident to the vertices of S. If $|S| = n$ we immediately have the required bound on $e(\mathcal{H})$, so assume $n > |S|$. We can delete the vertices of S from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n - |S|$ vertices. By induction, $\mathcal{H}'$ contains at most

$$\frac{r+1}{r}(n - |S| - 1)$$

hyperedges, so $\mathcal{H}$ contains less than

$$\frac{r+1}{r}(n - 1 - |S|) + |S| < \frac{r+1}{r}(n - 1)$$

hyperedges, as desired.

(2) There is no cut-hyperedge in $\mathcal{H}$. Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are non-empty disjoint sets V_1 and V_2 such that $V(\mathcal{H}) = V_1 \cup V_2$, and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between V_1 and V_2. So both hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge cycle of length $r + 2$ or longer. By induction,

$$e(\mathcal{H}[V_1]) \leqslant \frac{r+1}{r}(|V_1| - 1) \quad \text{and} \quad e(\mathcal{H}[V_2]) \leqslant \frac{r+1}{r}(|V_2| - 1).$$

In total,

$$e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \leqslant \frac{r+1}{r}(|V_1| + |V_2| - 2) + 1 < \frac{r+1}{r}(|V(\mathcal{H})| - 1),$$

as desired.

Consider an auxiliary bipartite graph B consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ in the other class, and the edges of B are defined as follows: $xh \in E(B)$ if and only if the vertex x is contained in the hyperedge h.

Then property (1) shows that Hall's condition holds in B. Therefore, there is a perfect matching in B. In other words, there exists an injection $f : V(\mathcal{H}) \to E(\mathcal{H})$ such that $v \in f(v)$.

Given an injection $f : V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge path of the form

$$v_1 f(v_1) v_2 f(v_2) \cdots v_{l-1} f(v_{l-1}) v_l,$$

where for each $1 \leqslant i \leqslant l - 1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \to E(\mathcal{H})$ is an injection for which the path

$$\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \cdots v_{l-1} \phi(v_{l-1}) v_l$$

is a longest path.

Claim 3.1. $\phi(v_l) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$.

Proof. First note that if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \ldots, v_{l-r-1}\}$, then the Berge cycle

$$v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \cdots v_l \phi(v_l) v_i$$

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is of length $r+2$ or longer, a contradiction. Further, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then $\mathcal{P}_\phi$ can be extended to a longer path

$$v_1\phi(v_1)v_2\phi(v_2)\cdots v_{l-1}\phi(v_{l-1})v_l\phi(v_l)v,$$

a contradiction. This completes the proof of the claim. □

By Claim 3.1, we know that

$$\phi(v_l) = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\} \setminus \{v_j\}$$

for some $l-r \leqslant j \leqslant l-1$.

Claim 3.2. *For any $i \in \{l-r, l-r+1, \dots, l\} \setminus \{j\}$, we have*

$$\phi(v_i) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}.$$

Proof. When $i = l$, we know the statement is true. Suppose

$$i \in \{l-r, l-r+1, \dots, l-1\} \setminus \{j\}.$$

We define a new injection $\psi : V(\mathcal{H}) \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{v_1, v_2, \dots, v_l\}$ and for every $v \in \{v_1, v_2, \dots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \geqslant k \geqslant i+1$.

Now consider the Berge path

$$v_1\phi(v_1)v_2\phi(v_2)\cdots v_i\phi(v_l)v_l\phi(v_{l-1})\cdots v_{i+2}\phi(v_{i+1})v_{i+1},$$

equivalently

$$v_1\psi(v_1)v_2\psi(v_2)\cdots v_i\psi(v_i)v_l\psi(v_l)\cdots v_{i+2}\psi(v_{i+2})v_{i+1}.$$

This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, note that the sets of last $r+1$ vertices of both paths are the same. Thus we can apply Claim 3.1 to conclude that

$$\phi(v_i) = \psi(v_{i+1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\},$$

as desired. □

Claim 3.2 shows that there are r hyperedges (each of size r) contained in the set

$$S := \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$$

of size $r+1$. We will apply Lemma 2.1 to S.

Claim 3.3. *The set $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ induces a block of $\partial_2(\mathcal{H})$.*

Proof. Since the set $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ contains $r \geqslant 3$ hyperedges every pair $x, y \in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block D of $\partial_2(\mathcal{H})$ containing S.

Suppose D contains a vertex $t \notin S$. Then since D is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between t and S which are vertex-disjoint besides t. Let $V(P_1) \cap S = \{u\}$ and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v.

On the other hand, by Lemma 2.1, there is a Berge path $\mathcal{P}'$ of length r between u and v consisting of the r hyperedges contained in S. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in S, while hyperedges of $\mathcal{P}'$ are contained in S). Therefore, $\mathcal{P} \cup \mathcal{P}'$ forms a Berge cycle of length $r+2$ or longer unless $\mathcal{P}$ consists of only

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one hyperedge, say h. Note that h contains a vertex $x \notin S$ and $u, v \in h$; moreover by property (2), h is not a cut-hyperedge of $\mathcal{H}$. So after deleting h from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{h\}$ is still connected, so there is a (shortest) Berge path $\mathcal{Q}$ in $\mathcal{H} \setminus \{h\}$ between x and a vertex $s \in S$. (Note that the hyperedges of $\mathcal{Q}$ are not contained in S, and as it is a shortest Berge path, both u and v do not appear among the 'defining' vertices of $\mathcal{Q}$.) The vertex s is different from either u or v, say $s \neq u$, without loss of generality. By Lemma 2.1, there is a Berge path $\mathcal{Q}'$ of length r between s and u (consisting of hyperedges contained in S). Then $\mathcal{Q}$, $\mathcal{Q}'$ and h form a Berge cycle of length at least $r+2$, a contradiction. Therefore, D contains no vertex outside S; thus S induces a block of $\partial_2(\mathcal{H})$, as required. □

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 3.3 shows that one of these blocks, say D_1, is induced by S. Let us contract the vertices of S to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length $r+2$ or longer, as well; moreover $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$. Thus, by induction, we have

$$e(\mathcal{H}') \leqslant \frac{r+1}{r}(|V(\mathcal{H}')| - 1).$$

Therefore,

$$\begin{aligned} e(\mathcal{H}) &\leqslant \frac{r+1}{r}(|V(\mathcal{H}')| - 1) + (r+1) \\ &= \frac{r+1}{r}(|V(\mathcal{H})| - r - 1) + (r+1) \\ &= \frac{r+1}{r}(|V(\mathcal{H})| - 1). \end{aligned}$$

Now if

$$e(\mathcal{H}) = \frac{r+1}{r}(|V(\mathcal{H})| - 1),$$

then we must have

$$e(\mathcal{H}') = \frac{r+1}{r}(|V(\mathcal{H}')| - 1)$$

and S must contain all $r+1$ subsets of size r (*i.e.* $\mathcal{H}[S] = \mathcal{H}[D_1] = K^r_{r+1}$). Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected, and for each block D_i (with $2 \leqslant i \leqslant p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i] = K^r_{r+1}$. This means that for every block D of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D] = K^r_{r+1}$, completing the proof.

4. Proof of Theorem 1.5 ($k = r+1$)

The proof is similar to that of Theorem 1.4 but there are many important differences.

We use induction on n. For the base cases, note that the statement of the theorem is trivially true if $1 \leqslant n \leqslant r$. Moreover, if $n = r+1$ then $e(\mathcal{H}) \leqslant r$, because otherwise $\mathcal{H} = K^r_{r+1}$, and then it is easy to see that there is a (Hamiltonian) Berge cycle of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, $e(\mathcal{H}) \leqslant r = n-1$. Moreover, equality holds if and only if $\partial_2(\mathcal{H}) = K_{r+1}$ and $\mathcal{H}$ consists of r hyperedges.

We will show the statement is true for n assuming it is true for all smaller values. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices having no Berge cycle of length $r+1$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

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(1) For any set $S \subseteq V(\mathcal{H})$ with $|S| \leqslant |V(\mathcal{H})| - 1 = n - 1$, the number of hyperedges of $\mathcal{H}$ incident to the vertices of S is at least $|S|$. Indeed, suppose there is a set $S \subset V(\mathcal{H})$ (*i.e.* $|S| \leqslant |V(\mathcal{H})| - 1$) with fewer than $|S|$ hyperedges incident to the vertices of S. We delete the vertices of S from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n - |S|$ vertices. By induction, $\mathcal{H}'$ contains at most $(n - |S| - 1)$ hyperedges, so $\mathcal{H}$ contains less than $(n - 1 - |S|) + |S| = (n - 1)$ hyperedges, as required.

(2) There is no cut-hyperedge in $\mathcal{H}$. Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are disjoint non-empty sets V_1 and V_2 such that $V(\mathcal{H}) = V_1 \cup V_2$ and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between V_1 and V_2. So the hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge cycle of length $r + 1$ or longer. Therefore, by induction, $e(\mathcal{H}[V_1]) \leqslant |V_1| - 1$ and $e(\mathcal{H}[V_2]) \leqslant |V_2| - 1$. In total,

$$e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \leqslant (|V_1| + |V_2| - 2) + 1 = |V(\mathcal{H})| - 1,$$

as desired.

Moreover, we claim that the equality $e(\mathcal{H}) = |V(\mathcal{H})| - 1$ cannot hold in this case (*i.e.* if there is a cut-hyperedge). Indeed, if equality holds, then we must have $e(\mathcal{H}[V_1]) = |V_1| - 1$ and $e(\mathcal{H}[V_2]) = |V_2| - 1$. Note that since $r \geqslant 3$, the hyperedge h either contains at least two vertices $x, y \in V_1$ or two vertices $x, y \in V_2$. Without loss of generality, assume the former is true. By induction, $\partial_2(\mathcal{H}[V_1])$ is connected, and for every block D of $\partial_2(\mathcal{H}[V_1])$, we have $D = K_{r+1}$ and the subhypergraph induced by D consists of r hyperedges. So by Lemma 2.1, there is a Berge path of length r (consisting of the r hyperedges induced by D) between any two vertices of a block D. Then it is easy to see that since $\partial_2(\mathcal{H}[V_1])$ is connected, there is a Berge path $\mathcal{P}$ of length at least r between any two vertices of V_1, so in particular between x and y. Then $\mathcal{P}$ together with h forms a Berge cycle of length $r + 1$ in $\mathcal{H}$, a contradiction.

Consider an auxiliary bipartite graph B consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ on the other class. Then property (1) shows that Hall's condition holds for all subsets of $V(\mathcal{H})$ of size up to $|V(\mathcal{H})| - 1$. Therefore, there is a matching in B that matches all the vertices in $V(\mathcal{H})$, except at most one vertex, say x. In other words, there exists an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ such that for every $v \in V(\mathcal{H}) \setminus \{x\}$, we have $v \in f(v)$. Given an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge path of the form $v_1 f(v_1) v_2 f(v_2) \cdots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \leqslant i \leqslant l - 1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \cdots v_{l-1} \phi(v_{l-1}) v_l$ is a longest path.

Because of the way $\mathcal{P}_\phi$ was constructed, it is also clear that $x \notin \{v_1, v_2, \ldots, v_{l-1}\}$. We consider two cases depending on whether v_l is equal to x or not.

Case 1. $v_l \neq x$. Our aim is to get a contradiction, and show that this case is impossible.

Claim 4.1. *If $v_l \neq x$, then $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$.*

Proof. If $v_l \neq x$, then we claim $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$. Indeed, if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \ldots, v_{l-r}\}$, then the Berge cycle

$$v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \cdots v_l \phi(v_l) v_i$$

is of length $r + 1$ or longer, a contradiction. Further, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \ldots, v_l\}$, then P_ϕ can be extended to a longer path

$$v_1 \phi(v_1) v_2 \phi(v_2), \ldots, v_{l-1} \phi(v_{l-1}) v_l \phi(v_l) v,$$

again a contradiction, proving that $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$. □

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Fix some $i \in \{l-r+1, l-r+2, \ldots, l-1\}$. Let us define a new injection $\psi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{x, v_1, v_2, \ldots, v_l\}$, and for every $v \in \{v_1, v_2, \ldots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \geqslant k \geqslant i+1$. Now consider the Berge path

$$\begin{aligned} &v_1\phi(v_1)v_2\phi(v_2)\cdots v_i\phi(v_l)v_l\phi(v_{l-1})\cdots v_{i+2}\phi(v_{i+1})v_{i+1} \\ &\quad = v_1\psi(v_1)v_2\psi(v_2)\cdots v_i\psi(v_i)v_l\psi(v_l)\cdots v_{i+2}\psi(v_{i+2})v_{i+1}. \end{aligned}$$

This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, $v_{i+1} \neq x$, so we can apply Claim 4.1 to conclude that

$$\psi(v_{i+1}) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\} = \phi(v_i).$$

But then $\phi(v_i) = \phi(v_l)$, a contradiction to the fact that ϕ was an injection.

Case 2. $v_l = x$.

Claim 4.2. $\phi(v_{l-1}) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_l\}$.

Proof. If $\phi(v_{l-1})$ contains a vertex $v \notin \{v_1, v_2, \ldots, v_l\}$, then we consider the Berge path

$$v_1\phi(v_1)v_2\phi(v_2), \ldots, v_{l-1}\phi(v_{l-1})v.$$

Since $v \neq x$, we get a contradiction by Case 1. Moreover, if $\phi(v_{l-1})$ contains a vertex v_i with $i \in \{1, 2, \ldots, l-r-1\}$, then the Berge cycle

$$v_i\phi(v_i)v_{i+1}\phi(v_{i+1})\cdots v_{l-1}\phi(v_{l-1})v_i$$

is of length $r+1$ or longer, a contradiction. This finishes the proof of the claim. □

By Claim 4.2, we know that

$$\phi(v_{l-1}) = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\} \setminus \{v_j\}$$

for some j with $l-r \leqslant j \leqslant l-2$. In the rest of the proof we fix this j.

Claim 4.3. *For any $i \in \{l-r, l-r+1, \ldots, l-1\} \setminus \{j\}$, we have*

$$\phi(v_i) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}.$$

Proof. When $i = l-1$, we know the statement is true by Claim 4.2. Suppose

$$i \in \{l-r, l-r+1, \ldots, l-2\} \setminus \{j\}.$$

Let us define a new injection $\psi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{v_1, v_2, \ldots, v_l\}$, and for every $v \in \{v_1, v_2, \ldots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_{l-1})$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l-1 \geqslant k \geqslant i+1$. Now consider the Berge path

$$\begin{aligned} &v_1\phi(v_1)v_2\phi(v_2)\cdots v_i\phi(v_{l-1})v_{l-1}\phi(v_{l-2})\cdots v_{i+1} \\ &\quad = v_1\psi(v_1)v_2\psi(v_2)\cdots v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\cdots v_{i+1}. \end{aligned}$$

Note that when $i = l-2$, the Berge path is simply

$$v_1\phi(v_1)v_2\phi(v_2)\cdots v_i\phi(v_{l-1})v_{l-1} = v_1\psi(v_1)v_2\psi(v_2)\cdots v_i\psi(v_i)v_{l-1}.$$

If $\psi(v_{i+1})$ contains a vertex $v \notin \{v_1, v_2, \ldots, v_l\}$, then the Berge path

$$v_1\psi(v_1)v_2\psi(v_2)\cdots v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\cdots v_{i+2}\psi(v_{i+2})v_{i+1}\psi(v_{i+1})v$$

has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, since $v \neq x$, we get a contradiction by Case 1.

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If $\psi(v_{i+1})$ contains a vertex $v_k \in \{v_1, v_2, \dots, v_{l-r-1}\}$, then one can see that the Berge cycle

$$v_k \psi(v_k) v_{k+1} \psi(v_{k+1}) \cdots v_{l-1} \psi(v_{l-1}) v_k$$

is of length $r+1$ or longer, a contradiction. Therefore, we have

$$\psi(v_{i+1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_l\}.$$

But we defined $\psi(v_{i+1}) = \phi(v_i)$, proving the claim. □

Note that Claim 4.3 shows that $r-1$ hyperedges of $\mathcal{H}$ are contained in a set

$$S := \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$$

of size $r+1$. The following claim shows that if we can find one more hyperedge of $\mathcal{H}$ contained in S, then S must induce a block of $\partial_2(\mathcal{H})$.

Claim 4.4. *Suppose $r \geqslant 3$. If a set S of size $r+1$ contains r hyperedges of $\mathcal{H}$ then it induces a block of $\partial_2(\mathcal{H})$.*

Proof. Since the set S contains at least three hyperedges, every pair $x, y \in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block D of $\partial_2(\mathcal{H})$ containing S.

Suppose D contains a vertex $t \notin S$. Then since D is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between t and S which are vertex-disjoint besides t. Let $V(P_1) \cap S = \{u\}$ and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges

$$\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$$

forms a Berge path $\mathcal{P}$ between u and v.

On the other hand, by Lemma 2.1, there is a Berge path $\mathcal{P}'$ of length r between u and v consisting of the r hyperedges contained in S. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in S, while hyperedges of $\mathcal{P}'$ are contained in S). Therefore, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length $r+1$ or longer, a contradiction. Therefore, D contains no vertex outside S; thus S induces a block of $\partial_2(\mathcal{H})$, as required. □

We will use the above claim several times later. At this point we need to distinguish the cases $r = 3$ and $r \geqslant 4$, since Lemma 2.2 only applies in the latter case.

The case $r \geqslant 4$

Since $r \geqslant 4$, by Claim 4.3 and Lemma 2.2 there is a Berge path of length $r-1$ between any two vertices of $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$. This will allow us to show the following. (Recall that j is defined just before Claim 4.3.)

Claim 4.5. $\phi(v_j) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\} = S$.

Proof. Suppose for a contradiction that $\phi(v_j)$ contains a vertex $v \notin S$. The hyperedge $\phi(v_j)$ contains at least two vertices from S, namely v_j and v_{j+1}. By property (2), $\phi(v_j)$ is not a cut-hyperedge of $\mathcal{H}$. So after deleting $\phi(v_j)$ from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{\phi(v_j)\}$ is still connected; so there is a (shortest) Berge path $\mathcal{Q}$ in $\mathcal{H} \setminus \{\phi(v_j)\}$ between v and a vertex $s \in S$ (note that the hyperedges of $\mathcal{Q}$ are not contained in S). The vertex s is different from either v_j or v_{j+1}, say $s \neq v_j$, without loss of generality. By Lemma 2.2, there is a Berge path $\mathcal{Q}'$ of length $r-1$ between s and v_j (consisting of the hyperedges contained in S). Then $\mathcal{Q}$, $\mathcal{Q}'$ and $\phi(v_j)$ form a Berge cycle of length at least $r+1$ in $\mathcal{H}$, a contradiction. □

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Claim 4.3 and Claim 4.5 together show that there are at least r hyperedges of $\mathcal{H}$ contained in S. If all $r+1$ subsets of S of size r are hyperedges of $\mathcal{H}$, then S induces K^r_{r+1} and it is easy to show that it contains a Berge cycle of length $r+1$, a contradiction. This means S contains exactly r hyperedges of $\mathcal{H}$. Then by Claim 4.4, we know that S induces a block of $\partial_2(\mathcal{H})$.

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 4.4 shows that one of these blocks, say D_1, is induced by S. Let us contract the vertices of S to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length $r+1$ or longer, as well; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$ and $e(\mathcal{H}') = e(\mathcal{H}) - r$. By induction, we have $e(\mathcal{H}') \leqslant |V(\mathcal{H}')| - 1$. Therefore,

$$e(\mathcal{H}) = e(\mathcal{H}') + r \leqslant (|V(\mathcal{H}')| - 1) + r = (|V(\mathcal{H})| - r - 1) + r = |V(\mathcal{H})| - 1.$$

If $e(\mathcal{H}) = |V(\mathcal{H})| - 1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')| - 1$ and S must contain exactly r hyperedges. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected, and for each block D_i (with $2 \leqslant i \leqslant p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i]$ contains exactly r hyperedges. This means that for every block D of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D]$ contains exactly r hyperedges, completing the proof in the case $r \geqslant 4$. □

The case r = 3

Recall that using Claim 4.3 we can find a set S of size 4 which contains two hyperedges of $\mathcal{H}$. Let $S = \{x, y, a, b\}$ and the two hyperedges be xab and yab. By property (2), xab is not a cut-hyperedge of $\mathcal{H}$. So after deleting xab from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{xab\}$ is still connected; so there is a (shortest) Berge path $\mathcal{Q}$ between x and $\{y, a, b\}$. If $\mathcal{Q}$ is of length at least 2, then it is easy to see that $\mathcal{Q}$ together with yab and xab form a Berge cycle of length at least 4, a contradiction. So $\mathcal{Q}$ consists of only one hyperedge, say h.

Our goal is to find a set of vertices which induces a block of $\partial_2(\mathcal{H})$, so that we can apply induction.

If $|h \cap \{y, a, b\}| = 2$ then h, xab, yab are three hyperedges of $\mathcal{H}$ contained in S, so by Claim 4.4, we can conclude that S induces a block of $\partial_2(\mathcal{H})$. (Note that S contains exactly $|S| - 1 = 3$ hyperedges of $\mathcal{H}$, otherwise it is easy to find a Berge cycle of length 4; this will be useful later.) So we can suppose $|h \cap \{y, a, b\}| = 1$. We consider two cases depending on whether $h \in \{xat, xbt\}$, or whether $h = xyt$ for some $t \notin S$.

Case 1. First suppose without loss of generality that $h = xat$ for some $t \notin S$. Consider the set $\mathcal{D}$ of all hyperedges of $\mathcal{H}$ containing the pairs xa, ab or xb and let D be the set of vertices spanned by them. For each pair of vertices $i, j \in \{x, a, b\}$, let $V_{ij} = \{v \mid ijv \in \mathcal{H}\} \setminus \{x, a, b\}$. We claim that the sets V_{xa}, V_{ab}, V_{xb} are pairwise disjoint. Suppose for the sake of contradiction that $t' \in V_{xa} \cap V_{ab}$. Then the hyperedges xat', abt', xab are contained in a set of four vertices $\{x, a, b, t'\}$. Thus by Claim 4.4, this set induces a block of $\partial_2(\mathcal{H})$ and we are done. Thus we can suppose $V_{xa} \cap V_{ab} = \emptyset$. Similarly $V_{ab} \cap V_{xb} = \emptyset$ and $V_{xa} \cap V_{xb} = \emptyset$. This shows that $|D| = 3 + |V_{xa}| + |V_{xb}| + |V_{ab}|$. On the other hand, $\mathcal{D}$ consists of $1 + |V_{xa}| + |V_{xb}| + |V_{ab}|$ hyperedges, so $|\mathcal{D}| = |D| - 2$.

We will now show that D induces a block of $\partial_2(\mathcal{H})$. Let D' be a (maximal) block of $\partial_2(\mathcal{H})$ containing D and suppose for the sake of contradiction that it contains a vertex $p \notin D$. Then since D' is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between p and D which are vertex-disjoint besides p. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v. If $uv \notin \{xa, ab, xb\}$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 between u and v consisting of the hyperedges of $\mathcal{D}$. Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand, if $uv \in \{xa, ab, xb\}$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$,

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but then puv should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair uv); moreover, it is easy to check that between u and v there is a Berge path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, D' contains no vertex outside D; so D induces a block of $\partial_2(\mathcal{H})$ (which contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Case 2. Finally, suppose $h = xyt$ for some $t \notin S$. Let $\mathcal{D}$ be the set of all hyperedges of $\mathcal{H}$ containing the pair xy plus the hyperedges xab and yab, and let D be the set of vertices spanned by the hyperedges of $\mathcal{D}$. Let $V_{xy} = \{v \mid xyv \in \mathcal{H}\}$. We claim that $a \notin V_{xy}$ and $b \notin V_{xy}$. Indeed suppose for the sake of contradiction that $a \in V_{xy}$. Then the hyperedges xab, yab, xya are contained in a set of four vertices $\{x, y, a, b\}$. So by Claim 4.4, this set induces a block of $\partial_2(\mathcal{H})$, and we are done. So $a \notin V_{xy}$. Similarly, we can conclude $b \notin V_{xy}$. Therefore, $|D| = |V_{xy}| + 4$. On the other hand, $|\mathcal{D}| = |V_{xy}| + 2$, so $|\mathcal{D}| = |D| - 2$.

We claim that D induces a block of $\partial_2(\mathcal{H})$. The proof is very similar to that of Case 1, but we still give it for completeness. Let D' be a (maximal) block of $\partial_2(\mathcal{H})$ containing D and suppose for the sake of contradiction that it contains a vertex $p \notin D$. Then since D' is 2-connected, there are two paths P_1, P_2 in $\partial_2(\mathcal{H})$ between p and D which are vertex-disjoint besides p. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge h_{xy} of $\mathcal{H}$ containing xy. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between u and v.

If $uv \neq xy$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 or 4 between u and v consisting of the hyperedges of $\mathcal{D}$. (Indeed if $u, v \in V_{xy}$, then $\mathcal{P}'$ is of length 4, otherwise it is of length 3.) Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand if $uv = xy$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$ but then puv should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair uv); moreover, it is easy to check that between u and v there is a Berge path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, D' contains no vertex outside D; so D induces a block of $\partial_2(\mathcal{H})$ (and contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. In Cases 1 and 2 we showed that one of these blocks, say $D_1 = D$, is such that $\mathcal{H}[D_1]$ contains $|D_1| - 2$ hyperedges of $\mathcal{H}$, otherwise, D_1 is a set of four vertices such that $\mathcal{H}[D_1]$ contains exactly $|D_1| - 1 = 3$ hyperedges of $\mathcal{H}$. In all these cases, note that $e(\mathcal{H}[D_1]) \leqslant |D_1| - 1$.

Let us contract the vertices of D_1 to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length 4 or longer, as well; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - |D_1| + 1$ and $e(\mathcal{H}') = e(\mathcal{H}) - e(\mathcal{H}[D_1])$. By induction, we have $e(\mathcal{H}') \leqslant |V(\mathcal{H}')| - 1$. Therefore,

$$\begin{aligned} e(\mathcal{H}) &= e(\mathcal{H}') + e(\mathcal{H}[D_1]) \\ &\leqslant |V(\mathcal{H}')| - 1 + |D_1| - 1 \\ &= (|V(\mathcal{H})| - |D_1| + 1) - 1 + |D_1| - 1 \\ &= |V(\mathcal{H})| - 1. \end{aligned}$$

If $e(\mathcal{H}) = |V(\mathcal{H})| - 1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')| - 1$ and $\mathcal{H}[D_1]$ must contain exactly $|D_1| - 1$ hyperedges. As noted before, this is only possible if D_1 has four vertices and induces exactly three hyperedges of $\mathcal{H}$. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected, and for each block D_i (with $2 \leqslant i \leqslant p$) of $\partial_2(\mathcal{H}')$, $D_i = K_4$ and $\mathcal{H}'[D_i]$ contains exactly three hyperedges. This means that for every block D of $\partial_2(\mathcal{H})$ we have $D = K_4$ and $\mathcal{H}[D]$ contains exactly three hyperedges of $\mathcal{H}$, completing the proof in the case $r = 3$.

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