Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2016.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

100.E (Chris Starr)

Let $ABC\ldots PQR$ be a regular 18-sided polygon. Lines QF and RH intersect at S; PH and QI intersect at T and RL and QI intersect at U. Prove that $RSTU$ is a parallelogram.

100.F (Osman Akar)

Let a, b, c, d be positive real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\frac{2 + c(a-b)}{a+b+2c} + \frac{2 + d(b-c)}{b+c+2d} + \frac{2 + a(c-d)}{c+d+2a} + \frac{2 + b(d-a)}{d+a+2b} \geqslant 2.$$

100.G (Stan Dolan)

Find, up to similarity, all triangles such that each angle (in degrees) and the square of each side are whole numbers.

100.H (M. N. Deshpande)

(a) For every integer $n \geqslant 2$, consider the isosceles triangle ABC in which $AB = AC = n^2 + n + 1$ and $BC = 2n^2 + 2n - 1$. Let D be on BC with $\angle ADB = 60°$. Show that triangle ABD has integer side-lengths.

(b) Find another example of a family of isosceles triangles in which AB and BC are quadratic functions of n which have the same property as (a).

Solutions and comments on **99.I**, **99.J**, **99.K**, **99.L** (November 2015).

99.I (Stan Dolan)

If x and y are integers such that $x^4 + x^2y^2 - y^4$ is a square, then prove that $x^4 + x^2y^2 - y^4$ is a fourth power.

Stan Dolan, the proposer of this intriguing number theory problem, gave the following ingenious solution using the method of descent.

We shall prove the stronger result that, if there are integers x, y and z such that either $x^4 + x^2y^2 - y^4 = z^2$ or $x^4 - 2x^2y^2 + 5y^4 = z^2$ then $|y| = 0$ or $|y| = |x|$.

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Suppose that we have a counterexample with $|y|$ minimal. Without loss of generality, we can assume that x, y and z are positive and pair-wise coprime. There are three cases:

(a) If $x^4 - 2x^2y^2 + 5y^4 = z^2$

Then $(x^2 - y^2 - z)(x^2 - y^2 + z) = -4y^4$. Let n be the g.c.d. of $x^2 - y^2 - z$ and $x^2 - y^2 + z$. Then n is an even factor of $2z$ such that n^2 divides $4y^4$. Hence, since y and z are coprime, $n = 2$.

Therefore $y = uv$ where u and v are coprime positive integers such that $\{x^2 - y^2 - z, x^2 - y^2 + z\} = \{2u^4, -2v^4\}$. Then $x^2 - y^2 = u^4 - v^4$ and so $u^4 + u^2v^2 - v^4 = x^2$. Now $v = \dfrac{y}{u} \leqslant y$ and so either $u = 1$ or $v < y$. In the latter case, minimality of y forces $u = v$ and then, by coprimeness, $u = v = 1$. So u is equal to 1 and therefore $1 + y^2 - y^4 = x^2$. However, $y^2 \leqslant y^4$ and so $x = y = 1$, a contradiction.

(b) If $x^4 + x^2y^2 - y^4 = z^2$, with y even

Let $y = 2t$. Then x and z are odd and $(x^2 + 2t^2 - z)(x^2 + 2t^2 + z) = 20t^4$. Let n be the g.c.d. of $x^2 + 2t^2 - z$ and $x^2 + 2t^2 + z$. Then n is an even factor of $2z$ such that n^2 divides $20t^4$. Hence, since $2t$ and z are coprime, $n = 2$.

Therefore $t = uv$ where u and v are coprime positive integers such that $\{x^2 + 2t^2 - z, x^2 + 2t^2 + z\} = \{2u^4, 10v^4\}$. Then $x^2 + 2t^2 = u^4 + 5v^4$ and so $u^4 - 2u^2v^2 + 5v^4 = x^2$. Now $v = \dfrac{y}{2u} < y$ and so, again, minimality and coprimeness force $u = v = 1$. Therefore $x = y = 2$, a contradiction.

(c) If $x^4 + x^2y^2 - y^4 = z^2$, with y odd

Then $(2x^2 + y^2 - 2z)(2x^2 + y^2 + 2z) = 5y^4$. Let n be the g.c.d. of $2x^2 + y^2 - 2z$ and $2x^2 + y^2 + 2z$. Then n is an odd factor of $4z$ such that n^2 divides $5y^4$. Hence, since y and z are coprime, $n = 1$.

Therefore $y = uv$ where u and v are coprime positive integers such that $\{2x^2 + y^2 - 2z, 2x^2 + y^2 + 2z\} = \{u^4, 5v^4\}$. Then $4x^2 + 2y^2 = u^4 + 5v^4$ and so $u^4 - 2u^2v^2 + 5v^4 = 4x^2$. Now $v = \dfrac{y}{u} \leqslant y$ and so, again, minimality and coprimeness force $u = 1$. Then $1 - 2y^2 + 5y^4 = 4x^2$, an equation for which we have already obtained a contradiction in part (a).

James Mundie resolved **99.I** by using elliptic curves. The substitutions $\dfrac{x^2}{y^2} = \dfrac{q - 3}{9}$, $\dfrac{xz}{y^3} = \dfrac{v}{27}$ transform the equation $x^4 + x^2y^2 - y^4 = z^2$ into the elliptic curve $v^2 = q^3 - 108q + 297$ (in its standard Weierstrass form). There are 6 rational points on this curve generated by the solution $(-6, 27)$; these correspond to the solutions noted above. The fact that $y = 0$ or $y = \pm x$ has to hold in this problem was also hinted at in comments made by Ángel Plaza and Lienhard Wimmer.

Correct solutions were received from: J. A. Mundie and the proposer Stan Dolan.

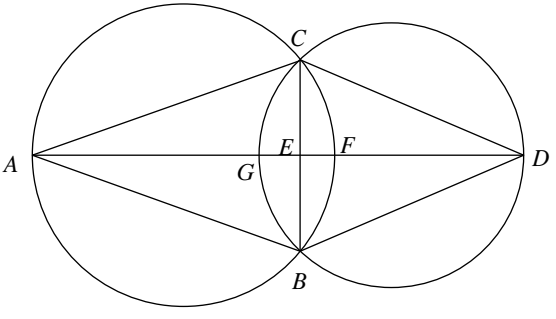
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99.J (M. N. Deshpande)

Triangle ABC is isosceles with $AB = AC$ and centroid G. Let D be the point (other than G) on the circumcircle of triangle BGC such that $BD = CD$. Show that the centroid of triangle BDC lies on the circumcircle of triangle ABC.

The most common solutions to this problem went along the following succinct lines.

In the Figure, let E be the midpoint of BC and F the point where AD meets the circumcircle of triangle ABC.



FIGURE

Then, by the intersecting chords theorem in each circle,

$$AE.EF = BE.EC = GE.ED$$

from which $\frac{GE}{AE} = \frac{EF}{ED}$. But $\frac{GE}{AE} = \frac{1}{3}$ since G is the centroid of triangle ABC, thus $\frac{EF}{ED} = \frac{1}{3}$ and F is indeed the centroid of triangle BDC.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, S. Dolan, M. G. Elliott, M. Fox, GCHQ Problem Solving Group, G. Howlett, J. A. Mundie, G. Strickland, A. Tee, A. J. W. Thorn, G. B. Trustrum, L. Wimmer and the proposer M. N. Deshpande.

99.K (Michael Fox)

The vertices B and C of a given triangle ABC are the centres of two rectangular hyperbolas whose respective asymptotes bisect the angles $\angle ABC$ and $\angle ACB$. Given that the curves touch at the point D and intersect at E and F prove that:

(i) D is concyclic with A, B and C;

(ii) EF is parallel to the Simson line of D with respect to triangle ABC.

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Although only four solutions were received, they were of high quality and any one could have been selected for publication. In the end, I opted for the solution provided by the proposer, Michael Fox, with his additional comments on the geometry of the configuration involved.

(i) *Lemma*: If the tangent at a point P on a rectangular hyperbola with centre O meets the asymptotes in M and N, then $PO = PM = PN$.

For the tangent at the point $P(t, 1/t)$ on $xy = 1$ is $x + t^2y = 2t$, meeting the asymptotes $x = 0$ and $y = 0$ in $M(2t, 0)$ and $N(0, 2/t)$. So the displacements **PO**, **PM** and **PN** are $\begin{bmatrix} -t \\ -1/t \end{bmatrix}$, $\begin{bmatrix} t \\ -1/t \end{bmatrix}$ and $\begin{bmatrix} -t \\ 1/t \end{bmatrix}$, which are clearly equal in magnitude.

In Figure 1, suppose that the common tangent at D meets BC at L and the asymptote BI at J. In ΔBLJ let the external angle at L be θ, and $\angle LBJ$ be β, where $\beta = \frac{1}{2}B$; then $\angle BJL = \theta - \beta$. Since, by the lemma, $DJ = DB$, we see that ΔDJB is isosceles, thus $\angle DBJ = \theta - \beta$. Now $\angle IBA = \beta$, so $\angle DBA = \theta$. Consequently $\angle DBA = \angle BLD = \theta$.

If the tangent at D meets the asymptote CI at K, and the external angle at L of ΔKLC is φ, a similar argument shows that $\angle ACD = \angle DLC = \varphi$.

Hence $\angle DBA + \angle ACD = \angle BLD + \angle DLC = \theta + \varphi = 180°$, so A, B, C and D are concyclic.

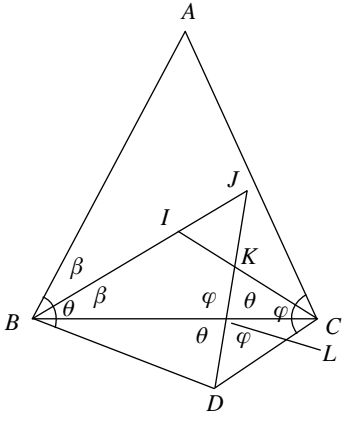


FIGURE 1

(ii) We tackle this part by proving that the tangent at D is perpendicular to EF and the Simson line of D.

Lemma: All rectangular hyperbolas that pass through the vertices of a given triangle also pass through its orthocentre.

For let the triangle be UVW, and consider a rectangular hyperbola that passes through them. Take its asymptotes as axes, with the curve lying in the first and third quadrants. We take its equation as $xy = 1$, and the

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vertices as $U\left(u,\ 1/u\right)$, $V\left(v,\ 1/v\right)$ and $W\left(w,\ 1/w\right)$. The gradient of VW is $-1/(vw)$; hence the gradient of the altitude through U is vw. If this altitude cuts the hyperbola again in $H\left(h,\ 1/h\right)$, then the gradient of UH is $-1/(uh)$; equating these gradients shows that $h = -1/(uvw)$. By the symmetry of this result, the point H is also on the altitudes through V and W, and is the orthocentre of ΔUVW. The position of the orthocentre is independent of the coordinate system, so all rectangular hyperbolas through U, V and W will pass through H.

In Figure 2, let the feet of the perpendiculars from D to BC, CA and AB be P, Q and R respectively, so PRQ is the Simson line of D. Quadrilateral $DPRB$ is cyclic, so $\angle DPQ = \angle DBR = \theta$. Let PQ cut DL at S. Then $\angle DPS = \angle DLP = \theta$, and triangles PDS, LDP are equiangular, thus $PS \perp DL$.

Now the two hyperbolas pass through D, E and F, and so pass through the orthocentre H of ΔDEF. But since the curves have double contact at D, this point must coincide with H. The relationship $DE \perp FH$ becomes $DE \perp FD$, so ΔDEF is right-angled at D. The common tangent is the limiting position of DH as $H \to D$, hence it is perpendicular to EF. Since both the Simson line and EF are perpendicular to this tangent. they are parallel.

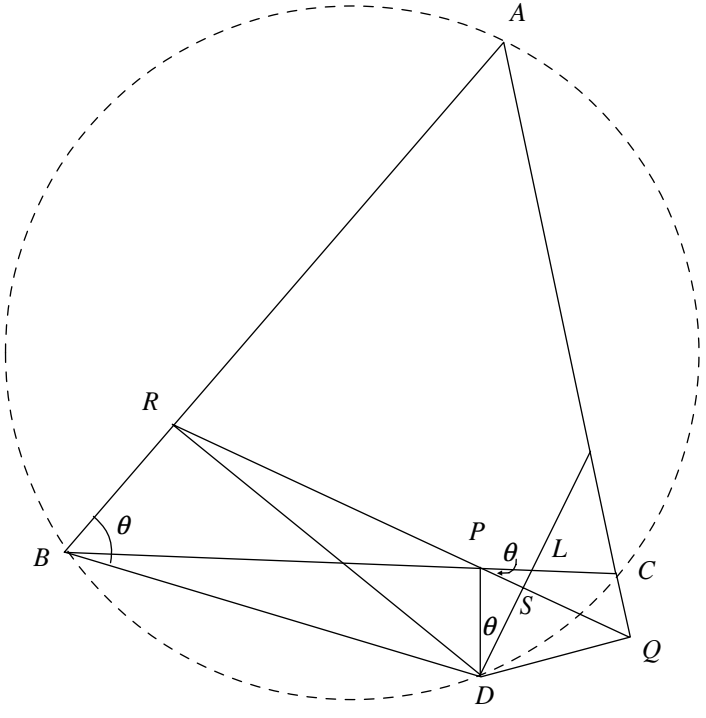


FIGURE 2

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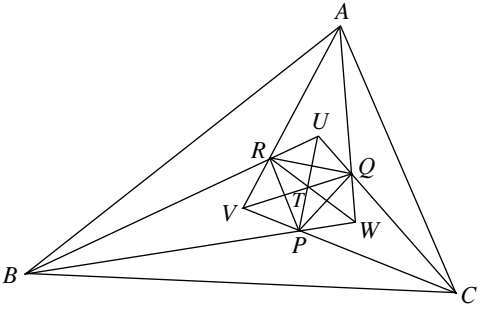
Notes: We can find D given A, B, C and the direction l of the common tangent: let a line through I parallel to l cut the asymptote BI' in J and CI' in J'. Then let K and K' be the midpoints of IJ and IJ' respectively. Then BK and CK' meet in the required position of D.

The diagram is very rich in results. The circumcircle of ΔABC is the nine-point circle of ΔDEF; the midpoint of EF lies on this circle and is diametrically opposite D, and the common tangent at D meets EF at a point on this circle. If D moves round the circle ABC then the common tangent sweeps out a deltoid, and the sides of ΔDEF always touch this curve as they move.

Correct solutions were received from: S. Dolan, M. G. Elliott, G. Howlett and the proposer Michael Fox.

99.L (Isaac Sofair)

In triangle ABC, the lines ARV and AQW trisect $\angle A$, the lines BPW and BRU trisect $\angle B$ and the lines CQU and CPV trisect $\angle C$, as shown in the Figure. It is well known that PQR is an equilateral triangle (the *Morley triangle*); it is also known that PU, QV and RW are concurrent at T.



Show that:

(i) Area UVW : Area PQR =

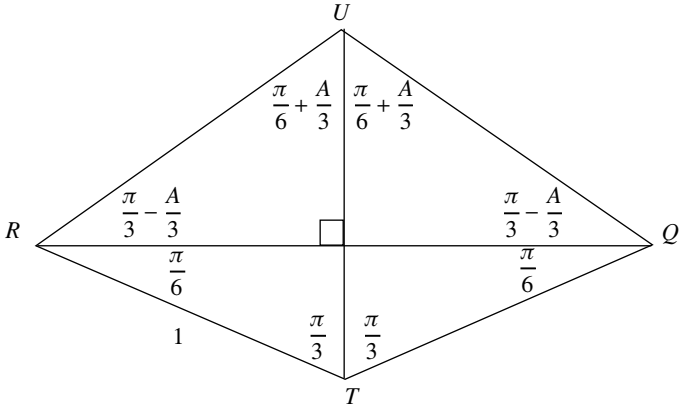
$$\frac{1}{8}+\frac{1}{\sqrt{3}}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}:\cos\frac{\pi-A}{3}\cos\frac{\pi-B}{3}\cos\frac{\pi-C}{3};$$

(ii) $$UV:VW:WU=\cos\frac{\pi-C}{3}\sqrt{1+8\cos\frac{C}{3}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}}:$$

$$\cos\frac{\pi-A}{3}\sqrt{1+8\cos\frac{A}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}}:\cos\frac{\pi-B}{3}\sqrt{1+8\cos\frac{B}{3}\sin\frac{\pi+C}{3}\sin\frac{\pi+A}{3}}.$$

Most solvers started by reprising the properties of the Morley configuration, leading to the Figure below (with analogous versions for $QWPT$ and $PVRT$).

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FIGURE

With $RT = QT = PT = 1$ we have, by the sine rule

$$UR = \frac{\sqrt{3}}{2\sin\left(\frac{\pi}{6}+\frac{A}{3}\right)} = \frac{\sqrt{3}}{2\cos\left(\frac{\pi}{3}-\frac{A}{3}\right)}$$

and

$$\frac{UT}{\sin\left(\frac{\pi}{2}-\frac{A}{3}\right)} = \frac{UR}{\sin\frac{\pi}{3}} \qquad \text{or} \qquad UT = \frac{\cos\frac{A}{3}}{\cos\left(\frac{\pi}{3}-\frac{A}{3}\right)}.$$

Stan Dolan, whose solution follows, then skilfully made repeated use of the trigonometrical identity:

$$4\cos x\cos y\cos z = 2\cos x\left[\cos(y+z)+\cos(y-z)\right]$$

$$= \cos(x+y+z)+\cos(-x+y+z)+\cos(x-y+z)+\cos(x+y-z). \quad (*)$$

(i) Using the expressions above for UT and VT,

$$\left(4\cos\frac{\pi-A}{3}\cos\frac{\pi-B}{3}\cos\frac{\pi-C}{3}\right)UT.VT$$

$$= 4\cos\frac{A}{3}\cos\frac{B}{3}\cos\frac{\pi-C}{3}$$

$$= 1+\cos\frac{2A}{3}+\cos\frac{2B}{3}+\cos\frac{2\pi-2C}{3} \qquad \text{by (*)}$$

$$= 1+\cos\frac{2A}{3}+\cos\frac{2B}{3}-2\cos\frac{2C}{3}+\sqrt{3}\cos\frac{4C-\pi}{6}.$$

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Summing for $UT.VT$, $VT.WT$ and $WT.UT$ gives

$$\begin{aligned}
&4\cos\frac{\pi-A}{3}\cos\frac{\pi-B}{3}\cos\frac{\pi-C}{3}(UT.VT+VT.WT+WT.UT)\\
&=\frac{3}{2}+\sqrt{3}\left(\cos\frac{\pi}{6}+\cos\frac{4A-\pi}{6}+\cos\frac{4B-\pi}{6}+\cos\frac{4C-\pi}{6}\right)\\
&=\frac{3}{2}+4\sqrt{3}\cos\frac{\pi-2A}{6}\cos\frac{\pi-2B}{6}\cos\frac{\pi-2C}{6} \qquad \text{by (*)}\\
&=\frac{3}{2}+4\sqrt{3}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\text{Area}\,UVW}{\text{Area}\,PQR} &= \frac{UT.VT\ +\ VT.WT\ +\ WT.UT}{3}\\
&= \frac{\frac{1}{8}+\frac{1}{\sqrt{3}}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}}{\cos\frac{\pi-A}{3}\cos\frac{\pi-B}{3}\cos\frac{\pi-C}{3}}.
\end{aligned}$$

(ii) Using the expressions above for UR and VR in the cosine rule for triangle URV,

$$UV^2 = UR^2 + RV^2 + 2UR.RV\cos\frac{\pi-C}{3},$$

we obtain

$$\begin{aligned}
&\frac{8}{3}\cos^2\frac{\pi-A}{3}\cos^2\frac{\pi-B}{3}UV^2\\
&=2\cos^2\frac{\pi-A}{3}+2\cos^2\frac{\pi-B}{3}+4\cos\frac{\pi-A}{3}\cos\frac{\pi-B}{3}\cos\frac{\pi-C}{3}\\
&=2+\cos\frac{2\pi-2A}{3}+\cos\frac{2\pi-2B}{3}+\cos\frac{2\pi}{3}+\cos\frac{2A}{3}+\cos\frac{2B}{3}+\cos\frac{2C}{3} \quad \text{by (*)}\\
&=\frac{1}{2}+\cos 0+\cos\frac{\pi-2A}{3}+\cos\frac{\pi-2B}{3}+\cos\frac{2C}{3}\\
&=\frac{1}{2}+4\cos\frac{\pi-2A}{3}\cos\frac{\pi-2B}{3}\cos\frac{C}{3} \qquad \text{by (*)}\\
&=\frac{1}{2}\left(1+8\cos\frac{C}{3}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\right).
\end{aligned}$$

Therefore

$$UV \propto \cos\frac{\pi-C}{3}\sqrt{1+8\cos\frac{C}{3}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}},$$

as required.

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10.1017/mag.2016.82 N.J.L.

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