Ergod. Th. & Dynam. Sys. (2013), **33**, 1891–1928 © *Cambridge University Press, 2013*
doi:10.1017/etds.2013.38

Bounded density shifts

BRETT STANLEY

School of Mathematics and Statistics, La Trobe University, Bundoora, Melbourne, Victoria 3083, Australia
(e-mail: bnstanley@students.latrobe.edu.au)

(*Received* 30 *July* 2012 *and accepted in revised form* 22 *April* 2013)

Abstract. We consider subshifts of the full shift of bi-infinite sequences with alphabet $\{0, 1, \ldots, n-1\}$ defined by not allowing the sum of finite words to exceed a value depending on its length. These shifts we call bounded density shifts. We study these shifts in detail and make a comparison on the similarities to but also differences from the well-known β-shifts.

1. *Introduction*

A subshift is a closed, invariant subset of $A^{\mathbb{Z}}$, where A is a finite alphabet. By giving each element of A a different weight from 0 to $n-1$ (where $|A| = n$), we can construct a subshift by requiring each subword of each element of the shift to have a maximum density dependent on its length. In this paper we provide a study of subshifts defined by this property. For simplicity, we assume that the alphabet is $\{0, 1, \ldots, n-1\}$ instead of having a bijective map between A and $\{0, 1, \ldots, n-1\}$. Each function $f : \mathbb{N}_0 \to [0, \infty)$ generates a bounded density shift Ψ_f, where $f(p)$ is the bound on the density of subwords of length p. Although these functions are fairly arbitrary, each bounded density shift has a canonical function which takes the minimum value possible for each point of the domain. The set of all canonical functions can be described as the set of all weakly increasing, sub-additive functions mapping to integers with 0 mapping to 0 (Theorem 2.9). Each shift contains a characteristic sequence which is maximal in Ψ_f with respect to the order $\preceq$ defined by $x_1x_2 \ldots \preceq y_1y_2 \ldots$ if $\sum_{r=1}^{m} x_i \leq \sum_{r=1}^{m} y_i$ for all $m \in \mathbb{N}$. The maximal sequences can be characterized in the same way as β-shifts but for a different order (Theorems 2.10, 1.12). Each canonical function determines a characteristic sequence and vice versa. It turns out that the set of characteristic sequences for transitive bounded density shifts is a proper subset of the set of maximal sequences of β-shifts. Additionally, Fekete's lemma may be used to prove that $\lim_{n\to\infty} f(n)/n$ exists. We call this the limiting gradient.

Similar to β-shifts, every transitive bounded density shift is mixing. There are also characterizations for various subclasses of shifts. The following theorems summarize the main results in this paper.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

THEOREM 1.1. *Let Ψ_f be a non-trivial bounded density shift and let f be canonical. The following are equivalent.*

(1) *Ψ_f has a dense set of periodic points.*
(2) *There exists a non-trivial periodic point in Ψ_f.*
(3) *f is bounded below by a strictly increasing linear function.*
(4) *The limiting gradient is positive.*
(5) *Ψ_f is coded.*

THEOREM 1.2. *Let Ψ_f be a bounded density shift and let f be canonical. The following are equivalent.*

(1) *Ψ_f is of finite type.*
(2) *f is a maximally extended canonical function.*
(3) $\min\{f(m)/m : m \in \mathbb{N}\}$ *exists.*
(4) $(\exists p \in \mathbb{N})(\forall m \in \mathbb{N})\, f(mp) = mf(p)$.
(5) $(\exists N \in \mathbb{N}_0)(\exists p \in \mathbb{N})(\forall m \geq N)\ f(m+p) = f(m) + f(p)$.

THEOREM 1.3. *Let Ψ_f be a bounded density shift, let f be canonical and let $M \in \mathbb{N}$. Then the following are equivalent.*

(1) *Ψ_f is specified with specification constant M.*
(2) $(\forall m,\ p \in \mathbb{N}_0)\ f(m) + f(p) \leq f(m+p+M)$.
(3) *0^M is intrinsically synchronizing.*

Additionally, Ψ_f is specified if and only if Ψ_f is transitive and f is bounded above by an affine function whose gradient is the limiting gradient.

THEOREM 1.4. *Let Ψ_f be a bounded density shift and let f be canonical. The following are equivalent.*

(1) *Ψ_f is sofic.*
(2) $(\exists p \in \mathbb{N})(\exists N, k \in \mathbb{N}_0)(\forall m \geq N)\ f(m+p) = f(m) + k$.
(3) *The characteristic sequence is eventually periodic.*

THEOREM 1.5. *Let Ψ_f be a non-trivial bounded density shift and let f be canonical. Then the following are equivalent.*

(1) *The characteristic sequence does not contain arbitrarily long subwords consisting entirely of 0s.*
(2) *The characteristic sequence is presented in some strongly connected presentation of Ψ_f.*
(3) *The left characteristic sequence is presented in some strongly connected presentation of Ψ_f.*

Figure 7 shows the various containments of the main subclasses. Unfortunately, there is no characterization of synchronized bounded density shifts. We also find upper and lower bounds for the entropy of coded bounded density shifts.

In §2, we characterize the set of canonical functions and characteristic sequences of bounded density shifts. In §§3–7, we prove Theorems 1.1–1.5. In §8, we link together canonical forms for bounded density shifts and β-shifts to give upper and lower bounds for the entropy of a bounded density shift.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

1.1. *Definitions.* In this paper, we assume that $\mathbb{N}$ is the set of positive integers and $\mathbb{N}_0$ is the set of non-negative integers. We also use the notation $[m..n] = \{m, m+1, \ldots, n\}$ and $(m..n] = \{m+1, m+2, \ldots, n\}$ and the other variants for sets of consecutive integers. Note that $[m, n]$ as a subscript of a word or string represents the integers between m and n rather than all real numbers.

Instead of the common definitions for (topological) *transitivity* and *mixing*, we use the equivalences in the following lemma.

LEMMA 1.6. *Let X be a shift. The following are equivalent.*

(1) *X is transitive if and only if for all $u, w \in L(X)$, there is a v such that $uvw \in L(X)$.*

(2) *X is mixing if and only if $\{|v| : uvw \in L(X)\}$ is co-finite in $\mathbb{N}_0$ for all $u, v \in L(X)$.*

It is obvious that mixing implies transitivity.

For the basics on alphabets and languages, see [**3**].

Let A be a finite alphabet. We endow A with the discrete topology and give $A^{\mathbb{Z}}$ the product topology. An element $x \in A^{\mathbb{Z}}$ is written $\ldots x_{-2}x_{-1}.x_0x_1x_2 \ldots$. A topology equivalent to the product topology on $A^{\mathbb{Z}}$ is given by the cylinder set basis $\mathcal{B} = \{C_i[w] : w \in A^*\}$, where the cylinders are $C_i[w] = \{x \in A^{\mathbb{Z}} : x_{[i,i+|w|)} = w\}$. The space $A^{\mathbb{Z}}$ is also a metric space with respect to this topology. The *shift* map σ on $A^{\mathbb{Z}}$ is defined by $\sigma(x) = \ldots x_{-1}x_0.x_1x_2x_3 \ldots$. A set $X \subseteq A^{\mathbb{Z}}$ is a *subshift* if it is closed and invariant under the shift map. For any subshift $X \subseteq A^{\mathbb{Z}}$, let $L(X)$ be the set of subwords of elements of X and let $L_k(X)$ be the subwords of length k for $k \geq 0$. We use the notation $x_{[i,\infty)}$ to indicate $x_ix_{i+1}x_{i+2} \ldots$ and similar notation for words. The notation w^∞ denotes an infinite concatenation of w's to the right. Similarly, ${}^\infty w$ denotes an infinite concatenation of w's to the left. The *reverse* of a word $w = w_1w_2 \ldots w_n$ is $w^R = w_n \ldots w_2w_1$.

A similar notion is to consider one-way infinite sequences of A indexed by $\mathbb{N}$. The notation and theory is very similar to the above two-way version. We use $C^+[w]$ to represent the set of sequences in $A^{\mathbb{N}}$ starting with w and a suitable basis is $\mathcal{B} = \{C^+[w] : w \in A^*\}$. The shift map is defined by $\sigma(x) = x_2x_3x_4 \ldots$ and the one-sided subshift with the same language as a two-sided subshift X is denoted by X^+. The *reverse* of a one-sided sequence $x = x_1x_2x_3 \ldots$ is defined as $x^R = \ldots x_3x_2x_1$. We use the index set $-\mathbb{N}$ for left-infinite sequences, so, for example, $x^R_{-2} = x_2$. The reverse of a two-sided sequence is defined in a similar way.

Finite state automata and presentations of shifts are mentioned throughout. For background, see [**3**]. The source, labelling and target maps of paths in presentations are s, λ and t respectively.

A *shift of finite type*, or SFT, is a subshift containing all bi-infinite sequences that do not contain a subword in a finite set of words. A shift is *sofic* if it is the semi-conjugate image of some shift of finite type. A shift is sofic if and only if its language is regular. A shift is sofic if and only if there are only finitely many follower sets of words in its language. A sofic shift is called *strictly sofic* if it is not of finite type. A subshift is *coded* if its language can be written as $(L^*)^{\leq_s}$ for some language L, where $(L^*)^{\leq_s}$ is the subword closure of L^*. Alternatively, a shift is coded if and only if there is a strongly connected presentation of itself. A word $v \in L(X)$ is *intrinsically synchronizing* if $uv, vw \in L(X)$ imply $uvw \in L(X)$. It is well known that if v is intrinsically synchronizing, then uvw is

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

intrinsically synchronizing. A subshift is *synchronized* if it is coded and has an intrinsically synchronizing word. A subshift X is *specified* if there is an $M \in \mathbb{N}$ such that for all $u, w \in L(X)$, there is a $v \in L_M(X)$ such that $uvw \in L(X)$. Every specified shift is mixing. Every mixing, sofic shift is specified and every specified shift is synchronized. The following describes the inclusions for mixing shifts:

$$\text{SFT} \subset \text{sofic} \subset \text{specified} \subset \text{synchronized} \subset \text{coded}.$$

Another important quantity of any subshift is the *entropy*. For any shift X, this is defined by

$$h(X) := \lim_{n\to\infty} \frac{\log_2(|L_n(X)|)}{n}.$$

The entropy of a subshift always exists and is finite [**3**].

For a shift X, let $\mathcal{F} = \{w \notin X : v \leq_s w \text{ and } v \neq w \implies v \in L(X)\}$. We call an element of $\mathcal{F}$ a minimal forbidden word.

LEMMA 1.7. *Let X be a shift and let $\mathcal{F}$ be a set of forbidden words in X containing the set of minimal forbidden words. Then $\mathcal{F}$ is a set of forbidden words generating X.*

LEMMA 1.8. *Let X be a shift and let $\mathcal{F}$ be a set of forbidden words in X containing the minimal forbidden words. If $\mathcal{F}$ is regular, then X is sofic.*

Proof. Lemma 1.7 shows that $\mathcal{F}$ is a list of forbidden words for X. Assume $\mathcal{F}$ is regular. We show that $L(X) = A^*\backslash A^*\mathcal{F}A^*$. If $w \in L(X)$, then every subword of w is not in $\mathcal{F}$. This gives $w \notin A^*\mathcal{F}A^*$, so $L(X) \subseteq A^*\backslash A^*\mathcal{F}A^*$. For the converse, let $w \in A^*$ such that $w \notin L(X)$. Then there is a subword v of w such that every proper subword of v is in $L(X)$. This gives $v \in \mathcal{F}$ upon which $w \in A^*\mathcal{F}A^*$. Hence $L(X) = A^*\backslash A^*\mathcal{F}A^*$. Since the regular languages are closed under concatenation and complementation, $L(X)$ is regular. Thus X is sofic. □

1.2. *β-shifts.* Let $\beta > 1$ be a real number and $x \in [0, 1]$. A representation of x in base β or a β-representation of x is a one-sided infinite sequence $y_{[1,\infty)}$ such that

$$x = \sum_{i=1}^{\infty} \frac{y_i}{\beta^i}.$$

There can be many representations of the same number. A β-expansion is a canonical β-representation defined by

$$y_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$

for each $i \in \mathbb{N}$, where $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$. The β-expansion of 1 defined this way is usually represented by s and is important to the theory of β-shifts. The theory of β-expansions was developed in [**4, 6**]. Let D_β be the β-expansion of all elements of $[0, 1)$. The one-sided β-shift S_β^+ is the closure of D_β. The two-sided shift S_β is the extension of S_β^+ to two-sided infinite sequences. The alphabet of any β-shift is $[0..\lceil \beta - 1 \rceil]$. Further information on β-shifts can be found in [**1, 2, 5, 7**].

The order $\leq$ represents the lexicographic order.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

We criticize the following theorem.

THEOREM 1.9. [**1, 4**] *A sequence s is a β-expansion of* 1 *if and only if $\sigma^n(s) < s$ for all $n \in \mathbb{N}$.*

When $s = 10^\infty$, note that $\sigma^n(s) < s$ for all $n \in \mathbb{N}$ yet the β corresponding to s is 1. It is implicitly assumed that $\beta > 1$ since base 1 makes no sense. However, a β-shift can still be constructed for $\beta = 1$ and it gives the one-sided shift $\{0^\infty\}$. In the following, we will include this case since it corresponds with a bounded density shift so the previous theorem will be taken to be correct.

THEOREM 1.10. [**1, 4**] *Let $\beta \geq 1$, let s be the β-expansion of* 1 *and let $y \in [0..\lceil \beta - 1 \rceil]^{\mathbb{N}}$. Then y is a β-expansion of some element of* $[0, 1)$ *if and only if $\sigma^n(y) < s$ for all $n \in \mathbb{N}_0$.*

Note the distinction in the previous two theorems over the values of n.

Every β-shift has a maximum element with respect to the lexicographic ordering. We restate a couple of theorems in terms of the maximum element instead of the β-expansion.

A β-expansion is said to be *finite* if it has only a finite number of non-zero characters.

LEMMA 1.11. *Let S_β be a β-shift and let s be the β-expansion of* 1.

(1) *If $s = w_{[1,n]}0^\infty$ is a finite expansion with $w_n \neq 0$, then the maximum element of S_β is $(w_{[1,n)}(w_n - 1))^\infty$.*
(2) *If s is not a finite expansion, then the maximum element is s.*

THEOREM 1.12. *A sequence x is the maximum of some β-shift if and only if*

$$(\forall n \in \mathbb{N})\ \sigma^n(x) \leq x$$

and x has infinitely many non-zeros.

THEOREM 1.13. (Essentially in [**1, 5**]) *Let S_β be a β-shift and let x be the maximum in S_β^+. Then $y \in S_\beta^+$ if and only if $\sigma^n(y) \leq x$ for all $n \in \mathbb{N}_0$.*

THEOREM 1.14. [**1**] *Let s be the β-expansion of* 1.

(1) *S_β is a subshift of finite type if and only if s is finite.*
(2) *S_β is sofic if and only if s is ultimately periodic.*
(3) *S_β is specified if and only if there is an $n \in \mathbb{N}$ such that all blocks of 0s in s have length less than n.*
(4) *S_β is synchronized if and only if there is a word $w \in L(S_\beta)$ such that w does not appear as a subword of s.*
(5) *Every β-shift is coded.*

THEOREM 1.15. *The entropy of any β-shift is* $\log(\beta)$.

2. *Bounded density shifts*

Let $f : \mathbb{N}_0 \to [0, \infty)$ be a function. We define

$$\Psi_f = \left\{ x \in (\mathbb{N}_0)^{\mathbb{Z}} : (\forall p \in \mathbb{N}_0)(\forall i \in \mathbb{Z}) \sum_{r=i}^{i+p-1} x_r \leq f(p) \right\}.$$

We call these *bounded density shifts*.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

The first thing one may notice is that the alphabet has not been restricted to a finite set. However, given a function $f \in [0, \infty)^{\mathbb{N}_0}$, we can see that $L_1(\Psi_f) \subseteq \{n \in \mathbb{N}_0 : n \leq f(1)\}$, which is finite. More precisely, it can be shown that $L_1(\Psi_f) = \{n \in \mathbb{N}_0 : (\forall m \in \mathbb{N})\, n \leq f(m)\}$. We will always assume $A = L_1(\Psi_f)$ for each bounded density shift. It is not hard to verify that Ψ_f is closed and invariant.

Many different functions will give the same shift. For example, taking the floor pointwise of any function will not change the shift. We define a relation on the set of functions in $[0, \infty)^{\mathbb{N}_0}$ by $f \sim g$ if and only if $\Psi_f = \Psi_g$. This is clearly an equivalence relation. For each equivalence class, we will find a canonical function whose image is as minimal as possible.

LEMMA 2.1. *Let $\{g_j : j \in J\}$ be a non-empty set of $\sim$-equivalent functions. Then the function f defined by $f(m) := \min\{g_j(m) : j \in J\}$ is $\sim$-equivalent to each g_j.*

The function f above is the canonical function for the shift Ψ_f. It is easy to see that $f(m) = \max\{\sum_{r=1}^{|w|} w_r : w \in L_m(\Psi_f)\}$ for each $m \in \mathbb{N}$. We remark that the canonical function must satisfy a number of properties. For example, we must have $f(\mathbb{N}_0) \subseteq \mathbb{N}_0$ since, as remarked before, we can take the floor pointwise of a function without changing the shift. We also must have f non-decreasing and $f(0) = 0$. We will characterize which functions are canonical shortly. Note that the alphabet is $A = [0..f(1)]$ for each bounded density shift Ψ_f.

Example 1.

(1) The identity function $f(n) = n$ gives rise to the full 2-shift.
(2) The function $g(n) = (n + 1)/2$ gives rise to the golden mean shift which can be defined by forbidding the word 11. Its canonical function is $f(n) = \lfloor (n + 1)/2 \rfloor$.
(3) The even shift defined by having an even amount of 0s between consecutive 1s is not a bounded density shift. This is easily seen by noting that 110 is in the language of the shift but 101 isn't.

The proofs of the following are routine.

LEMMA 2.2. *Let Ψ_f be a bounded density shift. Then:*

(1) *$ubv \in L(\Psi_f)$ and $a \leq b$ (where $a, b \in A$) $\implies uav \in L(\Psi_f)$;*
(2) *$uv \in L(\Psi_f) \implies u0v \in L(\Psi_f)$.*

LEMMA 2.3. *Let Ψ_f be a bounded density shift with f canonical. Then the following are equivalent.*

(1) $w \in L(\Psi_f)$.
(2) $^{\infty}0.w0^{\infty} \in \Psi_f$.
(3) $(\forall v \leq_s w) \sum_{r=1}^{|v|} v_r \leq f(|v|)$.

Definition. Let Ψ_f be a bounded density shift and let $u \in L(\Psi_f)$. Define a right infinite sequence $\overrightarrow{u}$ inductively by

$$\overrightarrow{u}_{i+1} = \max\{a \in A : u\,\overrightarrow{u}_{[1,i]}a \in L(\Psi_f)\}$$

for all $i \in \mathbb{N}_0$. Define $\overrightarrow{u} := u\,\overrightarrow{u}$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

We define the left infinite sequences $\overleftarrow{u}$ and $\overleftarrow{\overleftarrow{u}}$ in a similar way.

Note that $\overleftarrow{u} = (\overrightarrow{u^R})^R$ and $\overleftarrow{\overleftarrow{u}} = (\overrightarrow{u^R})^R$. Note also that the index set for these two is $-\mathbb{N}$, not $\mathbb{N}$.

We call $\overrightarrow{\overrightarrow{\varepsilon}}$ the (right) characteristic sequence of Ψ_f. We will see later that $\overrightarrow{\overrightarrow{\varepsilon}}$ is related to the canonical function and both characterize the shift.

Definition. For $x, y \in A^{\mathbb{N}}$, define the partial ordering $x \preceq y$ if

$$(\forall i \in \mathbb{N}) \sum_{r=1}^{i} x_r \leq \sum_{r=1}^{i} y_r.$$

For $u, v \in A^m$ for some $m \in \mathbb{N}$, define $u \preceq v$ if

$$(\forall i \leq m) \sum_{r=1}^{i} u_r \leq \sum_{r=1}^{i} v_r.$$

LEMMA 2.4. *Let Ψ_f be a bounded density shift with f canonical, let $m \in \mathbb{N}_0$ and let $u, v \in L_m(\Psi_f)$ with $u \preceq v$. Then:*

(1) $u\overrightarrow{u}_1 \preceq v\overrightarrow{v}_1$;

(2) $u\overrightarrow{u} \preceq v\overrightarrow{v}$.

Proof. To show (1), we only need to prove that $(\sum_{r=1}^{m} u_r) + \overrightarrow{u}_1 \leq (\sum_{r=1}^{m} v_r) + \overrightarrow{v}_1$. Let $p = \sum_{r=1}^{m} v_r - \sum_{r=1}^{m} u_r$.

Case 1: $\overrightarrow{u}_1 \leq p$. Then

$$\left(\sum_{r=1}^{m} u_r\right) + \overrightarrow{u}_1 \leq \left(\sum_{r=1}^{m} u_r\right) + p = \sum_{r=1}^{m} v_r \leq \left(\sum_{r=1}^{m} v_r\right) + \overrightarrow{v}_1.$$

Case 2: $\overrightarrow{u}_1 > p$. We first show that $va \in L(\Psi_f)$ where $a = \overrightarrow{u}_1 - p$. It is clear that $a \in A$ since $a \leq \overrightarrow{u}_1 \leq f(1)$. Now the only subwords of va that are not obviously in $L(\Psi_f)$ are of the form $v_{[i,m]}a$ for $i \in [1..m]$. Let $i \in [1..m]$. Then

$$\begin{aligned}
\left(\sum_{r=i}^{m} v_r\right) + a &= \left(\sum_{r=1}^{m} v_r\right) + a - \left(\sum_{r=1}^{i-1} v_r\right) \\
&\leq \left(\sum_{r=1}^{m} u_r\right) + p + a - \left(\sum_{r=1}^{i-1} u_r\right) \\
&= \left(\sum_{r=i}^{m} u_r\right) + \overrightarrow{u}_1 \\
&\leq f(|v_{[i,m]}a|) \quad \text{since } u\overrightarrow{u}_1 \in \Psi_f^+.
\end{aligned}$$

Hence $va \in L(\Psi_f)$. Then we have

$$\overrightarrow{v}_1 = \max\{b \in A : vb \in L(\Psi_f)\} \geq a.$$

This shows that $u\overrightarrow{u}_1 \preceq v\overrightarrow{v}_1$.

(2) can be proved by induction using (1). □

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

The following lemma and corollary are obvious.

LEMMA 2.5. *Let Ψ_f be a bounded density shift and let $u \in L(\Psi_f)$. Then $\overrightarrow{u}$ is the maximum element of $C^+[u]$ restricted to Ψ_f^+ with respect to $\preceq$.*

COROLLARY 2.6. *$\overrightarrow{\varepsilon}$ is the maximum element of Ψ_f^+.*

The following lemmas are routine.

LEMMA 2.7. *Let Ψ_f be a bounded density shift with f canonical. Then for all $i \in \mathbb{N}_0$, we have $f(i) = \sum_{r=1}^{i} \overrightarrow{\varepsilon}_r$.*

LEMMA 2.8. (Cf. Theorem 1.13) *Let Ψ_f be a bounded density shift with f canonical. Then:*

(1) *$y \in \Psi_f^+$ if and only if $\sigma^n(y) \preceq \overrightarrow{\varepsilon}$ for all $n \in \mathbb{N}_0$;*

(2) *$y \in \Psi_f$ if and only if $y_{[n,\infty)} \preceq \overrightarrow{\varepsilon}$ for all $n \in \mathbb{Z}$.*

THEOREM 2.9. *A function f is canonical for the bounded density shift Ψ_f if and only if:*

(1) *$f(0) = 0$;*

(2) *$(\forall m \in \mathbb{N}_0)\, f(m+1) \in f(m) + \mathbb{N}_0$;*

(3) *$(\forall m, n \in \mathbb{N})\, f(m+n) \le f(m) + f(n)$.*

Proof. Let f be canonical. Clearly (1) and (2) hold. For (3), we have

$$f(m+n) = \sum_{r=1}^{m+n} \overrightarrow{\varepsilon}_r = \sum_{r=1}^{m} \overrightarrow{\varepsilon}_r + \sum_{r=1}^{n} \overrightarrow{\varepsilon}_{m+r} \le f(m) + f(n)$$

for all $m, n \in \mathbb{N}$ by Lemma 2.7.

Conversely, let f satisfy (1)–(3) and let g be the canonical function for Ψ_f. We show $g(m) = f(m)$ for all $m \in \mathbb{N}_0$ by modified induction. The base case holds by (1) and the definition of the canonical function. For an $m \in \mathbb{N}$, assume that $g(p) = f(p)$ for all $p \le m$. By (2) there is an $a \in \mathbb{N}_0$ such that $f(m+1) = f(m) + a$. We aim to show that $\overrightarrow{\varepsilon}_{[1,m]}a \in L(\Psi_f)$. The only subwords that are not easily seen to be in $L(\Psi_f)$ are of the form $\overrightarrow{\varepsilon}_{[i,m]}a$ for some $i \in [1..m]$. Let $i \in [1..m]$. Then

$$\begin{aligned}
\left(\sum_{r=i}^{m} \overrightarrow{\varepsilon}_r\right) + a &= \left(\sum_{r=1}^{m} \overrightarrow{\varepsilon}_r\right) - \left(\sum_{r=1}^{i-1} \overrightarrow{\varepsilon}_r\right) + a \\
&= g(m) - g(i-1) + a \\
&= f(m) - f(i-1) + a \quad \text{by inductive assumption} \\
&= f(m+1) - f(i-1) \\
&\le f(i-1) + f(m-i+2) - f(i-1) \quad \text{by (3) or (1) when } i = 1 \\
&= f(|\overrightarrow{\varepsilon}_{[i,m]}a|),
\end{aligned}$$

giving $\overrightarrow{\varepsilon}_{[1,m]}a \in L(\Psi_f)$. Hence $\overrightarrow{\varepsilon}_{[1,m]}a \preceq \overrightarrow{\varepsilon}_{[1,m+1]}$ by Lemma 2.5, so $g(m) + a \le g(m+1)$ by Lemma 2.7. This gives

$$f(m+1) = f(m) + a = g(m) + a \le g(m+1)$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

by the inductive hypothesis. Since $g(m+1) \le f(m+1)$ by Lemma 2.1, the induction is complete. □

THEOREM 2.10. (Cf. Theorem 1.12) *A right infinite string x is a characteristic sequence of a bounded density shift if and only if*

$$(\forall m \in \mathbb{N})\ \sigma^m(x) \preceq x.$$

Proof. Let x be the characteristic sequence of the bounded density shift Ψ_f, where f is canonical. Let $m, i \in \mathbb{N}$. We need to show that $\sum_{r=1}^{i} \sigma^m(x)_r \le \sum_{r=1}^{i} x_r$. But

$$\begin{aligned}\sum_{r=1}^{i} \sigma^m(x)_r &= \sum_{r=1}^{m+i} x_r - \sum_{r=1}^{m} x_r \\ &= f(i+m) - f(m) \quad \text{by Lemma 2.7} \\ &\le f(i) + f(m) - f(m) \quad \text{by Theorem 2.9(3)} \\ &= \sum_{r=1}^{i} x_r.\end{aligned}$$

Conversely, let $\sigma^m(x) \preceq x$ for all $m \in \mathbb{N}$. Define f by $f(m) = \sum_{r=1}^{m} x_r$. We show that f is canonical for the shift Ψ_f on alphabet $A = [0..x_1]$ by Theorem 2.9. First, $f(0) = 0$ so (1) holds. For any $m \in \mathbb{N}$,

$$f(m+1) = \sum_{r=1}^{m+1} x_r = \left(\sum_{r=1}^{m} x_r\right) + x_{m+1} = f(m) + x_{m+1} \in f(m) + \mathbb{N}_0$$

so (2) holds. Now let $m, p \in \mathbb{N}$. Then

$$\begin{aligned}f(m+p) = \sum_{r=1}^{m} x_r + \sum_{r=m+1}^{m+p} x_r &= f(m) + \sum_{r=1}^{p} \sigma^m(x)_r \\ &\le f(m) + \sum_{r=1}^{p} x_r \quad \text{since } \sigma^m(x) \preceq x \\ &= f(m) + f(p),\end{aligned}$$

so (3) holds. Thus f is canonical. Hence we have $\sum_{r=1}^{i} x_r = f(i) = \sum_{r=1}^{i} \overrightarrow{\varepsilon}_r$ for all $i \in \mathbb{N}_0$ by Lemma 2.7, so that $x = \overrightarrow{\varepsilon}$. Hence x is a characteristic sequence. □

The following lemma is routine.

LEMMA 2.11. *For two canonical functions f and g, we have $\overrightarrow{\varepsilon}_f \preceq \overrightarrow{\varepsilon}_g$ if and only if $\Psi_f \subseteq \Psi_g$.*

From this, we can easily construct two bounded density shifts that are incomparable to each other. For example, choose $\overrightarrow{\varepsilon}_f = (20)^\infty$ and $\overrightarrow{\varepsilon}_g = (210)^\infty$. This shows that bounded density shifts are not totally ordered whereas β-shifts are.

LEMMA 2.12. (Fekete's lemma, see [**3**, Lemma 4.1.7]) *For a bounded density shift Ψ_f, where f is canonical, the limit $\lim_{m\to\infty} f(m)/m$ always exists and is equal to* $\inf\{f(m)/m : m \in \mathbb{N}\}$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

We call $\lim_{m\to\infty} f(m)/m$ the *limiting gradient*. Many distinct bounded density shifts can have the same limiting gradient. Examples can be constructed from Examples 5, 6 and 8.

LEMMA 2.13. *Let $g : [0, \infty) \to \mathbb{R}$ be a function differentiable on $\mathbb{R}^+$ satisfying:*
(1) $g(0) = 0$*;*
(2) g *is non-decreasing;*
(3) g' *is non-increasing.*
Then f defined by $f(n) := \lceil g(n) \rceil$ for all $n \in \mathbb{N}_0$ is canonical. Moreover,

$$\lim_{n\to\infty} \frac{f(n)}{n} = \lim_{n\to\infty} \frac{g(n)}{n}.$$

Proof. We use Theorem 2.9. Clearly, $f(0) = 0$, so (1) holds. It is easy to see that $f(n+1) \geq f(n)$ and, since $f(\mathbb{N}_0) \subseteq \mathbb{N}_0$, it is clear that (2) holds for all $n \in \mathbb{N}_0$. For (3), note that every tangent line of g is a global upper bound for g. This can be proved using the mean value theorem.

Let $m, n \in \mathbb{N}$ and assume without loss of generality that $m \leq n$. This implies that $g'(n) \leq g'(m)$. The tangent line at m is given by $h_m(x) = (x - m)g'(m) + g(m)$. Thus

$$0 = g(0) \leq h_m(0) = g(m) - mg'(m),$$

which gives $mg'(m) \leq g(m)$. Similarly, the tangent line at n is given by $h_n(x) = (x - n)g'(n) + g(n)$. Thus

$$g(m + n) \leq h_n(m + n) = mg'(n) + g(n).$$

This all gives

$$g(m + n) \leq mg'(n) + g(n) \leq mg'(m) + g(n) \leq g(m) + g(n).$$

Thus $\lceil g(m + n) \rceil \leq \lceil g(m) + g(n) \rceil$ which implies $\lceil g(m + n) \rceil \leq \lceil g(m) \rceil + \lceil g(n) \rceil$. Hence $f(m + n) \leq f(m) + f(n)$.

The moreover follows from the squeeze theorem for limits. □

The hypotheses of this lemma can be used to construct numerous examples illustrating the difference between a number of different classes of bounded density shifts.

Example 2.
(1) The identity function satisfies the hypotheses of Lemma 2.13 and gives rise to the full 2-shift as in Example 1.
(2) The function id/2 also satisfies the hypotheses of Lemma 2.13. Its ceiling gives the golden mean shift as in Example 1.
(3) The function defined $f(n) := \sqrt{n}$ satisfies the hypotheses of Lemma 2.13. The ceiling of this function has a limiting gradient of 0.

We call $\{^\infty 0.0^\infty\}$ the *trivial* bounded density shift.

A set $R \subseteq \mathbb{N}_0$ is said to be *syndetic* if there is a $k \in \mathbb{N}$ such that $[i..i + k) \cap R \neq \varnothing$ for every $i \in \mathbb{N}_0$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

THEOREM 2.14. *Let Ψ_f be a non-trivial bounded density shift with f canonical. Then the following are equivalent.*

(1) *Ψ_f is transitive.*
(2) *Ψ_f is mixing.*
(3) *f is not bounded above by a constant.*
(4) *$f(\mathbb{N}_0)$ is syndetic.*

Proof. (1) $\Longrightarrow$ (2). Let Ψ_f be transitive and let $u, w \in L(\Psi_f)$. Then there exists a $v \in L(\Psi_f)$ such that $uvw \in L(\Psi_f)$. By an induction on Lemma 2.2, we have $uv0^m w \in L(\Psi_f)$ for all $m \in \mathbb{N}_0$. Lemma 1.6 gives the result.

(1) $\Longrightarrow$ (4). Let Ψ_f be transitive. By Theorem 2.9(1) and (2), we see that $f(\mathbb{N}_0) \subseteq \mathbb{N}_0$. Since $1 \in L(\Psi_f)$, we can use transitivity and compactness to construct a right infinite sequence with an infinite number of 1s. This proves that $\vec{\varepsilon}$ has an infinite number of non-zero elements. Lemma 2.7 shows that $f(\mathbb{N}_0)$ is infinite and, in fact, syndetic, since $f(i+1) - f(i) \leq f(1)$ for each $i \in \mathbb{N}_0$.

(3) $\Longrightarrow$ (1). Assume f is not bounded above by a constant and let $u, w \in L(\Psi_f)$. Then there is an $m \in \mathbb{N}$ such that $f(m) \geq \sum_{r=1}^{|u|} u_r + \sum_{r=1}^{|w|} w_r$. Since f is non-decreasing, it is easily seen that $u0^m w \in L(\Psi_f)$.

The remaining implications (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (3) are obvious. □

COROLLARY 2.15. *Every bounded density SFT is transitive.*

Proof. Let Ψ_f be a bounded density shift with f canonical. First, the trivial bounded density shift is transitive so we can assume that Ψ_f is non-trivial. Let Ψ_f be an SFT with a set of allowed words W of some length n. Since $0^p 10^{n-p-1} \in W$ for all $p \leq n-1$, we have $(10^{n-1})^m \in L(\Psi_f)$ for all $m \in \mathbb{N}$ proving that f is not bounded by a constant. The result follows from Theorem 2.14. □

THEOREM 2.16. *A non-transitive bounded density shift is countable and strictly sofic.*

Proof. Let Ψ_f be a non-transitive bounded density shift where f is canonical. Then Corollary 2.15 proves that Ψ_f is not an SFT. By Theorem 2.14, f is eventually constant. Let $k = \max(f(\mathbb{N}_0))$ and choose $m \in \mathbb{N}$ so that $f(m) = k$. Create a finite state machine G by $V(G) = [0..k+1]$,

$$E(G) = \{(p, a, q) : p + a = q < k+1, a \in A \text{ and } p, q \in V(G)\}$$
$$\cup \{(p, a, k+1) : p + a \geq k+1, a \in A \text{ and } p \in V(G)\},$$

$I = 0$ and $F = \{k+1\}$. An example where $A = \{0, 1, 2\}$ and $k = 4$ is shown in Figure 1. It is easy to see that the vertices of G sum up the letters of each word up to $k+1$, so that $L(G) = \{w \in A^* : \sum_{r=1}^{|w|} w_r \geq k+1\}$. Let $M_w := A^* w A^*$ for all $w \in A^*$ and let

$$M = L(G) \cup \bigcup \{M_w : w \notin L(\Psi_f) \text{ and } |w| \leq m\}.$$

Note that this is a finite union. By Kleene's theorem, M is regular. We show that $M = A^* \backslash L(\Psi_f)$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

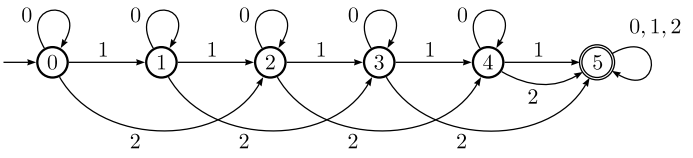


FIGURE 1. An example of the finite state machine constructed in Theorem 2.16 for $A = \{0, 1, 2\}$ and $k = 4$.

Let $u \in M$. If $u \in L(G)$, then $\sum_{r=1}^{|u|} u_r \geq k + 1 > k \geq f(|u|)$, so that $u \in A^*\backslash L(\Psi_f)$. Otherwise, there is a $w \in A^*\backslash L(\Psi_f)$ with $|u| \leq m$ such that $u \in M_w$. But then $w \leq_s u$, so $u \in A^*\backslash L(\Psi_f)$. Thus $M \subseteq A^*\backslash L(\Psi_f)$.

Conversely, let $u \in A^*\backslash L(\Psi_f)$. If $\sum_{r=1}^{|u|} u_r \geq k + 1$, then $u \in L(G)$. Otherwise, $\sum_{r=1}^{|u|} u_r \leq k$ and there is a $v \leq_s u$ such that $\sum_{r=1}^{|v|} v_r > f(|v|)$. Then

$$f(|v|) < \sum_{r=1}^{|v|} v_r \leq \sum_{r=1}^{|u|} u_r \leq k = f(m),$$

which implies that $|v| < m$ since f is non-decreasing. Hence $u \in M_v \subseteq M$.

This gives $M = A^*\backslash L(\Psi_f)$. Since the regular languages are closed under complements, $L(\Psi_f)$ is regular, so that Ψ_f is sofic. □

3. *Characterization of coded bounded density shifts*

We call ${}^{\infty}0.0^{\infty}$ the *trivial* periodic point.

THEOREM 3.1. *Let Ψ_f be a non-trivial bounded density shift and let f be canonical. The following are equivalent.*

(1) *Ψ_f has a dense set of periodic points.*
(2) *There exists a non-trivial periodic point in Ψ_f.*
(3) *f is bounded below by a strictly increasing linear function.*
(4) *The limiting gradient is positive.*
(5) *Ψ_f is coded.*

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3). Let ${}^{\infty}(av).(av)^{\infty}$ be a periodic point in Ψ_f with $a \neq 0$. Define $g(m) := m/(|v| + 1)$ for all $m \in \mathbb{N}$. Then

$$f(m) \geq \sum_{r=1}^{m} ((av)^{\infty})_r \geq \lceil m/(|v| + 1) \rceil \geq g(m)$$

for each m.

(3) $\Longrightarrow$ (1). Let f be bounded below by $g := c \cdot \mathrm{id}$ for some $c > 0$. We may assume that $c < 1$. Let $w \in L(\Psi_f)$ and let $|w| = k$. Choose a $p \in \mathbb{N}$ such that $g(k + p) \geq 2f(k)$ and define $x := {}^{\infty}(w0^p).(w0^p)^{\infty}$. We have

$$2f(k) \leq g(k + p) = g(k) + g(p) \leq f(k) + g(p),$$

so that $f(k) \leq g(p)$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

We show that $x \in \Psi_f$. Let v be a finite subword of x and choose $m \in \mathbb{N}$ to be the least element such that $|v| \leq m(p+k)$. We consider the case $m = 1$ first. If $|v| \leq p$, then $\sum_{r=1}^{|v|} v_r \leq f(|v|)$ is fairly obvious. If $p < |v| \leq p + k$, then

$$\sum_{r=1}^{|v|} v_r \leq \sum_{r=1}^{|w|} w_r \leq f(k) \leq g(p) \leq g(|v|) \leq f(|v|).$$

With the case $m = 1$ done, we assume that $m \geq 2$. We have

$$\begin{aligned}\sum_{r=1}^{|v|} v_r \leq m \cdot \sum_{r=1}^{|w|} w_r \leq mf(k) &= 2f(k) + (m-2)f(k) \\ &\leq g(p+k) + (m-2)g(p+k) \\ &= c(m-1)(p+k) \\ &\leq c|v| \\ &\leq f(|v|),\end{aligned}$$

using the fact that m was minimal for $c(m-1)g(p+k) \leq c|v|$. Hence $x \in \Psi_f$ and so Ψ_f has a dense set of periodic points.

(3) $\implies$ (4). Let f be bounded below by a strictly increasing linear function $g := c \cdot \mathrm{id}$. Then by Lemma 2.12,

$$\lim_{m\to\infty} f(m)/m = \inf\{f(m)/m : m \in \mathbb{N}\} \geq \inf\{cm/m : m \in \mathbb{N}\} = c > 0,$$

proving the limiting gradient is positive.

(4) $\implies$ (3). Define $g(p) := p \inf\{f(m)/m : m \in \mathbb{N}\}$. By Lemma 2.12 and assumption, g is a strictly increasing linear function. Clearly

$$f(p) = pf(p)/p \geq p \inf\{f(m)/m : m \in \mathbb{N}\} = g(p)$$

for all $p \in \mathbb{N}_0$.

(4) $\implies$ (5). Let γ be the limiting gradient. Define a function g by $g(u) := \lceil f(|u|)/\gamma \rceil$ for all $u \in A^*$ and define $P := \{0^{g(u)}u0^{g(u)} : u \in L(\Psi_f)\}$. We will show that $(P^*)^{\leq_s} = L(\Psi_f)$. It is easy to see that $L(\Psi_f) \subseteq (P^*)^{\leq_s}$. For the reverse inclusion, we will prove

$$(\forall n \in \mathbb{N})\ P^n \subseteq L(\Psi_f)$$

by induction. Recall Lemma 2.2(2) as it will be used many times here. Clearly the $n = 1$ case holds. So assume that $P^n \subseteq L(\Psi_f)$ for some $n \in \mathbb{N}$. Let $z \in P^{n+1}$. Then we may write $z = x^{(1)}x^{(2)} \dots x^{(n+1)}$, where each $x^{(i)} \in P$, and thus write $x^{(i)} = 0^{g(y^{(i)})}y^{(i)}0^{g(y^{(i)})}$ for each $i \in [1..n+1]$.

Let $u \leq_s z$. If $u \leq_s x^{(1)} \dots x^{(n)}0^{g(y^{(n+1)})}$ or $u \leq_s 0^{g(y^{(1)})}x^{(2)} \dots x^{(n+1)}$, then $u \in L(\Psi_f)$ by the inductive assumption. So we may assume that

$$0^{g(y^{(1)})}x^{(2)} \dots x^{(n)}0^{g(y^{(n+1)})} \leq_s u.$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Now we have

$$\begin{aligned}\sum_{r=1}^{|u|} u_r \le \sum_{s=1}^{n+1}\sum_{r=1}^{|y^{(s)}|} y_r^{(s)} &\le \sum_{s=1}^{n+1} f(|y^{(s)}|)\\ &\le \sum_{s=1}^{n+1} \gamma \cdot \left\lceil \frac{f(|y^{(s)}|)}{k} \right\rceil\\ &= \gamma \sum_{s=1}^{n+1} g(y^{(s)})\\ &\le f\left(\sum_{s=1}^{n+1} g(y^{(s)})\right) \quad \text{since } \frac{f(m)}{m} \ge \gamma \text{ for all } m\\ &\le f(|u|).\end{aligned}$$

Hence $u \in L(\Psi_f)$, so that $z \in L(\Psi_f)$. This completes the induction, which gives $P^* \subseteq L(\Psi_f)$, so that $(P^*)^{\le_s} \subseteq L(\Psi_f)$. Hence $(P^*)^{\le_s} = L(\Psi_f)$ proving that Ψ_f is coded.

(5) $\implies$ (1) is well known. □

COROLLARY 3.2. *A bounded density shift with a dense set of periodic points is mixing.*

Proof. This follows from Theorems 2.14 and 3.1. □

Example 3. Let $f(n) := \lceil \log_b(n+1) \rceil$ for all $n \in \mathbb{N}_0$ and any $b > 1$. We claim that each f is canonical and transitive but without a dense set of periodic points. It is clear that f is not bounded above by a constant, so it is transitive by Theorem 2.14. Also, $\log_b(n+1)$ as a function of n on the domain $[0, \infty)$ satisfies the hypotheses of Lemma 2.13, so f is canonical and

$$\lim_{n\to\infty} \frac{f(n)}{n} = \lim_{n\to\infty} \frac{\log_b(n+1)}{n} = 0.$$

Thus Ψ_f does not have a dense set of periodic points by Theorem 3.1.

4. *Characterization of bounded density shifts of finite type*

Definition. A canonical function f is said to be *maximally extended* if there is a $q \in \mathbb{N}_0$ such that, for every $n > q$,

$$f(n) = \min\{f(k) + f(n-k) : k \in [1..n)\}.$$

Given $f(0), f(1), \ldots, f(n-1)$, the value $\min\{f(k) + f(n-k) : k \in [1..n)\}$ always happens to be the largest value that $f(n)$ can take such that f can still be a canonical function. The reasoning for this comes from Theorem 2.9(3). The following lemma follows routinely from Theorem 2.9.

LEMMA 4.1. *Let f be a canonical function, let $q \in \mathbb{N}$ and let g be defined by*

$$g(m) := \begin{cases} f(m) & \text{if } m \le q,\\ \min\{f(k) + f(m-k) : k \in [0..m]\} & \text{if } m > q.\end{cases}$$

Then g is a canonical function.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

The following lemma is routine.

LEMMA 4.2. *Let $n \in \mathbb{N}_0$ and let f and g be canonical functions. Then $L_n(\Psi_f) = L_n(\Psi_g)$ if and only if $f(m) = g(m)$ for all $m \leq n$.*

THEOREM 4.3. *Let Ψ_f be a bounded density shift and let f be canonical. The following are equivalent.*

(1) *Ψ_f is of finite type.*
(2) *f is a maximally extended canonical function.*
(3) *$\min\{f(m)/m : m \in \mathbb{N}\}$ exists.*
(4) *$(\exists p \in \mathbb{N})(\forall m \in \mathbb{N})\ f(mp) = mf(p)$.*
(5) *$(\exists N \in \mathbb{N}_0)(\exists p \in \mathbb{N})(\forall m \geq N)\ f(m + p) = f(m) + f(p)$.*

Proof. (1) $\Longrightarrow$ (2). Let Ψ_f be a $(q-1)$-step SFT, so it has allowed words $L_q(\Psi_f)$ of length q. Define a function g by

$$g(m) := \begin{cases} f(m) & \text{if } m \leq q, \\ \min\{f(k) + f(m-k) : k \in [0..m]\} & \text{if } m > q. \end{cases}$$

We will show that $f = g$ which implies that f is maximally extended. Now $f(m) \leq f(k) + f(m-k)$ for all $k \leq m$, so that

$$f(m) \leq \min\{f(k) + f(m-k) : k \in [0..m]\} = g(m)$$

holds for every $m > q$. Clearly $f(m) \leq g(m)$ for $m \leq q$, so $f \leq g$. Lemma 4.1 proves that g is canonical, so that $L_q(\Psi_f) = L_q(\Psi_g)$ by Lemma 4.2. Also, we have

$$\begin{aligned} x \in \Psi_g &\Longrightarrow (\forall i \in \mathbb{Z})\ x_{[i,i+q)} \in L(\Psi_g) \\ &\Longrightarrow (\forall i \in \mathbb{Z})\ x_{[i,i+q)} \in L(\Psi_f) \quad \text{since } L_q(\Psi_f) = L_q(\Psi_g) \\ &\Longrightarrow x \in \Psi_f \quad \text{since } \Psi_f \text{ is a } (q-1)\text{-step SFT.} \end{aligned}$$

Thus $\Psi_g \subseteq \Psi_f$, which implies that $\vec{\varepsilon}_g \preceq \vec{\varepsilon}_f$ since $\vec{\varepsilon}_f$ is the maximum in Ψ_f. Thus Lemma 2.7 gives $g(m) \leq f(m)$ for all $m \in \mathbb{N}_0$. Hence $f = g$.

(2) $\Longrightarrow$ (3). Then there is a $q \in \mathbb{N}_0$ such that for all $m > q$

$$f(m) = \min\{f(k) + f(m-k) : k \in [1..m)\}. \tag{1}$$

Choose $p \leq q$ such that $f(p)/p = \min\{f(m)/m : m \leq q\}$. Note that p may not be unique. We show

$$(\forall m \in \mathbb{N})\ \frac{f(p)}{p} \leq \frac{f(m)}{m}$$

by modified induction. By choice of p this already holds for $m \leq q$, so assume that the induction hypothesis holds for all numbers up to and including some $m \geq q$. Let $k \in [1..m]$. Then $k, m + 1 - k \in [1..m]$, so that

$$\frac{f(p)}{p} \leq \frac{f(k)}{k} \quad \text{and} \quad \frac{f(p)}{p} \leq \frac{f(m+1-k)}{m+1-k}$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

hold by the inductive hypothesis. Then $kf(p) + (m+1-k)f(p) \leq pf(k) + pf(m+1-k)$, which implies that $(m+1)f(p) \leq p(f(k) + f(m+1-k))$. But this gives

$$(m+1)f(p) \leq p \min\{f(k) + f(m+1-k) : k \in [1..m]\}$$

since k was arbitrary. Thus $(m+1)f(p) \leq pf(m+1)$ by (1), so that

$$\frac{f(p)}{p} \leq \frac{f(m+1)}{m+1}$$

which completes the induction. Hence $f(p)/p = \min\{f(m)/m : m \in \mathbb{N}\}$.

(3) $\Longrightarrow$ (4). Let $f(p)/p = \min\{f(m)/m : m \in \mathbb{N}\}$ and let $m \in \mathbb{N}$. Then we have $f(p)/p \leq f(mp)/(mp)$ by assumption which gives $mf(p) \leq f(mp)$. The reverse inequality follows from an inductive application of Theorem 2.9(3) upon which we get $mf(p) = f(mp)$, completing the proof.

(4) $\Longrightarrow$ (5). We prove the contrapositive. Assume that for all $N \in \mathbb{N}_0$ and $p \in \mathbb{N}$ there exists an $m \geq N$ such that $f(m+p) \neq f(m) + f(p)$. Let $p \in \mathbb{N}$. We aim to show that there is an $m \in \mathbb{N}$ such that $f(mp) \neq mf(p)$. We have

$$(\forall N \in \mathbb{N})(\exists m \geq N)\ f(m+p) \neq f(m) + f(p).$$

Thus the set $\{m \in \mathbb{N} : f(m+p) \neq f(m) + f(p)\}$ is infinite so there is an $i \in [1..p]$ such that the set

$$R := \{m \in p\mathbb{N}_0 + i : f(m+p) \neq f(m) + f(p)\}$$

is infinite. But by Theorem 2.9(3), we see that

$$\begin{aligned} R &= \{m \in p\mathbb{N}_0 + i : f(m+p) < f(m) + f(p)\} \\ &= \{m \in p\mathbb{N}_0 + i : f(m+p) \leq f(m) + f(p) - 1\}. \end{aligned}$$

We now prove

$$(\forall k \in \mathbb{N}_0)\ f(i+kp) \leq f(i) + kf(p) - |R \cap [1..i+kp)| \tag{2}$$

by induction. We see the case $k = 0$ is trivial, so assume that $f(i+kp) \leq f(i) + kf(p) - |R \cap [1..i+kp)|$ holds for some $k \in \mathbb{N}_0$. Then

$$\begin{aligned} &f(i) + (k+1)f(p) - |R \cap [1..i+(k+1)p)| \\ &\quad = f(i) + kf(p) - |R \cap [1..i+kp)| + f(p) - |R \cap [i+kp..i+(k+1)p)| \\ &\quad \geq \begin{cases} f(i+kp) + f(p) & \text{if } i+kp \notin R \\ f(i+kp) + f(p) - 1 & \text{if } i+kp \in R \end{cases} \quad \text{by the induction hypothesis} \\ &\quad \geq f(i+(k+1)p) \end{aligned}$$

by Theorem 2.9(3) or the definition of R respectively. Hence the induction is complete. Since R is infinite, there is an $m \in \mathbb{N}_0$ such that $|R \cap [1..i+mp)| = pf(1) + 1$. Then $f(i+mp) \leq f(i) + mf(p) - pf(1) - 1$ by (2). We have $f(i) \leq pf(1)$ so we get $f(i+mp) \leq mf(p) - 1$. This gives $f(mp) \leq mf(p) - 1$ since f is non-decreasing upon which $f(mp) < mf(p)$. Thus we have found an $m \in \mathbb{N}$ such that $f(mp) \neq mf(p)$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

(5) $\Longrightarrow$ (1). Assume there are $N \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that, for all $m \geq N$,

$$f(m+p) = f(m) + f(p). \tag{3}$$

Let Σ be the SFT with allowed words $L_{N+p}(\Psi_f)$ of length $N+p$. Clearly $\Psi_f \subseteq \Sigma$. Let $x \in \Sigma$, $q \in \mathbb{N}$ and $i \in \mathbb{Z}$. Then $\sum_{r=i}^{i+q-1} x_r \leq f(q)$ for $q \leq N+p$ since $x_{[i,i+q)} \leq_s x_{[i,i+N+p)} \in L_{N+p}(\Psi_f)$. Otherwise, we can write $q = ap + N + b$ for some $a \in \mathbb{N}$ and $b \in [0..p)$. Then

$$\begin{aligned}\sum_{r=i}^{i+q-1} x_r &= \sum_{c\in[0..a)} \left(\sum_{r=i+cp}^{i+cp+p-1} x_r \right) + \sum_{r=i+ap}^{i+q-1} x_r \\ &\leq af(p) + f(N+b) \\ &= f(ap+N+b) \quad \text{by a simple induction on (3)} \\ &= f(q).\end{aligned}$$

Hence $x \in \Psi_f$. This gives $\Psi_f = \Sigma$, so that Ψ_f is an SFT. □

Examples of shifts of finite type are easy to come up with since it is only necessary to define some initial segment of a function and check that the initial section is canonical. The rest of the function can be constructed inductively using the formula

$$f(n) := \min\{f(k) + f(n-k) : k \in [1..n)\}.$$

Example 4.

(1) As mentioned in Example 1, the golden mean shift is of finite type. It can be defined by choosing $f(1) = f(2) = 1$. Its limiting gradient is $f(2)/2 = 1/2$.

(2) Choose f to be defined by $f(1) = 1$, $f(2) = f(3) = f(4) = 2$ and $f(5) = f(6) = 3$. The maximal extension of this function is very similar to the golden mean shift's function; it only varies at the point 2. This example shows that the N in Theorem 4.3(5) is necessary and cannot always be chosen to be 0.

(3) Let $k \in \mathbb{Q}^+$ and define a function $f(n) = \lceil kn \rceil$ for all $n \in \mathbb{N}_0$. Lemma 2.13 proves that f is canonical and the limiting gradient is k. Write $k = p/q$ for $p, q \in \mathbb{N}$. Now for all $m \in \mathbb{N}$, we have $f(mq) = kmq = mf(q)$ so Ψ_f is an SFT.

5. *Characterization of specified bounded density shifts*

Remember that $\overleftarrow{\varepsilon} = (\overrightarrow{\varepsilon})^R$.

THEOREM 5.1. *Let Ψ_f be a bounded density shift, let f be canonical and let $M \in \mathbb{N}$. Then the following are equivalent:*

(1) *Ψ_f is specified with specification constant M.*

(2) $(\forall m,\ p \in \mathbb{N}_0)\ f(m) + f(p) \leq f(m+p+M)$.

(3) *0^M is intrinsically synchronizing.*

Additionally, Ψ_f is specified if and only if Ψ_f is transitive and f is bounded above by an affine function whose gradient is the limiting gradient.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Proof. (1) $\Longrightarrow$ (3). Let $u0^M, 0^M w \in L(\Psi_f)$. By assumption, there is a $v \in L_M(\Psi_f)$ such that $uvw \in L(\Psi_f)$. By Lemma 2.2(1), we have $u0^M w \in L(\Psi_f)$. Hence (3) holds.

(3) $\Longrightarrow$ (2). Let $m, p \in \mathbb{N}_0$. Since $\overleftarrow{\varepsilon}_{[-m,-1]}0^M, 0^M\overrightarrow{\varepsilon}_{[1,p]} \in L(\Psi_f)$, we have $x := \overleftarrow{\varepsilon}_{[-m,-1]}0^M\overrightarrow{\varepsilon}_{[1,p]} \in L(\Psi_f)$. Thus

$$f(m) + f(p) = \sum_{r=-m}^{-1} \overleftarrow{\varepsilon}_r + \sum_{r=1}^{p} \overrightarrow{\varepsilon}_r = \sum_{r=1}^{m+M+p} x_r \leq f(m+p+M).$$

(2) $\Longrightarrow$ (1). Let $u, w \in L(\Psi_f)$. Let $x = u0^M w$. Then

$$\begin{aligned}\sum_{r=1}^{|u|+|w|+M} x_r = \sum_{r=1}^{|u|} u_r + \sum_{r=1}^{|w|} w_r &\leq f(|u|) + f(|w|)\\ &\leq f(|u| + |w| + M) \quad \text{by assumption}\\ &= f(|x|).\end{aligned}$$

Since each subword of x is also of this form or is trivially in $L(\Psi_f)$, we have $u0^M w \in L(\Psi_f)$. Hence Ψ_f is specified.

Now assume that Ψ_f is specified, so (2) holds for some $M \in \mathbb{N}$. It is easy to see that the trivial bounded density shift satisfies the required properties so we may assume that Ψ_f is non-trivial. Thus $f(1) \geq 1$. We prove

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\ f(m) \geq n$$

by induction. The case $n = 1$ was shown above so let $f(m) \geq n$ for some $m, n \in \mathbb{N}$. Then $f(m + 1 + M) \geq f(m) + f(1) \geq n + 1$, completing the induction. Hence $f(\mathbb{N}_0)$ is not bounded above by a constant so that Ψ_f is transitive by Theorem 2.14.

Suppose, by way of contradiction, that f is not bounded above by any affine function with gradient γ, where γ is the limiting gradient. In particular, f is not bounded above by $h := \gamma \cdot \mathrm{id} + (\gamma + 1)M$. Then there is a $p \in \mathbb{N}$ such that $f(p) > h(p) = \gamma p + (\gamma + 1)M$. Then, by assumption,

$$(\forall m \in \mathbb{N})\ f(m) + f(p) \leq f(m + p + M),$$

which implies

$$(\forall m \in \mathbb{N})\ f(m) + \gamma p + (\gamma + 1)M \leq f(m + p + M). \tag{4}$$

We can now prove

$$(\forall n \in \mathbb{N})\ f(n(M + p)) \geq n\gamma(M + p) + (n - 1)M \tag{5}$$

by induction on n. First, $f(M + p) \geq \gamma(M + p)$ holds by Lemma 2.12 so assume that (5) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}f((n + 1)(M + p)) &= f(n(M + p) + M + p)\\ &\geq \gamma p + (\gamma + 1)M + f(n(M + p)) \quad \text{by (4)}\\ &\geq \gamma(M + p) + M + n\gamma(M + p) + (n - 1)M\\ &\quad \text{by inductive assumption}\\ &= (n + 1)\gamma(M + p) + nM.\end{aligned}$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Thus the induction holds. Now for any $n \in \mathbb{N}$, we have $f(n(M+p)) \geq n\gamma(M+p) + (n-1)M$, so that

$$\frac{f(n(M+p))}{n(M+p)} \geq \gamma + \frac{(n-1)M}{n(M+p)}.$$

Hence

$$\gamma = \lim_{n\to\infty} \frac{f(n)}{n} = \lim_{n\to\infty} \frac{f(n(M+p))}{n(M+p)} \geq \gamma + \frac{M}{M+p} > \gamma,$$

which is a contradiction. Hence f is bounded above by an affine function with gradient γ.

Conversely, assume that f is bounded above by $h := \gamma \cdot \mathrm{id} + c$, where γ is the limiting gradient and $c \in \mathbb{R}$. First note that $\gamma \neq 0$ since, if it were 0, then there would be a constant upper bound for $f(\mathbb{N}_0)$, contradicting Theorem 2.14. Choose an $M \geq 2c/\gamma$ in $\mathbb{N}$ and let $m,\ p \in \mathbb{N}_0$. Then

$$\begin{aligned} f(m) + f(p) \leq h(m) + h(p) &= \gamma(m+p) + 2c \\ &\leq \gamma(m+p+M) \\ &\leq f(m+p+M) \end{aligned}$$

by Lemma 2.12. Hence (2) holds, so Ψ_f is specified. □

6. *Characterization of sofic bounded density shifts*

LEMMA 6.1. *Let Ψ_f be a bounded density shift, where f is canonical. If*

$$(\exists N \in \mathbb{N}_0)(\exists p, k \in \mathbb{N})(\forall m \geq N)\ f(m+p) = f(m) + k \tag{6}$$

then Ψ_f is specified.

Proof. Assume (6). By assumption we can see that

$$f(N) < f(N+p) < f(N+2p) < \cdots$$

so we can pick an $M \in \mathbb{N}$ such that $M = lp$ for some $l \in \mathbb{N}$, $f(M) \geq 2f(N+p)$ and $M \geq pf(N)/k$. We prove (2) of Theorem 5.1. Let $m, n \in \mathbb{N}_0$. If $m < N$ or $n < N$, then we may assume that $n < N$ without loss of generality. Then

$$\begin{aligned} f(m) + f(n) \leq f(m) + f(N) &\leq f(m) + \frac{Mk}{p} \\ &= f(m) + lk \\ &= f(m+M) \quad \text{by (6)} \\ &\leq f(m+n+M). \end{aligned}$$

If $m, n \geq N$, then we may write $m = N + ap + b$ and $n = N + cp + d$ for $a, c \in \mathbb{N}_0$ and $b, d \in [0..p)$. Then

$$\begin{aligned} f(m) + f(n) &= f(N+ap+b) + f(N+cp+d) \\ &= f(N+b) + f(N+d) + ak + ck \\ &\leq 2f(N+p) + ak + ck \end{aligned}$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

$$\begin{aligned}
&\leq f(M) + ak + ck \\
&\leq f(M + 2N + b + d) + ak + ck \\
&= f(M + 2N + b + d + ap + cp) \quad \text{by (6)} \\
&= f(M + m + n). \qquad \square
\end{aligned}$$

LEMMA 6.2. *Let Ψ_f be a specified bounded density shift with specification constant M and canonical function f. Let w be a minimal forbidden word. Then every non-empty subword v of w satisfies $\sum_{r=1}^{|v|} v_r + f(2M) > f(|v|)$.*

Proof. Let $v \leq_s w$ with $v \neq \varepsilon$. Then $w = xvy$ for some $x, y \in A^*$. Since Ψ_f is specified, we have

$$f(|x|) + f(|v|) + f(|y|) \leq f(|xv| + M) + f(|y|) \leq f(|xvy| + 2M) \tag{7}$$

by Theorem 5.1. Since $x, y \in L(\Psi_f)$, we have $\sum_{r=1}^{|x|} x_r \leq f(|x|)$ and $\sum_{r=1}^{|y|} y_r \leq f(|y|)$. Also $\sum_{r=1}^{|w|} w_r > f(|w|)$. This gives

$$\begin{aligned}
f(|v|) &\leq f(|w| + 2M) - f(|x|) - f(|y|) \quad \text{by (7)} \\
&\leq f(|w|) + f(2M) - f(|x|) - f(|y|) \\
&< \left(\sum_{r=1}^{|w|} w_r\right) + f(2M) - \left(\sum_{r=1}^{|x|} x_r\right) - \left(\sum_{r=1}^{|y|} y_r\right) \\
&= \left(\sum_{r=1}^{|v|} v_r\right) + f(2M). \qquad \square
\end{aligned}$$

We let $V(G)$ be the set of vertices of a graph G and $E(G)$ be its set of edges. An element (I, a, J) in $E(G)$ represents an edge from I to J labelled a. We rename the three projection maps to be the *source* map $s : E(G) \to V(G)$, *edge labelling* $\lambda : E(G) \to A$ and *target* map $t : E(G) \to V(G)$ respectively.

THEOREM 6.3. *Let Ψ_f be a bounded density shift and let f be canonical. The following are equivalent.*

(1) *Ψ_f is sofic.*

(2) $(\exists p \in \mathbb{N})(\exists N, k \in \mathbb{N}_0)(\forall m \geq N)\ f(m + p) = f(m) + k$.

(3) *The characteristic sequence is eventually periodic.*

Proof. (1) $\Longrightarrow$ (3). Let Ψ_f be sofic, let G be a finite, right resolving presentation of Ψ_f and let α be a right infinite path presenting $\overrightarrow{\varepsilon}$. It is easy to see that

$$\lambda(\alpha_{m+1}) = \max\{\lambda(e) : s(e) = s(\alpha_{m+1})\}, \tag{8}$$

since otherwise $\overrightarrow{\varepsilon}$ would not be the maximum element in Ψ_f^+. Since G is finite, there exists N, $p \in \mathbb{N}$ such that $s(\alpha_N) = s(\alpha_{N+p})$. From (8) it is easy to see that $\alpha_m = \alpha_{m+p}$ for all $m \geq N$. Hence $\overrightarrow{\varepsilon}_m = \overrightarrow{\varepsilon}_{m+p}$ for all $m \geq N$, so $\overrightarrow{\varepsilon}$ is eventually periodic.

(3) $\Longrightarrow$ (2). Let $\overrightarrow{\varepsilon}$ be eventually periodic. Then there are N, $p \in \mathbb{N}$ such that $\overrightarrow{\varepsilon}_{m+p} = \overrightarrow{\varepsilon}_m$ for all $m \geq N$. Choose $k = \sum_{r=N}^{N+p-1} \overrightarrow{\varepsilon}_r$. By a simple induction, we see

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

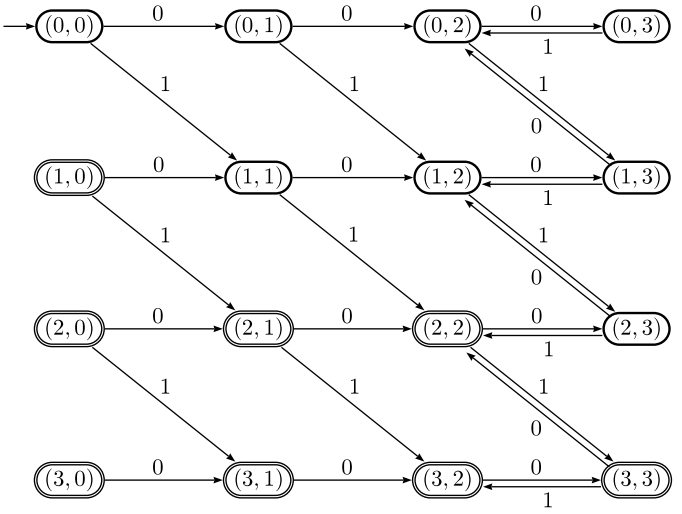


FIGURE 2. The finite state machine G from the proof of Theorem 6.3 for the golden mean shift. Here $N = p = 2$ and $k = 1$.

that $k = \sum_{r=m}^{m+p-1} \overrightarrow{\varepsilon}_r$ for all $m \geq N$. Hence

$$f(m+p) = \sum_{r=1}^{m} \overrightarrow{\varepsilon}_r + \sum_{r=m+1}^{m+p} \overrightarrow{\varepsilon}_r = f(m) + k$$

for all $m \geq N$ by Lemma 2.7.

(2) $\Longrightarrow$ (1). If $k = 0$, then f is clearly eventually constant. Each such shift is non-transitive by Theorem 2.14 and hence sofic by Theorem 2.16. So we may assume $k \neq 0$. By Lemma 6.1, Ψ_f is specified with some specification constant M. By choosing larger values for M and N, we can assume that $N = 2M$. When $p = 1$, we get the full $(k+1)$-shift which is sofic so we also assume $p > 1$. Construct a finite state automaton G as follows:

$$V(G) = \{(i, j) : i \in [0..f(1)(N + p - 1)] \text{ and } j \in [0..N + p)\},$$

$E(G) = E(G)_1 \cup E(G)_2$, where

$$E(G)_1 = \{((i, j), a, (i + a, j + 1)) : (i, j) \in V(G) \text{ and } j \neq N + p - 1 \text{ and } a \in A\}$$

and

$$E(G)_2 = \{((i, N + p - 1), a, (i + a - k, N)) : k - a \leq i \leq N + p - 1 \text{ and } a \in A\},$$

$I = (0, 0)$ and $F = \{(i, j) : i > f(j)\}$.

An example of the machine constructed is given for the golden mean shift in Figure 2.

Let $\mathcal{F}$ be the set of minimal forbidden words. We show that $\mathcal{F} \subseteq L(G) \subseteq A^* \backslash L(\Psi_f)$, proving that $L(G)$ is a regular set of forbidden words for Ψ_f and hence that Ψ_f is sofic by Lemma 1.8.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

This construction is designed to remember two pieces of information. The first coordinate of any vertex remembers the sum of the letters of any word. The second remembers the length of the word. However, once the length of a word is at least $N+p$, we can use the assumption (2) to subtract k from the sum of the word and p from the length. This is necessary for the construction to be finite.

Let $I_m(G)$ be the set of paths of length m in G that begin at the initial vertex I. To show $L(G) \subseteq A^* \backslash L(\Psi_f)$, we will first prove by induction: for all $m \in \mathbb{N}_0$,

$$(\forall \alpha \in I_m(G)) \sum_{r=1}^{m} \lambda(\alpha)_r = \pi_1(t(\alpha)) + ck \quad \text{and} \quad \pi_2(t(\alpha)) + cp = m, \tag{9}$$

where $c = 0$ for $m < N$ and $c = \lfloor (m-N)/p \rfloor$ for $m \geq N$. The value c here counts the number of edges that are in $E(G)_2$. Equation (9) will show the above assertion that the vertices count the letters of the word and its length.

The case $m = 0$ is trivial. Assume then that (9) holds for some $m \in \mathbb{N}_0$. Let $\alpha \in I_{m+1}(G)$ and let $\lambda(\alpha) = w$. Then $\sum_{r=1}^{m} w_r = \pi_1(t(\alpha_m)) + ck$ and $\pi_2(t(\alpha_m)) + cp = m$, where $c = 0$ if $m < N$ and $c = \lfloor (m-N)/p \rfloor$ if $m \geq N$.

Case 1: $m + 1 \neq N + (c+1)p$. This implies that $\lfloor (m+1-N)/p \rfloor = \lfloor (m-N)/p \rfloor = c$ if $m \geq N$. Also $\pi_2(s(\alpha_{m+1})) = \pi_2(t(\alpha_m)) = m - cp \neq N - 1 + p$, so that α_{m+1} must be in $E(G)_1$. Then

$$\alpha_{m+1} = \left(\left(\left(\sum_{r=1}^{m} w_r \right) - ck,\, m - cp \right),\, w_{m+1},\, \left(\left(\sum_{r=1}^{m+1} w_r \right) - ck,\, m - cp + 1 \right) \right).$$

Thus $\sum_{r=1}^{m+1} w_r = \pi_1(t(\alpha)) + ck$ and $\pi_2(t(\alpha)) + cp = m + 1$, where $c = \lfloor (m+1-N)/p \rfloor$.

Case 2: $m + 1 = N + (c+1)p$. Then $\pi_2(t(\alpha_m)) = m - cp = N + p - 1$, so that α_{m+1} must be in $E(G)_2$. Thus

$$\alpha_{m+1} = \left(\left(\left(\sum_{r=1}^{m} w_r \right) - ck,\, N + p - 1 \right),\, w_{m+1},\, \left(\left(\sum_{r=1}^{m+1} w_r \right) - (c+1)k,\, N \right) \right).$$

Note here that $m \geq N$ and $\lfloor (m+1-N)/p \rfloor = c + 1$. Then $\sum_{r=1}^{m+1} w_r = \pi_1(t(\alpha)) + (c+1)k$ and $\pi_2(t(\alpha)) + (c+1)p = N + (c+1)p = m + 1$, completing the induction.

Let $w \in L(G)$, so that there is an $\alpha \in P(G)$ accepting w. Let $t(\alpha) = (i, j)$. This means that $i > f(j)$ since $(i, j) \in F$. Thus, for $j \geq N$, we have

$$\begin{aligned} \sum_{r=1}^{|w|} w_r &= \pi_1(t(\alpha)) + ck \quad \text{by (9)} \\ &> f(j) + ck \\ &= f(j + cp) \quad \text{by an induction on (2) of the theorem} \\ &= f(|w|) \quad \text{by (9)}, \end{aligned}$$

where $c = \lfloor (|w| - N)/p \rfloor$. If $j < N$, then

$$\sum_{r=1}^{|w|} w_r = \pi_1(t(\alpha)) = i > f(j) = f(\pi_2(t(\alpha))) = f(|w|)$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

by (9). In both cases $w \notin L(\Psi_f)$, so $L(G) \subseteq A^* \backslash L(\Psi_f)$. Next we show that $\mathcal{F} \subseteq L(G)$. Let $w \in \mathcal{F}$ so that w is a minimal forbidden word. By Lemma 6.2,

$$(\forall v \leq_s w) v \neq \varepsilon \implies \left(\sum_{r=1}^{|v|} v_r\right) + f(N) > f(|v|). \tag{10}$$

Construct α by

$$\alpha_m = \left(\left(\sum_{r=1}^{m-1} w_r, m-1\right), w_m, \left(\sum_{r=1}^{m} w_r, m\right)\right)$$

if $m \leq N + p - 1$,

$$\alpha_m = \left(\left(\left(\sum_{r=1}^{m-1} w_r\right) - ck, m - 1 - cp\right), w_m, \left(\left(\sum_{r=1}^{m} w_r\right) - ck, m - cp\right)\right)$$

if $N + cp + 1 \leq m \leq N + (c+1)p - 1$ for some $c \in \mathbb{N}$, or

$$\alpha_m = \left(\left(\left(\sum_{r=1}^{m-1} w_r\right) - (c-1)k, m - 1 - (c-1)p\right), w_m, \left(\left(\sum_{r=1}^{m} w_r\right) - ck, m - cp\right)\right)$$

if $m = N + cp$ for some $c \in \mathbb{N}$, for all $m \leq |w|$.

We need to check this is a path in G accepting w. We first check that $\alpha_m \in E(G)$ for each m. Let $m \leq |w|$.

Case 1: $m \leq N + p - 1$. This case is trivial.

Case 2: $N + cp + 1 \leq m \leq N + (c+1)p - 1$ for some $c \in \mathbb{N}$. Then $N \leq m - 1 - cp \leq N + p - 2$ so that $m - 1 - cp \neq N + p - 1$. Then α_m can only be in $E(G)_1$ and it is sufficient to show that $s(\alpha_m), t(\alpha_m) \in V(G)$. But we have

$$\begin{aligned}
\pi_1(t(\alpha_m)) = \left(\sum_{r=1}^{m} w_r\right) - ck &\geq \pi_1(s(\alpha_m)) \\
&= \left(\sum_{r=1}^{m-1} w_r\right) - ck \\
&\geq \left(\sum_{r=1}^{N+cp} w_r\right) - ck \\
&> f(N + cp) - f(N) - ck \quad \text{by (10)} \\
&= 0 \quad \text{by (2)}.
\end{aligned}$$

Similarly, it can be shown that $\pi_1(s(\alpha_m)), \pi_1(t(\alpha_m)) \leq f(1)(N + p - 1)$.

This gives $0 \leq \pi_1(s(\alpha_m)) \leq (N + p - 1) f(1)$, and similarly for $\pi_1(t(\alpha_m))$. This shows that the first coordinates of $s(\alpha_m)$ and $t(\alpha_m)$ are allowed. The second coordinates follow from

$$0 \leq N \leq m - 1 - cp = \pi_2(s(\alpha_m)) \leq m - cp = \pi_2(t(\alpha_m)) \leq N + p - 1.$$

Hence $\alpha_m \in E(G)$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Case 3: $m = N + cp$ for some $c \in \mathbb{N}$. This is proven in a similar way to case 2.

It is straightforward to prove that α is a path in G. Clearly, $s(\alpha) = (0, 0) = I$, so that α starts at the initial state. We only need to check that it ends in a final state. If $|w| \leq N + p - 1$, then this is trivial; we must have $t(\alpha) = (\sum_{r=1}^{|w|} w_r, |w|)$ and $\sum_{r=1}^{|w|} w_r > f(|w|)$ implying that $t(\alpha) \in F$. Otherwise, $N + cp \leq |w| \leq N + (c+1)p - 1$ for some $c \in \mathbb{N}$ upon which $t(\alpha) = ((\sum_{r=1}^{|w|} w_r) - ck, |w| - cp)$. We have

$$\sum_{r=1}^{|w|} w_r - ck > f(|w|) - ck = f(|w| - cp),$$

which gives $t(\alpha) \in F$. Hence $w \in L(G)$. Since we have shown that $\mathcal{F} \subseteq L(G) \subseteq A^* \backslash L(\Psi_f)$ with $L(G)$ regular, Lemma 1.8 proves that Ψ_f is sofic. □

We now prove a couple of lemmas that will help to construct presentations of sofic bounded density shifts. We use $\downarrow$ to mean the downset with respect to $\preceq$. We write $F(u) = \{x \in A^{\mathbb{N}} : ux \in \Psi_f^+\}$.

LEMMA 6.4. *Let Ψ_f be a bounded density shift and let $u \in L(\Psi_f)$. Then $F(u) = \downarrow \overrightarrow{u} \cap \Psi_f^+$.*

Proof. Let $x \preceq \overrightarrow{u}$ and $x \in \Psi_f^+$. Let $|u| = n$. Then for all $i \leq n$, we have $u_{[i,n]}x \preceq u_{[i,n]}\overrightarrow{u}$. Since $x \in \Psi_f^+$, this shows $ux \in \Psi_f^+$.

Conversely, if $ux \in \Psi_f^+$, then $x \in \Psi_f^+$ and $ux \preceq u\overrightarrow{u}$ implying $x \preceq \overrightarrow{u}$. That is, $x \in F(u)$ implies $x \in \downarrow \overrightarrow{u} \cap \Psi_f^+$. □

The following lemma is a corollary of Lemma 6.4.

LEMMA 6.5. *Let Ψ_f be a bounded density shift and let $u, v \in L(\Psi_f)$. Then $F(u) = F(v) \iff \overrightarrow{u} = \overrightarrow{v}$.*

We can use the previous lemma and the follower set presentation to draw presentations of bounded density shifts. Two presentations are given.

Define f inductively by $f(0) = 0$, $f(1) = 1$, $f(2) = f(3) = 2$ and $f(n+4) = f(n) + 2$ for $n \geq 0$. This gives a canonical function which has 0, 1, 2, 2, 2, 3, 4, 4, 4, ... as its image and is a 3-step SFT. It can be determined that

$$\overrightarrow{\varepsilon} = (1100)^\infty, \qquad \overrightarrow{1} = (1001)^\infty,$$
$$\overrightarrow{11} = (0011)^\infty, \qquad \overrightarrow{110} = (0110)^\infty,$$
$$\overrightarrow{10} = (10)^\infty, \qquad \overrightarrow{101} = (01)^\infty,$$

and that every other $\overrightarrow{u}$ is equal to one of the above. Its minimal presentation is shown in Figure 3.

Define g inductively by $g(m) = m$ for $m \leq 3$, $g(4) = 3$ and $g(n+2) = g(n) + 1$ for $n \geq 3$. This gives a canonical function which has 0, 1, 2, 3, 3, 4, 4, 5, 5, ... as its image and is strictly sofic. Its minimal presentation is shown in Figure 4.

LEMMA 6.6. *If Ψ_f is a sofic bounded density shift, then the limiting gradient is rational.*

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

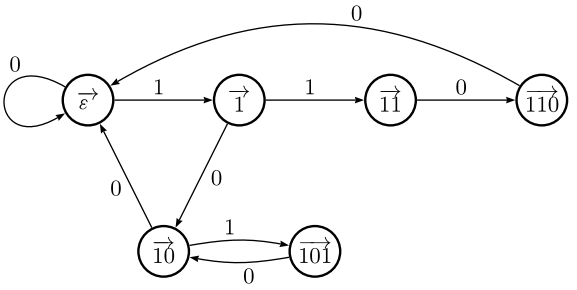


FIGURE 3. Presentation for the bounded density shift given by f.

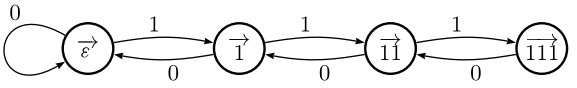


FIGURE 4. Presentation for the bounded density shift given by g.

Proof. Assume f is canonical. Theorem 6.3(2) gives $p \in \mathbb{N}$ and $N, k \in \mathbb{N}_0$ such that

$$(\forall m \geq N)\ f(m+p) = f(m) + k.$$

We claim that $\lim_{m\to\infty} f(m)/m = k/p$. We will use Lemma 2.12. By choosing a larger multiple of p and k, we may assume that $p \geq N$. We have $f(N) + k = f(N+p) \leq f(N) + f(p)$, so $k \leq f(p)$. Now for all $s \in \mathbb{N}$, we have

$$\frac{f(sp)}{sp} = \frac{f(p) + (s-1)k}{sp} \geq \frac{k + (s-1)k}{sp} = \frac{k}{p}.$$

Thus $\lim_{m\to\infty} f(m)/m = \lim_{s\to\infty} f(sp)/(sp) \geq k/p$.

We now show that $\inf\{f(m)/m : m \in \mathbb{N}\} \leq k/p$. Let $\delta > 0$ and choose $\nu = \max\{f(p)/\delta, p\}$. Let $m > \nu$. Then we may write $m = sp + t$ for some $s \in \mathbb{N}$ and $t \in [0..p)$. So

$$\frac{f(m)}{m} = \frac{f(sp+t)}{m} \leq \frac{f(p)}{m} + \frac{sk}{m} < \delta + \frac{k}{p}.$$

Hence $\inf\{f(m)/m : m \in \mathbb{N}\} \leq k/p$ and so $\lim_{m\to\infty} f(m)/m = k/p \in \mathbb{Q}$. □

The converse is false as can be seen in Example 6.

LEMMA 6.7. *For all $n \in \mathbb{N}_0$ and $k \in [0, n]$, there is a bounded density shift Ψ_f with alphabet $[0..n]$ and limiting gradient γ.*

Proof. Define a function $g : [0, \infty) \to \mathbb{R}$ by

$$g(m) := \begin{cases} (\gamma - n)m^2 + (2n - \gamma)m & \text{if } m < 1, \\ \gamma m - \gamma + n & \text{if } m \geq 1, \end{cases}$$

and a function $f : \mathbb{N}_0 \to [0, \infty)$ by $f(m) := \lceil \gamma m - \gamma + n \rceil$ for all $m \in \mathbb{N}$ and $f(0) := 0$. It can be seen that g is differentiable on $(0, \infty)$ and satisfies the hypotheses of Lemma 2.13, so f is canonical. Clearly, the limiting gradient is γ. Figure 5 shows the general curve of the graph. □

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

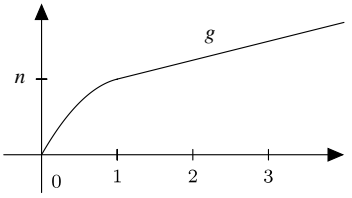


FIGURE 5. A typical example of the function g defined in Lemma 6.7.

We can now show that there are specified bounded density shifts which are not sofic.

Example 5. Every bounded density shift constructed in Lemma 6.7 is specified except for when $\gamma = 0$. To see this, note that, for all $m \in \mathbb{N}_0$,

$$f(m) = \lceil \gamma m - \gamma + n \rceil \leq \gamma m - \gamma + n + 1,$$

where the right side is an affine function with gradient γ. Since $\gamma \neq 0$, Ψ_f is transitive by Theorem 2.14, so that Ψ_f is specified by Theorem 5.1. For each irrational γ, Lemma 6.6 shows that Ψ_f is not sofic.

There are also bounded density shifts with dense periodic points that are not specified.

Example 6. Let $\gamma \in \mathbb{R}^+$ and $b > 1$. Define a function f by $f(n) := \lceil \gamma n + \log_b(n+1) \rceil$ for all $n \in \mathbb{N}_0$. By Lemma 2.13, it is easy to see that f is canonical. The limiting gradient is γ by Lemma 2.13, since

$$\lim_{n \to \infty} \frac{\gamma n + \log_b(n+1)}{n} = \gamma.$$

As $\gamma > 0$, Theorem 3.1 implies that Ψ_f has a dense set of periodic points. To check that Ψ_f is not specified, let $c \in \mathbb{R}$ and let $g := \gamma \cdot \mathrm{id} + c$ be a function on $\mathbb{N}_0$. Choose an $n > b^c - 1$ in $\mathbb{N}$. Then

$$f(n) \geq \gamma n + \log_b(n+1) > \gamma n + \log_b(b^c) = g(n),$$

so that f is not bounded above by g. Theorem 5.1 shows that Ψ_f is not specified.

7. *Characterization of characteristic sequences with bounded blocks of 0s*

LEMMA 7.1. *For a non-trivial bounded density shift Ψ_f, where f is canonical, the following statements are equivalent.*

(1) $(\forall m \in \mathbb{N})(\exists M \in \mathbb{N})(\forall p \in \mathbb{N})(\forall n \leq m) f(n) + f(p) \leq f(n+p+M)$.
(2) $(\forall m \in \mathbb{N})(\exists M \in \mathbb{N})(\forall p \in \mathbb{N}) f(m) + f(p) \leq f(m+p+M)$.
(3) $(\exists n, N \in \mathbb{N})(\forall p \in \mathbb{N}) f(n) + f(p) \leq f(n+p+N)$.
(4) $(\exists k, l \in \mathbb{N})(\forall p \in \mathbb{N}) f(p) + k \leq f(p+l)$.
(5) $(\exists l \in \mathbb{N})(\forall p \in \mathbb{N}) f(p) \neq f(p+l)$.
(6) *The characteristic sequence does not contain arbitrarily long subwords consisting entirely of 0s.*

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Proof. We can see that the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) hold trivially. Note that the assumption of non-triviality is needed for (3) $\Longrightarrow$ (4). Also (5) $\Longleftrightarrow$ (6) can be seen to hold by Lemma 2.7.

(5) $\Longrightarrow$ (4). Choose $k = 1$ and l as in (5). Let $p \in \mathbb{N}$. Then $f(p) \neq f(p+l)$ so that $f(p) < f(p+l)$. But this means that $f(p) + 1 \leq f(p+l)$.

(4) $\Longrightarrow$ (3). We choose $n = 1$. First, there is an $m \in \mathbb{N}$ such that $f(1) \leq km$. Now choose $N = ml$ and let $p \in \mathbb{N}$. It is easy to show that

$$(\forall i \in \mathbb{N}) f(p) + ki \leq f(p + li)$$

by induction. Thus we have

$$f(n) + f(p) = f(1) + f(p) \leq f(p) + km \leq f(p + lm) = f(p + N) \leq f(p + n + N).$$

(3) $\Longrightarrow$ (2). We can show this by induction. For $m = 1$, choose $M = n + N - 1$. Let $p' \in \mathbb{N}$. Then

$$f(1) + f(p') \leq f(n) + f(p') \leq f(n + p' + N) = f(1 + p' + M).$$

Now assume that (2) holds for some $m \in \mathbb{N}$. Choose $M' = 2M + m - 1$ and let $p' \in \mathbb{N}$. Then

$$\begin{aligned} f(m+1) + f(p') &\leq f(m) + f(p') + f(1) \\ &\leq f(m + p' + M) + f(1) \quad \text{by induction with } p = p' \\ &\leq f(m + p' + M) + f(m) \\ &\leq f(2m + p' + 2M) \quad \text{by induction with } p = m + p' + M \\ &= f(m + 1 + p' + M'). \end{aligned}$$

This completes the induction. Notice that (3) is only needed for the base case.

(2) $\Longrightarrow$ (1). Let $m \in \mathbb{N}$. Then for each $n \leq m$, there is an $M_n \in \mathbb{N}$ such that

$$(\forall p \in \mathbb{N}) f(n) + f(p) \leq f(n + p + M_n)$$

holds. Choose $M := \max\{M_n : n \leq m\}$, let $p \in \mathbb{N}$ and let $n \leq m$. Then

$$f(n) + f(p) \leq f(n + p + M_n) \leq f(n + p + M),$$

so (1) holds. □

This part of the theorem gives further equivalences of Lemma 7.1.

THEOREM 7.2. *Let Ψ_f be a non-trivial bounded density shift and let f be canonical. Then the following are equivalent.*

(1) *The characteristic sequence does not contain arbitrarily long subwords consisting entirely of 0s.*
(2) *The characteristic sequence is presented in some strongly connected presentation of Ψ_f.*
(3) *The left characteristic sequence is presented in some strongly connected presentation of Ψ_f.*

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Proof. (1) $\Longrightarrow$ (2). Assume that the characteristic sequence does not contain arbitrarily long subwords consisting entirely of 0s. As is the case for every element of Ψ_f^+, the path α defined by $\alpha_i = (F(\overrightarrow{\varepsilon}_{[1,i)}), \overrightarrow{\varepsilon}_i, F(\overrightarrow{\varepsilon}_{[1,i]}))$ presents $\overrightarrow{\varepsilon}$. So we only need to show that the follower set presentation is strongly connected. Since there is a path from $F(\varepsilon)$ to $F(u)$ for every $u \in L(\Psi_f)$, it is sufficient to show that there is a path from $F(u)$ to $F(\varepsilon)$ for every $u \in L(\Psi_f)$. Let $u \in L(\Psi_f)$ and let $|u| = m$. Then there is an $M \in \mathbb{N}$ such that

$$(\forall p \in \mathbb{N})(\forall n \leq m) f(n) + f(p) \leq f(n + p + M)$$

by Lemma 7.1. Let $x = u0^M \overrightarrow{\varepsilon}$. We first show that $x \in \Psi_f^+$. Let $i, j \in \mathbb{N}$. If $i + j - 1 \leq m + M$ or if $i > m$, then $x_{[i,i+j)} \in L(\Psi_f)$ since $x_{[i,i+j)} \leq_s u0^M$ or $x_{[i,i+j)} \leq_s 0^M \overrightarrow{\varepsilon}$ respectively. So we may assume $i \leq m$ and $i + j > m + M$. Then

$$\begin{aligned}\sum_{r=i}^{i+j-1} x_r &= \sum_{r=i}^{m} u_r + \sum_{r=1}^{i+j-1-m-M} \overrightarrow{\varepsilon}_r \\ &\leq f(m - i + 1) + f(i + j - 1 - m - M) \\ &\leq f(j)\end{aligned}$$

by assumption. Hence $x \in \Psi_f^+$. Let $v := u0^M$. We have $v\overrightarrow{v} \succeq x = v\overrightarrow{\varepsilon}$, since $v\overrightarrow{v}$ is the maximum element in $C^+[v] \cap \Psi_f^+$ by Lemma 2.5. This gives $\overrightarrow{v} \succeq \overrightarrow{\varepsilon}$. But also $\overrightarrow{v} \preceq \overrightarrow{\varepsilon}$, since $\overrightarrow{\varepsilon}$ is the maximum element of Ψ_f^+. Thus $\overrightarrow{v} = \overrightarrow{\varepsilon}$. Lemma 6.5 now gives $F(v) = F(\varepsilon)$. Thus there is a path labelled 0^M from $F(u)$ to $F(v) = F(\varepsilon)$. Hence the characteristic sequence is presented by a strongly connected presentation.

(2) $\Longrightarrow$ (3). Let G be a strongly connected presentation and let α be a path in G presenting $\overrightarrow{\varepsilon}$. Construct the reverse graph G^R by reversing all the edges; so $V(G^R) = V(G)$ and $E(G^R) = \{(I, a, J) : (J, a, I) \in E(G)\}$. For any $I, J \in V(G^R)$, there is a path β from J to I in G, so the reverse path β^R is a path from I to J in G^R. Thus G^R is strongly connected. In a similar way, it is easy to see that α^R is a left infinite path presenting $\overleftarrow{\varepsilon}$.

(3) $\Longrightarrow$ (1). Assume that G is a strongly connected presentation and α is a left infinite path presenting $\overleftarrow{\varepsilon}$ in G. Remember that α has index set $-\mathbb{N}$. We prove (4) of Lemma 7.1. First, there is a path β with $s(\beta) = t(\alpha)$ and $t(\beta) = s(\alpha_{-1})$, since G is strongly connected. Let $\lambda(\beta) = v$. Choose $k = f(1)$ and $l = |v| + 1$. Let $p \in \mathbb{N}$ so that we have $\overleftarrow{\varepsilon}_{[-p,-1]} v \overleftarrow{\varepsilon}_{-1} \in L(\Psi_f)$, since it is presented by $\alpha_{[-p,-1]}\beta\alpha_{-1}$. Then

$$\begin{aligned}f(p) + k &= \left(\sum_{r=1}^{p} \overrightarrow{\varepsilon}_r\right) + \overrightarrow{\varepsilon}_1 \\ &\leq \sum_{r=-p}^{-1} \overleftarrow{\varepsilon}_r + \sum_{r=1}^{l-1} v_r + \overleftarrow{\varepsilon}_{-1} \\ &\leq f(|\overleftarrow{\varepsilon}_{[-p,-1]} v \overleftarrow{\varepsilon}_{-1}|) \\ &= f(l + p).\end{aligned}$$

Hence $\overrightarrow{\varepsilon}$ does not contain arbitrarily long subwords consisting entirely of 0s. □

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Example 7. Recall that the functions defined by $f(n) := \lceil \gamma n + \log_b(n+1) \rceil$ are canonical, have dense periodic points and are not specified from Example 6. We will show that each f satisfies Lemma 7.1(5). Choose $l \geq 1/\gamma$ in $\mathbb{N}$ and let $p \in \mathbb{N}$. Then

$$\begin{aligned} f(p) &= \lceil \gamma p + \log_b(p+1) \rceil \\ &< \gamma p + \log_b(p+1) + 1 \\ &< \gamma p + \log_b(p+l+1) + \gamma l \\ &\leq \lceil \gamma(p+l) + \log_b(p+l+1) \rceil \\ &= f(p+l). \end{aligned}$$

Let $\mathrm{Pre}(u) = \{v \in A^* : (\exists w \in A^*)\ vw = u\}$ be the set of prefixes of u and $\mathrm{Suf}(u) = \{v \in A^* : (\exists w \in A^*)\ wv = u\}$ be the set of suffixes of u.

LEMMA 7.3. *Let Ψ_f be a bounded density shift. Then v is intrinsically synchronizing if and only if $\overleftarrow{v} \cdot v \overrightarrow{v} \in \Psi_f$.*

Proof. Assume f is canonical and let v be intrinsically synchronizing. Then for all $n \in \mathbb{N}$, we have $\overleftarrow{v}_{[-n,-1]}v,\ v\overrightarrow{v}_{[1,n]} \in L(\Psi_f)$, giving $\overleftarrow{v}_{[-n,-1]}v\overrightarrow{v}_{[1,n]} \in L(\Psi_f)$. Thus $\overleftarrow{v} \cdot v \overrightarrow{v} \in \Psi_f$.

Conversely, assume that $\overleftarrow{v} \cdot v \overrightarrow{v} \in \Psi_f$. Let $uv, vw \in L(\Psi_f)$. We must have $w \preceq \overrightarrow{v}_{[1,|w|]}$ and $u^R \preceq (\overleftarrow{v}_{[-|u|,-1]})^R = \overleftarrow{v}_{-1}\overleftarrow{v}_{-2}\ldots\overleftarrow{v}_{-|u|}$. Let $x \leq_s uvw$. If $x \leq_s uv$ or $x \leq_s vw$, then $x \in L(\Psi_f)$ by assumption. So we may assume that $x = u'vw'$ for some $u' \in \mathrm{Suf}(u)$ and $w' \in \mathrm{Pre}(w)$. Note that we have

$$\sum_{r=1}^{|u'|} u'_r = \sum_{r=1}^{|u'|} u_r^R \leq \sum_{r=-|u'|}^{-1} \overleftarrow{v}_r.$$

Then

$$\begin{aligned} \sum_{r=1}^{|x|} x_r &= \sum_{r=1}^{|u'|} u'_r + \sum_{r=1}^{|v|} v_r + \sum_{r=1}^{|w'|} w'_r \\ &\leq \sum_{r=-|u'|}^{-1} \overleftarrow{v}_r + \sum_{r=1}^{|v|} v_r + \sum_{r=1}^{|w'|} \overrightarrow{v}_r \\ &\leq f(|u'| + |v| + |w'|) \quad \text{as } \overleftarrow{v} \cdot v\overrightarrow{v} \in \Psi_f \\ &= f(|x|). \end{aligned}$$

Hence $x \in L(\Psi_f)$ showing that v is intrinsically sychronizing. □

THEOREM 7.4. *Let Ψ_f be a bounded density shift. Then there is a strongly connected presentation presenting both the left and right characteristic sequences if and only if Ψ_f is specified.*

Proof. Let Ψ_f be specified with specification constant M and assume f is canonical. We may assume that $M > 1$. Then $f(m) + f(p) \leq f(m + p + M)$ for all $m, p \in \mathbb{N}$ by Theorem 5.1 which clearly implies (2) of Lemma 7.1. Thus $\overrightarrow{\varepsilon}$ does not contain arbitrarily long subwords consisting entirely of 0s. By Theorem 7.2 there is a strongly connected

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

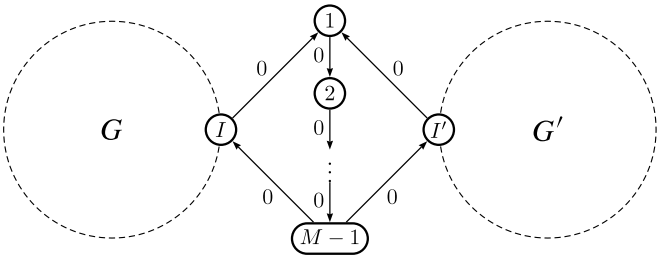


FIGURE 6. The presentation H of Theorem 7.4.

presentation G in which $\overrightarrow{\varepsilon}$ is presented and there is a strongly connected presentation G' in which $\overleftarrow{\varepsilon}$ is presented. We may assume that the vertices of G and G' are disjoint and are also disjoint from $[1..M)$. Fix an $I \in V(G)$ and $I' \in V(G')$. Construct a presentation H by

$$V(H) = V(G) \cup V(G') \cup [1..M)$$

and

$$E(H) = E(G) \cup E(G') \cup \{(I, 0, 1), (I', 0, 1), (M-1, 0, I), (M-1, 0, I')\}$$
$$\cup \{(i, 0, i+1) : i \in [1..M-2]\}.$$

Figure 6 gives the construction of H. It is clear that the right and left characteristic sequences are presented in H and that it is strongly connected, see Figure 6. Since we may place 0^M in between words in $L(\Psi_f)$ by Theorem 5.1, and since we may extend words by 0s, it is easy to see that H presents Ψ_f.

Conversely, let G be a strongly connected presentation of Ψ_f and let α be a left infinite path in G presenting $\overleftarrow{\varepsilon}$ and γ be a right infinite path in G presenting $\overrightarrow{\varepsilon}$. Then there is a path β with $s(\beta) = t(\alpha)$ and $t(\beta) = s(\gamma)$. Let $\lambda(\beta) = v$ and $|v| = M$. Then $\overleftarrow{\varepsilon} \cdot v \overrightarrow{\varepsilon} \in \Psi_f$, so that $\overleftarrow{\varepsilon} \cdot 0^M \overrightarrow{\varepsilon} \in \Psi_f$ by Lemma 2.2. It is easy to see that $\overleftarrow{0^M} = \overleftarrow{\varepsilon}$ and $\overrightarrow{0^M} = \overrightarrow{\varepsilon}$, so that $\overleftarrow{0^M} \cdot 0^M \overrightarrow{0^M} \in \Psi_f$. Hence 0^M is intrinsically synchronizing by Lemma 7.3 and so Ψ_f is specified by Theorem 5.1. □

THEOREM 7.5. *Let Ψ_f be a synchronized bounded density shift and assume that the characteristic sequence does not have arbitrarily long subwords consisting of 0s. Then Ψ_f is specified.*

Proof. By Theorem 7.2, there are strongly connected presentations G and G' presenting Ψ_f also presenting $\overrightarrow{\varepsilon}$ and $\overleftarrow{\varepsilon}$ in G and G' respectively. By assumption, there is also an intrinsically synchronizing word v. By constructing paths in these presentations, we can find $u, w \in L(\Psi_f)$ such that $\overleftarrow{\varepsilon} uv \in \Psi_f^-$ and $vw\overrightarrow{\varepsilon} \in \Psi_f^+$. Thus $\overleftarrow{\varepsilon}_{[-n,-1]}uv$, $vw\overrightarrow{\varepsilon}_{[1,n]} \in L(\Psi_f)$ for all $n \in \mathbb{N}$, so that $\overleftarrow{\varepsilon}_{[-n,-1]}uvw\overrightarrow{\varepsilon}_{[1,n]} \in L(\Psi_f)$ for every $n \in \mathbb{N}$. Thus $\overleftarrow{\varepsilon}.uvw\overrightarrow{\varepsilon} \in \Psi_f$. By Lemma 2.2, we have $\overleftarrow{\varepsilon}.0^M\overrightarrow{\varepsilon} \in \Psi_f$, where $M = |uvw|$. Thus $\overleftarrow{0^M} \cdot 0^M \overrightarrow{0^M} \in \Psi_f$, showing that 0^M is intrinsically synchronizing by Lemma 7.3. Hence Ψ_f is specified by Theorem 5.1. □

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

We now construct a function which gives a coded shift where the characteristic sequence has arbitrarily long blocks of 0s. We will construct a function f inductively. To do this we will start with $f(0) = 0$ and $f(1)$ to be any natural number. We will choose a number $\gamma \in (0, f(1))$ to be a lower bound for $f(n)/n$, so that Ψ_f will indeed be coded. This number will turn out to be the limiting gradient. For each step in the construction, the first thing we do is find and count the longest block of the domain which has constant image under the partially constructed function f. This will always be a consecutive set of natural numbers since we want f to be canonical. Say we have defined f on $[0..n]$ and the longest block with constant image under f is m. We check whether defining f on $[n+1..n+m]$ to be $f(n)$ will keep f strictly above the line $\gamma \cdot \mathrm{id}$. If it is, then we will define f this way which will mean that f now has an even larger block, at least length $m+1$, with the same image. If it is not, then we will define $f(n+1)$ to be the largest value that maintains the canonical form. The biggest problem this raises is how to know that there will always be arbitrarily long blocks on which f is constant. This will be answered by contradiction in the proof.

Example 8. We construct a function inductively. Choose $f(0) = 0$ and $f(1) \in \mathbb{N}$. Let $\gamma \in (0, f(1))$. Assume f is defined on the domain $[0..n]$. Define

$$S(p) := \max\{|j - i| + 1 : f(i) = f(j) \text{ and } i, j \leq p\}$$

for $p \in [0..n]$. Note that S is a non-decreasing function.

If $f(n)/(n + S(n)) \leq \gamma$, then define

$$f(n+1) := \min\{f(l) + f(n+1-l) : l \in [1..n]\}. \tag{11}$$

But if $f(n)/(n + S(n)) > \gamma$, then define

$$f(n+1) = f(n+2) = \cdots = f(n + S(n)) := f(n). \tag{12}$$

We will show that f is canonical, that the characteristic sequence has arbitrarily long blocks of 0s and that γ is the limiting gradient.

Now $f(0) = 0$. We rewrite (2) and (3) of Theorem 2.9 and prove these by modified induction: for all $n \in \mathbb{N}$,

$$f(n) \in f(n-1) + \mathbb{N}_0 \tag{13}$$

and

$$(\forall m \in [1..n))\, f(n) \leq f(m) + f(n-m). \tag{14}$$

We have $f(1) \in \mathbb{N} \subseteq f(0) + \mathbb{N}_0$ and $[1..1) = \varnothing$ so the base case holds. Assume that there is an $n \in \mathbb{N}$ such that (13) and (14) hold for all values in $[1..n]$.

Case 1: $f(n+1)$ was defined by (11). That is, $f(n+1) = \min\{f(l) + f(n+1-l) : l \in [1..n]\}$. We have

$$\begin{aligned} f(n+1) &= \min\{f(l) + f(n+1-l) : l \in [1..n]\} \\ &\geq \min\{f(l) + f(n-l) : l \in [1..n]\} \quad \text{by (13)} \\ &\geq \min\{f(n) : l \in [1..n]\} \quad \text{by (14) and } f(0) = 0 \text{ for } l = n \\ &= f(n). \end{aligned}$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

It is clear that $f(\mathbb{N}_0) \subseteq \mathbb{N}_0$, so we conclude that $f(n+1) \in f(n) + \mathbb{N}_0$. Now let $m \in [1..n]$. Then

$$f(n+1) = \min\{f(l) + f(n+1-l) : l \in [1..n]\} \leq f(m) + f(n+1-m),$$

so this case holds.

Case 2: $f(n+1)$ was defined by (12). That is, $f(n+1) = f(n)$. We clearly have $f(n+1) \in f(n) + \mathbb{N}_0$, so let $m \in [1..n]$. Then

$$f(n+1) = f(n) \leq f(m) + f(n-m) \leq f(m) + f(n+1-m)$$

using (14) and (13) when $m \neq n$. This inequality still holds when $m = n$ so this case is complete.

These are the only cases, so f is canonical.

Before the next part we use modified induction to prove

$$\frac{f(n)}{n} > \gamma \tag{15}$$

for all $n \in \mathbb{N}$. We have $f(1)/1 > \gamma$ by choice of γ, so assume there is an $n \in \mathbb{N}$ such that $f(m)/m > \gamma$ for all $m \leq n$. There are two cases. If $f(n+1)$ was defined by (11), then $f(n+1) = \min\{f(l) + f(n+1-l) : l \in [1..n]\}$, so there is an $l \in [1..n]$ such that $f(n+1) = f(l) + f(n+1-l)$. Thus

$$\begin{aligned}
\frac{f(n+1)}{n+1} &= \frac{f(l)}{n+1} + \frac{f(n+1-l)}{n+1} \\
&= \frac{l}{n+1} \cdot \frac{f(l)}{l} + \frac{n+1-l}{n+1} \cdot \frac{f(n+1-l)}{n+1-l} \\
&> \frac{l}{n+1} \cdot \gamma + \frac{n+1-l}{n+1} \cdot \gamma \quad \text{by the inductive hypothesis} \\
&= \gamma.
\end{aligned}$$

Otherwise, $f(n+1)$ was defined by (12) so there is an $l \in \mathbb{N}_0$ such that $f(l)/(l+S(l)) > \gamma$ and $l < n+1 \leq l + S(l)$. We had $f(n+1) := f(l)$ so

$$\frac{f(n+1)}{n+1} = \frac{f(l)}{n+1} \geq \frac{f(l)}{l+S(l)} > \gamma.$$

Hence the induction is complete.

Next we show that the characteristic sequence has arbitrarily long blocks of 0s. Suppose that this is false. Then there is an $M \in \mathbb{N}$ such that, for every $p \in \mathbb{N}_0$, we have $f(p) \neq f(p+M)$ by Lemma 7.1. We may assume that M is as small as possible so there is a $q \in \mathbb{N}$ such that $f(q) = f(q+M-1)$. Then, for all $r \geq M+q-1$,

$$S(r) = \max\{|j-i|+1 : f(i) = f(j) \text{ and } i, j \leq r\} \geq |q+M-1-q|+1 = M.$$

If $S(r) > M$, then there would be $i < j$ in $\mathbb{N}$ with $f(i) = f(j)$ and $j - i + 1 \geq M+1$. This would imply that $f(i) = f(i+M)$ since $i < i+M \leq j$. Thus we must have $S(r) = M$ for all $r \geq M+q-1$.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

Note that we must have $f(q+M) > f(q+M-1)$ so as not to contradict $f(q) \neq f(q+M)$. Also, there cannot be an $r \geq M+q-1$ such that $f(r)/(r+S(r)) > \gamma$ since, by definition of f, we would have

$$f(r+1) = \cdots = f(r+S(r)) := f(r).$$

As $S(r) = M$ this would give $f(r) = f(r+M)$ contradicting our assumption. Note that we need to check the condition $f(r)/(r+S(r)) \leq \gamma$ for every $r \geq q+M-1$ since $f(q+M) > f(q+M-1)$ and since we will be defining f by (11) for this domain. This is to say that we cannot skip the checking of this condition for some of the domain because (12) defines many values at once.

So, for every $r \geq M+q-1$, we have $f(r)/(r+S(r)) \leq \gamma$. Thus, by definition of f, we have

$$f(r+1) = \min\{f(l) + f(r+1-l) : l \in [1..r]\}.$$

By Theorem 4.3, Ψ_f is an SFT or, equivalently, $\min\{f(n)/n : n \in \mathbb{N}\} := \gamma'$ exists. Then $\gamma' > \gamma$ by (15). Choose an $n \in \mathbb{N}$ such that $n > \max\{M+q-1, \gamma M/(\gamma'-\gamma)\}$. Since $n > M+q-1$, we have $S(n) = M$. Now $f(n)/n \geq \gamma' > \gamma + \gamma M/n$ since $n > \gamma M/(\gamma'-\gamma)$. This implies

$$\frac{f(n)}{n+S(n)} > \frac{\gamma n + \gamma M}{n+S(n)} = \gamma.$$

But we showed that $f(n)/(n+S(n)) \leq \gamma$ earlier, giving us a contradiction. Hence the characteristic sequence has arbitrarily long blocks of 0s.

Finally we show that γ is the limiting gradient. For all $n \in \mathbb{N}$, define

$$r_n := \min\{m \in \mathbb{N} : f(m) = f(m+n-1)\}.$$

Note that the r_n must exist since the characteristic sequence contains arbitrarily long blocks of 0s. Also note that we must have

$$f(r_n - 1) < f(r_n), \tag{16}$$

$$S(r_n) < n \tag{17}$$

for $n \neq 1$ and

$$S(r_n) = S(r_n - 1) \tag{18}$$

for all $n \in \mathbb{N}$. Combining the definition of r_n with (17) gives

$$f(r_n) = f(r_n + S(r_n)) \tag{19}$$

for $n \neq 1$. Choose a $q \in \mathbb{N}$ such that $q \geq f(1)/\gamma$ and $r_{q-1} < r_q$. This implies $S(r_q) = q-1$. Note that $f(1)/\gamma > 1$ so $q \geq 2$. We will be proving statements about values greater than or equal to q so this means we won't need to worry about the case $n = 1$ in (17) and (19).

Let $p \geq q$. We have $f(r_p - 1) < f(r_p)$ by (16) so $f(r_p)$ was defined by (11). Thus we had $f(r_p-1)/(r_p-1+S(r_p-1)) \leq \gamma$, which implies

$$\frac{f(r_p-1)}{\gamma} \leq r_p - 1 + S(r_p-1) = r_p - 1 + S(r_p) \tag{20}$$

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

by (18). Thus

$$\begin{aligned}\frac{f(r_p + S(r_p))}{\gamma} &= \frac{f(r_p)}{\gamma} \quad \text{by (19)} \\ &\le \frac{f(r_p - 1)}{\gamma} + \frac{f(1)}{\gamma} \quad \text{since } f \text{ is canonical} \\ &\le r_p - 1 + S(r_p) + q \quad \text{by (20)} \\ &\le r_p + S(r_p) + q. \end{aligned} \tag{21}$$

Now $f(r_p)/(r_p + S(r_p)) = f(r_p + S(r_p))/(r_p + S(r_p)) > \gamma$ by (19) and (15) so we had defined

$$f(r_p + 1) = f(r_p + 2) = \cdots = f(r_p + S(r_p)) := f(r_p)$$

by (12).

Since $f(r_q) = f(r_q + S(r_q))$ by (19), we must have

$$S(r_q + S(r_q)) \ge |r_q + S(r_q) - r_q| + 1 = S(r_q) + 1.$$

Thus $S(r_p + S(r_p)) \ge S(r_q + S(r_q)) \ge S(r_q) + 1 = q$, so that

$$\begin{aligned}\frac{f(r_p + S(r_p))}{r_p + S(r_p) + S(r_p + S(r_p))} &\le \frac{f(r_p + S(r_p))}{r_p + S(r_p) + q} \\ &= \frac{f(r_p + S(r_p))}{\gamma} \cdot \frac{\gamma}{r_p + S(r_p) + q} \\ &\le \frac{\gamma(r_p + S(r_p) + q)}{r_p + S(r_p) + q} \quad \text{by (21)} \\ &= \gamma. \end{aligned}$$

Then $f(r_p + S(r_p) + 1)$ is defined by (11); that is,

$$f(r_p + S(r_p) + 1) = \min\{f(l) + f(r_p + S(r_p) + 1 - l) : l \in [1..r_p + S(r_p)]\}.$$

Thus there exists an $l \in [1..r_p + S(r_p)]$ such that

$$f(r_p + S(r_p) + 1) = f(l) + f(r_p + S(r_p) + 1 - l).$$

We now claim that $f(r_p + S(r_p) + 1) > f(r_p + S(r_p))$. Suppose, for a contradiction, that $f(r_p + S(r_p) + 1) = f(r_p + S(r_p))$ which by (19) implies that $f(r_p + S(r_p) + 1) = f(r_p)$. We have $S(r_p) + 1 \le p \le r_p$ by (17), so that $l + (r_p + S(r_p) + 1 - l) \le 2r_p$ which implies $l \le r_p$ or $r_p + S(r_p) + 1 - l \le r_p$. By symmetry, we will assume that $l \le r_p$. Then

$$\begin{aligned} f(r_p + S(r_p) + 1 - l) &= f(r_p + S(r_p) + 1) - f(l) \\ &= f(r_p) - f(l) \\ &\le f(r_p - l) \quad \text{since } f \text{ is canonical,} \end{aligned}$$

which gives $f(r_p + S(r_p) + 1 - l) = f(r_p - l)$ since f is non-decreasing. Thus $f(r_p + S(r_p) + 1 - l) = f(r_p - l) \le f(r_p - 1) < f(r_p)$ by (16), which implies $r_p + S(r_p) + 1 - l < r_p$ since f is non-decreasing, which is a contradiction. Thus $f(r_p + S(r_p) + 1) \ne f(r_p + S(r_p))$. By (17) we get $r_p + S(r_p) + 1 \le r_p + p$, which gives $f(r_p + S(r_p) + 1) \le f(r_p + p)$. This implies that $f(r_p) < f(r_p + S(r_p) + 1) \le f(r_p + p)$, which

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

gives $r_{p+1} > r_p$. In fact, this gives

$$r_{p+1} \geq r_p + p$$

for $p \geq q$. To see this, note that if $r_{p+1} = r_p + m$ for some $m \in [1..p)$, then as

$$f(r_p) = f(r_p + 1) = \cdots = f(r_p + p - 1),$$

we would have

$$f(r_{p+1} - 1) = f(r_p + m - 1) = f(r_p + m) = f(r_{p+1}),$$

which contradicts (16). Thus the r_p grow at least quadratically which shows $\lim_{n\to\infty} n/r_n = 0$.

Finally we can show that $\lim_{n\to\infty} f(n)/n = \gamma$. We do this by showing that $\lim_{n\to\infty} f(r_n)/r_n = \gamma$. By (15) we have $\lim_{n\to\infty} f(r_n)/r_n \geq \gamma$. Note that $r_n + S(r_n) < r_n + n \leq r_{n+1}$ by (17). This implies that $S(r_n + S(r_n)) \leq S(r_{n+1})$, which gives $S(r_n + S(r_n)) \leq n$ by (17). Then, for all $n \geq q$, we have

$$\frac{f(r_n + S(r_n))}{r_n + S(r_n) + n} = \frac{f(r_n + S(r_n))}{r_n + S(r_n) + S(r_n + S(r_n))} \leq \gamma.$$

This gives $f(r_n) \leq \gamma(r_n + n - 1 + n)$ by (19) and (17), which implies $f(r_n)/r_n \leq \gamma + k(2n-1)/r_n$ for $n \geq q$. Thus

$$\lim_{n\to\infty} \frac{f(r_n)}{r_n} \leq \lim_{n\to\infty} \left(\gamma + \frac{\gamma(2n-1)}{r_n}\right) = \gamma$$

since $\lim_{n\to\infty} n/r_n = 0$. This shows that $\lim_{n\to\infty} f(r_n)/r_n = \gamma$, proving that the limiting gradient is γ. Then Ψ_f is coded by Theorem 3.1.

Figure 7 is a diagram of the various classes of shifts. Whilst the coded plus synchronized but not specified contains many different bounded density shifts, the border between coded and synchronized is unknown.

8. *Connections with β-shifts*

Recall that we use $\leq$ for the lexicographical ordering.

LEMMA 8.1. *If there are two sequences x and y satisfying $\sigma^n(y) \preceq x$ for all $n \in \mathbb{N}$, then they satisfy $\sigma^n(y) \leq x$ for all $n \in \mathbb{N}$.*

Proof. Assume that $\sigma^n(y) \preceq x$ for all $n \in \mathbb{N}$ and let $n \in \mathbb{N}$. We aim to show that $\sigma^n(y) \leq x$ or, equivalently, $y_{[n+1,\infty)} \leq x$. We show this by proving

$$(\forall m \in \mathbb{N}) y_{[n+1,n+m]} \leq x_{[1,m]}$$

by induction. Note that $\sum_{r=n+1}^{n+m} y_r \leq \sum_{r=1}^{m} x_r$ for all $m \in \mathbb{N}$ since $\sigma^n(y) \preceq x$. Choosing $m = 1$ proves the base case. So assume that $y_{[n+1,n+m]} \leq x_{[1,m]}$ for some $m \in \mathbb{N}$. If $y_{[n+1,n+m]} < x_{[1,m]}$ then it is obvious that $y_{[n+1,n+m+1]} \leq x_{[1,m+1]}$ so we will assume that $y_{[n+1,n+m]} = x_{[1,m]}$. Then

$$\sum_{r=n+1}^{n+m+1} y_r \leq \sum_{r=1}^{m+1} x_r$$

implies $y_{n+m+1} \leq x_{m+1}$. Hence $y_{[n+1,n+m+1]} \leq x_{[1,m+1]}$. This completes the induction and the proof. □

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

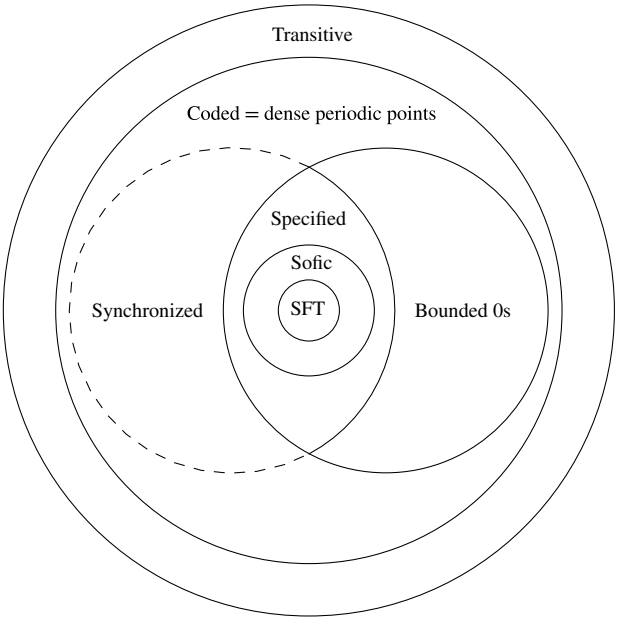


FIGURE 7. Classes of transitive bounded density shifts.

The next corollary follows immediately.

COROLLARY 8.2. *If a sequence x satisfies $\sigma^n(x) \preceq x$ for all $n \in \mathbb{N}$ then it satisfies $\sigma^n(x) \leq x$ for all $n \in \mathbb{N}$.*

THEOREM 8.3. *If x is a characteristic sequence for a transitive bounded density shift, then x is the maximum sequence of a β-shift.*

Proof. Use Theorem 2.14 to show that x has infinitely many non-zero letters and Corollary 8.2 to show $\sigma^n(x) \leq x$ for all $n \in \mathbb{N}$ and complete with Theorem 1.12. □

The converse is false. For $x = 210(20)^\infty$, we see that x is the maximum sequence of some β-shift, yet $\sigma^3(x) \not\preceq x$, so it is not canonical for a bounded density shift.

LEMMA 8.4. *Let Ψ_f be a transitive bounded density shift and let x be its characteristic sequence. Let S_β be the β-shift with maximum sequence x. Then $\Psi_f^+ \subseteq S_\beta^+$.*

Proof. Let $y \in \Psi_f^+$. Then $\sigma^n(y) \preceq x$ for all $n \in \mathbb{N}_0$. Lemma 8.1 shows that $\sigma^n(y) \leq x$ for all $n \in \mathbb{N}_0$ and Theorem 1.13 proves that $y \in S_\beta^+$. □

THEOREM 8.5. *Let Ψ_f be a coded bounded density shift and let x be its characteristic sequence. Let S_β be the β-shift with maximum sequence x. Then the entropy of Ψ_f is non-zero and bounded above by* $\log(\beta)$.

Proof. Assume f is canonical. We have $h(\Psi_f) \leq \log(\beta)$ from Lemma 8.4 and Theorem 1.15.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

We will actually show that the entropy is bounded below by $\log(\beta)/m$, where $m = \lceil 2f(1)/\gamma \rceil$ and γ is the limiting gradient of Ψ_f.

For any $n \in \mathbb{N}$, let $u \in L_n(S_\beta)$. Define $w := u_1 0^{m-1} u_2 0^{m-1} \dots u_n 0^{m-1}$. We show that $w \in L_{mn}(\Psi_f)$. Let $x \leq_s w$ with $|x| = p$. Since there are at least $m-1$ 0s between any two non-zeros, we have at most $\lceil p/m \rceil$ non-zeros in x. When $p = 0$, it is trivial to show $\sum_{r=1}^{p} x_r \leq f(p)$ so we will assume $p > 0$. If $p \leq m$, then there is at most one non-zero, so we have $\sum_{r=1}^{p} x_r \leq f(1) \leq f(p)$. Otherwise, $p > m$ so we have

$$\begin{aligned}
\sum_{r=1}^{p} x_r &\leq f(1) \cdot \left\lceil \frac{p}{m} \right\rceil \quad \text{as there are at most } \left\lceil \frac{p}{m} \right\rceil \text{ non-zeros} \\
&\leq f(1)\left(\frac{p}{m} + 1\right) \\
&\leq f(1)\left(\frac{p\gamma}{2f(1)} + 1\right) \quad \text{since } m \geq \frac{2f(1)}{\gamma} \\
&= \frac{p\gamma}{2} + f(1) \\
&\leq \frac{p\gamma}{2} + \frac{p\gamma}{2} \quad \text{since } f(1) \leq \frac{\gamma m}{2} \leq \frac{\gamma p}{2} \\
&= p\gamma \\
&\leq f(p) \quad \text{since } \gamma \leq \frac{f(p)}{p}.
\end{aligned}$$

Thus $w \in L_{mn}(\Psi_f)$. It is clear that for different u we end up with different w, so we have $|L_n(S_\beta)| \leq |L_{mn}(\Psi_f)|$. This implies

$$\frac{1}{m}\log(\beta) = \lim_{n\to\infty} \frac{\log(|L_n(S_\beta)|)}{mn} \leq \lim_{n\to\infty} \frac{\log(|L_{mn}(\Psi_f)|)}{mn}.$$

This proves that $\lim_{n\to\infty} \log(|L_n(\Psi_f)|)/n \geq \log(\beta)/m$ so the entropy is indeed non-zero. □

9. *Open questions*

Question 1. Is there a nice characterization of synchronized bounded density shifts. In particular, are any of the shifts in Example 8 synchronized?

Question 2. Suppose, for a sofic shift Ψ_f, that we have determined values of p, N and k as in Theorem 6.3(2). Is it always true that $\sigma^N(\overrightarrow{u})$ is periodic with period p for any $u \in L(\Psi_f)$? Also, does $\sum_{r=N+1}^{N+p} \overrightarrow{u}_r = k$?

Since each characteristic sequence is a β-representation for some β, we can ask the following question.

Question 3. What is the measure of the set of β that are characteristic sequences for bounded density shifts? What is the measure for each class in Figure 7? It is easily seen that the number of sofic bounded density shifts is countable so the set of β with an ultimately periodic expansion has measure 0.

Question 4. Are all transitive bounded density shifts intrinsically ergodic?

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press

REFERENCES

[1] F. Blanchard. β-expansions and symbolic dynamics. *Theoret. Comput. Sci.* **65**(2) (1989), 131–141.
[2] V. Climenhaga and D. J. Thompson. Intrinsic ergodicity beyond specification: β-shifts, S-gap shifts, and their factors. *Israel J. Math.* **192**(2) (2012), 785–817.
[3] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
[4] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **11**(3–4) (1960), 401–416.
[5] D. Pyo Chi and D. Kwon. Sturmian words, β-shifts and transcendence. *Theoret. Comput. Sci.* **321**(2–3) (2004), 395–404.
[6] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **8**(3–4) (1957), 477–493.
[7] K. Thomsen. On the structure of beta shifts. *Contemp. Math.* **385** (2005), 321–332.

https://doi.org/10.1017/etds.2013.38 Published online by Cambridge University Press