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ON THE CONTINUITY OF PICKANDS CONSTANTS

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Abstract

For a non-negative separable random field $Z(t)$, $t \in \mathbb{R}^d$, satisfying some mild assumptions, we show that $H_Z^\delta = \lim_{T\to\infty} (1/T^d)\mathbb{E}\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d} Z(t)\} < \infty$ for $\delta \geq 0$, where $0\mathbb{Z}^d := \mathbb{R}^d$, and prove that H_Z^0 can be approximated by H_Z^δ if δ tends to 0. These results extend the classical findings for Pickands constants H_Z^δ, defined for $Z(t) = \exp(\sqrt{2}B_\alpha(t) - |t|^{2\alpha})$, $t \in \mathbb{R}$, with B_α a standard fractional Brownian motion with Hurst parameter $\alpha \in (0, 1]$. The continuity of H_Z^δ at $\delta = 0$ is additionally shown for two particular extensions of Pickands constants.

Keywords: Pickands constants; discrete approximation; locally stationary Gaussian random fields; max-stable random fields; extremal index

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1. Introduction

The discrete Pickands constant H_Z^δ is defined for a given positive δ by

$$H_Z^\delta = \lim_{n\to\infty} \frac{1}{n\delta}\mathbb{E}\left\{\sup_{1\leq k\leq n} Z(k\delta)\right\} = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\right\} \in (0, \infty),$$

where $Z(t) = \exp(\sqrt{2}B_\alpha(t) - |t|^{2\alpha})$ and B_α is a standard fractional Brownian motion (fBm) with Hurst parameter $\alpha \in (0, 1]$. When $\delta = 0$, interpreting $0\mathbb{Z}$ as $\mathbb{R}$, H_Z^0 can be defined in an analogous way, that is,

$$H_Z^0 = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left\{\sup_{t\in[0,T]} Z(t)\right\} \in (0, \infty),$$

which is the classical Pickands constant appearing in the tail asymptotics of the distribution of the supremum for a wide class of Gaussian processes; see e.g. [27], [28], and [13]. Pickands' pioneering method (see [27]) for the approximation of the tail distribution of the supremum for

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a stationary Gaussian process relies strongly on a discretisation approach. A crucial element in Pickands' methodology is the fact that

$$\lim_{\delta \downarrow 0} H_Z^\delta = H_Z^0. \tag{1.1}$$

Notably, Pickands' first attempt to prove (1.1) contains a gap; a correct proof is given in [3, Theorem B3]. See also the comments after [3, Lemma A3].

A systematic study of Pickands constants started with [13, Theorem 1, Proposition 2] showing that

$$H_Z^0 = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} Z(t)}{\eta \sum_{t\in\eta\mathbb{Z}} Z(t)} \right\} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} Z(t)}{\int_{t\in\mathbb{R}} Z(t)\lambda(\mathrm{d}t)} \right\} \tag{1.2}$$

is valid for any $\eta > 0$, where $\lambda(\,\cdot\,)$ is the Lebesgue measure on $\mathbb{R}$.

The two expressions in (1.2) paved the way for simulations of Pickands constants and inspired new formulas for extremal indices of stationary time series; see e.g. [12], [19], [11], [8], [30], [20], and [37].

Pickands constants for general Gaussian processes were considered first in [6]; see also [7]. Later Dębicki and Hashorva [9,19] discussed extensions to general random fields (RFs), as we briefly outline next. Therefore let $Z(t)$, $t \in \mathbb{R}^d$, be a separable, non-negative RF such that

$$\mathbb{E}\{Z(t)\} = 1 \quad \text{for all } t \in \mathbb{R}^d, \tag{1.3}$$

and define the corresponding Pickands constant by

$$H_Z^\delta = \lim_{T\to\infty} T^{-d} \mathbb{E}\left\{ \sup_{t\in[0,T]^d \cap \delta\mathbb{Z}^d} Z(t) \right\}, \quad \delta \geq 0, \tag{1.4}$$

where we set $\delta\mathbb{Z}^d := \mathbb{R}^d$ when $\delta = 0$. If $\delta = 0$, in order to avoid degenerated cases, we shall assume further that

$$\mathbb{E}\left\{ \sup_{t\in[0,T]^d} Z(t) \right\} \in (0, \infty) \quad \text{for all } T > 0. \tag{1.5}$$

Definition (1.4) might not be valid for a general Z since the limit might not exist. However, if for all compact sets $K \subset \mathbb{R}^d$

$$\mathbb{E}\left\{ \sup_{t\in K} Z(t+c) \right\} = \mathbb{E}\left\{ \sup_{t\in K} Z(t) \right\} \quad \text{for all } c \in \mathbb{R}^d, \tag{1.6}$$

then, as we shall show in Section 2, the constant H_Z^δ is well-defined and finite for any $\delta \geq 0$. Notably, we know from [19], [9], and [20] that the limit in (1.4) exists if there is a stationary max-stable process $Y(t)$, $t \in \mathbb{R}^d$, with unit Fréchet marginals, which has spectral process Z in its de Haan representation (see e.g. [18] and [14])

$$Y(t) = \max_{i\geq 1} \Gamma_i^{-1} Z^{(i)}(t), \quad t \in \mathbb{R}^d. \tag{1.7}$$

Here $\Gamma_i = \sum_{k=1}^i \mathcal{V}_k$, with $\mathcal{V}_k$, $k \geq 1$ mutually independent unit exponential random variables (RVs) being independent of $\{Z^{(i)}\}_{i=1}^\infty$, which are independent copies of Z. The finite-dimensional distributions (FIDIs) of Y are given by

$$\mathbb{P}\{Y(t_1) \leq x_1, \ldots, Y(t_n) \leq x_n\} = \mathrm{e}^{-\mathbb{E}\{\max_{i\leq n} Z(t_i)/x_i\}}, \quad x_i > 0,\ t_i \in \mathbb{R}^d,\ i \leq n \tag{1.8}$$

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and hence

$$(\mathbb{P}\{Y(t_1) \le x_1, \ldots, Y(t_n) \le x_n\})^m = \mathbb{P}\{mY(t_1) \le x_1, \ldots, mY(t_n) \le x_n\}$$

for all $m > 0$, which shows that the FIDIs of Y are max-stable. Clearly, if Y is stationary, then $\sup_{t\in K} Y(t+c)$ has the same law as $\sup_{t\in K} Y(t)$ for all $c \in \mathbb{R}^d$, and hence if Y has locally bounded sample paths, then (1.8) implies (1.5) and further (1.6) is valid.

Pickands constants are closely related to extremal indices of the max-stable stationary RF Y. Indeed, under the assumption (1.5) and the finiteness of Pickands constants, the separability of Y implies

$$-\ln \mathbb{P}\left\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d} Y(t) \le rT^d\right\} = \frac{1}{rT^d}\mathbb{E}\left\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d} Z(t)\right\} \to \frac{1}{r}H_Z^\delta \tag{1.9}$$

as $T \to \infty$ for all $r > 0$. Thus, by definition, the extremal index of the stationary RF $Y(t)$, $t \in \delta\mathbb{Z}^d$, is equal to $\delta^d H_Z^\delta \in [0, 1]$ for any $\delta > 0$. Clearly, (1.9) is an approximation of the distribution of the supremum of Y. Such approximations are known for general stationary RFs. A prominent example is Y being a symmetric α-stable RF; see [35], [36], [40], [32], and the references therein.

A natural question that arises here is the relevance of general Pickands constants, both in extreme value theory of RFs and in stochastic modelling. As shown in [10], Pickands constants determined by $Z(t) = \exp(W(t) - \sigma_W^2(t)/2)$, where W is a centered Gaussian RF with stationary increments and variance function σ_W^2, appear naturally in risk and queueing theory. Moreover, as advocated in [20], Pickands constants related to a general non-Gaussian RF Z have appeared in the literature in numerous papers. In that context, considering a general RF Z is important since it unifies the study of extremal indices and Pickands constants.

The second question is: for what general Z does the limit (1.1) hold? The answer to this question is presented in Section 2. Such a result has the following two immediate consequences.

(A) Pickands' discretisation method for the study of extremes of Gaussian RFs can be utilised also for cases where the limiting constants are determined by general Gaussian RFs with stationary increments.

(B) The calculation of H_Z^0 can be carried out by simulating H_Z^δ for small $\delta > 0$. Therefore it is also interesting to derive tractable formulas for H_Z^δ as given by (2.3).

The organisation of the rest of the paper is as follows. In Section 2 we first discuss the finiteness and continuity of discrete Pickands constants (in Theorem 2.1) and then derive a formula corresponding to (1.2) (in Proposition 2.1), which shows in particular how to approximate H_Z^0 using the FIDIs of Z. Section 3 is concerned with two extensions. The first one is motivated by results related to symmetric α-stable RFs derived in [35, 36], whereas the second extension is motivated by the constant

$$\mathcal{H}_Z^\delta = \lim_{T\to\infty} T^{-d} \int_0^1 \mathbb{E}\left\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d} Z_z(t)\right\} \mathrm{d}z \tag{1.10}$$

defined in [5] for $\delta = 0$. Here Z_z, $z \in [0, 1]$ is equal in law to $\exp(W_z(t) - \sigma_z^2(t)/2)$, with W_z a centered Gaussian RF with stationary increments, variance function σ_z^2 and almost surely

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continuous sample paths. In Theorem 3.1 we establish the continuity of $\mathcal{H}_Z^\delta$ at $\delta = 0$. Such a result is crucial for the applications of Pickands' discretisation method in the study of extremes of locally stationary Gaussian RFs. It is also of certain relevance for the simulations of those constants. We have relegated all the proofs to Section 4.

2. Main results

Our first result establishes the finiteness of Pickands constants and (1.1) for a general RF Z under some weak restrictions. In particular, our result is satisfied for Z such that $Y(t)$, $t \in \mathbb{R}^d$, defined in (1.7) is a stochastically continuous stationary max-stable RF with locally bounded sample paths. Hereafter we shall suppose that all RFs are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the following, for a given stochastically continuous RF $U(t)$, $t \in \mathcal{T}$, we shall suppose that it is also separable and jointly measurable. In view of [16], a separable and jointly measurable version of a stochastically continuous RF exists.

Theorem 2.1. *If $Z(t)$, $t \in \mathbb{R}^d$, is a non-negative stochastically continuous RF such that (1.3), (1.5), and (1.6) hold, then the constants defined in (1.4) are finite for all $\delta \geq 0$ and further $\lim_{\delta \downarrow 0} H_Z^\delta = H_Z^0$.*

In the following,

$$S_\delta(Z) = \int_{t \in \delta \mathbb{Z}^d} Z(t) \lambda_\delta(\mathrm{d}t), \quad \delta \geq 0,$$

where $\lambda_0(\mathrm{d}t) = \lambda(\mathrm{d}t)$ is the Lebesgue measure on $\mathbb{R}^d$ and for $\delta > 0$ $\lambda_\delta(\mathrm{d}t) = \delta^d \lambda(\mathrm{d}t)$ with $\lambda(\mathrm{d}t)$ the counting measure on $\delta \mathbb{Z}^d$.

Clearly $S_\delta(Z)$ is an RV for any $\delta > 0$. If $\delta = 0$ (recall that $0\mathbb{Z}^d$ simply denotes $\mathbb{R}^d$), since we consider Z to be jointly measurable, then supposing that (1.5) holds and using that Z is non-negative (recall that we assume the probability space is complete), it follows that $S_0(Z)$ is an RV; see [16, Theorems 2.7 and 2.8] for details of the case $d = 1$.

Corollary 2.1. *Suppose that Z satisfies the assumptions of Theorem 2.1 and further*

$$\mathbb{P}\left\{ \sup_{t \in \mathbb{R}^d} Z(t) > 0 \right\} = 1.$$

If $S_\eta(Z) = \infty$ almost surely for some $\eta \geq 0$, then $H_Z^\delta = 0$ for all $\delta \geq 0$. Conversely, if $H_Z^\delta = 0$ for some $\delta \geq 0$, then $S_\eta(Z) = \infty$ almost surely for all $\eta \geq 0$.

Remark 2.1.

(i) The assumption $\mathbb{P}\{\sup_{t \in \mathbb{R}^d} Z(t) > 0\} = 1$ in Corollary 2.1 cannot be removed. Taking for instance $Z(t) = V/p$, $t \in \mathbb{R}^d$, with V a Bernoulli RV such that $\mathbb{P}\{V = 1\} = p \in (0, 1)$, we get $H_Z^\delta = 0$ for all $\delta \geq 0$. However, $S_1(Z) = 0$ with probability $1 - p > 0$.

(ii) Let $Z(t)$, $t \in \mathbb{R}^d$, be a stationary RF satisfying the assumptions of Theorem 2.1. By the definition and the fact that Z and $\tilde{Z} = 1 + Z$ are both stationary, the corresponding Pickands constants exist and we simply have $H_Z^\delta = H_{\tilde{Z}}^\delta$ for all $\delta \geq 0$. Clearly $S_1(\tilde{Z}) = \infty$ and $\mathbb{P}\{\sup_{t \in \mathbb{R}^d} \tilde{Z}(t) > 0\} = 1$, hence Corollary 2.1 implies $H_Z^\delta = 0$ and moreover $S_\eta(Z) = \infty$, $\eta \geq 0$ if further $\mathbb{P}\{\sup_{t \in \mathbb{R}^d} Z(t) > 0\} = 1$.

In the rest of this section we consider $Y(t)$, $t \in \mathbb{R}^d$, a stationary max-stable RF as in Section 1 with spectral RF Z and de Haan representation (1.7). Suppose next that Z has sample paths

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almost surely in the space $D = D(\mathbb{R}^d, [0, \infty))$ of generalised càdlàg functions $f\colon \mathbb{R}^d \to [0, \infty)$ equipped with the σ-field $\mathcal{D} = \sigma(\pi_t, t \in \mathcal{T}_0)$ generated by the projection maps $\pi_t\colon \pi_t f = f(t)$, $f \in D$, and $\mathcal{T}_0$ a dense subset of $\mathbb{R}^d$. See e.g. [21] and [4] for the definition and properties of generalised càdlàg functions. In view of [19, Theorem 6.9] (take $\alpha = 1$ and $L = B^{-1}$ therein), the stationarity of Y is equivalent to the validity of

$$\mathbb{E}\{Z(h)F(Z)\} = \mathbb{E}\{Z(0)F(B^h Z)\} \tag{2.1}$$

for all $h \in \mathbb{R}^d$ and all 0-homogeneous measurable functionals $F\colon D \mapsto [0, \infty)$ (0-homogeneous means $F(cf) = F(f)$, $\forall c > 0, f \in D$) with $B^h Z(\cdot) = Z(\cdot - h)$, $h \in \mathbb{R}^d$. For discrete max-stable processes, (2.1) is stated in [30, eq. (5.2)]; see also [37] for other equivalent formulations. We note in passing that (2.1) is initially derived for Z as in Example 2.1 below in [12, Lemma 5.2].

Clearly, if Z is stationary, then Y is stationary too. However, this instance is not interesting, since as shown in Remark 2.1, (ii) $H_Z^0 = 0$ in this case. We discuss below two other examples such that max-stable RF Y is stationary.

Example 2.1. It is known from [23] and [19] that if $X(t) = W(t) - \mathrm{Var}(W(t))/2$, $t \in \mathbb{R}^d$, with W a centered Gaussian RF with stationary increments and almost surely continuous sample paths, then both (1.3) and (1.6) hold with $Z(t) = \mathrm{e}^{X(t)}$. Using for instance [39, Theorem 1], we have that (1.5) holds. Moreover, in view of [24], the corresponding max-stable RF Y is stationary.

Example 2.2. Let $L(t)$, $t \in \mathbb{R}^d$, be a non-negative deterministic measurable function such that $\int_{\mathbb{R}^d} L(t)\lambda(\mathrm{d}t) = 1$, with $\lambda(\mathrm{d}t)$ the Lebesgue measure on $\mathbb{R}^d$. It follows easily that $Z(t) = L(t - N)/p(N)$, $t \in \mathbb{R}^d$, with N an $\mathbb{R}^d$-valued RV having a positive density function $p(t) > 0$, $t \in \mathbb{R}^d$ satisfies (1.6). Note that for this case condition (1.5) reads as

$$\int_{x \in \mathbb{R}^d} \sup_{t \in [0,T]^d} L(t - x)\lambda(\mathrm{d}x) < \infty \quad \text{for all } T > 0. \tag{2.2}$$

As shown in [20, Theorem 1, eq. (3.5)], without any further assumption on the max-stable stationary RF Y, for $\delta = \eta > 0$ we have

$$H_Z^\delta = \mathbb{E}\left\{ Z(0) \frac{\sup_{t \in \delta\mathbb{Z}^d} Z(t)}{S_\eta(Z)} \right\} \in [0, \infty). \tag{2.3}$$

In the next result we show that the above holds under some weak assumptions also for $\delta = 0$. It turns out that under (2.4) below we can also obtain the first formula in (1.2).

Proposition 2.1. *Let $Z(t)$, $t \in \mathbb{R}^d$, be a non-negative RF with almost surely sample paths in D. If (1.3), (1.5), and (2.1) hold and $H_Z^0 > 0$, then (2.3) holds for $\delta = \eta = 0$. Moreover, (2.3) holds also for $\delta = 0$, $\eta > 0$ or $\delta > 0$, $\eta = k\delta$, $k \in \mathbb{N}$ if also*

$$\{S_0(Z) < \infty\} \subset \{S_\eta(B^r Z) \in (0, \infty)\} \quad \textit{for all } r \in \delta\mathbb{Z}^d \tag{2.4}$$

almost surely.

Remark 2.2. It is shown in the proof of Proposition 2.1 that (2.4) is implied by the assumption $\mathbb{P}\{Z(0) > 0\} = 1$, which is satisfied for the choice of Z as in Example 2.1. In particular, Proposition 2.1 extends Theorems 2 and 3 of [11] and Proposition 2 of [13]. Moreover, as discussed in Example 2.3 below, condition (2.4) cannot be removed.

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Example 2.3. Let $L(t)$, $t \in \mathbb{R}^d$, be as in Example 2.2 and suppose further that $L(t) > 0$, $t \in \mathbb{R}^d$, (2.2) holds and $L \in D(\mathbb{R}^d, [0, \infty))$. Then the claim of Theorem 2.1 follows. Moreover, Proposition 2.1 implies for all $\eta > 0$

$$H_Z^0 = \int_{\mathbb{R}^d} L(s) \frac{\sup_{t\in\mathbb{R}^d} L(t+s)}{\eta^d \sum_{t\in\eta\mathbb{Z}^d} L(t+s)} \lambda(\mathrm{d}s) = \sup_{t\in\mathbb{R}^d} L(t), \tag{2.5}$$

where the last equality follows by (2.3) with $\delta = 0$ (recall that $\int_{\mathbb{R}^d} L(t)\lambda(\mathrm{d}t) = 1$). It is mentioned in [13] that for $d = 1$ and L the standard Gaussian density on $\mathbb{R}$, the above identity could be verified numerically. As shown by Dmitry Zaporozhets (personal communications) the last two equalities in (2.5) can be derived by utilising the translation invariance of both the Lebesgue measure and the counting measure, respectively, and applying the Fubini–Tonelli theorem. Specifically, setting $\eta = 1$ for simplicity,

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{L(s)}{\sum_{t\in\mathbb{Z}^d} L(t+s)} \lambda(\mathrm{d}s) &= \sum_{r\in\mathbb{Z}^d} \int_{r+[0,1)^d} \frac{L(s)}{\sum_{t\in\mathbb{Z}^d} L(t+s)} \lambda(\mathrm{d}s) \\ &= \sum_{r\in\mathbb{Z}^d} \int_{[0,1)^d} \frac{L(r+s)}{\sum_{t\in\mathbb{Z}^d} L(r+t+s)} \lambda(\mathrm{d}s) \\ &= \int_{[0,1)^d} \sum_{r\in\mathbb{Z}^d} \frac{L(r+s)}{\sum_{t\in\mathbb{Z}^d} L(t+s)} \lambda(\mathrm{d}s) = 1.\end{aligned}$$

Our initial proof of Proposition 2.1 was asymptotic in nature. A modification of Zaporozhets' arguments and (2.1) led to the current proof of Proposition 2.1.

Note that if we take for instance $L(t) = \mathbf{1}\{t \in [0, 1]\}$, $d = 1$, with $\mathbf{1}\{\cdot\}$ the indicator function, then H_Z^0 cannot be given by (2.3) with $\eta > 2$, since the first equality in (2.5) gives $H_Z^0 = \infty$, which is a contradiction to the fact that $H_Z^0 < \infty$. For this choice of η we have that condition (2.4) is not satisfied.

3. Two extensions

Motivated by [36] and [35], we first consider an alternative definition of Pickands constants $H^\delta_{|f|,m}$ defined with respect to a collection of functions $f_s \colon E \to \mathbb{R}$, with $(E, \mathcal{E})$ a measurable space equipped with some σ-finite measure m. In the second part of this section we discuss the constant $\mathcal{H}^\delta_Z$ defined in (1.10), also proving its continuity at $\delta = 0$.

3.1. Behaviour of $H^\delta_{|f|,m}$ at $\delta = 0$

Let m be a σ-finite measure on some measurable space $(E, \mathcal{E})$ and $f = f_s(z)$, $s \in \mathbb{R}^d$, $z \in E$ real functions, where $f_s \in \mathcal{L}^1(m)$ ($\mathcal{L}^\alpha(m)$ is the set of all functions g such that $\int_E |g(x)|^\alpha m(\mathrm{d}x) < \infty$ and $\alpha > 0$). We now define for any $\delta = 1/n$, $n \in \mathbb{N}$, or $\delta = 0$

$$H^\delta_{|f|,m} = \lim_{T\to\infty} \frac{1}{T^d} \int_E \sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d} |f_t(z)| m(\mathrm{d}z),$$

where $0\mathbb{Z}^d$ equals $\mathbb{Q}_*^d$, with $\mathbb{Q}_*$ the set of dyadic rational numbers $\{k/2^n \colon k \in \mathbb{Z}, n \in \mathbb{N}\}$. Clearly the above limit is in general not defined. In order to include the case $\delta = 0$, we shall assume further that

$$\int_E \sup_{t\in K\cap\mathbb{Q}_*^d} |f_t(z)| m(\mathrm{d}z) < \infty \tag{3.1}$$

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and, similarly to (1.6), we have

$$\int_E \sup_{t\in K\cap\mathbb{Q}_*^d} |f_{t+c}(z)|m(\mathrm{d}z) = \int_E \sup_{t\in K\cap\mathbb{Q}_*^d} |f_t(z)|m(\mathrm{d}z) \quad \text{for all } c\in\mathbb{Q}_*^d \tag{3.2}$$

for all compact sets $K\subset\mathbb{R}^d$.

If L is as in Example 2.2, then taking $f_t(z)=L(z-t)$, $t, z\in\mathbb{R}^d$, $E=\mathbb{R}^d$ equipped with the Borel σ-field and $m(\mathrm{d}z)$ the Lebesgue measure on $\mathbb{R}^d$, we have that (3.1) holds and further (3.2) is satisfied since $m(\mathrm{d}z)$ is shift-invariant.

Example 3.1. Let $Y(t)$, $t\in\mathbb{R}^d$, be a symmetric α-stable stationary RF with locally bounded sample paths, $\alpha\in(0,2)$, and representation

$$Y(t)=\int_E f_t(z)M(\mathrm{d}z), \quad t\in\mathbb{R}^d,$$

where M is a symmetric α-stable random measure on E with control measure m and $f_t\in\mathcal{L}^\alpha(m)$, $t\in\mathbb{R}^d$; see [36]. It follows that both (3.1) and (3.2) hold; see [36] for the case $d=1$ and [33] for the case $d>1$.

With the same arguments as in the proof of Theorem 2.1, we have that under (3.1) and (3.2), constant $H^\delta_{|f|,m}$ is finite, non-negative, and further

$$\lim_{n\to\infty} H^{2^{-n}}_{|f|,m} = H^0_{|f|,m}. \tag{3.3}$$

Remark 3.1. For Y as in Example 3.1, explicit formulas for $H^\delta_{|f|,m}$ are derived in [36], [35], and [34]. Utilising the relation between α-stable and max-stable processes (see [22]), the aforementioned formulas imply (2.3) when $\delta=\eta\geq 0$.

3.2. Continuity of $\mathcal{H}^\delta_Z$ at $\delta=0$

Let $W_z(t)$, $t\in\mathbb{R}^d$, $z\in[0,1]$, be a centered Gaussian RF with stationary increments, almost surely continuous sample paths, and variance function σ_z^2, $z\in[0,1]$, such that $\sigma_z^2(0)=0$ for all $z\in[0,1]$. We formulate below the assumptions on the variance functions σ_z^2 imposed in Theorem 3.1 below. Specifically, we shall assume that

$$\lim_{w\to z}\sigma_w(t)=\sigma_z(t) \quad \text{for all } z\in[0,1],\ t\in\mathbb{R}^d \tag{3.4}$$

and further, for some C_0, C_∞ positive and ν_0, $\nu_\infty\in(0,2]$,

$$\limsup_{\|t\|\to 0}\frac{\sigma_z^2(t)}{\|t\|^{\nu_0}}\leq C_0 \quad \text{and} \quad \limsup_{\|t\|\to\infty}\frac{\sigma_z^2(t)}{\|t\|^{\nu_\infty}}\leq C_\infty \tag{3.5}$$

hold for all $z\in[0,1]$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Theorem 3.1. *If* (3.4) *and* (3.5) *are satisfied, then for any* $\delta\geq 0$ *and* $Z_z(t)=\exp\left(W_z(t)-\sigma_z^2(t)/2\right)$,

$$\mathcal{H}^\delta_Z=\lim_{T\to\infty}T^{-d}\int_0^1\mathbb{E}\left\{\sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d}Z_z(t)\right\}\mathrm{d}z=\int_0^1 H^\delta_{Z_z}\,\mathrm{d}z\in[0,\infty), \tag{3.6}$$

and furthermore $\lim_{\delta\downarrow 0}\mathcal{H}^\delta_Z=\mathcal{H}^0_Z$.

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Remark 3.2.

(i) A sufficient condition for $H^{\delta}_{Z_z}$, $z \in [0,1]$ to be positive is

$$\lim_{\|t\|\to\infty} \frac{\sigma_z^2(t)}{\ln\|t\|} > 8d \tag{3.7}$$

(see [9]). If the above holds for $z \in [0,1]$ on a set with non-zero Lebesgue measure, then under the assumptions of Theorem 3.1 we have that $\mathcal{H}^{\delta}_{Z} > 0$ for any $\delta \geq 0$.

(ii) In order to define RFs W_z, $z \in [0,1]$ with stationary increments, we need to determine variance function $\sigma_z^2(t) = \mathrm{Var}(W_z(t+s) - W_z(t))$, $s, t \in \mathbb{R}^d$, which is a negatively (or conditionally negatively) definite function. One example is to start with a negatively definite variance function $\sigma^2(t)$, $t \in \mathbb{R}^d$, and then define $\sigma_z(t) = q(z)\sigma(t)$ for some continuous function q not identical to zero. Such σ_z clearly satisfies (3.4) and if $\sigma^2(t) \leq C\|t\|^{\nu}$, $t \in \mathbb{R}^d$, then (3.5) also holds. Another example is to consider σ_z as in [5], namely

$$\sigma_z^2(t) = \|t\|^{\lambda} r_z(t/\|t\|), \quad \lambda \in (0,2], \ t \in \mathbb{R}^d, \tag{3.8}$$

where r_z is a non-negative function defined on the unit sphere on $\mathbb{R}^d$ determined by the norm $\|\cdot\|$.

4. Proofs

Proof of Theorem 2.1. In view of (1.5) and (1.6),

$$\mathbb{E}\left\{\sup_{t\in K\cap\delta\mathbb{Z}^d} Z(t+c)\right\} = \mathbb{E}\left\{\sup_{t\in K\cap\delta\mathbb{Z}^d} Z(t)\right\} < \infty \tag{4.1}$$

holds for all $c \in \mathbb{R}^d$, $\delta \geq 0$, and all compact sets $K \subset \mathbb{R}^d$. Next, assume for notational simplicity that $d = 1$. In light of (4.1), $\mathbb{E}\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\}$, $T > 0$, is a subadditive function of T for any $\delta \geq 0$, implying that H^{δ}_{Z} is a finite non-negative constant (recall that Z is non-negative) in view of Fekete's lemma. Further, to show the second part of the theorem it is enough to take $T = a_\delta n$, where $a_\delta = \lfloor 1/\delta \rfloor \delta$ with $\lfloor x \rfloor$ the integer part of $x > 0$ and $n \in \mathbb{N}$ $\delta \in (0, 1/2)$. Utilising (4.1) to derive the last equality in the next calculations, we have

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{t\in[0,T]} Z(t)\right\} - \mathbb{E}\left\{\max_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\right\} \\
&= \mathbb{E}\left\{\max_{1\leq i\leq n} \sup_{t\in[a_\delta(i-1),a_\delta i]} Z(t) - \max_{1\leq i\leq n} \max_{t\in[a_\delta(i-1),a_\delta i]\cap\delta\mathbb{Z}} Z(t)\right\} \\
&\leq \mathbb{E}\left\{\max_{1\leq i\leq n}\left(\sup_{t\in[a_\delta(i-1),a_\delta i]} Z(t) - \max_{t\in[a_\delta(i-1),a_\delta i]\cap\delta\mathbb{Z}} Z(t)\right)\right\} \\
&\leq \sum_{i=1}^{n} \mathbb{E}\left\{\sup_{t\in[a_\delta(i-1),a_\delta i]} Z(t) - \max_{t\in[a_\delta(i-1),a_\delta i]\cap\delta\mathbb{Z}} Z(t)\right\} \\
&= n\mathbb{E}\left\{\sup_{t\in[0,a_\delta]} Z(t) - \max_{t\in[0,a_\delta]\cap\delta\mathbb{Z}} Z(t)\right\}.
\end{aligned} \tag{4.2}$$

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The assumption that Z is stochastically continuous implies that any dense subset of $\mathbb{R}^d$ is a separant for the separable RF Z (see e.g. [31, Theorem 2.9]) and therefore, since $\lim_{\delta \downarrow 0} a_\delta = 1$, we have the following convergence in probability:

$$\max_{t\in[0,a_\delta]\cap\delta\mathbb{Z}} Z(t) \to \sup_{t\in[0,1]} Z(t)$$

as $\delta \downarrow 0$. Hence, by (1.5) and the dominated convergence theorem,

$$\lim_{\delta\downarrow 0} \mathbb{E}\left\{\max_{t\in[0,a_\delta]\cap\delta\mathbb{Z}} Z(t)\right\} = \mathbb{E}\left\{\sup_{t\in[0,1]} Z(t)\right\} < \infty. \tag{4.3}$$

Consequently (4.2) implies

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left\{\max_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\right\} &\le \frac{1}{T}\mathbb{E}\left\{\sup_{t\in[0,T]} Z(t)\right\} \\ &\le a_\delta^{-1}\mathbb{E}\left\{\sup_{t\in[0,a_\delta]} Z(t) - \max_{t\in[0,a_\delta]\cap\delta\mathbb{Z}} Z(t)\right\} + \frac{1}{T}\mathbb{E}\left\{\max_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\right\}.\end{aligned}$$

Since H_Z^δ exists and is finite for all $\delta \ge 0$, we have

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sup_{t\in[0,T]} Z(t)\right\} = \lim_{n\to\infty}\frac{1}{na_\delta}\mathbb{E}\left\{\sup_{t\in[0,na_\delta]} Z(t)\right\} = H_Z^0$$

and

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} Z(t)\right\} = \lim_{n\to\infty}\frac{1}{na_\delta}\mathbb{E}\left\{\sup_{t\in[0,na_\delta]\cap\delta Z} Z(t)\right\} = H_Z^\delta \le H_Z^0 < \infty.$$

Thus, letting n to infinity, we obtain from the above inequalities

$$0 \le H_Z^0 - H_Z^\delta \le a_\delta^{-1}\mathbb{E}\left\{\sup_{t\in[0,a_\delta]} Z(t) - \max_{t\in[0,a_\delta]\cap\delta\mathbb{Z}} Z(t)\right\}, \tag{4.4}$$

which together with (4.3) yields $\lim_{\delta\downarrow 0} H_Z^\delta = H_Z^0$. □

Proof of Corollary 2.1. First, let us notice that $H_Z^\delta = 0$ for some $\delta \ge 0$ if and only if $S_\delta(Z) = \infty$ almost surely, which is a direct implication of [14], [22], and [34, 35, 36] ; this has already been discussed in [17], [9], and [37] for the case $d = 1$, and [20] for the d-dimensional discrete setup. Note in passing that in [14] Z is such that $\mathbb{P}\{\sup_{t\in\mathbb{R}^d} Z(t) > 0\} = 1$. So if for some $\eta > 0$ we have $H_Z^\eta = 0$, then $S_{\eta/k}(Z) = \infty$ almost surely for any $k \in N$ and $H_Z^{\eta/k} = 0$. Consequently, Theorem 2.1 implies that $\lim_{k\to\infty} H_Z^{\eta/k} = H_Z^0 = 0$ and hence $0 \le H_Z^\eta \le H_Z^0 = 0$ for all $\eta \ge 0$. Conversely, if for some $\eta \ge 0$ we have $S_\eta(Z) = \infty$, then from the above equivalence $H_Z^\eta = 0$ for all $\eta \ge 0$, hence the proof is complete. □

Proof of Proposition 2.1. The claim for $\delta = \eta = 0$ follows as in [9], where $d = 1$ is considered and its proof is therefore omitted.

Now let $D = D(\mathbb{R}^d, [0, \infty))$ be the space of generalised càdlàg functions $f\colon \mathbb{R}^d \mapsto [0, \infty)$, which can be equipped with a metric that turns it into a Polish space. The corresponding Borel σ-field in D denoted by $\mathcal{D}$ agrees with the σ-field $\sigma(\pi_t, t \in \mathcal{T}_0)$ for any $\mathcal{T}_0$ a dense subset of

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$\mathbb{R}^d$; see e.g. [4, Theorem 7.1]. Let $\Theta(t)$, $t \in \mathbb{R}^d$, denote an RF with almost surely sample paths in D defined by

$$\mathbb{P}\{\Theta \in A\} = \mathbb{E}\{Z(0)\mathbf{1}(Z/Z(0) \in A)\}/\mathbb{E}\{Z(0)\} \quad \text{for all } A \in \mathcal{D},$$

where we interpret 0:0 as 0 and set $\mathbf{1}(x \in A)$ equal to 1 or 0 if $x \in A$ or $x \notin A$, respectively. Note that $\mathbb{E}\{Z(0)\}$ equals 1, but we leave it since the same formula is applied to Z_* below. Note further that since D is Polish, by [38, lemma on page 1276] we can realise both Z and Θ in the same complete non-atomic probability space that we assume for notational simplicity below. By (2.1), for any measurable functional $F\colon D \mapsto [0, \infty]$ that is 0-homogeneous,

$$\mathbb{E}\{\Theta(h)F(\Theta)\} = \mathbb{E}\{\mathbf{1}(Z(0) \neq 0)Z(h)F(Z)\} = \mathbb{E}\{\mathbf{1}(\Theta(-h) \neq 0)F(B^h\Theta)\}. \tag{4.5}$$

In view of Corollary 2.1, the assumption $H_Z^0 > 0$ implies $\mathbb{P}\{S_\eta(Z) < \infty\} = p_\eta > 0$ for all $\eta \geq 0$. If $p_0 = 1$, then $\mathbb{P}\{S_0(\Theta) < \infty\} = 1$ also follows, and thus

$$\{S_\eta(\Theta) < \infty\} \subset \{S_0(\Theta) < \infty\} \tag{4.6}$$

almost surely.

Assume next that $p_0 \in (0, 1)$. The RF $Z_*(t) = Z(t) \mid S_0(Z) = \infty$, $t \in \mathbb{R}^d$ has almost surely sample paths in D and satisfies (2.1). Let Θ_* denote the corresponding RF of Z_* defined by the change of measure as above. We have that Θ_* has the same law as $\Theta \mid S_0(\Theta) = \infty$. Since $S_0(Z_*) = \infty$ with probability 1, applying Corollary 2.1, we obtain $S_\eta(Z_*) = \infty$ almost surely and thus $S_\eta(\Theta_*) = \infty$ almost surely, that is,

$$\mathbb{P}\{S_\eta(\Theta) = \infty, S_0(\Theta) = \infty\}/\mathbb{P}\{S_0(\Theta) = \infty\} = 1$$

(implying $\{S_0(\Theta) = \infty\} \subset \{S_\eta(\Theta) = \infty\}$ almost surely), and hence (4.6) holds. If $\mathbb{P}\{S_0(Z) < \infty\} = 0$, then by Corollary 2.1 $\mathbb{P}\{S_\eta(Z) < \infty\} = 0$ for all $\eta > 0$, which in turn implies $\mathbb{P}\{S_\eta(\Theta) < \infty\} = 0$. The assumption $p_0 = 0$ also implies $\mathbb{P}\{S_0(\Theta) < \infty\} = 0$, hence again (4.6) holds.

Next suppose that $\mathbb{P}\{S_0(\Theta) < \infty\} > 0$. Since then $\mathbb{P}\{S_0(Z) < \infty\} > 0$, we can define $Z_*(t) = Z(t) \mid S_0(Z) < \infty$, $t \in \mathbb{R}^d$, as above. The corresponding Θ_* has the same law as $\Theta \mid S_0(\Theta) < \infty$. Since $\mathbb{P}\{S_0(Z_*) < \infty\} = \mathbb{P}\{S_0(\Theta_*) < \infty\} = 1$, by [37, Theorem 2.8],

$$\mathbb{P}\left\{\lim_{\|t\| \to \infty, t \in \eta\mathbb{Z}^d} \Theta_*(t) = 0\right\} = 1,$$

with $\|\cdot\|$ some norm on $\mathbb{R}^d$. The latter is equivalent to $\mathbb{P}\{S_\eta(\Theta_*) < \infty\} = 1$; see [20, Condition A2, A4]. As in the proof of [20, Lemma A.2], $\mathbb{P}\{S_0(\Theta) < \infty\} = 0 \iff p_0 = 0$ and thus the reverse inclusion to (4.6) holds, implying

$$\{S_\eta(\Theta) < \infty\} = \{S_0(\Theta) < \infty\} \quad \text{for all } \eta > 0 \tag{4.7}$$

almost surely. Since further $\mathbb{P}\{S_\eta(\Theta) > 0\} = 1$ for all $\eta \geq 0$, which follows from $\mathbb{P}\{\Theta(0) = 1\} = 1$ and the fact that Θ has paths in D almost surely, we have almost surely for all $\delta, \eta \in [0, \infty)$

$$\frac{1}{S_\eta(\Theta)} = \frac{1}{S_\eta(\Theta)} \frac{S_\delta(\Theta)}{S_\delta(\Theta)} \Theta(0). \tag{4.8}$$

Now set $S_\eta(f) = \int_{\eta\mathbb{Z}^d} f(t)\lambda_\eta(\mathrm{d}t)$, $\eta \geq 0$, and

$$M_\eta(f) = \sup_{t \in \eta\mathbb{Z}^d} f(t), \quad M_0(f) = \sup_{t \in \mathcal{T}_0} f(t), \quad f \in D, \ \eta > 0.$$

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Both maps $S_\eta(\,\cdot\,)$ and $M_\eta(\,\cdot\,)$, $\eta \geq 0$, are measurable and by the separability of Z we have that $M_0(Z)$ has the same law as $\sup_{t\in\mathbb{R}^d} Z(t)$. Hence, also using the definition of Θ and (4.8), by the Fubini–Tonelli theorem, for all $\delta \neq \eta$, $\delta \geq 0$, $\eta > 0$ we have

$$\begin{aligned}
&\mathbb{E}\left\{Z(0)\frac{\sup_{t\in\delta\mathbb{Z}^d} Z(t)}{S_\eta(Z)}\right\}\\
&\quad= \mathbb{E}\left\{\frac{\sup_{t\in\delta\mathbb{Z}^d} \Theta(t)}{S_\eta(\Theta)}\right\}\\
&\quad= \mathbb{E}\left\{\frac{\sup_{t\in\delta\mathbb{Z}^d} \Theta(t)}{S_\eta(\Theta)}\frac{S_\delta(\Theta)}{S_\delta(\Theta)}\Theta(0)\right\}\\
&\quad= \int_{\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\frac{\Theta(0)}{S_\eta(\Theta)}\Theta(s)\right\}\lambda_\delta(\mathrm{d}s)\\
&\quad= \sum_{i\in\eta\mathbb{Z}^d}\int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\frac{\Theta(0)}{S_\eta(\Theta)}\Theta(i+r)\right\}\lambda_\delta(\mathrm{d}r),
\end{aligned} \tag{4.9}$$

where in the last equality we also used the translation invariance of λ_δ. Now, by (4.5), for all $i, r \in \mathbb{R}^d$,

$$\mathbb{E}\left\{\Theta(i+r)\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\frac{\Theta(0)}{S_\eta(\Theta)}\right\} = \mathbb{E}\left\{\frac{M_\delta(B^{r+i}\Theta)}{S_\delta(B^{r+i}\Theta)}\frac{\Theta(-i-r)}{S_\eta(B^{r+i}\Theta)}\right\}. \tag{4.10}$$

Note in passing that $\Theta(-i-r)\mathbf{1}(\Theta(-i-r)\neq 0) = \Theta(-i-r)$ almost surely.

If $\delta = 0$, $\eta > 0$ or $\delta > 0$ and $\eta = k\delta$, $k \in \mathbb{N}$, then almost surely, for all $i \in \eta\mathbb{Z}^d$, $r \in \delta\mathbb{Z}^d$,

$$M_\delta(B^{r+i}\Theta) = M_\delta(\Theta), \quad S_\delta(B^{r+i}\Theta) = S_\delta(\Theta), \quad S_\eta(B^{r+i}\Theta) = S_\eta(B^r\Theta), \tag{4.11}$$

and from (2.4)

$$\begin{aligned}
S_\delta(\Theta) < \infty &= \mathbb{P}\{S_0(\Theta) < \infty\}\\
&= \mathbb{E}\{Z(0)\mathbf{1}\{S_0(Z) < \infty\}\}\\
&= \mathbb{E}\{Z(0)\mathbf{1}\{S_0(Z) < \infty,\, S_\eta(B^rZ) \in (0,\infty)\}\}\\
&= \mathbb{E}\{Z(0)\mathbf{1}\{S_\delta(Z) < \infty,\, S_\eta(B^rZ) \in (0,\infty)\}\}\\
&= \mathbb{E}\{Z(0)\mathbf{1}\{S_\delta(Z/Z(0)) < \infty,\, S_\eta(B^rZ/Z(0)) \in (0,\infty)\}\}\\
&= \mathbb{P}\{S_\delta(\Theta) < \infty,\, S_\eta(B^r\Theta) \in (0,\infty)\} \quad \text{for all } r \in \delta\mathbb{Z}^d,
\end{aligned}$$

where the third-to-last line follows from (4.7). Consequently, almost surely

$$\frac{S_\eta(B^r\Theta)}{S_\eta(B^r\Theta)}\mathbf{1}\{S_\delta(\Theta) < \infty\} = \mathbf{1}\{S_\delta(\Theta) < \infty\}. \tag{4.12}$$

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Hence, for these choices of δ and η, (4.9)–(4.12) and the Fubini–Tonelli theorem yield

$$\begin{aligned}\mathbb{E}\left\{Z(0)\frac{\sup_{t\in\delta\mathbb{Z}^d} Z(t)}{S_\eta(Z)}\right\} &= \int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\sum_{i\in\eta\mathbb{Z}^d}\frac{\Theta(-r-i)}{S_\eta(B^r\Theta)}\right\}\lambda_\delta(\mathrm{d}r)\\ &= \eta^{-d}\int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\frac{S_\eta(B^r\Theta)}{S_\eta(B^r\Theta)}\mathbf{1}\{S_\delta(\Theta)<\infty\}\right\}\lambda_\delta(\mathrm{d}r)\\ &= \eta^{-d}\int_{r\in[0,\eta)^d\cap\delta\mathbb{Z}^d} \mathbb{E}\left\{\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\mathbf{1}\{S_\delta(\Theta)<\infty\}\right\}\lambda_\delta(\mathrm{d}r)\\ &= \mathbb{E}\left\{\frac{M_\delta(\Theta)}{S_\delta(\Theta)}\right\}\\ &= \mathbb{E}\left\{Z(0)\frac{\sup_{t\in\delta\mathbb{Z}^d} Z(t)}{S_\delta(Z)}\right\},\end{aligned}$$

and hence the claim follows from (2.3).

Note that if $\mathbb{P}\{Z(0)>0\}=1$, by (2.1) for all $t\in\mathbb{R}^d$,

$$\mathbb{E}\{Z(t)\}=\mathbb{E}\{Z(t)\mathbf{1}\{Z(0)>0\}\}=\mathbb{E}\{Z(0)\mathbf{1}\{Z(-t)>0\}\}=\mathbb{E}\{Z(0)\},$$

and hence $\mathbb{P}\{Z(t)>0\}=\mathbb{P}\{\Theta(t)>0\}=1$ since $\mathbb{E}\{Z(t)\}=\mathbb{E}\{Z(0)\}=1$. Consequently, applying (4.5) and utilising (4.7) for all $r\in\mathbb{R}^d$, $\eta>0$,

$$\begin{aligned}&\mathbb{E}\{Z(-r)\mathbf{1}\{S_0(Z)<\infty, 0<S_\eta(B^rZ)<\infty\}\}\\ &\quad=\mathbb{E}\{\Theta(-r)\mathbf{1}\{S_0(\Theta)<\infty, 0<S_\eta(B^r\Theta)<\infty\}\}\\ &\quad=\mathbb{E}\{\mathbf{1}\{S_0(\Theta)<\infty, 0<S_\eta(\Theta)<\infty\}\}\\ &\quad=\mathbb{E}\{\mathbf{1}\{S_0(\Theta)<\infty\}\}\\ &\quad=\mathbb{E}\{Z(0)\mathbf{1}\{S_0(Z)<\infty\}\}\\ &\quad=\mathbb{E}\{Z(-r)\mathbf{1}\{S_0(Z)<\infty\}\}\\ &\quad\le 1,\end{aligned}$$

and thus (2.4) follows. The reason that the indicator function did not appear when we applied (4.5) in the above calculations is that $\Theta(t)>0$ almost surely for all $t\in\mathbb{R}^d$. □

Proof of Theorem 3.1. Let $z\in[0,1]$ be fixed. For simplicity we consider the case $d=1$ and set $X_z(t)=W_z(t)-\sigma_z^2(t)/2$. We now show that for any positive T the function

$$A_T(z)=\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X_z(t)}\right\}$$

is continuous in z and thus integrable for any $\delta\ge 0$. First note that, by (1.5),

$$\mathbb{E}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X_z(t)}\right\}=1+\int_0^\infty \mathrm{e}^s\mathbb{P}\left\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} X_z(t)>s\right\}\mathrm{d}s<\infty.$$

Since σ_z determines the covariance function of X_z, then (3.4) implies that FIDIs of X_{z+h} converge weakly to those of X_z as $h\to 0$. Moreover, for some $\varepsilon>0$ and all $|h|<\varepsilon$, by (3.5),

$$\mathrm{Var}(X_{z+h}(s)-X_{z+h}(t))=\sigma^2_{z+h}(t-s)\le C\|t-s\|^{\nu_0}$$

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for some $C > 0$ and all t, $s \in [0, T]$. Consequently, $X_{z+h}(t)$, $t \in [0, T]$, is tight (with respect to h) and converges weakly in the space of real-valued continuous functions on $[0,T]$ equipped with the uniform topology; see [29, Proposition 9.7]. Hence, by the continuous mapping theorem for almost all $s \in \mathbb{R}$,

$$\lim_{h\to 0} \mathbb{P}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} X_{z+h}(t) > s \right\} = \mathbb{P}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} X_z(t) > s \right\}.$$

Consequently, for all $T > 0$,

$$\lim_{h\to 0} \mathbb{E}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X_{z+h}(t)} \right\} = \mathbb{E}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X_z(t)} \right\} = A_T(z) < \infty,$$

implying the claim that $A_T(z)$, $z \in [0, 1]$, is continuous in z. The derivation above rests on the application of the dominated convergence theorem that can be justified by the Borell–TIS inequality. Indeed, for all $z \in [0, 1]$,

$$\sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X_z(t)} \le \sup_{t\in[0,T],v\in[0,1]} \mathrm{e}^{W_v(t)}$$

and

$$\mathbb{E}\left\{ \sup_{t\in[0,T],v\in[0,1]} \mathrm{e}^{W_v(t)} \right\} = 1 + \int_0^\infty \mathrm{e}^s \mathbb{P}\left\{ \sup_{t\in[0,T],v\in[0,1]} W_v(t) > s \right\} \mathrm{d}s.$$

By the Borell–TIS inequality and (3.5),

$$\mathbb{P}\left\{ \sup_{t\in[0,T],v\in[0,1]} W_v(t) > s \right\} \le \exp\left(-\frac{(s-Q_1)^2}{2Q_2} \right)$$

for sufficiently large s and Q_1, Q_2 positive constants, which justifies the limit above. Since by assumption (3.5), for some $C > 0$ and all $z \in [0, 1]$, we have

$$\sigma_z^2(t) \le C(\|t\|^{\nu_0} + \|t\|^{\nu_\infty}) =: \sigma^2(t),$$

then for the Gaussian process

$$X(t) = W_0(t) + W_\infty(t) - \sigma^2(t)/2, \quad t \in [0, T],$$

with W_0, W_∞ mutually independent centered Gaussian processes with continuous sample paths, stationary increments and variance function $C\|t\|^{\nu_0}$, $C\|t\|^{\nu_\infty}$, respectively, applying [9, Theorem 3.1] we obtain for any $z \in [0, 1]$

$$H^\delta_{Z_z} \le H^\delta_{\widetilde{Z}} \in (0, \infty),$$

where $\widetilde{Z}(t) = \exp(X(t))$, $t \in [0, T]$, and

$$\mathbb{E}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X_z(t)} \right\} \le \mathbb{E}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} \mathrm{e}^{X(t)} \right\} < \infty.$$

Consequently, by the measurability of $A_T(z)$, $z \in [0, 1]$, and the dominated convergence theorem,

$$\mathcal{H}^\delta_Z = \lim_{T\to\infty} T^{-d} \int_0^1 \mathbb{E}\left\{ \sup_{t\in[0,T]^d\cap\delta\mathbb{Z}^d} \mathrm{e}^{X_z(t)} \right\} \mathrm{d}z = \int_0^1 H^\delta_{Z_z} \,\mathrm{d}z \le H^\delta_{\widetilde{Z}} < \infty.$$

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Thus, in view of Theorem 2.1, we obtain

$$\lim_{\delta \downarrow 0} \mathcal{H}_Z^\delta = \int_0^1 \lim_{\delta \downarrow 0} H_{Z_z}^\delta \, \mathrm{d}z = \int_0^1 H_{Z_z}^0 \, \mathrm{d}z \geq 0,$$

which completes the proof. □

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