A NOTE ON INVERSE SEMIGROUPS

By W. D. MUNN and R. PENROSE

Received 13 November 1954

The concept of an 'inverse semigroup' or 'generalized group' has recently been introduced, independently, by Preston (4) and Vagner (5). In the present note we give several alternative definitions for such a system, some of which appear to be weaker than that originally given.

A *semigroup* is a non-empty set S which is closed with respect to an associative binary operation. An element e of S is termed an *idempotent* if $e^2 = e$. A *left ideal* L of S is a non-empty subset of S such that $SL \subseteq L$, and a *right ideal* R of S is a non-empty subset of S such that $RS \subseteq R$. A *two-sided ideal* (or, simply, an *ideal*) of S is a subset which is both a left and a right ideal.

Following Green (3), we shall say that the elements a, b of S are $\mathfrak{l}$-*equivalent* or $\mathfrak{r}$-*equivalent* according as a and b generate the same left or right ideal* of S respectively, and we write $a \equiv b(\mathfrak{l})$ or $a \equiv b(\mathfrak{r})$. If a left or right ideal is generated by a single element, it is termed *principal*.

A semigroup S is said to be *regular* if for any given element $a \in S$ there is at least one solution of the equation $axa = a$ in S. We observe that in a regular semigroup the equations

$$axa = a, \quad xax = x$$

have at least one common solution. For suppose u is any solution of $axa = a$. Write $v = uau$. Then $ava = auaua = aua = a$, and $vav = uauauau = uau = v$; hence v is a common solution of the given equations.

An *inverse semigroup* is a regular semigroup in which all the idempotents commute. This is essentially the definition given by Preston and Vagner. Alternative definitions are provided by the following result:

Theorem. *The following conditions on a semigroup S are equivalent:*

C1. *S is regular, $\mathfrak{l}$-equivalent idempotents commute, and $\mathfrak{r}$-equivalent idempotents commute.*

C2. *Each principal left ideal and each principal right ideal of S is generated by one and only one idempotent.*†

C3. *S is regular, and for any idempotent $e \in S$ the equations $exe = e$, $xex = x$ have no common idempotent solution other than e.*

C4. *For any element $a \in S$ the equations $axa = a$, $xax = x$ have a unique common solution.*

C5. *S is an inverse semigroup.*

* For example, the left ideal of S generated by a is $Sa \cup a$.

† That the left and right properties of S in C1 and C2 are in fact independent can be seen by considering a semigroup S whose multiplication is given by $ab = b$ for all $a, b \in S$.

https://doi.org/10.1017/S030500410003036X Published online by Cambridge University Press

We require five lemmas.

LEMMA 1. *If two* l*-equivalent* (r*-equivalent*) *idempotents of a semigroup* S *commute, then they are equal.*

Proof. Let e, f be l-equivalent idempotents. Then there exists $u \in S$ such that $e = uf$. Hence $ef = uf^2 = uf = e$. Similarly $fe = f$. Thus if $ef = fe$ we have $e = f$. The proof for r-equivalent idempotents is similar.

LEMMA 2. *If each principal left* (*right*) *ideal of a semigroup* S *is generated by at least one idempotent, then* S *is regular. Conversely each principal left ideal and each principal right ideal of a regular semigroup is generated by an idempotent.*

(This is Theorem 6 of (3), but we give the proof for the sake of completeness.)

Proof. Suppose that for a given $a \in S$ we can find an idempotent e such that $e \equiv a(\mathrm{l})$. Then there exist elements $u, v \in S$ such that $e = ua$, $a = ve$. Hence $ae = ve^2 = ve = a$ and so $aua = a$. A similar argument holds for principal right ideals.

Conversely if u is any solution of the equation $axa = a$, then $ua \equiv a(\mathrm{l})$ and $au \equiv a(\mathrm{r})$. But ua and au are idempotents and the result follows.

LEMMA 3. *If each principal left ideal and each principal right ideal of a semigroup* S *is generated by one and only one idempotent, then for any* $a \in S$ *the equations* $axa = a$, $xax = x$ *have a unique common solution.*

Proof. By Lemma 2, S is regular. Let u, v be common solutions of $axa = a$, $xax = x$. Then we have $ua \equiv a(\mathrm{l})$, $va \equiv a(\mathrm{l})$ and so $ua \equiv va(\mathrm{l})$. But ua and va are idempotents and so $ua = va$. Similarly $au = av$. Thus we have

$$u = (ua)u = (va)u = v(au) = v(av) = v,$$

as required.

LEMMA 4. *Let* S *be a regular semigroup in which for any given idempotent* e *the equations* $exe = e$, $xex = x$ *have no common idempotent solution other than* e. *Then each principal left ideal and each principal right ideal is generated by one and only one idempotent.*

Proof. Since S is regular it follows from Lemma 2 that each principal left ideal and each principal right ideal is generated by at least one idempotent. Let e and f be l-equivalent idempotents. Then, as in the proof of Lemma 1, we have $ef = e$, $fe = f$. Hence $efe = e^2 = e$ and $fef = f^2 = f$. Thus from our hypothesis we have $e = f$. Similarly, if e and f are r-equivalent idempotents, then $e = f$.

LEMMA 5. *Let* S *be a semigroup in which for any given element* a *the equations* $axa = a$, $xax = x$ *have a unique common solution. Then* S *is an inverse semigroup.*

Proof. Let e and f be any two idempotents of S, and let g be the unique common solution of the equations

$$(ef)x(ef) = ef, \quad x(ef)x = x. \tag{1}$$

Then $$(ef)fg(ef) = (ef)g(ef) = ef \quad \text{and} \quad fg(ef)fg = f.g(ef)g = fg,$$

whence, by the uniqueness of g, we have $g = fg$. Similarly, it can be shown that $g = ge$. Thus $g^2 = (ge)(fg) = g$, and so g is a solution of the equations $xgx = x$, $gxg = g$. But ef is also a solution of these equations since g is the solution of equations (1).

https://doi.org/10.1017/S030500410003036X Published online by Cambridge University Press

Hence $g = ef$, which shows that ef is an idempotent. In the same way we can show that fe is an idempotent.

Thus $(ef)fe(ef) = (ef)^2 = ef$ and $fe(ef)fe = (fe)^2 = fe$,

and so fe is a solution of equations (1). Hence $ef = fe$ as required.

We now complete the proof of the theorem. Lemmas 1 and 2 show that C1 implies C2. Also C2 implies C4 by Lemma 3, and C4 implies C5 by Lemma 5. But *a fortiori* C5 implies C1, and so C1, C2, C4, C5 are equivalent. Further C3 implies C2 by Lemma 4, and *a fortiori* C4 implies C3. Hence C1, C2, C3, C4, C5 are equivalent.

Preston and Vagner have proved that C5 implies C4.

The proof of Lemma 5 is a generalization of part of that given by Croisot to establish the result that a semigroup in which the equation $axa = a$ has a unique solution for any a, must be a group ((2), Theorem 5). We give a stronger result:

COROLLARY 1. *A regular semigroup in which for any idempotent e the equation $exe = e$ has no idempotent solution other than e is a group.*

Proof. Let S be a semigroup with the above property. Then the equations $exe = e$, $xex = x$ have the unique common idempotent solution e for any given idempotent e of S; that is, S satisfies C3. Hence S is an inverse semigroup. Thus if e and f are any two idempotents of S, then ef is an idempotent, and so the equation $(ef)x(ef) = ef$ has a unique idempotent solution. But e and f are such solutions, and so $e = f$. Finally, let a be any element of S, and let u be any solution of $axa = a$. Then $au = ua = e$, the unique idempotent of S, and $ea = ae = a$. Hence S is a group.

A *semilattice* is a commutative semigroup of idempotents. Now let S be a semigroup which is expressible as a union of disjoint groups G_α, where α belongs to a semilattice Y. Then Clifford (1) has termed S a *semilattice of groups* if the multiplication in S is such that $G_\alpha G_\beta \subseteq G_{\alpha\beta}$, where $\alpha\beta$ is the product of the elements α, β in Y. We then have

COROLLARY 2. *If S is a semilattice of groups, then all the idempotents of S commute.*

Proof. With the above notation we see that two elements are l-equivalent (r-equivalent) if and only if they belong to the same group G_α, say. But G_α contains one and only one idempotent, and so S satisfies C2. Thus S is an inverse semigroup.

This result was previously obtained by Clifford ((1), Theorem 8) and by Croisot ((2), p. 375).

Let M be an ideal of a semigroup S. Then the semigroup $S - M$ is defined to be the set obtained from S by collapsing M to a single element 0, say, the remaining elements of S being unchanged. Multiplication ($\circ$) in $S - M$ is then defined by the rules

(i) $a \circ b = ab$ if $a, b \neq 0$ and $ab \notin M$,

(ii) $a \circ b = 0$ if $a, b \neq 0$ and $ab \in M$,

and (iii) $a \circ 0 = 0 = 0 \circ a$ for all $a \in S - M$.

COROLLARY 3. *Let M be an ideal of a semigroup S. Then S is an inverse semigroup if and only if M and $S - M$ are inverse semigroups.*

Proof. We use C4. Let S be an inverse semigroup and a any element of S. Let u be the unique common solution of the equations $axa = a$, $xax = x$. Then $u \in M$ if and only if $a \in M$. It follows that M and $S - M$ also satisfy C4.

https://doi.org/10.1017/S030500410003036X Published online by Cambridge University Press

Conversely suppose M and $S-M$ satisfy C4. Then if a is any element of S, the equations $axa = a$, $xax = x$ have a common solution u, say. But $u \in M$ if and only if $a \in M$. Hence u is uniquely determined by a; that is, S satisfies C4.

REFERENCES

(1) Clifford, A. H. Bands of semigroups. *Proc. Amer. math. Soc.* 5 (1954), 499–504.
(2) Croisot, R. Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples. *Ann. sci. Éc. norm. sup., Paris*, (3), 70 (1953), 361–79.
(3) Green, J. A. On the structure of semigroups. *Ann. Math., Princeton* (2), 54 (1951), 163–72.
(4) Preston, G. B. Inverse semi-groups. *J. Lond. math. Soc.* 29 (1954), 396–403.
(5) Vagner, V. V. Generalized groups. *C.R. Acad. Sci. U.R.S.S.* 84 (1952), 1119–22 (Russian).

St John's College
Cambridge

Note added in Proof. The result of Lemma 5 has also recently been obtained by A. E. Liber, On the theory of generalized groups. *C.R. Acad. Sci. U.R.S.S.* 97 (1954), 26–28 (Russian).

https://doi.org/10.1017/S030500410003036X Published online by Cambridge University Press