# SMOOTH SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EVOLUTION EQUATIONS

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Abstract

The purpose of the paper is to introduce a new linearization method for local well-posedness of nonlinear evolution equations. This approach is based on an implicit function theorem of Nash–Moser type. The technique is illustrated by an application to a general class of fully nonlinear parabolic partial differential equations of arbitrary order on $\mathbb{R}^n$. The estimates required by the Nash–Moser technique are derived for the higher order Sobolev norms of the solutions of the linearized parabolic equation using semigroup theory and elliptic theory. In particular, *a priori* estimates, resolvent estimates and commutator estimates are involved. The general method based on a combination of Nash–Moser techniques with semigroup theory is applicable to other problems and has already been used to prove short-time solvability for some nonlinear Schrödinger type equation. This approach might be useful in other situations as well since it compensates for a loss of derivatives in the estimates of the solutions of the linearized equation.

## 0. *Introduction*

The aim of this paper is to present a general new linearization method for the proof of local well-posedness of nonlinear evolution equations. This approach is based on an implicit function theorem of Nash–Moser type. The technique is illustrated by a general example in the parabolic case. There is given a new proof for short-time (and almost global) solvability for a class of fully nonlinear parabolic whole space problems of arbitrary order. The linearized parabolic equation is solved using semigroup theory.

The central linearization result (Theorem 4.4) is applicable to a much greater class of problems. For example, in [**18**] is given an application of Theorem 4.4 to a specific nonlinear Schrödinger type equation appearing in plasma physics; see Remark 6.12 for details and further applications. It should be emphasized that the assumptions of Theorem 4.4 allow a loss of derivatives in the estimates of the linearized problem. Hence Theorem 4.4 might be a valuable method also to be applied in other situations where a loss of derivatives would cause difficulties in a Banach space setting; notice that such an obstacle is overcome by a smoothing procedure in the proof of Nash–Moser type implicit function theorems involving Fréchet spaces.

Local and global existence theorems for linear and nonlinear parabolic whole space problems and boundary value problems have been proved in the literature by a lot of authors using various methods. We shall not give a detailed survey of such results. We only mention some articles where additional references can be found. The classical linear theory of parabolic equations and systems is for instance investigated in [**4**, **7**, **17**, **21**, **38**]. Early results on short-time solvability of nonlinear second order equations are contained in references [**1**] through [**8**] of the survey paper [**27**]. Subsequently, nonlinear parabolic problems have been studied by many authors, for

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example in [**7**, **12**, **16**, **17**] or, more recently, in [**14**, **19**, **20**, **36**, **40**]. In numerous papers and monographs linear semigroup theory has been applied to the solution of linear and nonlinear parabolic problems; we refer to [**1**, **5**, **23–25**, **28**, **37**, **39**, **44**].

In the literature, so-called Nash–Moser techniques have been used to solve ordinary and partial differential equations in several situations, for instance to obtain small global solutions in [**13**] or periodic solutions in [**6**, **15**, **33**] or local solutions in [**9**, III, Theorem 2.1.1]; the local result in [**9**] is based on an inverse function theorem.

In a different way from such known methods, in this paper an implicit function theorem of Nash–Moser type is applied to prove short-time solvability of evolution equations. The underlying simple idea can roughly be described as follows (see Section 4 for details). Consider a nonlinear Cauchy problem of the form $\{u_t = \mathscr{F}(u), t\in[0,a]; u(0) = \phi\}$ where $a > 0$ is unknown and the initial value $\phi$ is fixed. Using the transformation $t = as$ in the time variable we get the equivalent problem $\{u_t = a\mathscr{F}(u), t\in[0,1]; u(0) = \phi\}$. This problem can be written in the form $\Phi(u,a,\phi) := (u_t - a\mathscr{F}(u), u(0)-\phi) = (0,0)$ with a mapping $\Phi$ involving $a$ as an argument; observe that $\Phi(\phi,0,\phi) = (0,0)$. An application of an implicit function theorem near $a = 0$, $u = \phi$ gives a solution $u = \Psi(a,\phi)$ and hence yields by fixing some suitable small positive $a$ the local well-posedness of the nonlinear problem provided the linearized problem meets the necessary assumptions.

In this paper we give in Theorem 6.10 an application of Theorem 4.4 and prove under weak assumptions the local well-posedness in $H^\infty(\mathbb{R}^n)$ of the Cauchy problem for fully nonlinear parabolic partial differential equations of arbitrary order on $\mathbb{R}^n$; here $H^\infty(\mathbb{R}^n)$ denotes the intersection of all Sobolev spaces $H^k(\mathbb{R}^n)$.

For instance, as a very particular case, we obtain the following theorem.

THEOREM 0.1. *Let $\mathscr{F}$ be a nonlinear partial differential operator of order $m$ given by*

$$\mathscr{F}(u)(x) = F(\{\partial^\alpha u(x)\}_{|\alpha|\leqslant m}), \quad u\in H^\infty(\mathbb{R}^n)$$

*where $m$ is even and $F$ is a real $C^\infty$-map with $F(0) = 0$. Let $\phi\in H^\infty(\mathbb{R}^n)$ and assume that the derivative $\mathscr{F}'(\phi)$ is elliptic (in the sense of Definition* 6.1*). Then the initial value problem*

$$\begin{cases} u_t = \mathscr{F}(u), \quad t\in[0,a] \\ u(0) = \phi \end{cases} \tag{0.1}$$

*has a unique solution $u\in C^1([0,a], H^\infty(\mathbb{R}^n))$ for some suitable small $a > 0$.*

More generally, we consider nonlinear partial differential operators of the form

$$\mathscr{F}(t,u)(x) = F(t,x,\{\partial^\alpha u(x)\}_{|\alpha|\leqslant m})$$

of even order $m$ and prove in Theorem 6.10 a local existence result similar to Theorem 0.1 for the Cauchy problem $\{u_t = \mathscr{F}(t,u), u(0) = \phi\}$ under suitable mild assumptions on $F$. Although the required conditions $(A_1)$, $(A_2)$, $(A_3)$ seem not to have been used in this form in the literature before, it is quite within the bounds of probability that results similar to Theorem 6.10 can also be proved using more standard techniques (see the above references). However, the proof of Theorem 6.10 using an $H^\infty$-setting and an implicit function theorem is new in the literature and seems to be valuable also due to its applicability to other types of problems.

The proof of Theorem 4.4 is based on an application of the generalized implicit function theorem of Nash–Moser type [**30**, 4.2] and includes the $C^k$-dependence from

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the initial value. When verifying the assumptions of Theorem 4.4 in the proof of Theorem 6.10 the higher order Sobolev norms of the solutions of the linearized parabolic problem are shown to satisfy the delicate estimates required by the implicit function theorem of Nash–Moser type [**30**, 4.2] and thus by Theorem 4.4; Corollary 6.9 stating these estimates seems to be of independent interest.

In order to solve the linearized problem we make use of the theory of linear evolution equations involving semigroups and evolution systems (cf. [**28**, **39**]); the fundamental *a priori* estimates for elliptic operators [**3**] and a classical result of Agmon [**2**] are applied. A central part of this paper is devoted to the proof of the required higher order estimates in Sobolev spaces for the solutions of the linearized parabolic problem. As an important tool commutator estimates are derived for linear differential operators which can comfortably be formulated by means of the weighted multiseminorms $[\ ]_{m,k}$ introduced in [**30**]; furthermore, a generalized version of Gronwall's lemma is applied.

This paper is organized as follows. In Section 1 some preliminary notations and results are stated. The implicit function theorem of Nash–Moser type (Theorem 1.3) is taken from [**30**, 4.2] and is a generalization of [**22**]. The weighted multiseminorms $[\ ]_{m,k}$ (cf. [**30**]) are defined in Definition 1.1; these are a comfortable and important tool when working with estimates in the whole of this paper. The generalization of Gronwall's lemma (Lemma 1.4) is adapted to estimates in Fréchet spaces; Lemma 1.4 is proved in [**31**, 3.6].

In Section 2 the basic assumptions are stated such that a nonlinear partial differential operator is defined and differentiable as a map $\mathscr{F}:[0,T]\times(U\subset H^\infty(\Omega))\longrightarrow H^\infty(\Omega)$. Using the expressions $[\ ]_{m,k}$ we prove estimates in Sobolev spaces for $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{F}''$ and also for the powers $(\mathscr{F}'(u))^p$ and the commutators $\partial^\alpha\mathscr{F}'(u)^p-\mathscr{F}'(u)^p\,\partial^\alpha$. This generalizes some well-known inequalities which are called Moser estimates in [**40**].

Section 3 contains estimates for linear elliptic partial differential operators as required by the implicit function theorem of Nash–Moser type [**30**, 4.2]. Some of the results in Sections 2 and 3 are obtained for a single linear operator by means of well-known arguments. The new ingredient is the use of the weighted multiseminorms $[\ ]_{m,k}$ which quantifies in several estimates the dependence of the appearing constants from the coefficients of the linear problem. This symbolic calculus provides an easy and completely formal way to prove the higher order estimates required by the Nash–Moser technique.

In Section 4 the Nash–Moser implicit function theorem (Theorem 1.3) is applied to get the linearization of the nonlinear initial value problem. A central idea (a suitable transformation in the time variable) is taken from [**31**]. Theorem 4.4 gives a sufficient condition on the linearized problem such that the nonlinear problem is locally solvable. This result does not need parabolicity and applies to arbitrary nonlinear evolution equations.

Section 5 is concerned with the linearized parabolic problem. We assume the parabolic conditions $(P_0)$, $(P_1)$, $(P_2)$, $(P_3)$ (generalizing [**28**, 5.6.1; **39**, 5.1.1, 5.1.2]) and obtain the existence of analytic semigroups and evolution systems in the Sobolev spaces $H^{m,p}(\mathbb{R}^n)$ for any $p$; this implies the existence of the solutions of the linearized problem as required by Theorem 4.4. The necessary estimates are proved in Theorem 5.8 using the basic $L_2$-estimates for semigroups and evolution systems and applying the commutator estimates of the previous sections and the generalized Gronwall's lemma (Lemma 1.4).

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In Section 6 the nonlinear parabolic problem is solved in Theorem 6.10. The proof is based on the fundamental *a priori* estimates for elliptic operators of arbitrary order due to Agmon, Douglis and Nirenberg [**3**] and on a classical observation of Agmon [**2**] on the resolvents of elliptic operators; since we need a precise statement on the dependence of the constants from the given data we include for $\Omega = \mathbb{R}^n$ elementary proofs of these results. Under suitable ellipticity assumptions, conditions $(P_0)$, $(P_1)$, $(P_2)$, $(P_3)$ of Section 5 are verified and the local nonlinear existence theorem is derived.

As a consequence of the estimates proved in Corollary 6.9 a result on almost global solvability is formulated in Corollary 6.11. Some remarks on further applications are given in Remark 6.12.

## 1. *Preliminaries*

We shall always consider Fréchet spaces $E$ which are equipped with some fixed fundamental sequence $|\ |_0 \leqslant |\ |_1 \leqslant |\ |_2 \leqslant \dots$ of seminorms defining the topology. If $E, F$ are such Fréchet spaces then the product $E \times F$ is endowed with the seminorms $|(x,y)|_k = \max\{|x|_k, |y|_k\}$, and for an integer $n \geqslant 0$ the Fréchet space $C^n([a,b], E)$ of all $E$-valued $n$-times continuously differentiable functions on $[a,b]$ is equipped with

$$|f|_k = \sup_{i=0,\dots,n} \sup_{t\in[a,b]} |f^{(i)}(t)|_k, \quad f \in C^n([a,b], E), \quad k = 0, 1, 2, \dots$$

In particular, we write $C([a,b], E) = C^0([a,b], E)$ for the space of all continuous functions.

A continuous nonlinear map $\Phi: (U \subset E) \longrightarrow F$ between Fréchet spaces, $U$ open, is called a $C^1$-map if the derivative $\Phi'(x)\, y = \lim_{t\to 0}(1/t)\,(\Phi(x+ty) - \Phi(x))$ exists for all $x \in U$, $y \in E$ and is continuous as a map $\Phi': U \times E \longrightarrow F$. $\Phi$ is called a $C^2$-map if it is $C^1$ and the second derivative $\Phi''(x)\{y_1, y_2\} = \lim_{t\to 0}(1/t)\,(\Phi'(x+ty_2)\, y_1 - \Phi'(x)\, y_1)$ exists and is continuous as a map $\Phi'': U \times E^2 \longrightarrow F$. Similar definitions apply to higher derivatives $\Phi^{(n)}$; $\Phi$ is called $C^\infty$ if it is $C^n$ for all $n$. Given a function of two variables $\Phi = \Phi(x,y)$ we can also consider the partial derivatives $\Phi_x$ and $\Phi_y$ where for example $\Phi_x(x,y)\, z = \lim_{t\to 0}(1/t)\,(\Phi(x+tz, y) - \Phi(x,y))$. One-dimensional derivatives $\Phi_t$, $t \in \mathbb{R}$ are alternatively considered as a map $\Phi_t: U \times \mathbb{R} \longrightarrow F$ or as a map $\Phi_t: U \longrightarrow F$, respectively. For these notions we refer to [**9**, I.3].

A main part of this paper consists of estimates for differential operators formulated by means of the following weighted multiseminorms $[\ ]_{m,k}$ (cf. [**30**, 1.1]).

DEFINITION 1.1. Let $p, q \geqslant 0$ be integers, $p + q \geqslant 1$, and let $E_1, \dots, E_p, F_1, \dots, F_q$ be linear spaces, each of them equipped with a sequence $|\ |_0 \leqslant |\ |_1 \leqslant |\ |_2 \leqslant \dots$ of seminorms. For any integer $m, k \geqslant 0$, $x_1 \in E_1, \dots, x_p \in E_p$, $y_1 \in F_1, \dots, y_q \in F_q$ we put

$$[x_1, \dots, x_p; y_1, \dots, y_q]_{m,k} = \sup\{1 \cdot |x_{k_1}|_{m+i_1} \cdot \dots \cdot |x_{k_r}|_{m+i_r} \cdot |y_1|_{m+j_1} \cdot \dots \cdot |y_q|_{m+j_q}\}$$

where the 'sup' is running over all $i_1, \dots, i_r, j_1, \dots, j_q \geqslant 0$ and $1 \leqslant k_1, \dots, k_r \leqslant p$ with $0 \leqslant r \leqslant k$ and $i_1 + \dots + i_r + j_1 + \dots + j_q \leqslant k$ (for $r = 0$ the $|x|$-terms are omitted). For $q = 0$ we write $[x_1, \dots, x_p]_{m,k}$ (where the $|y|$-terms are omitted), and for $p = 0$ we write $[; y_1, \dots, y_q]_{m,k}$. For $m = 0$ we write $[\dots]_k = [\dots]_{0,k}$ in all cases.

Observe that the expression $[x_1, \dots, x_p; y_1, \dots, y_q]_{m,k}$ is a seminorm separately in each component $y_j$ while it is 'completely nonlinear' in the $x_i$-components. The terms

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$[\,]_{m,k}$ are increasing in $m$ and in $k$. For the purely nonlinear terms (that is $q = 0$) we have $[x_1, \dots, x_p]_{m,0} = 1$ and $[x_1, \dots, x_p]_{m,k} \geqslant 1$ for all $m, k$, and for $p = 1$ we have explicitly

$$[x]_{m,k} = \sup\{1 \cdot |x|_{m+i_1} \cdot \dots \cdot |x|_{m+i_r} : 0 \leqslant r \leqslant k; i_1, \dots, i_r \geqslant 0; i_1 + \dots + i_r \leqslant k\}.$$

For properties of the terms $[\,]_{m,k}$ we refer to [**30**, 1.7]; we shall often apply the rules

$$[x]_{m,k}[x]_{m,i} \leqslant [x]_{m,k+i} \quad \text{and} \quad [x]_{m,k}[x;y]_{m,i} \leqslant [x;y]_{m,k+i}.$$

The following Theorem 1.3 on implicit functions of Nash–Moser type is proved in [**30**, 4.2]. The spaces are assumed to satisfy a technical smoothing property $(S_\Omega)_t$ (defined in [**29**, 3.3], cf. [**30**, 2.2]) and property (DN) of Vogt (cf. [**42**]). Recall that a Fréchet space $(E, (|\,|_n)_{n=0}^\infty)$ has property (DN) if and only if there is an integer $b$ such that for any integer $n$ there is an integer $k$ and a constant $C > 0$ such that

$$|x|_n^2 \leqslant C|x|_b |x|_{n+k}, \quad x \in E.$$

For the purposes of this paper it is enough to have the following sufficient condition for property $(S_\Omega)_t$; the proof of Lemma 1.2 is given in [**31**, 2.5]. For the proof of Theorem 4.4 it is important that both $E$ and $C^n([0,1], E)$ have the smoothing property $(S_\Omega)_t$.

LEMMA 1.2. *Let $E$ be a Fréchet space equipped with a fundamental sequence of seminorms $|\,|_0 \leqslant |\,|_1 \leqslant |\,|_2 \leqslant \dots$ which are induced by semiscalar products such that*

$$|\phi|_k^{*2} \leqslant C_k |\phi|_{k-1}^* |\phi|_{k+1}^*, \quad \phi \in E', \quad k = 1, 2, \dots$$

*for constants $C_k > 0$ (where $|\phi|_k^* := \sup\{|\phi(x)| : |x|_k \leqslant 1\} \in [0, \infty]$, $\phi \in E'$). Then the spaces $E$ and $C^n([0,1], E)$, $n = 0, 1, 2, \dots$, have the smoothing property $(S_\Omega)_t$.*

For instance, Lemma 1.2 applies to the space $E = H^\infty(\mathbb{R}^n)$ (cf. Section 2) by means of [**26**, 14.12]. Notice that the assumption of Lemma 1.2 is called 'property ($\Omega$) in standard form' by Vogt in [**43**].

THEOREM 1.3 (on implicit functions [**30**, 4.2]). *Let $E, F, G$ be Fréchet spaces having the smoothing property $(S_\Omega)_t$ and (DN). Let $U \subset E$, $V \subset F$ be open and $x_0 \in U$, $y_0 \in V$. Let $f : U \times V \longrightarrow G$ be a $C^2$-map, $f = f(x, y)$, and $f(x_0, y_0) = 0$. Assume that the partial derivative $f_y(w) : F \longrightarrow G$ is bijective for any $w \in U \times V$ and that there is a fixed $d \geqslant 0$ such that for any $k$ there is a constant $c_k > 0$ such that the following estimates hold:*

$$|f'(w)\,x|_k \leqslant c_k [w; x]_{d,k}, \qquad w \in U \times V, x \in E \times F \tag{1.1}$$

$$|f_y(w)^{-1} z|_k \leqslant c_k [w; z]_{d,k}, \qquad w \in U \times V, z \in G \tag{1.2}$$

$$|f''(w)\{x, x\}|_k \leqslant c_k [w; x, x]_{d,k}, \quad w \in U \times V, x \in E \times F. \tag{1.3}$$

*Then there exist open neighbourhoods $U_0 \subset U$ of $x_0$, $V_0 \subset V$ of $y_0$ and a $C^2$-map $h : U_0 \longrightarrow V_0$ such that $\{(x, y) \in U_0 \times V_0 : f(x, y) = 0\} = \{(x, h(x)) : x \in U_0\}$. In addition, if $f$ is $C^n$ then $h$ is $C^n$ as well, $2 \leqslant n \leqslant \infty$.*

We shall need the following generalization of Gronwall's lemma (proved in [**31**, 3.6]).

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LEMMA 1.4. *Let $T > 0, B_0 \geqslant 1$, and let $0 \leqslant D_0 \leqslant D_1 \leqslant \dots$ be a sequence of constants. Then there exists a sequence of constants $C_k \geqslant 0, k = 0, 1, \dots$ (where $C_k$ only depends on $k$, $D_k$, $T$, $B_0$) such that the following holds. Let $u_k \in C([0, T], \mathbb{R})$, $k = 0, 1, \dots$ satisfy*

$$u_k(t) \leqslant A_k + D_k \sum_{i=0}^{k} \int_0^t B_{k-i}\, u_i(s)\, ds$$

*for all $t \in [0, T]$ and $k = 0, 1, \dots$ where $0 \leqslant A_k \leqslant A_{k+1}$ $(k \geqslant 0)$ and $B_k \geqslant 1$ $(k \geqslant 1)$ are sequences such that $B_i B_j \leqslant B_{i+j}$ for all $i, j \geqslant 0$. Then we have*

$$u_k(t) \leqslant C_k \sum_{i=0}^{k} B_{k-i} A_i$$

*for all $t \in [0, T]$ and all $k$.*

## 2. *Nonlinear partial differential operators in $H^\infty(\Omega)$*

Let $\Omega$ be a nonempty open subset of $\mathbb{R}^n$. Let $H^k(\Omega)$ denote the Hilbert space consisting of all (real or complex valued) functions $u \in L_2(\Omega)$ whose derivatives $\partial^\alpha u$, in the sense of distributions, of order $|\alpha| = \alpha_1 + \dots + \alpha_n \leqslant k$ are in $L_2(\Omega)$; the norm in $H^k(\Omega)$ is

$$|u|_k = \left( \sum_{|\alpha| \leqslant k} \int_\Omega |\partial^\alpha u(x)|^2\, dx \right)^{1/2}, \quad u \in H^k(\Omega).$$

The space $H^\infty(\Omega) = \bigcap_{k=0}^\infty H^k(\Omega)$ is a Fréchet space equipped with the norms $(|\ |_k)_{k=0}^\infty$. Let $C^k_{\mathrm{BU}}(\Omega)$ denote the Banach space of all functions $u \in C^k(\Omega)$ whose derivatives $\partial^\alpha u$ of order $|\alpha| \leqslant k$ are bounded and uniformly continuous on $\Omega$, endowed with the norm

$$|u|_k^\infty = \sup_{|\alpha| \leqslant k} \sup_{x \in \Omega} |\partial^\alpha u(x)|, \quad u \in C^k_{\mathrm{BU}}(\Omega).$$

The Fréchet space $C^\infty_{\mathrm{BU}}(\Omega) = \bigcap_{k=0}^\infty C^k_{\mathrm{BU}}(\Omega)$ is equipped with the norm system $(|\ |_k^\infty)_{k=0}^\infty$. For $u \in C^k_{\mathrm{BU}}(\Omega)$ all partial derivatives $\partial^\alpha u, |\alpha| \leqslant k$, are naturally defined on $\bar{\Omega}$; if $\Omega$ is bounded then we write $C^k(\bar{\Omega}) = C^k_{\mathrm{BU}}(\Omega)$ and $C^\infty(\bar{\Omega}) = C^\infty_{\mathrm{BU}}(\Omega)$ as usual. In the remainder of this section we assume that $\Omega$ is uniformly regular of class $C^\infty$ in the sense of Amann [**5**, Chapter III, p. 642]. In this case there exist continuous imbeddings $H^{k+d}(\Omega) \hookrightarrow C^k_{\mathrm{BU},0}(\Omega)$ where $d = [n/2] + 1 > n/2$ and $C^k_{\mathrm{BU},0}(\Omega)$ denotes the closed subspace of $C^k_{\mathrm{BU}}(\Omega)$ consisting of all functions $u$ such that $\partial^\alpha u$ vanishes at infinity for $|\alpha| \leqslant k$ (cf. [**5**, Theorem 11.5, p. 647]). In fact, the existence of such imbeddings is the only assumption on $\Omega$. We then have for all $k = 0, 1, 2, \dots$ with suitable constants $c_k > 0$ the inequalities

$$|u|_k^\infty \leqslant c_k |u|_{k+d}, \quad u \in H^{k+d}(\Omega).$$

Consequently, the space $H^\infty(\Omega)$ is an algebra, and for the product in $H^\infty(\Omega)$ we obtain

$$|u_1 \cdot \dots \cdot u_p|_k \leqslant c_{k,p} \sup_{0 \leqslant i_1 + \dots + i_p \leqslant k} |u_1|_{i_1}^\infty \cdot \dots \cdot |u_{p-1}|_{i_{p-1}}^\infty |u_p|_{i_p} \leqslant c'_{k,p}[; u_1, \dots, u_p]_{d,k}$$

for all $u_1, \dots, u_p \in H^\infty(\Omega)$ and all $p, k$ with suitable constants $c_{k,p}$, $c'_{k,p} > 0$.

For instance, the above assumptions on $\Omega$ are fulfilled for $\Omega = \mathbb{R}^n$, for any open half space $\Omega$ in $\mathbb{R}^n$ or for any open bounded set in $\mathbb{R}^n$ with smooth boundary.

We say that a vector function $u = (u_1, \dots, u_M)$ belongs to $H^\infty(\Omega, \mathbb{R}^M)$ if each coordinate $u_j$ is in $H^\infty(\Omega)$; we put $|u|_k = (\sum_{j=1}^M |u_j|_k^2)^{1/2}$ and $|u|_k^\infty = \sup_{j=1,\dots,M} |u_j|_k^\infty$.

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We consider nonlinear smooth (systems of) partial differential operators on $H^\infty(\Omega)$. We fix integers $m \geqslant 0$, $M$, $P \geqslant 1$ and put $I(m) = \{\alpha \in \mathbb{N}_0^n : |\alpha| \leqslant m\}$. We choose an open set $A \subset (\mathbb{R}^M)^{I(m)}$; if $\Omega$ is unbounded then we assume that $0 \in A$. The set

$$U = \{u \in H^\infty(\Omega, \mathbb{R}^M) : \{\partial^\alpha u(x)\}_{|\alpha| \leqslant m} \in A \text{ for all } x \in \bar{\Omega}\}$$

is open in $H^\infty(\Omega, \mathbb{R}^M)$. Let $T > 0$ and let $F : [0, T] \times \Omega \times A \longrightarrow \mathbb{R}^P$ be a continuous map, $F = F(t, x, u)$. Throughout this paper we shall assume the following conditions:

$(A_1)$ The partial derivatives $\partial_t^i \partial_x^\alpha \partial_u^\beta F$ exist and are continuous for $i = 0, 1, 2$ and every $\alpha, \beta$ and are bounded on $[0, T] \times \Omega \times A_0$ for any bounded subset $A_0 \subset A$.

$(A_2)$ For every $\alpha \in \mathbb{N}_0^n$, $i = 0, 1, 2$ and every constant $C > 0$ we have

$$\sup_{t \in [0, T]} \sup_{u \in U, |u|_{m+d} \leqslant C} \int_\Omega |\partial_t^i \partial_x^\alpha F(t, w, \{\partial^\gamma u(w)\}_{|\gamma| \leqslant m})|^2 \, dw < +\infty.$$

REMARK 2.1. (i) Since $|u|_m^\infty \leqslant c_m |u|_{m+d}$ a sufficient condition for $(A_2)$ is that

$$\sup_{t \in [0, T]} \sup_{\phi \in C(\Omega, A_0)} \|\partial_t^i \partial_x^\alpha F(t, w, \phi(w))\|_{L_2(\Omega, dw)} < +\infty$$

for any $\alpha \in \mathbb{N}_0^n$, $i = 0, 1, 2$ and any bounded subset $A_0 \subset A$. For instance, if $\Omega$ is bounded then any $F \in C^\infty([0, T] \times \mathbb{R}^n \times (\mathbb{R}^M)^{I(m)})$ satisfies $(A_1)$ and $(A_2)$.

(ii) If $A$ is convex and $0 \in A$ then any $F$ satisfying $(A_1)$ and $\partial_t^i \partial_x^\alpha F(t, x, 0) = 0$ for any $\alpha \in \mathbb{N}_0^n$, $i = 0, 1, 2$ and all $t, x$ has property $(A_2)$. For instance, for a convex $A$ any $F \in C^\infty(A)$ independent of $t, x$ with $F(0) = 0$ satisfies $(A_1)$ and $(A_2)$.

(iii) Condition $(A_2)$ for $\alpha = 0$, $i = 0$ is necessary for the estimate in Lemma 2.2 for $k = 0$.

We consider the map $\mathscr{F} : [0, T] \times (U \subset H^\infty(\Omega, \mathbb{R}^M)) \longrightarrow H^\infty(\Omega, \mathbb{R}^P)$ defined by

$$\mathscr{F}(t, u)(x) = F(t, x, \{\partial^\alpha u(x)\}_{|\alpha| \leqslant m}), \quad u \in U, \quad t \in [0, T], \quad x \in \Omega.$$

Lemmas 2.2, 2.3 and 2.4 contain $H^\infty(\Omega)$-estimates for $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ as required by implicit function theorems of Nash–Moser type (cf. [**40**, 13.3.9]). Corresponding $C^\infty(\bar{\Omega})$-estimates for sup-norms and a bounded $\Omega$ are given in [**11**, **30**].

LEMMA 2.2. *Assume conditions* $(A_1)$ *and* $(A_2)$. *Then the map* $\mathscr{F} : [0, T] \times U \longrightarrow H^\infty(\Omega, \mathbb{R}^P)$ *is defined and continuous, and for any* $C > 0$ *there are constants* $C_k > 0$ *such that*

$$|\mathscr{F}(t, u)|_k \leqslant C_k [u]_{m+d, k}$$

*for all* $u \in U$ *with* $|u|_{m+d} \leqslant C$ *and all* $t \in [0, T]$.

*Proof.* Condition $(A_2)$ yields $\mathscr{F}(t, u) \in L_2(\Omega)$ and gives the estimate for $k = 0$. We choose $\alpha \in \mathbb{N}_0^n \setminus \{0\}$, $x \in \Omega$, and put $g(x) = (x, \{\partial^\alpha u(x)\}_{|\alpha| \leqslant m})$ and $Q = n + M|I(m)|$. Omitting $t$ in the notation, the chain rule gives for $\mathscr{F}(u) = F \circ g$ the formula

$$\partial^\alpha (F \circ g)(x) = \sum_{\beta \in \mathbb{N}_0^Q, \, 1 \leqslant |\beta| \leqslant |\alpha|} \partial^\beta F(g(x)) \, A_{\beta, \alpha}(x)$$

where the functions $A_{\beta, \alpha}(x)$ are for $\beta = (\beta_1, \ldots, \beta_Q) \in \mathbb{N}_0^Q \setminus \{0\}$ defined by

$$A_{\beta, \alpha}(x) = \sum c_{\alpha, \beta, j_1, \ldots, j_{|\beta|}} \partial^{j_1} g_1(x) \ldots \partial^{j_{\beta_1}} g_1(x) \ldots \partial^{j_{\beta_1 + \ldots + \beta_{Q-1}+1}} g_Q(x) \ldots \partial^{j_{|\beta|}} g_Q(x)$$

with suitable constants $c_{\alpha, \beta, j_1, \ldots, j_{|\beta|}}$, the sum running over all $j_1, \ldots, j_{|\beta|} \in \mathbb{N}_0^n \setminus \{0\}$ with $j_1 + \ldots + j_{|\beta|} = \alpha$; for $\beta_i = 0$ the corresponding factors containing $g_i$ are omitted. The formula shows that $\mathscr{F}(t, u) \in C^\infty(\Omega, \mathbb{R}^P)$ for any $t$, and for $|u|_m^\infty \leqslant C'$ we get

$$|\mathscr{F}(t, u)|_k \leqslant c_k \Big(1 + \sup_{1 \leqslant |\beta| \leqslant |\alpha| \leqslant k} \|\partial^\beta F(t, x, \{\partial^\gamma u(x)\}_{|\gamma| \leqslant m}) \, A_{\beta, \alpha}(x)\|_{L_2(\Omega, dx)}\Big).$$

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For $\partial^\beta = \partial_x^\beta$ the term $A_{\beta,\alpha}(x)$ is constant and $\|\partial_x^\beta F(t,x,\{\partial^\gamma u(x)\}_{|\gamma|\leqslant m})\|_{L_2(\Omega,dx)} \leqslant c_k$ for all $t,u$ by means of $(A_2)$. For $\partial^\beta = \partial_x^{\beta'}\partial_u^{\beta''}$ with $\beta'' \neq 0$ we obtain from $(A_1)$

$$\|A_{\beta,\alpha}\|_{L_2} \leqslant c_k \sup_{1\leqslant i_1+\ldots+i_r\leqslant k,\,1\leqslant r\leqslant k} |u|^\infty_{m+i_1}\ldots|u|^\infty_{m+i_{r-1}}|u|_{m+i_r} \leqslant c'_k[u]_{m+d,k}.$$

This proves that $\mathscr{F}(t,u)\in H^\infty(\Omega,\mathbb{R}^P)$ and the desired inequalities. The above estimates and formulae also imply the continuity of $\mathscr{F}$ and thus the assertion. □

Subsequently we will use the following conventions: the expressions $[u]_{m,k}$ are defined by the norms $|u|_i$, and the terms $[u]^\infty_{m,k}$ are specified by the norms $|u|^\infty_i$; the terms $[u;v]_{m,k}$ are determined by the norms $|u|_i$ and $|v|_j$; however, the expressions $[u;v]^\infty_{m,k}$ are defined by the norms $|u|^\infty_i$ and $|v|_j$; more explicitly, we put

$$[u;v]^\infty_{m,k} = \sup\{|u|^\infty_{m+i_1}\cdot\ldots\cdot|u|^\infty_{m+i_r}|v|_{m+j}: 0\leqslant r\leqslant k, 0\leqslant i_1+\ldots+i_r+j\leqslant k\}.$$

LEMMA 2.3. *We assume* $(A_1)$ *and* $(A_2)$. *Then the derivative*

$$\mathscr{F}': ([0,T]\times U)\times(\mathbb{R}\times H^\infty(\Omega,\mathbb{R}^M)) \longrightarrow H^\infty(\Omega,\mathbb{R}^P)$$

*exists and is continuous, and for* $t\in[0,T]$, $s\in\mathbb{R}$, $u\in U$, $v\in H^\infty(\Omega,\mathbb{R}^M)$ *we have*

$$\begin{aligned}\mathscr{F}'(t,u)(s,v) &= \frac{d\mathscr{F}}{dt}(t,u)s + \frac{\partial\mathscr{F}}{\partial u}(t,u)v\\ &= F_t(t,\cdot,\{\partial^\alpha u(\cdot)\})s + \sum_{|\alpha|\leqslant m} F_{\partial^\alpha u}(t,\cdot,\{\partial^\alpha u(\cdot)\})\partial^\alpha v.\end{aligned}$$

*In addition, for any* $C>0$ *there are constants* $C_k>0$ *such that for any* $t\in[0,T]$, $s\in\mathbb{R}$ *and all* $u\in U$ *with* $|u|^\infty_m\leqslant C$ *and all* $v\in H^\infty(\Omega,\mathbb{R}^M)$ *we have*

$$\left|\frac{\partial\mathscr{F}}{\partial u}(t,u)v\right|_k \leqslant C_k[u;v]^\infty_{m,k}$$

$$|\mathscr{F}'(t,u)(s,v)|_k \leqslant C_k[(t,u);(s,v)]_{m+d,k}.$$

*Proof.* It is obviously enough to consider the derivative $\partial\mathscr{F}/\partial u$. In the proof we omit $t$ in the notation because all estimates are uniform with respect to $t$. We first consider the case $m=0$ and $\mathscr{F}_0(u)(x)=F(x,u(x))$. We get $\mathscr{F}_0'(u)v = F_u(\cdot,u)v$ since for small $|h|\neq 0$ and $x\in\Omega$ the Taylor formula gives

$$\begin{aligned}&\left(\frac{1}{h}\right)(F(x,u(x)+hv(x))-F(x,u(x)))-F_u(x,u(x))v(x)\\ &\qquad = h\int_0^1 (1-s)F_{uu}(x,u(x)+shv(x))\{v(x),v(x)\}\,ds\end{aligned}$$

which tends to 0 in $H^\infty(\Omega,\mathbb{R}^P)$ as $h$ tends to 0. Since the map $u\longmapsto\{\partial^\alpha u\}_{|\alpha|\leqslant m}$ is linear and thus differentiable, the desired formula for $\partial\mathscr{F}/\partial u$ follows from the chain rule;

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this gives also the continuity of $\mathscr{F}'$. To prove the required estimates we first notice that

$$|F_{\partial^\alpha u}(t,\cdot,\{\partial^\alpha u(\cdot)\})|_k^\infty \leqslant c_k[u]_{m,k}^\infty$$

for all $u\in U$ with $|u|_m^\infty \leqslant C$ which follows from the chain rule stated in Lemma 2.2. We deduce that

$$\begin{aligned}\left|\frac{\partial\mathscr{F}}{\partial u}(t,u)\,v\right|_k &\leqslant \sum_{|\alpha|\leqslant m}|F_{\partial^\alpha u}(t,\cdot,\{\partial^\alpha u(\cdot)\})\,\partial^\alpha v|_k\\ &\leqslant c_k\sum_{j=0}^{k}|F_{\partial^\alpha u}(t,\cdot,\{\partial^\alpha u(\cdot)\})|_{k-j}^\infty|\partial^\alpha v|_j\\ &\leqslant c_k'\sum_{j=0}^{k}[u]_{m,k-j}^\infty|v|_{m+j}\leqslant c_k'[u;v]_{m,k}^\infty\leqslant c_k''[u;v]_{m+d,k}\end{aligned}$$

for all $u\in U$ with $|u|_m^\infty\leqslant C$ and all $v$. This proves the assertion. □

LEMMA 2.4. *Assuming* $(A_1)$ *and* $(A_2)$ *the second derivative*

$$\mathscr{F}'':([0,T]\times U)\times(\mathbb{R}\times H^\infty(\Omega,\mathbb{R}^M))\times(\mathbb{R}\times H^\infty(\Omega,\mathbb{R}^M))\longrightarrow H^\infty(\Omega,\mathbb{R}^P)$$

*exists, is continuous, and for* $t\in[0,T]$, $s_1,s_2\in\mathbb{R}$, $u\in U$, $v_1,v_2\in H^\infty(\Omega,\mathbb{R}^M)$ *we have*

$$\begin{aligned}\mathscr{F}''(t,u)\{(s_1,v_1),(s_2,v_2)\} &= \frac{d^2\mathscr{F}}{dt^2}(t,u)\,s_1s_2+\frac{d}{dt}\frac{\partial\mathscr{F}}{\partial u}(t,u)\,(s_2v_1+s_1v_2)\\ &\quad+\frac{\partial^2\mathscr{F}}{\partial u^2}(t,u)\{v_1,v_2\}=F_{tt}(t,\cdot,\{\partial^\alpha u(\cdot)\})\,s_1s_2\\ &\quad+\sum_{|\alpha|\leqslant m}F_{t,\partial^\alpha u}(t,\cdot,\{\partial^\alpha u(\cdot)\})\,\partial^\alpha(s_2v_1+s_1v_2)\\ &\quad+\sum_{|\alpha|\leqslant m}\sum_{|\beta|\leqslant m}F_{\partial^\alpha u\partial^\beta u}(t,\cdot,\{\partial^\alpha u(\cdot)\})\,\partial^\alpha v_1\,\partial^\beta v_2.\end{aligned}$$

*In addition, for any* $C>0$ *there are constants* $C_k>0$ *such that for all* $u\in U$ *with* $|u|_m^\infty\leqslant C$ *and all* $v,v_1,v_2\in H^\infty(\Omega,\mathbb{R}^M)$ *and* $t\in[0,T], s\in\mathbb{R}$ *we have*

$$\left|\frac{\partial^2\mathscr{F}}{\partial u^2}(t,u)\{v_1,v_2\}\right|_k\leqslant C_k[u;v_1,v_2]_{m+d,k}$$

$$|\mathscr{F}''(t,u)\{(s,v),(s,v)\}|_k\leqslant C_k[(t,u);(s,v),(s,v)]_{m+d,k}.$$

*Proof.* For a fixed $(s_1,v_1)$ we apply Lemma 2.3 to $\mathscr{F}'(t,u)(s_1,v_1)$ and obtain the existence and continuity of $\mathscr{F}''$ and the required formulae. The estimates are obtained from

$$\begin{aligned}\left|\frac{\partial^2\mathscr{F}}{\partial u^2}(t,u)\{v_1,v_2\}\right|_k &\leqslant \sum_{|\alpha|\leqslant m}\sum_{|\beta|\leqslant m}|F_{\partial^\alpha u\partial^\beta u}(t,\cdot,\{\partial^\alpha u(\cdot)\})\,\partial^\alpha v_1\,\partial^\beta v_2|_k\\ &\leqslant c_k\sum_{\substack{|\alpha|\leqslant m\\|\beta|\leqslant m}}\sum_{0\leqslant i\leqslant j\leqslant k}|F_{\partial^\alpha u\partial^\beta u}(t,\cdot,\{\partial^\alpha u(\cdot)\})|_{k-j}^\infty|\partial^\alpha v_1|_{j-i}^\infty|\partial^\beta v_2|_i\\ &\leqslant c_k'\sum_{0\leqslant i\leqslant j\leqslant k}[u]_{m,k-j}^\infty|v_1|_{m+j-i}^\infty|v_2|_{m+i}\leqslant c_k''[u;v_1,v_2]_{m+d,k}.\end{aligned}$$ □

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REMARK 2.5. (i) The above results also apply to complex valued function spaces $H^\infty(\Omega,\mathbb{C}) \cong H^\infty(\Omega,\mathbb{R}^2)$ considered as *real* Fréchet spaces. For instance, the map

$$\mathscr{F}: H^\infty(\Omega,\mathbb{C}) \longrightarrow H^\infty(\Omega,\mathbb{C}), \quad \mathscr{F}(u) = \Delta|u|^2 = \Delta(u\cdot\bar{u})$$

is a $C^2$-map (even $C^\infty$), and $\mathscr{F}'(u)v = 2\,\mathrm{Re}\,\Delta(\bar{u}v)$ is $\mathbb{R}$-linear.

(ii) If the conditions in $(A_1)$ and $(A_2)$ are satisfied for all $i = 0, 1, \ldots$ then $\mathscr{F}$ is a $C^\infty$-map.

If $P = M$ in the situation of Lemma 2.3 then for any $t\in[0,T]$ and $u\in U$ the linear map $\partial\mathscr{F}/\partial u(t,u): H^\infty(\Omega,\mathbb{R}^M) \longrightarrow H^\infty(\Omega,\mathbb{R}^M)$ is defined and continuous, and we can consider the $p$-fold composition $((\partial\mathscr{F}/\partial u)(t,u))^p$ for any $p \geqslant 1$.

LEMMA 2.6. *If $P = M$ in the situation of Lemma* 2.3 *then for any $C > 0$ and all integers $k \geqslant 0$, $p \geqslant 1$ there are constants $C_{k,p} > 0$ such that*

$$\left|\left(\frac{\partial\mathscr{F}}{\partial u}(t,u)\right)^p v\right|_k \leqslant C_{k,p}[u;v]^\infty_{m,k+(p-1)m}$$

*for all $t\in[0,T]$ and all $u\in U$ with $|u|^\infty_m \leqslant C$ and all $v\in H^\infty(\Omega,\mathbb{R}^M)$.*

*Proof.* The case $p = 1$ is proved in Lemma 2.3. If the assertion is proved for $p \geqslant 1$ then we find

$$\begin{aligned}\left|\left(\frac{\partial\mathscr{F}}{\partial u}(t,u)\right)^{p+1} v\right|_k &= \left|\frac{\partial\mathscr{F}}{\partial u}(t,u)\frac{\partial\mathscr{F}}{\partial u}(t,u)^p v\right|_k \leqslant c_k\sum_{j=0}^{k}[u]^\infty_{m,k-j}\left|\frac{\partial\mathscr{F}}{\partial u}(t,u)^p v\right|_{m+j}\\ &\leqslant c_{k,p}\sum_{j=0}^{k}[u]^\infty_{m,k-j}[u;v]^\infty_{m,m+j+(p-1)m} \leqslant c_{k,p}[u;v]^\infty_{m,k+pm}.\end{aligned}$$

Using the notation $j_- = \min(0,j)$ for an integer $j$, Lemma 2.6 implies for all $p \geqslant 0$ (including the case $p = 0$) the estimates

$$\left|\left(\frac{\partial\mathscr{F}}{\partial u}(t,u)\right)^p v\right|_k \leqslant C_{k,p}\sum_{i=(k+(p-1)m)_-}^{k+(p-1)m}[u]^\infty_{m,k+(p-1)m-i}|v|_{m+i}.$$

The expressions $[\,]_{m,k}$ are a suitable tool to prove inequalities for commutators of linear partial differential operators. The first simple observation is that in the situation of Lemma 2.3 the commutator $\partial_i\circ\partial\mathscr{F}/\partial u(t,u) - \partial\mathscr{F}/\partial u(t,u)\circ\partial_i$, $i = 1, \ldots, n$, is a linear differential operator of order $m$ (and not of order $m+1$) since

$$\left\{\partial_i\circ\frac{\partial\mathscr{F}}{\partial u}(t,u) - \frac{\partial\mathscr{F}}{\partial u}(t,u)\circ\partial_i\right\}v = \sum_{|\alpha|\leqslant m}\partial_i\{F_{\partial^\alpha u}(t,\cdot,\{\partial^\alpha u(\cdot)\})\}\,\partial^\alpha v.$$

Therefore, we get for all $u\in U$ with $|u|^\infty_m \leqslant C$ with some suitable $c > 0$ the estimates

$$\left|\left\{\partial_i\frac{\partial\mathscr{F}}{\partial u}(t,u) - \frac{\partial\mathscr{F}}{\partial u}(t,u)\,\partial_i\right\}v\right|_0 \leqslant c[u]^\infty_{m,1}|v|_m. \qquad \square$$

More general commutator estimates are contained in the next two lemmas. The crucial point in the following estimates is that the highest order of the derivatives of the linear term $v$ appearing on the right-hand side is, as in the easiest case above, always one lower than the trivially expected order; this is not true for the nonlinear terms $u$. Lemmas 2.7 and 2.8 are sharp which is important for the remainder of this paper. Notice that estimates of this form are called Moser-estimates in [**40**, 13.3.7].

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LEMMA 2.7. *We suppose the assumptions of Lemma* 2.3. *Then for any* $C > 0$, *any integer* $k \geqslant 0$ *and any multi-index* $\gamma \in \mathbb{N}_0^n$ *with* $|\gamma| = q \geqslant 1$ *there exists a constant* $C_{k,q} > 0$ *such that*

$$\left| \left\{ \partial^\gamma \frac{\partial \mathscr{F}}{\partial u}(t,u) - \frac{\partial \mathscr{F}}{\partial u}(t,u)\, \partial^\gamma \right\} v \right|_k \leqslant C_{k,q} \sum_{j=0}^{k+q-1} [u]^\infty_{m,k+q-j} |v|_{m+j}$$

*for all* $u \in U$ *with* $|u|^\infty_m \leqslant C$ *and all* $v \in H^\infty(\Omega, \mathbb{R}^M)$.

*Proof.* We write $A_u = \partial \mathscr{F} / \partial u(t,u)$. The case $|\gamma| = q = 1$ and $\partial^\gamma = \partial_i$ follows from

$$\begin{aligned} |\{\partial_i A_u - A_u \partial_i\} v|_k &\leqslant c_k \sum_{|\alpha| \leqslant m} \sum_{j=0}^{k} |F_{\partial^\alpha u}(t, \cdot, \{\partial^\alpha u(\cdot)\})|^\infty_{k-j+1} |v|_{m+j} \\ &\leqslant c'_k \sum_{j=0}^{k} [u]^\infty_{m,k-j+1} |v|_{m+j}. \end{aligned}$$

The general case $\gamma = \partial_{i_1} \dots \partial_{i_q}$ is then obtained from the estimates

$$\begin{aligned} |\{\partial^\gamma A_u - A_u \partial^\gamma\} v|_k &= \left| \sum_{j=1}^{q} \partial_{i_1} \dots \partial_{i_{j-1}} (\partial_{i_j} A_u - A_u \partial_{i_j}) \partial_{i_{j+1}} \dots \partial_{i_q} v \right|_k \\ &\leqslant \sum_{j=1}^{q} |\{\partial_{i_j} A_u - A_u \partial_{i_j}\} (\partial_{i_{j+1}} \dots \partial_{i_q} v)|_{k+j-1} \\ &\leqslant c'_{k,q} \sum_{j=1}^{q} \sum_{i=0}^{k+j-1} [u]^\infty_{m,k+j-i} |v|_{m+i+q-j} \\ &\leqslant c_{k,q} \sum_{l=0}^{k+q-1} [u]^\infty_{m,k+q-l} |v|_{m+l}. \end{aligned}$$

□

LEMMA 2.8. *We assume that* $P = M$ *in the situation of Lemma* 2.3. *Then for any* $C > 0$, *any* $k \geqslant 0$, $p \geqslant 1$ *and any* $\gamma \in \mathbb{N}_0^n$ *with* $|\gamma| = q \geqslant 1$ *there is a constant* $C_{k,p,q} > 0$ *such that*

$$\left| \left\{ \partial^\gamma \left( \frac{\partial \mathscr{F}}{\partial u}(t,u) \right)^p - \left( \frac{\partial \mathscr{F}}{\partial u}(t,u) \right)^p \partial^\gamma \right\} v \right|_k \leqslant c_{k,p,q} \sum_{j=0}^{k+q+(p-1)m-1} [u]^\infty_{m,k+q+(p-1)m-j} |v|_{m+j}$$

*for all* $u \in U$ *with* $|u|^\infty_m \leqslant C$ *and all* $v \in H^\infty(\Omega, \mathbb{R}^M)$.

*Proof.* We write $A_u = \partial \mathscr{F} / \partial u(t,u)$. The case $p = 1$ is proved in Lemma 2.7. We assume that $p \geqslant 2$. Applying Lemmas 2.6 and 2.7 we obtain the assertion from the estimates

$$\begin{aligned} |\{\partial^\gamma A_u^p - A_u^p \partial^\gamma\} v|_k &= \left| \sum_{j=0}^{p-1} \{A_u^j (\partial^\gamma A_u - A_u \partial^\gamma) A_u^{p-j-1}\} v \right|_k \\ &\leqslant c_{k,p} \sum_{j=0}^{p-1} \sum_{i=(k+(j-1)m)_-}^{k+(j-1)m} [u]^\infty_{m,k+(j-1)m-i} |(\partial^\gamma A_u - A_u \partial^\gamma) A_u^{p-j-1} v|_{m+i} \\ &\leqslant c_{k,p,q} \sum_{j=0}^{p-1} \sum_{i=(k+(j-1)m)_-}^{k+(j-1)m} \sum_{l=0}^{q+m+i-1} [u]^\infty_{m,k+(j-1)m-i} [u]^\infty_{m,q+m+i-l} |A_u^{p-j-1} v|_{m+l} \\ &\leqslant c'_{k,p,q} \sum_{j=0}^{p-1} \sum_{i=(k+(j-1)m)_-}^{k+(j-1)m} \sum_{l=0}^{q+m+i-1} \sum_{r=(l+(p-j-1)m)_-}^{l+(p-j-1)m} [u]^\infty_{m,k+q+(p-1)m-r} |v|_{m+r} \\ &\leqslant c''_{k,p,q} \sum_{r=0}^{k+q+(p-1)m-1} [u]^\infty_{m,k+q+(p-1)m-r} |v|_{m+r}. \end{aligned}$$

□

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Since the proofs of Lemmas 2.6, 2.7 and 2.8 are based on abstract arguments and do not use the particular form of $\partial \mathscr{F}/\partial u(t,u)$, the same proofs give also the following result.

COROLLARY 2.9. *Let* $U \subset \{u \in H^\infty(\Omega, \mathbb{R}^M) : |u|_m^\infty \leqslant C\}$ *where* $C > 0$, $m \geqslant 0$ *are fixed. Let* $A_u : H^\infty(\Omega, \mathbb{R}^M) \longrightarrow H^\infty(\Omega, \mathbb{R}^M)$, $u \in U$, *be a family of linear operators satisfying*

$$|A_u v|_k \leqslant c_k [u; v]_{m,k}^\infty \tag{2.1}$$

$$|(\partial_i A_u - A_u \partial_i) v|_k \leqslant c_k \sum_{j=0}^{k} [u]_{m,k-j+1}^\infty |v|_{m+j} \tag{2.2}$$

*for all* $u \in U$, $v \in H^\infty(\Omega, \mathbb{R}^M)$ *and* $i = 1, \ldots, n$ *with suitable constants* $c_k > 0$. *Then*

$$|A_u^p v|_k \leqslant c_{k,p} [u; v]_{m,k+(p-1)m}^\infty \tag{2.3}$$

$$|(\partial^\gamma A_u^p - A_u^p \partial^\gamma) v|_k \leqslant c_{k,p,q} \sum_{j=0}^{k+q+(p-1)m-1} [u]_{m,k+q+(p-1)m-j}^\infty |v|_{m+j} \tag{2.4}$$

*holds for all* $p \geqslant 1$, $k \geqslant 0$, $|\gamma| = q \geqslant 1$, $u \in U$, $v \in H^\infty(\Omega, \mathbb{R}^M)$ *with suitable constants* $c_{k,p}, c_{k,p,q} > 0$ (*where* (2.3) *follows from* (2.1) *without assuming* (2.2)).

## 3. *Some estimates for linear operators in* $H^\infty(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be an open set. We write $H^\infty = H^\infty(\Omega, \mathbb{R}^M)$. Let $A_u : H^\infty \longrightarrow H^\infty$, $u \in U$, be a family of linear operators satisfying

$$|A_u z|_k \leqslant c_k [u; z]_{m,k}^\infty, \quad u \in U, z \in H^\infty \tag{3.1}$$

$$|(\partial_i A_u - A_u \partial_i) z|_k \leqslant c_k \sum_{j=0}^{k} [u]_{m,k-j+1}^\infty |z|_{m+j}, \quad u \in U, z \in H^\infty \tag{3.2}$$

where $U$ is a subset of $\{u \in H^\infty(\Omega, \mathbb{R}^M) : |u|_m^\infty \leqslant C\}$ and $m \geqslant 1$, $C > 0$ are fixed.

LEMMA 3.1. *Let* $A_u : H^\infty \longrightarrow H^\infty$ *satisfy* (3.1), (3.2) *and*

$$|z|_m \leqslant C_0(|A_u z|_0 + |z|_0), \quad u \in U, z \in H^\infty \tag{3.3}$$

*for some fixed* $C_0 > 0$. *Then for any* $0 \leqslant j \leqslant m$ *there is a constant* $C_j > 0$ *such that*

$$|z|_{m+j} \leqslant C_j \sum_{i=0}^{j} [u]_{m,j-i}^\infty (|A_u z|_i + |z|_0), \quad u \in U, z \in H^\infty.$$

*Proof.* The case $j = 0$ holds by assumption. If the assertion is already proved for $0 \leqslant j-1 \leqslant m-1$ then we choose $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m+j$ and $\alpha = \beta + \gamma$ where $|\beta| = m$, $|\gamma| = j$. Then the desired estimates are obtained from

$$\begin{aligned}
|\partial^\alpha z|_0 = |\partial^\beta \partial^\gamma|_0 &\leqslant |\partial^\gamma z|_m \\
&\leqslant C_0(|(A_u \partial^\gamma - \partial^\gamma A_u) z|_0 + |\partial^\gamma A_u z|_0 + |z|_m) \\
&\leqslant c_j \left( \sum_{i=0}^{j-1} [u]_{m,j-i}^\infty |z|_{m+i} + |A_u z|_j \right) \\
&\leqslant c_j' \left( \sum_{i=0}^{j-1} [u]_{m,j-i}^\infty \sum_{l=0}^{i} [u]_{m,i-l}^\infty (|A_u z|_l + |z|_0) + |A_u z|_j \right) \\
&\leqslant c_j'' \sum_{l=0}^{j} [u]_{m,j-l}^\infty (|A_u z|_l + |z|_0).
\end{aligned}$$

□

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LEMMA 3.2. *Let the linear operators $A_u: H^\infty \longrightarrow H^\infty, u \in U$, satisfy* (3.1), (3.2), (3.3). *Then for any $p \geqslant 1$ and $0 \leqslant j \leqslant m$ there is $c_{p,j} > 0$ such that for all $u \in U, z \in H^\infty$ we have*

$$|z|_{pm+j} \leqslant c_{p,j}\left\{\sum_{i=0}^{j}[u]^\infty_{m,j-i}(|A_u^p z|_i+|z|_0)+\sum_{i=0}^{m-1}\sum_{k=1}^{p-1}[u]^\infty_{m,(p-k)m+j-i}(|A_u^k z|_i+|z|_i)\right\}.$$

*Proof.* The case $p=1$ is proved in Lemma 3.1. If the estimate is proved for $p \geqslant 1$ and $j=m$ then the assertion follows for $p+1$ and $j=0$ since $|A_u^p z|_m \leqslant C_0(|A_u^{p+1} z|_0+|A_u^p z|_0)$ and

$$|z|_{pm+m} \leqslant c_{p,m}\left\{\sum_{i=0}^{m}[u]^\infty_{m,m-i}(|A_u^p z|_i+|z|_0)+\sum_{i=0}^{m-1}\sum_{k=1}^{p-1}[u]^\infty_{m,(p+1-k)m-i}(|A_u^k z|_i+|z|_i)\right\}.$$

Next we assume that the estimate is proved for $p \geqslant 2$ and $0 \leqslant j \leqslant m-1$. We choose $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = pm+j+1$ and $\alpha = \beta+\gamma$ with $|\beta| = pm+j$, $|\gamma|=1$ and get

$$\begin{aligned}|\partial^\alpha z|_0 &= |\partial^\beta \partial^\gamma z|_0 \leqslant |\partial^\gamma z|_{pm+j}\\ &\leqslant c_{p,j}\left\{\sum_{i=0}^{j}[u]^\infty_{m,j-i}(|(\partial^\gamma A_u^p - A_u^p\partial^\gamma)z|_i+|A_u^p z|_{i+1}+|z|_1)+\dots\right.\\ &\quad\left.+\sum_{i=0}^{m-1}\sum_{k=1}^{p-1}[u]^\infty_{m,(p-k)m+j-i}(|(\partial^\gamma A_u^k - A_u^k\partial^\gamma)z|_i+|A_u^k z|_{i+1}+|z|_{i+1})\right\}.\end{aligned}$$

We have to estimate six terms. The second and third terms are clear; the fifth and sixth terms are clear for $i=0,\dots,m-2$, and are for $i=m-1$, $k=1,\dots,p-1$ estimated by

$$\begin{aligned}&[u]^\infty_{m,(p-k-1)m+j+1}(|A_u^k z|_m+|z|_m)\\ &\quad\leqslant C_0[u]^\infty_{m,(p-k-1)m+j+1}(|A_u^{k+1}z|_0+|A_u^k z|_0+|A_u z|_0+|z|_0)\\ &\quad\leqslant C_0'\left\{[u]^\infty_{m,j+1}(|A_u^p z|_0+|z|_0)+\sum_{l=1}^{p-1}[u]^\infty_{m,(p-l)m+j+1}|A_u^i z|_0\right\}.\end{aligned}$$

It remains to estimate the commutator terms $|(\partial^\gamma A_u^k - A_u^k\partial^\gamma)z|_i$ for $k=1,\dots,p$ and $i=0,\dots,j$ if $k=p$ or for $i=0,\dots,m-1$ if $1 \leqslant k \leqslant p-1$. For such $i,k$ we get

$$[u]^\infty_{m,(p-k)m+j-i}|(\partial^\gamma A_u^k - A_u^k\partial^\gamma)z|_i \leqslant c_{i,k}\sum_{l=0}^{(k-1)m+i}[u]^\infty_{m,(p-1)m+j+1-l}|z|_{m+l}.$$

Since $m+l \leqslant pm+j$ we may apply the hypothesis of the induction. Let $l=(q-1)m+r$ with $1 \leqslant q \leqslant p$ where $0 \leqslant r \leqslant j$ if $q=p$ or $0 \leqslant r \leqslant m-1$ if $q \leqslant p-1$. We find that

$$\begin{aligned}&[u]^\infty_{m,(p-1)m+j+1-l}|z|_{m+l}\\ &\quad\leqslant c_{p,j}[u]^\infty_{m,(p-1)m+j+1-l}\left\{\sum_{s=0}^{r}[u]^\infty_{m,r-s}(|A_u^q z|_s+|z|_0)+\dots\right.\\ &\qquad\left.+\sum_{s=0}^{m-1}\sum_{t=1}^{q-1}[u]^\infty_{m,(q-t)m+r-s}(|A_u^t z|_s+|z|_s)\right\}\\ &\quad\leqslant c'_{p,j}\left\{\sum_{s=0}^{r}[u]^\infty_{m,(p-q)m+j+1-s}(|A_u^q z|_s+|z|_0)+\dots\right.\\ &\qquad\left.+\sum_{s=0}^{m-1}\sum_{t=1}^{q-1}[u]^\infty_{m,(p-t)m+j+1-s}(|A_u^t z|_s+|z|_s)\right\}.\end{aligned}$$

This proves the assertion. □

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COROLLARY 3.3. *Let $A_u, B_u : H^\infty \longrightarrow H^\infty, u \in U$, be families of linear operators such that $(B_u)$ satisfy* (3.1) *and $(A_u)$ satisfy* (3.1), (3.2) *and*

$$|z|_m \leqslant C|A_u z|_0, \quad u \in U, \quad z \in H^\infty \tag{3.4}$$

*for some constant $C > 0$. Then for any $k \geqslant 0$ there is a constant $C_k > 0$ such that*

$$|B_u z|_k \leqslant C_k \sum_{i=0}^{k} [u]^\infty_{m,k-i} |A_u z|_i, \quad u \in U, \quad z \in H^\infty.$$

*Proof.* The assumption on $(B_u)$ implies that

$$|B_u z|_k \leqslant c_k [u; z]^\infty_{m,k} \leqslant c_k \sum_{n=0}^{k} [u]^\infty_{m,k-n} |z|_{m+n}.$$

Lemma 3.1 gives for $0 \leqslant n \leqslant m$ with $0 \leqslant n \leqslant k$ the estimate

$$[u]^\infty_{m,k-n} |z|_{m+n} \leqslant c_n \sum_{i=0}^{n} [u]^\infty_{m,k-i} (|A_u z|_i + |A_u z|_0)$$

and hence the claim in the case $0 \leqslant k \leqslant m$. For $n = (p-1)m + j$ with $p \geqslant 2$ and $0 \leqslant j \leqslant m$ an application of (2.3) and Lemma 3.2 yields

$$\begin{aligned} |z|_{pm+j} \leqslant c_{p,j} \Bigg\{ & \sum_{i=0}^{j} [u]^\infty_{m,j-i} [u; A_u z]^\infty_{m,i+(p-2)m} + \dots \\ & + \sum_{i=0}^{m-1} \sum_{l=2}^{p-1} [u]^\infty_{m,(p-l)m+j-i} [u; A_u z]^\infty_{m,i+(l-2)m} + \sum_{i=0}^{m-1} [u]^\infty_{m,(p-1)m+j-i} |A_u z|_i \Bigg\} \end{aligned}$$

which implies for $0 \leqslant n \leqslant k$ and $k \geqslant m$ the inequality

$$[u]^\infty_{m,k-n} |z|_{m+n} \leqslant c_k \left\{ [u; A_u z]^\infty_{m,k-m} + \sum_{i=0}^{m-1} [u]^\infty_{m,k-i} |A_u z|_i \right\}.$$

Since $[u; A_u z]^\infty_{m,k-m} \leqslant \sum_{i=m}^{k} [u]^\infty_{m,k-i} |A_u z|_i$ we obtain the assertion. □

COROLLARY 3.4. *Let $A_u : H^\infty \longrightarrow H^\infty$ satisfy* (3.1), (3.2), (3.4). *Then for any $k \geqslant 0$ there is a constant $C_k > 0$ such that*

$$|z|_{m+k} \leqslant C_k \sum_{i=0}^{k} [u]^\infty_{m,k-i} |A_u z|_i, \quad u \in U, \quad z \in H^\infty.$$

*Proof.* This follows from Corollary 3.3 considering $B_u = \partial^\alpha$ for $|\alpha| \leqslant m$. □

We shall need the following two observations. In Lemma 3.5 and Corollary 3.6 we assume that $\Omega = \mathbb{R}^n$.

LEMMA 3.5. *Let $Az(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x) \partial^\alpha z(x)$ be a linear differential operator with coefficients $a_\alpha \in C^\infty_{\mathrm{BU}}(\mathbb{R}^n)$ satisfying for some $j \geqslant 0$ and some constant $C_j > 0$ the estimate*

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$$|z|_{m+j} \leqslant C_j |Az|_j, \quad z \in H^{m+j}. \tag{3.5}$$

*Then $z \in H^{m+j}$ and $Az \in H^{j+1}$ implies that $z \in H^{m+j+1}$.*

*Proof.* Fix $z \in H^{m+j}$ with $Az \in H^{j+1}$ and put $u = \partial^\alpha z$ for some $|\alpha| = m+j$. It is enough to show that for $u_h = u(\cdot + he_k)$, $k = 1, \ldots, n$, the limit $\lim_{h \to 0} (u_h - u)/h$ exists in $L_2(\mathbb{R}^n)$ (cf. [**26**, 14.18]). Note that $A(z_h) = (Az)_h + \sum_{|\alpha| \leqslant m} (a_\alpha - (a_\alpha)_h)(\partial^\alpha z)_h$. Since

$$\left| \frac{u_{h_1} - u}{h_1} - \frac{u_{h_2} - u}{h_2} \right|_0 \leqslant \left| \frac{z_{h_1} - z}{h_1} - \frac{z_{h_2} - z}{h_2} \right|_{m+j} \leqslant C_j \left| \frac{A(z_{h_1}) - Az}{h_1} - \frac{A(z_{h_2}) - Az}{h_2} \right|_j$$

it suffices to show that the limit $\lim_{h \to 0} (A(z_h) - Az)/h$ exists in $H^j$. We have

$$\frac{A(z_h) - Az}{h} = \frac{(Az)_h - Az}{h} + \sum_{|\beta| \leqslant m} \frac{a_\beta - (a_\beta)_h}{h} \partial^\beta z_h.$$

The first summand is convergent in $H^j$ for $h \longrightarrow 0$ since $Az \in H^{j+1}$, the second summand is convergent in $H^j$ for $h \longrightarrow 0$ since $(a_\beta - (a_\beta)_h/h \longrightarrow -\partial_k a_\beta$ in $C^\infty_{\mathrm{BU}}(\mathbb{R}^n)$ and $\partial^\beta z_h \longrightarrow \partial^\beta z$ in $H^j$ observing that the product $C^\infty_{\mathrm{BU}}(\mathbb{R}^n) \times H^j \longrightarrow H^j$ is continuous. □

COROLLARY 3.6. *Let $A$ be a differential operator as in Lemma* 3.5 *considered as an unbounded operator in $L_2(\mathbb{R}^n)$ with domain $\mathscr{D}(A) = H^m$. Assume that for any $j \geqslant 0$ there is a constant $C_j > 0$ such that* (3.5) *holds. Then $\mathscr{D}(A^p) = H^{pm}$ for any integer $p \geqslant 1$.*

*Proof.* The case $p = 1$ is evident. Assume that $\mathscr{D}(A^p) = H^{pm}$ is already proved for some $p \geqslant 1$. The inclusion $H^{(p+1)m} \subset \mathscr{D}(A^{p+1})$ is clear. Let $x \in \mathscr{D}(A^{p+1})$; hence $x \in H^{pm}$ and $Ax \in H^{pm}$. We show by induction that $x \in H^{pm+j}$ for any $0 \leqslant j \leqslant m$. This is clear for $j = 0$; if the case $0 \leqslant j \leqslant m-1$ is proved then we get $x \in H^{pm+j}$ and $Ax \in H^{pm} \subset H^{(p-1)m+j+1}$ which implies that $x \in H^{pm+j+1}$ by means of Lemma 3.5. □

## 4. *Linearization of nonlinear evolution equations*

In this section we put $E = H^\infty(\Omega, \mathbb{R}^M)$ and $P = M$ and consider for some fixed $T > 0$ a map $\mathscr{F} : [0, T] \times (U \subset E) \longrightarrow E$ defined in some open set $U \subset E$ as in Section 2 by

$$\mathscr{F}(t, u)(x) = F(t, x, \{\partial^\alpha u(x)\}_{|\alpha| \leqslant m}).$$

We always assume $(A_1)$, $(A_2)$. The purpose of this section is to solve the Cauchy problem

$$\begin{cases} u_t = \mathscr{F}(t, u), & t \in [0, a] \\ u(0) = \phi. \end{cases} \tag{4.1}$$

More precisely, for a given $\phi \in E$ we are looking for a solution $u$ for some suitable small $a > 0$ where by a solution we mean a function $u \in C^1([0, a], E)$ such that $u(t) \in U$ for all $t \in [0, a]$ and (4.1) is satisfied. For a fixed $a > 0$ we can carry out the

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transformation $t = as$, $z(s) = u(as)$. Writing $J = [0, 1]$ we obtain the transformed Cauchy problem

$$\begin{cases} z_s = a\mathscr{F}(as, z), \quad s \in J \\ z(0) = \phi. \end{cases} \tag{4.2}$$

REMARK 4.1. If $u \in C^1([0, a], E)$ is a solution of (4.1) then $z(s) = u(as)$ gives a solution $z \in C^1(J, E)$ of (4.2). Conversely, if $z \in C^1(J, E)$ solves (4.2) then $u(t) = z(t/a)$ solves (4.1).

We formulate the Cauchy problem by means of a mapping. We note that for any $u \in C(J, E)$ the map $t \longmapsto \mathscr{F}(t, u(t))$ belongs to $C(J, E)$; this follows from Lemma 2.2. The set

$$W = \{u \in C^1(J, E) : u(t) \in U \text{ for all } t \in J\}$$

is open in $C^1(J, E)$ and we can define the map

$$\Phi : ((W \times (-\sqrt{T}, \sqrt{T}) \times U) \subset C^1(J, E) \times \mathbb{R} \times E) \longrightarrow C(J, E) \times E,$$
$$\Phi(u, a, \phi) = (u_t - a^2\mathscr{F}(a^2 t, u(t)), u(0) - \phi).$$

REMARK 4.2. (i) For any $\phi_0 \in U$ we have $\Phi(\phi_0, 0, \phi_0) = (0, 0)$ where an element of $E$ is also considered as a constant function in $C^1(J, E)$.

(ii) For $u \in W$, $a \in (-\sqrt{T}, \sqrt{T})$, $\phi \in U$ the equation $\Phi(u, a, \phi) = (0, 0)$ is equivalent to

$$\begin{cases} u_t = a^2\mathscr{F}(a^2 t, u), \quad t \in J \\ u(0) = \phi. \end{cases} \tag{4.3}$$

(iii) For a given $\phi_0 \in E$ we want to apply the Nash–Moser implicit function theorem (Theorem 1.3) in order to find a map $\Psi$ such that the equation $\Phi(u, a, \phi) = (0, 0)$ is uniquely solved by $u = \Psi(a, \phi)$ for all $(a, \phi)$ in some neighbourhood of $(0, \phi_0)$. We then can fix some suitable small $a > 0$, and the above transformation $t \longmapsto t/a^2$ gives the desired solution of (4.1). For the application of the implicit function theorem we have to show the bijectivity of the linearized map $\Phi_u(u, a, \phi)$ including the required estimates.

LEMMA 4.3. *The function $\Phi$ is a $C^2$-map with the partial derivatives*

$$\Phi_u(u, a, \phi)\, z = \left(z_t - a^2 \frac{\partial \mathscr{F}}{\partial u}(a^2 t, u(t))\, z(t), z(0)\right)$$

$$\Phi_{uu}(u, a, \phi)\{z_1, z_2\} = \left(-a^2 \frac{\partial^2 \mathscr{F}}{\partial u^2}(a^2 t, u(t))\{z_1(t), z_2(t)\}, 0\right)$$

*for $(u, a, \phi) \in W \times (-\sqrt{T}, \sqrt{T}) \times U$, $z, z_1, z_2 \in C^1(J, E)$. For any $C > 0$ there are constants $c_k > 0$ such that for all $k$ we have the estimates*

$$|\Phi'(u, a, \phi)\, w|_k \leqslant c_k[(u, a, \phi); w]_{m+d, k}$$
$$|\Phi''(u, a, \phi)\{w, w\}|_k \leqslant c_k[(u, a, \phi); w, w]_{m+d, k}$$

*for all $(u, a, \phi) \in W \times (-\sqrt{T}, \sqrt{T}) \times U$ with $|u|_{m+d}^{C(J, E)} \leqslant C$ and $w \in C^1(J, E) \times \mathbb{R} \times E$.*

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*Proof.* The continuity of $\Phi$ follows from the proof of Lemma 2.2 since the estimates obtained there are uniform with respect to $t$. For the same reason, Lemmas 2.3 and 2.4 imply that $\Phi$ is a $C^2$-map where $\Phi_u$, $\Phi_{uu}$ are as above. For the first derivative we obtain the formulae

$$\Phi'(u,a,\phi)(z,b,\psi) = \Phi_u(u,a,\phi)z + \Phi_a(u,a,\phi)b + \Phi_\phi(u,a,\phi)\psi$$

$$\Phi_a(u,a,\phi)b = \left(-2ab\mathscr{F}(a^2t,u(t)) - 2a^3bt\frac{d\mathscr{F}}{dt}(a^2t,u(t)),0\right)$$

$$\Phi_\phi(u,a,\phi)\psi = (0,-\psi)$$

which imply the desired estimates for $\Phi'$ as an application of Lemmas 2.2 and 2.3. Analogous formulae are obtained for the second partial derivatives $\Phi_{ua}$, $\Phi_{aa}$ of $\Phi$ implying the desired estimates for $\Phi''$ as a consequence of Lemmas 2.2, 2.3 and 2.4. This proves the assertion. □

THEOREM 4.4. *Let $E = H^\infty(\mathbb{R}^n)$. Let $\phi_0 \in E$ be fixed, let $U$ be an open neighbourhood of $\phi_0$ in $E$, and let $W \subset C^1(J,E)$ be defined as above. Assume that there exist $\delta > 0$, $b \geqslant 0$ and constants $c_k > 0$ such that for every $0 < a < \delta$, $0 \leqslant t_0 < \delta$ and $u \in W$ the problem*

$$\begin{cases} z_t(t) = a\dfrac{\partial\mathscr{F}}{\partial u}(t_0 + at, u(t))z(t) + f(t), \quad t \in J \\ z(0) = g \end{cases} \tag{4.4}$$

*has for any $f \in C(J,E)$ and $g \in E$ a unique solution $z \in C^1(J,E)$ satisfying*

$$|z|_k \leqslant c_k[u;(f,g)]_{b,k}.$$

*Then there exist $a > 0$ and an open neighbourhood $U_0 \subset U$ of $\phi_0$ such that the Cauchy problem* (4.1) *has for any $\phi \in U_0$ a unique solution $u \in C^1([0,a],E)$. Moreover, the solution map $\Psi : U_0 \longrightarrow C^1([0,a],E)$ is a $C^2$-map; if $\mathscr{F}$ is $C^n$ then $\Psi$ is $C^n$ as well, $2 \leqslant n \leqslant \infty$.*

*Proof.* The space $E = H^\infty(\mathbb{R}^n)$ satisfies the assumptions of Lemma 1.2 by means of [**26**, 14.12] and the spaces $E$, $C(J,E)$, $C^1(J,E)$ have therefore properties $(S_\Omega)_t$ and (DN). Hence the assumptions on the spaces in Theorem 1.3 are fulfilled. From Lemma 4.3 we conclude that $\Phi$ satisfies hypotheses (1.1) and (1.3) of Theorem 1.3 and $\Phi(\phi_0,0,\phi_0) = (0,0)$. Applying the assumption to $t_0 = 0$ we see that the linear map $\Phi_u(u,a,\phi) : C^1(J,E) \longrightarrow C(J,E) \times E$ is bijective for any $u \in W$, $a \in (-\sqrt{\delta},\sqrt{\delta})$, $\phi \in U$ observing that for $a = 0$ the problem (4.4) is uniquely solved by $z(t) = g + \int_0^t f(s)\,ds$. We get the estimates

$$|\Phi_u(u,a,\phi)^{-1}(f,g)|_k \leqslant c_k[(u,a,\phi);(f,g)]_{b,k}.$$

Therefore, the implicit function theorem (Theorem 1.3) yields open neighbourhoods $W_0 \subset W$ of $\phi_0$ in $C^1(J,E)$ and $U_0 \subset U$ of $\phi_0$ in $E$; further $0 < \delta_0 \leqslant \sqrt{\delta}$ and there is a $C^2$-map $\Psi_1 : (-\delta_0,\delta_0) \times U_0 \longrightarrow W_0$ such that the equation $\Phi(u,a,\phi) = (0,0)$ is uniquely solved in $W_0$ by $u = \Psi_1(a,\phi)$ for any $a \in (-\delta_0,\delta_0)$ and $\phi \in U_0$; hence $u = \Psi_1(a,\phi)$ is a solution of (4.3) which is unique in $W_0$. We can thus choose a fixed small $a > 0$ and a $C^2$-map $\Psi : U_0 \longrightarrow C^1([0,a],E)$, $\Psi(\phi)(t) = \Psi_1(\sqrt{a},\phi)(t/a)$, such that $u = \Psi(\phi)$ is a solution of (4.1) in $[0,a]$. This proves the existence part.

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To show uniqueness we choose $W_0$ as above and assume that $u_1, u_2$ are solutions of the problem $\{u_t = \mathscr{F}(t,u),\ t\in[0,a^2];\ u(0)=\phi_0\}$ for some $a>0$. Then $z_j(s)=u_j(a^2 s)$ solves (4.3). Since $u_j(0)=\phi_0\in W_0$ we can choose $a>0$ so small that $z_1, z_2\in W_0$ and thus $z_1=z_2$; this shows that $u_1=u_2$ in a small neighbourhood of 0. This local uniqueness implies global uniqueness. We assume that $u_1(t_1)\neq u_2(t_1)$ for some $0<t_1<\delta$ and put $t_0=\inf\{t_1>0: u_1(t_1)\neq u_2(t_1)\}$. Then $u_1(t_0)=u_2(t_0)$ by means of continuity. However, applying the assumption to $0\leqslant t_0<\delta$ and using the transformation $t=t_0+as$ we obtain from the above considerations also the local unique solvability of the initial value problem $\{u_t=\mathscr{F}(t,u), u(t_0)=\psi\}$ for $t$ near $t_0$ and $\psi=u_1(t_0)=u_2(t_0)$ and thus $u_1=u_2$ near $t_0$ which is a contradiction. The theorem is proved. □

## 5. *Linear parabolic evolution equations*

Let $H^\infty = H^\infty(\mathbb{R}^n)$, $J=[0,1]$. Let $U$ be an open subset of $\{u\in H^\infty : |u|_m^\infty \leqslant C\}$ for some $C>0$ and $W=\{u\in C^1(J,H^\infty): u(t)\in U \text{ for all } t\in J\}$. We consider a family of linear operators $A_u(t): H^\infty \longrightarrow H^\infty$, $t\in J$, $u\in W$ and consider the problem

$$\begin{cases} z_t(t)+aA_u(t)z(t)=f(t), & t\in J \\ z(0)=g \end{cases} \tag{5.1}$$

for $a>0, f\in C(J,H^\infty), g\in H^\infty$. Throughout this section we assume that the operators $A_u(t)$ satisfy the following conditions $(P_0)$, $(P_1)$, $(P_2)$, $(P_3)$ which are generalizations of related assumptions in Pazy [**28**, 5.6.1] or Tanabe [**39**, 5.1, 5.2] to the case of higher derivatives. If the operators $A_u(t)$ depend on additional parameters then we suppose the following properties to hold uniformly with respect to these parameters.

$(P_0)$ $A_u(t)$ satisfy the estimates (3.1), (3.2), (3.4) uniformly with respect to $t\in J$.

$(P_1)$ $A_u(t)$ is a closed operator in $X=L_2(\mathbb{R}^n)$ with domain $\mathscr{D}(A_u(t))=H^m(\mathbb{R}^n)$ for any $u\in W$, $t\in J$, and $\mathscr{D}(A_u(t)^p)=H^{pm}(\mathbb{R}^n)$ for any $p\geqslant 1$ and $u\in W$, $t\in J$.

$(P_2)$ The resolvent $R(\lambda: A_u(t))=(\lambda I - A_u(t))^{-1}$ exists for any complex $\lambda$ with $\operatorname{Re}\lambda\leqslant 0$, and there is a constant $C>0$ such that

$$\|R(\lambda: A_u(t))\|_{L(X)} \leqslant \frac{C}{|\lambda|+1}, \quad \operatorname{Re}\lambda\leqslant 0, \quad t\in J, \quad u\in W.$$

$(P_3)$ For any $k$ there is $C_k>0$ such that for any $s,t,\tau\in J$, $u\in W$, $z\in H^\infty$ we have

$$|(A_u(t)-A_u(s))z|_k \leqslant C_k|t-s|\sum_{i=0}^{k}[u]_{m,k-i}^{\infty}\,|A_u(\tau)z|_i.$$

Remark 5.1. (i) Since $0\in\rho(A_u(\tau))$ condition $(P_3)$ applied to $k=0$ yields

$$(P_4)\quad \|(A_u(t)-A_u(s))A_u(\tau)^{-1}\|_{L(X)}\leqslant C_0|t-s|, \quad s,t,\tau\in J.$$

In [**28**, 5.6.1] (cf. [**39**]) is considered a family of linear operators $A(t)$ on a Banach space $X$ with common dense domain $\mathscr{D}$ and it is proved that $(P_2)$, $(P_4)$ imply that the initial value problem $\{z_t+A(t)z(t)=0\ (0\leqslant s<t\leqslant 1),\ z(s)=x\}$ has for any $x\in X$ a unique solution $z(t)=U(t,s)x$. The operators $U(t,s)\in L(X)$ are called an evolution system which is a two-parameter family of bounded linear operators $U(t,s)$, $0\leqslant s\leqslant t\leqslant 1$, on a Banach space $X$ satisfying $U(s,s)=I$, $U(t,r)U(r,s)=U(t,s)$ for all $0\leqslant s\leqslant r\leqslant t\leqslant 1$ such that $(t,s)\longmapsto U(t,s)$ is strongly continuous for $0\leqslant s\leqslant t\leqslant 1$. For properties of evolution systems see [**28**, Chapter 5; **39**, Chapter 4].

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(ii) By (3.1) we can consider $A_u(t)$ also as an operator in $X = H^{pm} := H^{pm}(\mathbb{R}^n)$ with domain $H^{(p+1)m}$. Then $A_u(t)$ is closed in view of Corollary 3.4, and Lemma 3.2 gives the estimates

$$|z|_{pm} \leqslant c_p(u) \sum_{k=1}^{p} |A_u(t)^k z|_0 \leqslant \frac{c_p'(u)}{|\lambda|+1} \sum_{k=1}^{p} |(\lambda I - A_u(t)) A_u(t)^k z|_0$$

$$= \frac{c_p'(u)}{|\lambda|+1} \sum_{k=1}^{p} |A_u(t)^k (\lambda I - A_u(t)) z|_0 \leqslant \frac{c_p''(u)}{|\lambda|+1} |(\lambda I - A_u(t)) z|_{pm}$$

for any $z \in H^{(p+1)m}$ and $\operatorname{Re}\lambda \leqslant 0$. Hence $\lambda I - A_u(t)$ is injective with closed range as a map $H^{(p+1)m} \longrightarrow H^{pm}$, and it is also surjective since the resolvent $R(\lambda : A_u(t))$ is bijective $H^0 \longrightarrow H^m$ and maps $\mathscr{D}(A_u(t)^p)$ into $\mathscr{D}(A_u(t)^{p+1})$, and hence $H^{pm}$ into $H^{(p+1)m}$. Therefore, $A_u(t)$ satisfies $(P_2)$, $(P_4)$ also as an operator in $X = H^{pm}$, and [**28**, 5.6.1] yields an evolution system $U_u(t,s)$ which is continuous $H^{pm} \longrightarrow H^{pm}$, $p \geqslant 1$.

(iii) We have to treat carefully the number $a > 0$ in problem (5.1) since we cannot expect that $(aA_u(t))$ is a family of elliptic operators with uniform ellipticity constant for all small $a > 0$. For instance, condition $(P_2)$ gives for $A_{u,a}(t) = aA_u(t)$ the estimate

$$\|R(\lambda : A_{u,a}(t))\|_{L(H^0)} \leqslant \frac{C}{|\lambda|+a}, \quad \operatorname{Re}\lambda \leqslant 0, t \in J, u \in W, a > 0.$$

We will apply the following corollaries (Proposition 5.2 and Corollaries 5.3 and 5.4) of known results on $C_0$-semigroups [**28**, Chapter 1; **39**, Chapter 3] and on analytic semigroups [**28**, 2.5; **39**, 3.3].

PROPOSITION 5.2. *We assume* $(P_0)$, $(P_1)$, $(P_2)$, $(P_3)$ *and put* $A_{u,a}(t) = aA_u(t)$ *for* $a > 0$.

(i) *Let* $X = H^0$. *There are fixed constants* $0 < \delta < \pi/2$ *and* $M > 0$ (*only depending on* $C > 0$ *in* $(P_2)$) *such that* $\rho(-A_{u,a}(t)) \supset \Sigma_\delta := \{\lambda : |\arg\lambda| < (\pi/2)+\delta\} \cup \{0\}$ *and*

$$\|R(\lambda : -A_{u,a}(t))\|_{L(X)} \leqslant \frac{M}{|\lambda|}, \quad \lambda \in \Sigma_\delta \backslash \{0\}, t \in J, u \in W, a > 0.$$

(ii) *The operator* $-A_{u,a}(t)$ *is the infinitesimal generator of a* $C_0$*-semigroup* $\{S_t^{u,a}(s)\}_{s \geqslant 0}$ *in* $X$ (*which is in fact an analytic semigroup*) *given by*

$$S_t^{u,a}(s) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda s} R(\lambda : -A_{u,a}(t))\, d\lambda$$

*where* $\Gamma$ *is a smooth curve in* $\Sigma_\delta$ *running from* $\infty e^{-i\theta}$ *to* $\infty e^{i\theta}$ *for* $\pi/2 < \theta < (\pi/2)+\delta$. *The integral converges for* $s > 0$ *in the uniform operator topology of* $L(X)$.

(iii) *The operator* $-A_{u,a}(t)$ *generates for any* $p \geqslant 0$ *an analytic semigroup* $S_t^{u,a}(s)$ *in* $X = H^{pm}$ *defined as in* (i) *and* (ii). *For* $p = 0, 1$ *the constants* $\delta, M$ *can be chosen uniformly for* $t \in J$, $u \in W$, $a > 0$ *while for* $p \geqslant 2$ *these constants depend on* $p, u$. *The restriction of the semigroup from* $H^{pm}$ *to* $H^{qm}$, $q \geqslant p$, *does not depend on* $p$.

(iv) *We have* $S_t^{u,a}(s)(H^0) \subset H^\infty$ *for any* $s > 0$, *and for any* $x \in H^{(p+1)m}$ *and* $s > 0$ *we have* $d/ds\, S_t^{u,a}(s)\, x = -A_{u,a}(t)\, S_t^{u,a}(s)\, x = -S_t^{u,a}(s)\, A_{u,a}(t)\, x$ *in* $H^{pm}$.

*Proof.* (i) follows from [**28**, Theorem 2.5.2, $(b) \Rightarrow (c)$]. Now (ii) follows from [**28**, 1.7.7] or [**39**, 3.3]. We get (iii) from Remark 5.1(ii) and [**28**, 1.7.7], and (iv) is a consequence of [**28**, 1.2.4, 2.4.2]. □

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LEMMA 5.3 (cf. [**28**, 5.6.1; **39**, 3.3]). *There is a constant $C > 0$ such that*
(i) $|S_t^{u,a}(s)\,x|_j \leqslant C|x|_j$;
(ii) $|A_{u,a}(t)\,S_t^{u,a}(s)\,x|_j \leqslant (C/s)|x|_j$;

*for any $j = 0, \dots, m$, $u \in W$, $t \in J$, $s > 0$, $a > 0$, $x \in H^j$.*

*Proof.* It is enough to prove (i) and (ii) for $j = 0$ and $j = m$; the other cases follow by means of interpolation (cf. [**41**, 2.4.1]). From Proposition 5.2(iii) we know that the constants $\delta, M$ in Proposition 5.2(i) can be chosen uniformly for any $t, u, a$ and the cases $X = H^0$ and $X = H^m$. The above integral representation for $S_t^{u,a}(s)$ and the estimates used in [**28**, 1.7.7] thus yield the first estimate (i) for $j = 0$ and $j = m$. The estimates of the resolvents in Proposition 5.2(i) also give (cf. the proof of [**28**, 2.5.2, $(c) \Rightarrow (d)$]) the estimates

$$\|A_{u,a}(t)\,S_t^{u,a}(s)\|_{L(H^j)} = \|(S_t^{u,a})'(s)\|_{L(H^j)} \leqslant \frac{M}{|\cos\theta|}\frac{1}{s}, \quad s > 0$$

for $j = 0$ and for $j = m$ where $\pi/2 < \theta < (\pi/2) + \delta$. This gives (ii) and thus the assertion. □

LEMMA 5.4. *For any $k \geqslant m$ there is a constant $C_k > 0$ such that*
(i) $|S_t^{u,a}(s)\,x|_k \leqslant C_k[u; x]^\infty_{m,k-m}$;
(ii) $|A_{u,a}(t)\,S_t^{u,a}(s)\,x|_k \leqslant (C_k/s)[u; x]^\infty_{m,k-m}$;

*for any $u \in W$, $t \in J$, $s > 0$, $a > 0$, $x \in H^\infty$.*

*Proof.* We observe that $A_u(t)^k\,S_t^{u,a}(s)\,x = S_t^{u,a}(s)\,a^{-k}A_{u,a}(t)^k\,x = S_t^{u,a}(s)\,A_u(t)^k\,x$ for any $x \in H^\infty$. Thus Lemma 3.2 and Lemma 5.3(i) give for $k = pm + j$, $p \geqslant 1$, $0 \leqslant j \leqslant m$ the estimates

$$\begin{aligned}
&|S_t^{u,a}(s)\,x|_{pm+j} \\
&\quad \leqslant c_{p,j}\left\{\sum_{i=0}^{j}[u]^\infty_{m,j-i}|A_u(t)^p\,S_t^{u,a}(s)\,x|_i + \sum_{i=0}^{m-1}\sum_{k=1}^{p-1}[u]^\infty_{m,(p-k)m+j-i}|A_u(t)^k\,S_t^{u,a}(s)\,x|_i\right\} \\
&\quad \leqslant c'_{p,j}\left\{\sum_{i=0}^{j}[u]^\infty_{m,j-i}|A_u(t)^p\,x|_i + \sum_{i=0}^{m-1}\sum_{k=1}^{p-1}[u]^\infty_{m,(p-k)m+j-i}|A_u(t)^k\,x|_i\right\} \\
&\quad \leqslant c''_{p,j}\left\{\sum_{i=0}^{j}[u]^\infty_{m,j-i}[u; x]^\infty_{m,(p-1)m+i} + \sum_{i=0}^{m-1}\sum_{k=1}^{p-1}[u]^\infty_{m,(p-k)m+j-i}[u; x]^\infty_{m,(k-1)m+i}\right\} \\
&\quad \leqslant c'''_{p,j}[u; x]^\infty_{m,k-m}
\end{aligned}$$

which proves part (i). The same proof and Lemma 5.3(ii) give part (ii). □

The following result is based on [**28**, 5.6.1, 5.6.8] (cf. [**39**, 5.2]).

PROPOSITION 5.5. *Let $A_u(t)$ satisfy conditions $(P_0), (P_1), (P_2), (P_3)$.*
(i) *Let $p \geqslant 0, a > 0, u \in W, 0 \leqslant s < 1, g \in H^{pm}$. Then the initial value problem*

$$\begin{cases} z_t(t) + A_{u,a}(t)\,z(t) = 0, & s < t \leqslant 1 \\ z(s) = g \end{cases}$$

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*has a unique solution* $z \in C^1([s,1], H^{pm})$, *and* $U_{u,a}(t,s)g = z(t)$ *defines an evolution system* $U_{u,a}(t,s) \in L(H^{pm})$ *with* $U_{u,a}(t,s)(H^{pm}) \subset H^{(p+1)m}$, $0 \leqslant s \leqslant t \leqslant 1$. *The restriction of* $U_{u,a}(t,s)$ *from* $H^{pm}$ *to* $H^{qm}$, $q \geqslant p$, *does not depend on* $p$.

(ii) *There is a fixed* $C > 0$ *such that for any* $0 \leqslant s \leqslant t \leqslant 1$, $a > 0$, $u \in W$ *we have*

$$|U_{u,a}(t,s)x_0| \leqslant C|x|_0, \quad x \in H^0.$$

*Proof.* (i) follows from [**28**, 5.6.1, 5.6.8] by means of Remark 5.1(ii).

For fixed $u, a$ we get (ii) from [**28**, 5.6.1]. To show that $C$ can be chosen independently of $u, a$ we recall the construction of $U_{u,a}(t,s)$ in [**28**]. For that we define inductively $R_1(t,s) = (A_{u,a}(s) - A_{u,a}(t))\, S_s^{u,a}(t-s)$ and $R_{m+1}(t,s) = \int_s^t R_1(t,\tau) R_m(\tau,s)\, d\tau$. Then $R(t,s) = \sum_{m=1}^{\infty} R_m(t,s)$ and $U_{u,a}(t,s) = S_s^{u,a}(t-s) + \int_s^t S_\tau^{u,a}(t-\tau) R(\tau,s)\, d\tau$ exist in $L(H^0)$ for $0 \leqslant s < t \leqslant 1$. We obtain

$$\|R_1(t,s)\|_{L(H^0)} \leqslant \|(A_{u,a}(t) - A_{u,a}(s))\, A_{u,a}(s)^{-1}\|_{L(H^0)} \|A_{u,a}(s)\, S_s^{u,a}(t-s)\|_{L(H^0)} \leqslant C$$

by means of $(P_3)$ and Lemma 5.3(ii). This implies that $\|R_m(t,s)\|_{L(H^0)} \leqslant C^m (t-s)^{(m-1)}/(m-1)!$ and $\|R(t,s)\| \leqslant C_1$ (cf. [**28**, (6.23), (6.26)]), and the assertion follows from Lemma 5.3(i). □

We suppose in the following the assumptions of Proposition 5.5. The next lemma is the key result of this section based on the generalized Gronwall's lemma (Lemma 1.4).

LEMMA 5.6. *For any* $k \geqslant 0$ *there exists a constant* $C_k > 0$ *such that*

$$|A_u(t)\, U_{u,a}(t,s)\, z|_k \leqslant C_k [u;z]_{m,k}^{\infty}$$

*for any* $0 \leqslant s \leqslant t \leqslant 1$, $a > 0$, $u \in W$, $z \in H^{\infty}$.

*Proof.* We fix $z \in H^{\infty}$ and consider the function $\psi(s) = S_t^{u,a}(t-s)\, U_{u,a}(s,\tau)\, z$ for $0 \leqslant \tau < s < t \leqslant 1$. Then $\psi$ is differentiable as a function $\psi : (\tau, t) \longrightarrow H^{\infty}$ ([**28**, p. 159], cf. Propositions 5.2(iv) and 5.5(i)) with $\psi'(s) = S_t^{u,a}(t-s)\, (A_{u,a}(t) - A_{u,a}(s))\, U_{u,a}(s,\tau)\, z$. This implies that

$$U_{u,a}(t,\tau)\, z = S_t^{u,a}(t-\tau)\, z + \int_\tau^t S_t^{u,a}(t-s)\, (A_{u,a}(t) - A_{u,a}(s))\, U_{u,a}(s,\tau)\, z\, ds.$$

We obtain from Lemmas 5.3 and 5.4 and $(P_3)$ the estimates

$$\begin{aligned}
&|A_u(t)\, U_{u,a}(t,\tau)\, z|_k \\
&\quad \leqslant |S_t^{u,a}(t-\tau)\, A_u(t)\, z|_k + \int_\tau^t |A_{u,a}(t)\, S_t^{u,a}(t-s)\, (A_u(t) - A_u(s))\, U_{u,a}(s,\tau)\, z|_k\, ds \\
&\quad \leqslant c_k \sum_{i=0}^{k} [u]_{m,k-i}^{\infty} |A_u(t)\, z|_i + \frac{c_k}{t-s} \int_\tau^t \sum_{i=0}^{k} [u]_{m,k-i}^{\infty} |(A_u(t) - A_u(s))\, U_{u,a}(s,\tau)\, z|_i\, ds \\
&\quad \leqslant c_k' [u;z]_{m,k}^{\infty} + c_k' \sum_{i=0}^{k} \int_\tau^t [u]_{m,k-i}^{\infty} |A_u(s)\, U_{u,a}(s,\tau)\, z|_i\, ds.
\end{aligned}$$

We apply the generalized Gronwall's lemma (Lemma 1.4) with $A_i = c_i' [u;z]_{m,i}^{\infty}$ and $B_i = [u]_{m,i}^{\infty}$ observing that $B_i B_j \leqslant B_{i+j}$ for all $i, j$ and obtain

$$|A_u(t)\, U_{u,a}(t,\tau)\, z|_k \leqslant C_k \sum_{i=0}^{k} [u]_{m,k-i}^{\infty} [u;z]_{m,i}^{\infty} \leqslant C_k [u;z]_{m,k}^{\infty}.$$ □

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COROLLARY 5.7. *For any $k \geqslant 0$ there is a constant $C_k > 0$ such that*
(i) $|U_{u,a}(t,s)\,z|_k \leqslant C_k |z|_k$, $0 \leqslant k \leqslant m$;
(ii) $|U_{u,a}(t,s)\,z|_k \leqslant C_k [u;z]^{\infty}_{m,k-m}$, $k \geqslant m$;
*for any $0 \leqslant s \leqslant t \leqslant 1$, $a > 0$, $u \in W$, $z \in H^{\infty}$.*

*Proof.* From Corollary 3.4 and Lemma 5.6 we obtain for $k \geqslant m$ the estimates

$$|U_{u,a}(t,s)\,z|_k \leqslant c_k \sum_{i=0}^{k-m} [u]^{\infty}_{m,k-m-i} |A_u(t)\,U_{u,a}(t,s)\,z|_i$$
$$\leqslant c'_k \sum_{i=0}^{k-m} [u]^{\infty}_{m,k-m-i} [u;z]^{\infty}_{m,i} \leqslant c'_k [u;z]^{\infty}_{m,k-m}$$

proving (ii) and (i) for $k = m$. Using interpolation we get (i) from Proposition 5.5(ii). □

We get the main result of this section from the above considerations and applying [**28**, 5.7.1] (cf. [**39**, 5.2]).

THEOREM 5.8. *Let $A_u(t)$ satisfy the conditions* $(P_0)$, $(P_1)$, $(P_2)$, $(P_3)$. *Then for any $u \in W$, $a > 0$ and $f \in C(J, H^{\infty})$, $g \in H^{\infty}$ the initial value problem*

$$\begin{cases} z_t(t) + aA_u(t)\,z(t) = f(t), & t \in J \\ z(0) = g \end{cases} \tag{5.2}$$

*has a unique solution $z \in C^1(J, H^{\infty})$ given by $z(t) = U_{u,a}(t,0)\,g + \int_0^t U_{u,a}(t,s) f(s)\,ds$. Moreover, for any $k \geqslant 0$ there is a constant $C_k > 0$ such that*

$$|z|_k \leqslant C_k [u;(f,g)]^{\infty}_{m,k}$$
$$|z|_k \leqslant C_k [u;(f,g)]_{m+d,k}$$

*for $u \in W$, $a > 0$, $f \in C(J, H^{\infty})$, $g \in H^{\infty}$ (where $|z|_k$ denotes the norms in $C^1(J, H^{\infty})$).*

*Proof.* The uniqueness follows from Proposition 5.5. To prove the existence statement we may assume that $g = 0$ since the case $f = 0$ is contained in Proposition 5.5. It is thus sufficient to show that $v(t) = \int_0^t U_{u,a}(t,s) f(s)\,ds$ belongs to $C^1(J, H^{\infty})$ and satisfies $v_t(t) + aA_u(t)\,v(t) = f(t)$ for $t \in J$. This follows from the proof of [**28**, 5.7.1] inserting the following additional argument. In [**28**, 5.7.1, formula (7.9)] the existence of the limit

$$\lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} A_{u,a}(t)\,S_t^{u,a}(t-\sigma) f(\sigma)\,d\sigma, \quad t > 0$$

in $H^0$ is shown under the assumption that $f: J \longrightarrow H^0$ is Hölder continuous. In our situation we have $f \in C(J, H^{\infty})$ and this limit exists in $H^{pm}$ for any $p$ since

$$|A_{u,a}(t)\,S_t^{u,a}(t-\sigma) f(\sigma)|_k \leqslant c_k(u)\,|f|_{k+m}$$

(allowing a loss of derivatives instead of requiring the Hölder continuity of $f$). Note that $z$ is $C^1$ near 0 in view of (5.2). It remains to show the estimates. From Corollary 5.7 we obtain for $0 \leqslant k \leqslant m$ and $t \in J$ the estimate $|z(t)|_k \leqslant c_k(|g|_k + |f|_k)$, and for $k \geqslant m$ we get

$$|z(t)|_k \leqslant |U_{u,a}(t,0)\,g|_k + \int_0^t |U_{u,a}(t,s) f(s)|_k\,ds$$
$$\leqslant c_k([u;g]^{\infty}_{m,k-m} + [u;f]^{\infty}_{m,k-m}).$$

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Inserting (5.2) we finally obtain

$$\begin{aligned}|z_t(t)|_k &\leqslant |aA_u(t)\,z(t)|_k + |f(t)|_k \\ &\leqslant c_k[u; z(t)]^\infty_{m,k} + |f|_k \\ &\leqslant c_k \sum_{i=0}^{k} [u]^\infty_{m,k-i}\,|z|_{m+i} + |f|_k \\ &\leqslant c'_k \sum_{i=0}^{k} [u]^\infty_{m,k-i}([u; g]^\infty_{m,i} + [u; f]^\infty_{m,i}) + |f|_k \\ &\leqslant c''_k [u; (f, g)]^\infty_{m,k}.\end{aligned}$$

The second estimate follows for $d$ as above from the Sobolev imbedding theorem. □

## 6. *Nonlinear parabolic evolution equations*

Let $H^\infty = H^\infty(\mathbb{R}^n, \mathbb{R})$, $T > 0$. We consider a real nonlinear differential operator $\mathscr{F}: [0, T] \times (U \subset H^\infty) \longrightarrow H^\infty$ of even order $m$ defined as in Section 2 with $M = P = 1$ by

$$\mathscr{F}(t, u)(x) = F(t, x, \{\partial^\alpha u(x)\}_{|\alpha| \leqslant m}).$$

We assume that $F: [0, T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n \times A \longrightarrow \mathbb{R}$ satisfies conditions $(A_1)$, $(A_2)$ of Section 2 and the following condition:

$(A_3)$ For any bounded subset $A_0 \subset A$ and any $|\alpha| \leqslant m$ the limit $\lim_{|x| \to \infty} F_{\partial^\alpha u}(t, x, z)$ exists uniformly for $(t, z) \in [0, T] \times A_0$.

For instance, if $F$ is independent of $x$ then $(A_3)$ is trivially true. We consider for some fixed $\phi \in H^\infty$ the problem

$$\begin{cases} u_t = \mathscr{F}(t, u), \quad t \in [0, a] \\ u(0) = \phi. \end{cases} \tag{6.1}$$

We are looking for a local solution $u \in C^1([0, a], H^\infty)$ for some suitable small $a > 0$. By means of Theorem 4.4 we have to solve the linearized problem

$$\begin{cases} z_t(t) = a \dfrac{\partial \mathscr{F}}{\partial u}(t_0 + bt, u(t))\, z(t) + f(t), \quad t \in J \\ z(0) = g \end{cases} \tag{6.2}$$

for every $u$ in some open neighbourhood $W$ of $\phi$ in $C^1(J, H^\infty)$, $J = [0, 1]$, for every small $a > 0$, $b > 0$, $t_0 \geqslant 0$, and for every $f \in C(J, H^\infty)$, $g \in H^\infty$. Note that

$$\frac{\partial \mathscr{F}}{\partial u}(t, u)\, v = \sum_{|\alpha| \leqslant m} F_{\partial^\alpha u}(t, \cdot, \{\partial^\gamma u(\cdot)\}_{|\gamma| \leqslant m})\, \partial^\alpha v.$$

For $u \in W$, $t \in J$ and $0 \leqslant b \leqslant T/2$, $0 \leqslant t_0 \leqslant T/2$ we shall use the abbreviation

$$B_u(t) = \frac{\partial \mathscr{F}}{\partial u}(t_0 + bt, u(t))$$

omitting $b$, $t_0$ in the notation; all assumptions, estimates and results are understood to hold uniformly with respect to $b, t_0$.

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DEFINITION 6.1. A linear real differential operator $B = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, \partial^\alpha$ of even order $m \geqslant 2$ with coefficients $a_\alpha \in C^\infty_{\mathrm{BU}}(\mathbb{R}^n, \mathbb{R})$ is called elliptic with ellipticity constant $\mu > 0$ if

$$-(-1)^{m/2} \sum_{|\alpha|=m} a_\alpha(x)\, \xi^\alpha \geqslant \mu |\xi|^m, \quad x \in \mathbb{R}^n, \xi \in \mathbb{R}^n.$$

The operator $B$ is called elliptic if there exists such a positive constant $\mu$.

REMARK 6.2. Let $d = [n/2]+1$ as above, and let $\phi \in H^\infty$ be fixed. Assume that there is $\mu > 0$ such that the operators $B_\phi(t)$ are elliptic with the same ellipticity constant $2\mu > 0$ for every $0 \leqslant b,\ t_0 \leqslant T/2,\ t \in J$. Then there exists by means of $(A_1)$ a number $\delta > 0$ such that $B_u(t)$ is elliptic with ellipticity constant $\mu$ for any $u \in W$, $0 \leqslant b,\ t_0 \leqslant T/2,\ t \in J$ where

$$U = \{\psi \in H^\infty : |\phi - \psi|_{m+d+1} < \delta\}$$
$$W = \{u \in C^1(J, H^\infty) : u(t) \in U \text{ for any } t \in J\}.$$

In the remainder of this section we fix $\phi \in H^\infty$ and assume the situation of Remark 6.2. We use the well-known *a priori* estimates for elliptic operators due to Agmon, Douglis and Nirenberg [**3**]; the case $\Omega = \mathbb{R}^n$ is formulated in Amann [**5**, Theorem 12.1]. Since we need the precise dependence of the constants from the given data we include an easy proof which is obtained with some modifications from Schechter [**35**] where the case of bounded domains is considered. The case of complex coefficients is important for Theorem 6.5.

THEOREM 6.3. *Let $B = B(x, \partial) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, \partial^\alpha$ be a differential operator of even order $m$ with complex coefficients $a_\alpha \in C^\infty_{\mathrm{BU}}(\mathbb{R}^n, \mathbb{C})$ such that $\lim_{|x| \to \infty} a_\alpha(x)$ exists for all $|\alpha| \leqslant m$. Assume that there is $\mu > 0$ such that $|\sum_{|\alpha|=m} a_\alpha(x)\, \xi^\alpha| \geqslant \mu |\xi|^m$ for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$. Then*

$$|z|_m \leqslant C_0(|z|_0 + |Bz|_0), \quad z \in H^m$$

*holds for some constant $C_0 > 0$ depending only on the data $1/\mu$, $m$, $n$, $|\alpha_\alpha|_1^\infty$ and on a function $R(\varepsilon)$ such that $|a_\alpha(x) - a_\alpha(y)| \leqslant \varepsilon$ for $|x|, |y| \geqslant R(\varepsilon)$ and $|\alpha| \leqslant m$ where $\varepsilon > 0$.*

*Proof.* The assertion follows for constant coefficient operators $B$ from [**35**, Lemma 2–8] (with $C_0$ depending on $1/\mu$ and $(|a_\alpha|)_{|\alpha| \leqslant m-1}$). Using a partition of unity and a compactness argument it is thus enough to show that any $x^0 \in \mathbb{R}^n \cup \{\infty\}$ has a neighbourhood $V$ such that the estimate holds for any $z \in \mathscr{D}(V) = \{z \in C^\infty(\mathbb{R}^n) : \operatorname{supp} z \subset\subset V\}$ (cf. [**35**, Lemma 9–17]) where a neighbourhood of $\infty$ contains a set $\{x \in \mathbb{R}^n : |x| > R\}$ for some $R > 0$. Considering $B(x, \partial)$ as a perturbation of the constant coefficient operator $B(x^0, \partial)$ (where $B(\infty, \partial) := B(x^1, \partial)$ for some large enough $|x^1|$) it is sufficient to show that for any $\varepsilon > 0$ and $x^0 \in \mathbb{R}^n \cup \{\infty\}$ there is a neighbourhood $V$ of $x^0$ such that $|(B(x, \partial) - B(x^0, \partial))\, z|_0 \leqslant \varepsilon |z|_m$ for any $z \in \mathscr{D}(V)$ (cf. [**35**, Lemma 9–18]). This is clear for $x^0 \in \mathbb{R}^n$ since $a_\alpha$ is continuous, and follows for $x^0 = \infty$ from the existence of the limit in the assumption. □

COROLLARY 6.4. *In the situation of Remark 6.2 there exists a fixed constant $C_0 > 0$ such that*

$$|z|_m \leqslant C_0(|z|_0 + |B_u(t)\, z|_0), \quad z \in H^m$$

*for any $0 \leqslant b \leqslant T/2$, $0 \leqslant t_0 \leqslant T/2$, $t \in J$, $u \in W$.*

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*Proof.* The coefficients $a_\alpha(x)$ of $B_u(t)$ satisfy $|a_\alpha|_1^\infty \leqslant c[u]_{m,1}^\infty \leqslant c'[u]_{m+d,1} \leqslant c''$ uniformly for $t \in J$, $u \in W$ (cf. the proof of Lemma 2.3). Thus the assertion follows from $(A_3)$ observing that for $b_\alpha = \partial^\alpha u$, $u \in U$, $|\alpha| \leqslant m$ we have $\|\hat{b}_\alpha\|_{L_1} \leqslant C_1 |b_\alpha|_d \leqslant C_1 |u|_{m+d} \leqslant C_2$ and hence $\lim_{|x|\to\infty} |\partial^\alpha u(x)| = 0$ uniformly for $|\alpha| \leqslant m$ and $u \in U$ by means of [**26**, 14.5]. □

We need the following result of Agmon [**2**, Theorem 2.1] (cf. [**28**, 7.3.2; **39**, 3.8]).

THEOREM 6.5. *Let $B = B(x, \partial)$ be a real linear elliptic operator as in Definition* 6.1 *considered as an unbounded operator in $L_2$ with domain $\mathscr{D}(B) = H^m$. Assume that $\lim_{|x|\to\infty} a_\alpha(x)$ exists for $|\alpha| \leqslant m$. Then there exist constants $\gamma > 0$ and $C_1 > 0$ which depend on the same data as the constant $C_0$ in Theorem* 6.3 *such that:*

(i) *The resolvent set $\rho(B)$ contains the set $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant \gamma\}$.*

(ii) *For every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant \gamma$ we have $|\lambda| \cdot |z|_0 \leqslant C_1 |\lambda z - Bz|_0$, $z \in H^m$.*

*Proof.* To prove (ii) we consider for $-\pi/2 \leqslant \theta \leqslant \pi/2$ the elliptic operator in $n+1$ variables $B_\theta = B - (-1)^{m/2} e^{i\theta} \partial_t^m$ which satisfies the assumptions of Theorem 6.3. We choose $\zeta \in C^\infty(\mathbb{R})$ such that $\zeta(t) = 1$ for $|t| \leqslant \frac{1}{2}$ and $\zeta(t) = 0$ for $|t| \geqslant 1$. For $z \in H^m(\mathbb{R}^n)$ and $r > 0$ we put $w(x, t) = \zeta(t) e^{irt} z(x)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Then

$$B_\theta w = \zeta(t) e^{irt} (B - r^m e^{i\theta}) z - (-1)^{m/2} e^{i(\theta + rt)} \sum_{j=0}^{m-1} \binom{m}{j} \zeta^{(m-j)}(t) (ir)^j z.$$

Since $\zeta(t) = 1$ for $|t| \leqslant \frac{1}{2}$ we have $|w|_m \geqslant r^m |z|_0$. Hence Theorem 6.3 applied to $w$ gives

$$|r^m e^{i\theta}| \cdot |z|_0 \leqslant |w|_m \leqslant C_0 (|w|_0 + |B_\theta w|_0) \leqslant C_0' \left( |(B - r^m e^{i\theta}) z|_0 + \sum_{j=0}^{m-1} r^j |z|_0 \right).$$

This gives (ii) for $\lambda = r^m e^{i\theta}$ and $\operatorname{Re} \lambda \geqslant \gamma$ if $\gamma$ is chosen large enough.

From Theorem 6.3 we obtain that $B$ is a closed operator, and $\lambda I - B$ is injective with closed range by means of (ii) for any $\operatorname{Re} \lambda \geqslant \gamma$. The same argument, applied to the adjoint operator, shows that $\bar{\lambda} I - B^*$ is injective and thus that $\lambda I - B$ is invertible. Note that the adjoint $B^*$ of $B$ is the formal adjoint (cf. [**21**, Chapter 2]) with the domain $\mathscr{D}(B^*) = H^m$; this follows from local regularity (cf. [**21**, Chapter 2, Theorem 3.2]) by means of the *a priori* estimate Theorem 6.3. Thus (i) is proved. □

COROLLARY 6.6. *There exist fixed constants $\gamma > 0$ and $C_1 > 0$ such that* (i) *and* (ii) *hold.*

(i) *$\rho(B_u(t)) \supset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geqslant \gamma\}$ for all $0 \leqslant b$, $t_0 \leqslant T/2$, $t \in J$, $u \in W$.*

(ii) *For every $0 \leqslant b$, $t_0 \leqslant T/2$, $t \in J$, $u \in W$ and all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant \gamma$ we have*

$$|\lambda| \cdot |z|_0 \leqslant C_1 |\lambda z - B_u(t) z|_0, \quad z \in H^m.$$

We now choose $C_0$, $C_1$ and $\gamma$ as in Corollaries 6.4 and 6.6, fix some number $\omega > \gamma$ and put

$$A_u(t) = -B_u(t) + \omega I = -\frac{\partial \mathscr{F}}{\partial u}(t_0 + bt, u(t)) + \omega I$$

for $0 \leqslant b$, $t_0 \leqslant T/2$, $t \in J$, $u \in W$ where $I$ denotes the identity. We can show that the family of operators $A_u(t)$ satisfies the assumptions of Section 5 (cf. [**28**, 7.6.1]).

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THEOREM 6.7. *The operators $A_u(t)$ satisfy conditions* $(P_0), (P_1), (P_2), (P_3)$.

*Proof.* The estimates (3.1) and (3.2) follow from Lemmas 2.3 and 2.7, and Corollary 6.6(ii) applied to $\lambda = \omega$ gives (3.4); this proves $(P_0)$. The operator $A_u(t)$ is closed in $L_2$ with the domain $\mathscr{D}(A_u(t)) = H^m$ since $B_u(t)$ is closed; thus Corollaries 3.4 and 3.6 imply $(P_1)$. Further, Corollary 6.6(i) shows that the resolvent set $\rho(A_u(t))$ contains the set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant 0\}$, and Corollary 6.6(ii) implies the estimates

$$|z|_0 \leqslant \frac{C_1}{|\omega-\lambda|}|(\omega-\lambda)z - B_u(t)z|_0 \leqslant \frac{C_1'}{|\lambda|+1}|(\lambda I - A_u(t))z|_0, \quad z \in H^m,$$

for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \leqslant 0$. This proves $(P_2)$. Finally, $(P_3)$ follows from Corollary 3.3 applied to the operators $(A_u(t) - A_u(s))/(t-s)$ which satisfy (3.1) uniformly for all $s \neq t$; this is true since

$$\frac{A_u(t)-A_u(s)}{t-s}z = -\sum_{|\alpha|\leqslant m}\frac{a_\alpha(t)-a_\alpha(s)}{t-s}\partial^\alpha z = -\sum_{|\alpha|\leqslant m}\left\{\int_0^1 a_\alpha'(s+\tau(t-s))\,d\tau\right\}\partial^\alpha z$$

where $a_\alpha(t) = F_{\partial^\alpha u}(t_0+bt, \cdot, \{\partial_x^\alpha u(t)(\cdot)\})$. For $a_\alpha \in C^1(J, C^\infty_{\mathrm{BU}}(\mathbb{R}^n))$ we get the inequalities $|a_\alpha'(t)|_k^\infty \leqslant c_k[u]_{m,k}^\infty$ (cf. the proof of Lemma 2.3), and the above representation yields in view of Lemma 2.3 the estimates $|(A_u(t)-A_u(s)/(t-s)z|_k \leqslant C_k[u;z]_{m,k}^\infty$ where the expressions $[u;z]_{m,k}^\infty$ are defined by means of the norms $|u|_j^\infty$ in $C^1(J, H^\infty)$ (it should be emphasized that derivatives $u_t$ appear). The theorem is proved. □

We may apply Theorem 5.8 and obtain the following.

COROLLARY 6.8. *For any $u \in W$, $0 \leqslant a$, $t_0 \leqslant T/2$, $f \in C(J, H^\infty)$, $g \in H^\infty$, the problem*

$$\begin{cases} v_t(t) = a\dfrac{\partial \mathscr{F}}{\partial u}(t_0+at, u(t))\,v(t) - a\omega v(t) + e^{-a\omega t}f(t), \quad t \in J \\ v(0) = g \end{cases}$$

*has a unique solution $v \in C^1(J, H^\infty)$ which satisfies for suitable $C_k > 0$ the estimates*

$$|v|_k \leqslant C_k[u;(f,g)]_{m+d,k}.$$

*Proof.* This follows from Theorems 5.8 and 6.7. □

COROLLARY 6.9. *For any $u \in W$, $0 \leqslant a$, $t_0 \leqslant T/2$, $f \in C(J, H^\infty)$, $g \in H^\infty$, the problem*

$$\begin{cases} z_t(t) = a\dfrac{\partial \mathscr{F}}{\partial u}(t_0+at, u(t))\,z(t) + f(t), \quad t \in J \\ z(0) = g \end{cases} \tag{6.3}$$

*has a unique solution $z \in C^1(J, H^\infty)$ which satisfies for suitable $C_k > 0$ the estimates*

$$|z|_k \leqslant C_k[u;(f,g)]_{m+d,k}.$$

*Proof.* This follows from Corollary 6.8 when we put $z(t) = e^{a\omega t}v(t)$.

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THEOREM 6.10. *Let $\mathscr{F}$ be a nonlinear partial differential operator*

$$\mathscr{F}(t,u)(x) = F(t,x,\{\partial^\alpha u(x)\}_{|\alpha|\leqslant m})$$

*of even order $m$ defined as in Section 2 where $F$ satisfies $(A_1)$, $(A_2)$ and $(A_3)$. Let $\phi \in H^\infty$, and assume that there is $\delta > 0$ such that $\partial\mathscr{F}/\partial u(t,\phi)$ is elliptic for $0 \leqslant t \leqslant \delta$. Then there exist an open neighbourhood $U$ of $\phi$ in $H^\infty$ and a number $a > 0$ such that for any $\psi \in U$ the initial value problem*

$$\begin{cases} u_t = \mathscr{F}(t,u), & t \in [0,a] \\ u(0) = \psi \end{cases} \tag{6.4}$$

*has a unique solution $u \in C^1([0,a], H^\infty)$. The solution map $\psi \longmapsto u$ is $C^2$ considered as a map $\Psi : (U \subset H^\infty) \longrightarrow C^1([0,a], H^\infty)$; if $\mathscr{F}$ is $C^n$ then $\Psi$ is $C^n$ as well, $2 \leqslant n \leqslant \infty$.*

*Proof.* This is obtained from Theorem 4.4 and Corollary 6.9. □

It is easy to verify the assumptions of Theorem 6.10 in concrete situations. For instance, Theorem 0.1 follows immediately from Theorem 6.10.

The estimates obtained in Corollary 6.9 also imply results on almost global solvability.

COROLLARY 6.11. *We assume in the situation of Theorem 0.1 that $F(0) = 0$ and that $\mathscr{F}'(0)$ is elliptic. Then for any $T > 0$ there exist $s \geqslant 0$ and $\varepsilon > 0$ such that for any $\phi \in H^\infty(\mathbb{R}^n)$ with $|\phi|_s \leqslant \varepsilon$ there is a unique solution $u \in C^1([0,T], H^\infty)$ of problem (0.1).*

*Proof.* We consider the map $\Phi : C^1([0,T], H^\infty) \longrightarrow C([0,T], H^\infty) \times H^\infty$ defined by $\Phi(u) = (u_t - \mathscr{F}(u), u(0))$ with the derivative $\Phi'(u)z = (z_t - \mathscr{F}'(u(t))z(t), z(0))$. Since Corollary 6.9 holds with $J = [0,T]$ and $a = 1$ we may apply the Nash–Moser inverse function theorem [**30**, 4.1]. Hence the map $\Phi$ is a $C^\infty$-diffeomorphism near 0, and there exist $s, \varepsilon$ such that the equation $\Phi(u) = (0,\phi)$ is solvable if $|\phi|_s \leqslant \varepsilon$. □

REMARK 6.12. (i) It is almost evident that the techniques of this paper may be applied to other classes of nonlinear evolution equations. Since the *a priori* estimates (Theorem 6.3) and the resolvent estimates (Theorem 6.5) are well known for elliptic boundary value problems, a generalization to parabolic boundary value problems on bounded domains is possible and carried out in [**32**]. In addition, parabolic systems or degenerate parabolic problems or free boundary value problems may be considered as well.

(ii) Theorem 4.4 is applied in [**18**] to prove the local well-posedness in $H^\infty(\mathbb{R})$ of the Cauchy problem for the nonlinear Schrödinger type equation ($\kappa$ is a real constant)

$$iu_t = -\Delta u + \kappa(\Delta(|u|^2)) \cdot u$$

where differently from previously known results the nonlinearity appears in the highest order space derivatives. This equation has been studied in plasma physics in the theory of superfluids (cf. the references in [**18**]). Obviously, also more general nonlinear Schrödinger type equations may be considered.

(iii) In general, one might say that the methods of this paper are applicable to some nonlinear evolution equation provided that a suitable semigroup theory is available for the corresponding class of linearized equations. For instance, this is the case for linear equations in the hyperbolic case as considered in [**28**, 5.3] (cf. [**39**, 4]).

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(iv) In future applications of the method proposed in this paper it may be helpful that the required estimates in Theorem 4.4 permit a fixed loss of derivatives.

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