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Singular measures for class P-circle homeomorphisms with several break points

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Abstract. Let f be a class P-homeomorphism of the circle with break point singularities, that is, differentiable except at some singular points where the derivative has a jump. Let f have irrational rotation number and Df be absolutely continuous on every continuity interval of Df. We prove that if the product of the f-jumps along any subset of break points is distinct from 1 then the invariant measure μ_f is singular with respect to the Haar measure. This result generalizes previous results obtained by Dzhalilov and Khanin, Dzhalilov, Akhadkulov, Dzhalilov–Liousse and Mayer. Moreover, we prove that if the rotation number $\rho(f)$ is irrational of bounded type then (a) if the product of the f-jumps on some orbit is distinct from 1 then the invariant measure μ_f is singular with respect to the Haar measure m, and (b) if the product of the f-jumps on each orbit is equal to 1 and $D^2 f \in L^p(S^1)$ for some $p > 1$ then μ_f is equivalent to the Haar measure.

1. *Introduction*

Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and $p : \mathbb{R} \longrightarrow S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1. The homeomorphism f admits a lift $\widehat{f}$: $\mathbb{R} \longrightarrow \mathbb{R}$ that is an increasing homeomorphism of $\mathbb{R}$ such that $p \circ \widehat{f} = f \circ p$. Conversely, the projection of such a homeomorphism of $\mathbb{R}$ is an orientation preserving homeomorphism of S^1. Let $x \in S^1$. We call the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ the *orbit* of x by f.

Historically, the study of the dynamics of circle homeomorphisms was initiated by Poincaré [**18**], who introduced the rotation number of a homeomorphism f of S^1 as

$$\rho(f) = \lim_{n \to +\infty} \frac{\widehat{f}^n(\widehat{x}) - \widehat{x}}{n} \pmod 1$$

where $\widehat{x} \in \mathbb{R}$ is such that $p(\widehat{x}) = x$. Poincaré shows that this limit exists and does not depend on x or on the lift $\widehat{f}$ of f.

We say that f is semi-conjugate to the rotation $R_{\rho(f)}$ if there exists an orientation preserving surjective continuous map $h : S^1 \longrightarrow S^1$ of degree 1 such that $h \circ f = R_{\rho(f)} \circ h$.

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Poincaré's theorem. Let f be a homeomorphism of S^1 with irrational rotation number $\rho(f)$. Then f is semi-conjugate to the rotation $R_{\rho(f)}$.

A natural question is whether the semi-conjugation h could be improved to be a conjugation, that is, to be a homeomorphism. In this case, we say that f is topologically conjugate to the rotation $R_{\rho(f)}$. In this direction, Denjoy [**5**] proves the following.

Denjoy's theorem. [**5**] Every C^2-diffeomorphism f of S^1 with irrational rotation number $\rho(f)$ is topologically conjugate to the rotation $R_{\rho(f)}$.

Denjoy asked whether or not C^2-diffeomorphisms f of S^1 are ergodic with respect to the Haar measure m (f is said to be ergodic with respect to m if any f-invariant measurable set A has measure $m(A)$ equal to 0 or 1). Simultaneously, Herman and Katok gave a positive answer to this question.

Herman–Katok's theorem. [**10, 13**] Every C^2-diffeomorphism f with irrational rotation number is ergodic with respect to the Haar measure m.

It is well-known that a homeomorphism f of S^1 with irrational rotation number preserves a unique normalized measure on S^1, denoted by μ_f. If $\widehat{h}$ is the lift of h, by taking $\widehat{h}(0) = 0$, the conjugating homeomorphism h is unique and related to μ_f by $\widehat{h}(\widehat{x}) = \mu_f([0, \widehat{x}])$, $\widehat{x} \in [0, 1[$. Uniqueness of μ_f implies that μ_f is either singular, or absolutely continuous with respect to m; in this second case, h is an absolutely continuous function. Recall that μ_f is said to be *singular* with respect to the Haar measure m on S^1 if there exists a measurable subset E of S^1 such that $\mu_f(E) = 1$ and $m(E) = 0$. In fact, if μ_f is absolutely continuous with respect to the Haar measure m and f is a C^2-diffeomorphism, μ_f is necessarily equivalent to m as a consequence of Herman–Katok's theorem above.

Definition. A real number $\alpha \in]0, 1[$ is called *Diophantine with exponent* $\delta \geq 0$ if there is a constant $c(\alpha)$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^{2+\delta}} \quad \text{for any } \frac{p}{q} \in \mathbb{Q}. \tag{1}$$

A number that is not Diophantine is called a *Liouville number*.

Every real number $\alpha \in]0, 1[$ has a continued fraction expansion represented by

$$\alpha = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cdots}} := [a_0, a_1, \ldots, a_n, \ldots],$$

where the positive integers a_m, $m \in \mathbb{N}$, are called *partial quotients* of α. When $(a_m)_{m\in\mathbb{N}}$ is bounded, α is said to be of *bounded type*. This is equivalent to the fact that (1) holds with $\delta = 0$.

Katznelson and Ornstein (see also [**15, 20**]) proved the following.

Katznelson–Ornstein's theorem. [**13**] Let f be a C^1-diffeomorphism of S^1 with irrational rotation number $\alpha = \rho(f)$. If Df is absolutely continuous, $D\mathrm{Log}Df \in L^p(S^1)$ for some $p > 1$ and α is of bounded type then μ_f is equivalent to the Haar measure m.

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The condition of boundedness of $\rho(f)$ is essential (see [**14**]); there exist C^∞-diffeomorphisms with Liouville rotation numbers and admitting invariant measures singular with respect to the Haar measure m.

We refer the reader to the books [**4, 17**] for a thorough account on circle homeomorphisms.

Other classes of circle homeomorphisms commonly referred to as *the class P-homeomorphisms* are known to satisfy the conclusion of Denjoy's theorem (Corollary 2.6) (see also [**12**, Ch. VI], [**13**]) and, with additional regularity, of Herman–Katok's theorem (see [**3**]). Notice that Katznelson–Ornstein's theorem [**13**] cannot be extended in general to class P. The study of the regularity of the invariant measures of class P-homeomorphisms then arises naturally.

Class P-homeomorphisms

Definition 1.1. An orientation preserving homeomorphism f of S^1 is called a *class P-homeomorphism* if it is differentiable except at finitely many points, the so called *break points* of f, at which left and right derivatives (denoted, respectively, by Df_- and Df_+) exist and such that the derivative $Df : S^1 \longrightarrow \mathbb{R}_+^*$ has the following properties:

- there exist two constants $0 < a < b < +\infty$ such that $a < Df(x) < b$, for every x where Df exists;
- $a < Df_+(c) < b$ and $a < Df_-(c) < b$ at the break points c;
- $\log Df$ has bounded variation on S^1.

We point out that the third condition implies the other two.

The ratio

$$\sigma_f(c) := \frac{Df_-(c)}{Df_+(c)}$$

is called the *f-jump* in c.

Denote by:

- $C(f) = \{c_0, c_1, c_2, \ldots, c_p\}$ the set of break points of f in S^1;
- $c_{p+1} := c_0$;
- $\pi_{s,O_f(c)}(f) := \prod_{x \in O_f(c)} \sigma_f(x) = \prod_{x \in C(f) \cap O_f(c)} \sigma_f(x)$ for every $c \in S^1$;
- $\pi_s(f)$ the product of f-jumps at the break points of f

$$\pi_s(f) = \prod_{c \in C(f)} \sigma_f(c).$$

The total variation of $\log Df$ we denote by $V = \mathrm{Var}(\log Df)$. We have

$$V := \sum_{j=0}^{p} \mathrm{Var}_{[c_j, c_{j+1}]} \log Df + |\log(\sigma_f(c_j))| < +\infty.$$

In this case, V is the total variation of $\log Df$, $\log Df_-$, $\log Df_+$.

We notice the following properties.

- If f is a class P-homeomorphism of S^1 which is C^1 on S^1 then f is a C^1-diffeomorphism.

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- If f is a class P-homeomorphism of S^1 then $D^2 f \in L^1(S^1)$: indeed, this follows from [**11**, Theorem 18.14, p. 284] since Df has bounded variation on S^1.
- If f is a class P-homeomorphism of S^1 and Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$ then $f \in C^1(S^1 \backslash C(f))$.

Among the simplest examples of class P-homeomorphisms, there are:

- C^2-diffeomorphisms;
- piecewise linear PL-homeomorphisms.

An orientation preserving homeomorphism f of S^1 is called a PL-*homeomorphism* if f is differentiable except at finitely many break points $(c_i)_{1 \le i \le p}$ of S^1 such that the derivative Df is constant on each $]c_i, c_{i+1}[$.

Denote by:

- $\mathcal{P}(S^1)$ the set of class P-homeomorphisms of S^1;
- $\mathrm{PL}_+(S^1)$ the set of piecewise linear PL-homeomorphisms of S^1.

We notice that $\mathcal{P}(S^1)$ is a group which contains $\mathrm{PL}_+(S^1)$ (cf. [**1**]).

Definition 1.2. [**2**] Let $f \in \mathcal{P}(S^1)$. We say that f has the *(D)-property* if the product of f-jumps at the break points of each orbit is trivial, that is $\pi_{s, O_f(c)}(f) = 1$, for every $c \in C(f)$.

In particular, if f has the (D)-property, then $\pi_s(f) = 1$. Conversely, if all break points belong to the same orbit and $\pi_s(f) = 1$ then f has the (D)-property.

The invariant measures of PL-homeomorphisms with two break points were first studied by Herman in [**12**] and by Liousse [**16**] for generic PL-homeomorphisms with *several break points* and with irrational rotation numbers of bounded type.

Herman's theorem. [**12**] Let $f \in \mathrm{PL}_+(S^1)$ with irrational rotation number. If f has *two break points* then f preserves a measure equivalent to the Haar measure m if and only if the two break points belong to the same orbit.

Liousse's theorem. [**16**, Theorems 1 and 2] Let $f \in \mathrm{PL}_+(S^1)$ with irrational rotation number α. Then:

(i) if f has the (D)-property then the measure μ_f is equivalent to the Haar measure m;
(ii) if f does not satisfy the (D)-property, α is of *bounded type* and the logarithms of slopes of f are *rationally independent* then the measure μ_f is singular with respect to the Haar measure m.

Invariant measures of general (non-PL) class P-homeomorphisms with *one break point* have been studied by Dzhalilov and Khanin in [**6**]. The situation of their properties is the opposite of the $C^{2+\varepsilon}$ diffeomorphism case, $\varepsilon > 0$.

In the next step, invariant measures of class P-homeomorphisms with *two break points* have been recently studied by Dzhalilov *et al* in [**8**].

Invariant measures of general class P-homeomorphisms with *several break points* have been studied by Akhadkulov [**3**], Dzhalilov [**7**] in the case where the break points are *lying on the same orbit* and by the authors of [**1**] for class P-homeomorphisms *having the (D)-property*.

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Akhadkulov's theorem. [**3**, Theorems 2 and 3] Let $f \in \mathcal{P}(S^1)$ with irrational rotation number α. Suppose that Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$ and $D^2 f \in L^p(S^1)$ for some $p > 1$.

(i) If the break points of f belong to the same orbit and $\pi_s(f) \neq 1$, then the measure μ_f is singular with respect to the Haar measure m.

(ii) If α is of bounded type, the break points of f belong to the same orbit and $\pi_s(f) = 1$ then the measure μ_f is equivalent to the Haar measure m.

Assertion (ii) of Akhadkulov's theorem is a simple generalization of a result of Dzhalilov in [**7**] obtained when $f \in C^{2+\varepsilon}(S^1 \backslash C(f))$, for some $\varepsilon > 0$.

In this paper, we answer completely the question of the nature of the invariant measures μ_f in the general case. We show that the nature of μ_f depends not on whether the break points of f belong to the same orbit or not, but whether products of f-jumps are trivial or not. The principal results of our paper are the following.

THEOREM A. *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number. Suppose that Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$.*

If the product of the f-jumps along any subset of break points is distinct from 1 *then the invariant measure μ_f is singular with respect to the Haar measure m.*

THEOREM B. *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number α. Suppose that Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$. Assume that α is of bounded type.*

(a) *If f does not have the (D)-property then the invariant measure μ_f is singular with respect to the Haar measure m.*

(b) *If f has the (D)-property and $D^2 f \in L^p(S^1)$ for some $p > 1$ then the measure μ_f is equivalent to the Haar measure m.*

COROLLARY 1.3. *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number α. Suppose that f has two break points x_1 and x_2 and Df is absolutely continuous on every connected interval of $S^1 \backslash \{x_1, x_2\}$. Then the measure μ_f is singular with respect to the Haar measure m if one of the following conditions holds:*

(i) $\sigma_f(x_1)\sigma_f(x_2) \neq 1$;

(ii) *α is of bounded type, x_1 and x_2 are on distinct orbits and $\sigma_f(x_1)\sigma_f(x_2) = 1$.*

COROLLARY 1.4. *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number α. Suppose that f has one break point c and that Df is absolutely continuous on every connected interval of $S^1 \backslash \{c\}$. Then the measure μ_f is singular with respect to the Haar measure m.*

Remark 1.

(1) Corollaries 1.3 and 1.4 generalize Dzhalilov–Khanin's theorem [**6**] and Dzhalilov–Liousse–Mayer's theorem [**8**, Theorems 1.4 and 1.5] for all class P-homeomorphisms f with Df absolutely continuous on every connected interval of $S^1 \backslash C(f)$.

(2) Theorem B implies in particular assertion (ii) of Akhadkulov's theorem.

(3) Assertion (b) of Theorem B extends Katznelson–Ornstein's theorem to the case of class P homeomorphisms.

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Remark 2. Note that if α is not of bounded type, Theorem B is not true in general; there exists a piecewise C^∞ class P-homeomorphism f of S^1 with Liouville rotation number and with the following properties:

- f has exactly two break points x_1 and x_2;
- x_1 and x_2 belong to the same f-orbit and the product of f-jumps is $\sigma_f(x_1)\sigma_f(x_2) = 1$;
- f admits an invariant measure *singular* with respect to the Haar measure m.

Indeed, by [**14**], there exists a C^∞-diffeomorphism g with Liouville rotation number such that μ_g is a singular measure with respect to the Haar measure m. It suffices to take the homeomorphism $f := h \circ g \circ h^{-1}$ with h a piecewise C^∞ class P-homeomorphism of S^1 with one break point.

COROLLARY 1.5. *Let $f \in \mathrm{PL}_+(S^1)$ have irrational rotation number α of bounded type. Then the following are equivalent:*

(i) *f has the (D)-property;*

(ii) *the measure μ_f is equivalent to the Haar measure m.*

Corollary 1.5 completes assertion (ii) of Liousse's theorem and that of Herman's theorem for the PL case.

The main analytic tools for proving the main theorem are cross-ratios. This technique has been introduced by Yoccoz in [**23**] for discussing real analytic circle homeomorphisms with critical points; it has been intensively used by Swiatek in [**21**], recently by Teplinskii in [**22**] for the study of circle homeomorphisms and by Navas in the context of group actions on the circle [**17**].

2. *Notation and preliminary results*

2.1. *Some properties.*

LEMMA 1. [**2**, Proposition 2.1] *Let $f,\ g \in \mathcal{P}(S^1)$. If f and g are conjugated by $h \in \mathcal{P}(S^1)$ then $\pi_{s,O_f(c)}(f) = \pi_{s,O_g(h(c))}(g)$, for every $c \in S^1$.*

We have also shown the following proposition (stated slightly differently).

PROPOSITION 2.1. [**1**, Proposition 3.1] *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number. Suppose that Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$. Then, there exists a piecewise polynomial class P-homeomorphism K of S^1 such that: $F = K \circ f \circ K^{-1} \in \mathcal{P}(S^1)$ with $C(F) \subset \{K(x_0), \ldots, K(x_p)\}$ where $x_0, \ldots, x_p \in C(f)$ are on pairwise distinct orbits.*

In particular, we have the following corollary.

COROLLARY 2.2. *Let f, K and F be as in Proposition 2.1.*

(i) *If f does not satisfy the (D)-property, then there exists $0 \le i \le p$ such that $K(x_i)$ is the unique break point of F in its orbit.*

(ii) *If f satisfies the (D)-property then F is a C^1-diffeomorphism with DF absolutely continuous on S^1.*

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Proof. From the hypothesis, $f \in \mathcal{P}(S^1)$ with $f \in C^1(S^1 \backslash C(f))$.

(i) If f does not satisfy the (D)-property, there exists a break point $c \in C(f)$ such that $\pi_{s,O_f(c)}(f) \neq 1$. Since the product of jumps of break points on the same orbit is invariant by a conjugation in $\mathcal{P}(S^1)$ (Lemma 1), we see that

$$\pi_{s,O_f(c)}(f) = \sigma_F(K(c)).$$

This implies that $\sigma_F(K(c)) \neq 1$ and therefore $K(c)$ is the unique break point of F in its orbit.

(ii) If f satisfies the (D)-property then

$$\pi_{s,O_f(c)}(f) = 1 = \sigma_F(K(c)) \quad \text{for every } c \in C(f).$$

Thus, F is a C^1-diffeomorphism. As $F = K \circ f \circ K^{-1}$ and K is polynomial then DF is absolutely continuous on S^1. □

2.2. *Dynamical partitions.* Let f be a homeomorphism of S^1 with irrational rotation number $\alpha = \rho(f)$. We identify α to its lift $\widehat{\alpha}$ in $]0, 1[$. Let $(a_n)_{n\in\mathbb{N}}$ be the partial quotients of α in the continued fractions expansion

$$\alpha = [a_0, a_1, \ldots, a_n, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cdots}}.$$

For $n \in \mathbb{N}$, the fractions $[a_0, a_1, \ldots, a_n]$ are written in the form of irreducible fractions p_n/q_n. The sequence p_n/q_n converges to α and we say that p_n/q_n are *rational approximations* of α. Their denominators q_n satisfy the following recursion relation:

$$q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 2, \quad q_0 = 1, \quad q_1 = a_1.$$

Let $x_0 \in S^1$ be fixed. Denote by:

$$\Delta_0^{(n)}(x_0) = \begin{cases} [x_0,\ f^{q_n}x_0] & \text{if } n \text{ is even,} \\ [f^{q_n}(x_0),\ x_0] & \text{if } n \text{ is odd,} \end{cases}$$

$$\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0)), \quad i \in \mathbb{Z}.$$

In all the following, we deal with the case n odd (the case n even is obtained by reversing the orientation of S^1).

We have then the following lemma.

LEMMA 2.3. (See [**19**]) *The segments*

$$\Delta_i^{(n-1)}(x_0) = f^i(\Delta_0^{(n-1)}(x_0)), \quad 0 \leq i < q_n$$

and

$$\Delta_j^{(n)}(x_0) = f^j(\Delta_0^{(n)}(x_0)), \quad 0 \leq j < q_{n-1}$$

cover S^1 and their interiors are mutually disjoint.

The partition denoted by

$$\xi_n(x_0) := \{\Delta_i^{(n-1)}(x_0),\ \Delta_j^{(n)}(x_0) : 0 \leq i < q_n,\ 0 \leq j < q_{n-1}\}$$

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is called the nth *dynamical partition* of the point x_0. It is defined by the order of points $x_0, \ldots, f^{q_{n-1}+q_n-1}(x_0)$. The process of passing from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$ is described by keeping intact the elements $\Delta_j^{(n)}(x_0)$, while each element $\Delta_i^{(n-1)}(x_0)$, $0 \le i < q_n$, is partitioned into $a_{n+1}+1$ sub-segments; since $q_{n+1} = a_{n+1}q_n + q_{n-1}$,

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{a_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

For every $n \ge 1$, $\xi_{n+1}(x_0)$ is finer than $\xi_n(x_0)$ in the following sense: every element $\xi_n(x_0)$ is a union of elements $\xi_{n+1}(x_0)$.

LEMMA 2.4. *Let $n \ge 1$ be an integer and $y_1 \in S^1$. Set*

$$y_2 = f^{q_{n-1}}(y_1), \quad y_3 = f^{q_{n-1}}(y_2), \quad \Delta_i^{(n-1)}(y_1) = f^i([y_1, y_2])$$

and

$$\Delta_j^{(n-1)}(y_2) = f^j([y_2, y_3]), \quad 0 \le i, j < q_n.$$

Then $\Delta_i^{(n-1)}(y_1) \cap \Delta_j^{(n-1)}(y_2)$ has non-empty interior if and only if one of the following conditions holds:

(1) *$j = i - q_{n-1}$ and $q_{n-1} \le i < q_n$; in this case we have*

$$\Delta_i^{(n-1)}(y_1) = \Delta_j^{(n-1)}(y_2);$$

(2) *$j = q_n - q_{n-1} + i$ and $0 \le i < q_{n-1}$; in this case we have*

$$\Delta_j^{(n-1)}(y_2) = f^{q_n}(\Delta_i^{(n-1)}(y_1)).$$

Proof. Let $0 \le i, j < q_n$ and set

$$X := \Delta_i^{(n-1)}(y_1) \cap \Delta_j^{(n-1)}(y_2).$$

We have

$$\Delta_j^{(n-1)}(y_2) = f^j([y_2, f^{q_{n-1}}(y_2)]) = f^{j+q_{n-1}}([y_1, y_2]).$$

In the case (1), we have $\Delta_i^{(n-1)}(y_1) = \Delta_j^{(n-1)}(y_2)$. In the case (2), write $y = f^i(y_1)$. Then we have

$$\Delta_i^{(n-1)}(y_1) = \Delta_0^{(n-1)}(y)$$

and

$$\Delta_j^{(n-1)}(y_2) = f^{q_n}(f^i([y_1, y_2])) = f^{q_n}(\Delta_0^{(n-1)}(y)).$$

By the dynamical partition $\xi_n(y)$, we get

$$X = \Delta_0^{(n-1)}(y) \cap f^{q_n}(\Delta_0^{(n-1)}(y)) = [y, f^{q_{n-1}+q_n}(y)].$$

In either case, X has non-empty interior.

Conversely, suppose that X has non-empty interior. Since

$$\Delta_j^{(n-1)}(y_2) = f^{j+q_{n-1}}([y_1, y_2]),$$

we see that

$$X = f^i([y_1, y_2] \cap f^{j+q_{n-1}-i}([y_1, y_2])).$$

As the segments $(f^k[y_1, y_2])_{0 \le k < q_n}$ are disjoint (except at the endpoints) by Lemma 2.3, we obtain $j + q_{n-1} - i = 0$ or $|j + q_{n-1} - i| \ge q_n$.

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If $q_{n-1} \le i < q_n$ then $-q_n < -i < j + q_{n-1} - i \le j < q_n$ and hence $|j + q_{n-1} - i| < q_n$, which is possible only for $j + q_{n-1} - i = 0$, that is the condition (1). If $0 \le i < q_{n-1}$ then $j + q_{n-1} - i = |j + q_{n-1} - i| \ne 0$ and so $|j + q_{n-1} - i| \ge q_n$. Write $j + q_{n-1} = q_n + i + s$ where $0 \le s < q_{n-1}$ and $y = f^i(y_1)$. We have $\Delta_i^{(n-1)}(y_1) = \Delta_0^{(n-1)}(y)$ and $\Delta_j^{(n-1)}(y_2) = f^{q_n}(\Delta_s^{(n-1)}(y))$. By the dynamical partition $\xi_n(y)$, we get

$$f^{q_n}(\Delta_s^{(n-1)}(y)) \subset \Delta_s^{(n)}(y) \cup \Delta_s^{(n-1)}(y).$$

Since $\Delta_0^{(n-1)}(y) \cap [\Delta_s^{(n)}(y) \cup \Delta_s^{(n-1)}(y)]$ has non-empty interior, so by Lemma 2.3, we get necessarily $s = 0$, that is the condition (2). □

LEMMA 2.5. *Let $n \ge 1$ be an integer and $y_1 \in S^1$. Set $y_2 = f^{q_{n-1}}(y_1)$, $y_3 = f^{q_{n-1}}(y_2)$. Then for every $x \in S^1 \setminus O_f(y_1)$, there exists a unique $0 \le i = i_n(x) < q_n$ such that $x \in f^i(]y_1, y_2[) \cup f^{\varphi_n(i)}(]y_2, y_3[)$, where $\varphi_n : [0, q_n[\longrightarrow [0, q_n[$ is a bijective map given by:*

$$\varphi_n(i) = \begin{cases} q_n - q_{n-1} + i & \text{if } 0 \le i < q_{n-1}, \\ i - q_{n-1} & \text{if } q_{n-1} \le i < q_n. \end{cases}$$

Proof. Existence: let us consider the dynamical partition $\xi_n(y_1)$. For every $x \in S^1$, either there exists $0 \le i = i_n(x) < q_n$ such that $x \in \Delta_i^{(n-1)}(y_1) = f^i([y_1, y_2])$ or there exists $0 \le i = i_n(x) < q_{n-1}$ such that $x \in \Delta_i^{(n)}(y_1)$. In the first case, we have $x \in f^i([y_1, y_2])$. In the second case, we have $x \in f^i([f^{q_n}(y_1), y_1])$. Since

$$f^i([f^{q_n}(y_1), y_1]) = f^{\varphi_n(i)}([f^{q_{n-1}}(y_1), f^{q_{n-1}-q_n}(y_1)]) = f^{\varphi_n(i)}([y_2, f^{-q_n}(y_2)])$$

and as $f^{-q_n}(y_2) \in]y_2, y_3[$, we see that $f^i([f^{q_n}(y_1), y_1]) \subset f^{\varphi_n(i)}([y_2, y_3])$ and therefore, $x \in f^{\varphi_n(i)}([y_2, y_3])$. In fact, $x \in f^{\varphi_n(i)}(]y_2, y_3[)$ since $x \notin O_f(y_1)$.

Uniqueness: if there are $0 \le i, j < q_n$ such that

$$x \in (f^i(]y_1, y_2[) \cup f^{\varphi_n(i)}(]y_2, y_3[)) \cap (f^j(]y_1, y_2[) \cup f^{\varphi_n(j)}(]y_2, y_3[)),$$

we distinguish three cases:

- $x \in f^i(]y_1, y_2[)$ and $x \in f^j(]y_1, y_2[)$: by taking $x_0 = y_1$, Lemma 2.3 implies that $i = j$;
- $x \in f^{\varphi_n(i)}(]y_2, y_3[)$ and $x \in f^{\varphi_n(j)}(]y_2, y_3[)$: by taking $x_0 = y_2$, Lemma 2.3 implies that $\varphi_n(i) = \varphi_n(j)$ and then $i = j$ since φ_n is bijective;
- $x \in f^i(]y_1, y_2[)$ and $x \in f^{\varphi_n(j)}(]y_2, y_3[)$: Lemma 2.4 implies that $\varphi_n(j) = \varphi_n(i)$ and so $i = j$. □

2.3. *Useful inequalities.* The proofs of the following results are based on the dynamical partition of S^1.

Finzi's distortion lemma. [**9**] Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. For all $z \in S^1$, $x, y \in [z, f^{q_{n-1}}(z)]$ and for all integers $0 \le k < q_n$, we have:

$$e^{-V} \le \frac{Df^k(x)}{Df^k(y)} \le e^V.$$

Notice that Finzi's distortion lemma is used for the proof of Herman–Katok's theorem.

The following lemma plays a key role for studying metrical properties of the homeomorphism f.

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Denjoy's inequality. Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. For all $x \in S^1$ and all $n \in \mathbb{N}$, we have $e^{-V} \leq Df^{q_n}(x) \leq e^V$ where $V = \mathrm{Var}(\log Df)$.

The proof of this inequality results from Finzi's distortion lemma. It can be also obtained by applying Denjoy–Koksma's inequality to the function $\log Df$ (cf. Herman [**12**]).

Recall that the length of an interval $[a, b] \subset S^1$ is defined by

$$l([a, b]) := \begin{cases} \widehat{b} - \widehat{a} & \text{if } \widehat{a} < \widehat{b} < 1, \\ 1 + \widehat{b} - \widehat{a} & \text{if } 0 \leq \widehat{b} < \widehat{a} < 1. \end{cases}$$

Using Denjoy's inequality, one has the following corollary.

COROLLARY 2.6. (Geometric inequality) [**12**] *There exists a constant $C > 0$ such that for any $x_0 \in S^1$, $n \geq 1$ and any element $\Delta^{(n)}$ of the dynamical partition $\xi_n(x_0)$, we have $l(\Delta^{(n)}) \leq C\lambda^n$ where $\lambda = (1 + e^{-V})^{-\frac{1}{2}} < 1$.*

From Corollary 2.6 it follows that every orbit of every $x \in S^1$ is dense in S^1 and this implies the following generalization of the classical Denjoy theorem.

COROLLARY 2.7. (Denjoy's theorem: the class P) [**12**] *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number $\alpha = \rho(f)$. Then f is topologically conjugate to the irrational rotation R_α.*

In the following lemma we have to compare different iterates of intervals.

LEMMA 2.8. *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number $\alpha = \rho(f)$. Let $n \in \mathbb{N}^*$ and $z_1 \in S^1$. Set $z_2 = f^{q_{n-1}}(z_1)$, $z_3 = f^{q_{n-1}}(z_2)$. Then for any segments K_1, $K_2 \subset [z_1, z_3]$, one has*

$$e^{-2V} \frac{l(K_1)}{l(K_2)} \leq \frac{l(f^j(K_1))}{l(f^j(K_2))} \leq e^{2V} \frac{l(K_1)}{l(K_2)}$$

for all $j = 0, \ldots, q_n$.

Proof. If $j = q_n$, Lemma 2.8 is a consequence of Denjoy's inequality. We suppose that $0 \leq j < q_n$. Let $\mathcal{L}$ be the set of segments in $[z_1, z_2]$ or in $[z_2, z_3]$. If $K \in \mathcal{L}$ then $K \subset [z, f^{q_{n-1}}(z)]$ with $z = z_1$ or $z = z_2$.

Since

$$\frac{l(f^j(K))}{Df^j(z_2)} = \int_K \frac{Df^j(t)}{Df^j(z_2)}\, dm(t),$$

we get by Finzi's distortion lemma applied to f on $[z, f^{q_{n-1}}(z)]$,

$$e^{-V} \leq \frac{Df^j(t)}{Df^j(z_2)} \leq e^V \quad \text{for every } t \in K.$$

By integrating this last inequality, we obtain

$$e^{-V} l(K) \leq \frac{l(f^j(K))}{Df^j(z_2)} \leq e^V l(K). \tag{2.1}$$

Let K_1, $K_2 \subset [z_1, z_3]$ be two segments. We distinguish three cases.

(a) We have $K_1 \in \mathcal{L}$ and $K_2 \in \mathcal{L}$. In this case, the inequality (2.1) implies

$$e^{-V} l(K_1) \leq \frac{l(f^j(K_1))}{Df^j(z_2)} \leq e^V l(K_1)$$

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and

$$e^{-V} l(K_2) \le \frac{l(f^j(K_2))}{Df^j(z_2)} \le e^V l(K_2),$$

and therefore

$$e^{-2V} \frac{l(K_1)}{l(K_2)} \le \frac{l(f^j(K_1))}{l(f^j(K_2))} \le e^{2V} \frac{l(K_1)}{l(K_2)}.$$

(b) We have $K_1 \notin \mathcal{L}$ and $K_2 \in \mathcal{L}$. In this case, take $K_1 = K_1' \cup K_1''$ where $K_1' \subset [z_1, z_2]$ and $K_1'' \subset [z_2, z_4]$ are two segments, then $K_1' \in \mathcal{L}$ and $K_1'' \in \mathcal{L}$. By (a), we see that

$$e^{-2V} \frac{l(K_1')}{l(K_2)} \le \frac{l(f^j(K_1'))}{l(f^j(K_2))} \le e^{2V} \frac{l(K_1')}{l(K_2)}$$

and

$$e^{-2V} \frac{l(K_1'')}{l(K_2)} \le \frac{l(f^j(K_1''))}{l(f^j(K_2))} \le e^{2V} \frac{l(K_1'')}{l(K_2)}.$$

Adding these inequalities, we get

$$e^{-2V} \frac{l(K_1)}{l(K_2)} \le \frac{l(f^j(K_1))}{l(f^j(K_2))} \le e^{2V} \frac{l(K_1)}{l(K_2)}.$$

(c) We have $K_1 \notin \mathcal{L}$ and $K_2 \notin \mathcal{L}$ In this case, take $K_2 = K_2' \cup K_2''$ where $K_2' \subset [z_1, z_2]$ and $K_2'' \subset [z_2, z_3]$ are two segments. By applying (b) to K_1 and K_2' (respectively K_2''), we obtain the inequality of the lemma. □

2.4. *Some cross-ratio tools.* Let us introduce the notion of cross-ratio distortion with respect to a homeomorphism.

Definition 2.9. Let x_1, x_2, x_3 be real numbers such that $x_1 < x_2 < x_3$.

(i) The *cross-ratio* (Cr) of the triple (x_1, x_2, x_3) is the real number defined as

$$\mathrm{Cr}(x_1, x_2, x_3) = \frac{x_2 - x_1}{x_3 - x_2}.$$

(ii) The *cross-ratio distortion* (Dcr) of the triple (x_1, x_2, x_3) with respect to a strictly increasing function $F : \mathbb{R} \longrightarrow \mathbb{R}$ is the real number defined as

$$\mathrm{Dcr}_F(x_1, x_2, x_3) = \frac{\mathrm{Cr}(F(x_1), F(x_2), F(x_3))}{\mathrm{Cr}(x_1, x_2, x_3)} = \frac{F(x_2) - F(x_1)}{x_2 - x_1} \frac{x_3 - x_2}{F(x_3) - F(x_2)}.$$

Notice that:

- the cross-ratio is invariant by translations:

$$\mathrm{Cr}(x_1 + a, x_2 + a, x_3 + a) = \mathrm{Cr}(x_1, x_2, x_3), \quad a \in \mathbb{R};$$

- the cross-ratio distortion is multiplicative with respect to composition: for two functions F and G on $\mathbb{R}$, we have

$$\mathrm{Dcr}_{G \circ F}(x_1, x_2, x_3) = \mathrm{Dcr}_G(F(x_1), F(x_2), F(x_3)) \mathrm{Dcr}_F(x_1, x_2, x_3).$$

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Let $z_1, z_2, z_3 \in S^1$ have lifts $\widehat{z_1}, \widehat{z_2}, \widehat{z_3}$ and $z_1 \prec z_2 \prec z_3 \prec z_1$ in the anti-clockwise order on S^1. Define

$$\widetilde{z_i} := \begin{cases} \widehat{z_i} & \text{if } \widehat{z_1} < \widehat{z_i} < 1, \\ 1 + \widehat{z_i} & \text{if } 0 \le \widehat{z_i} < \widehat{z_1}, \end{cases}$$

where $i = 2, 3$. Obviously we have $\widetilde{z_1} < \widetilde{z_2} < \widetilde{z_3}$. The vector $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3}) \in \mathbb{R}^3$ is called the *lifted vector* of $(z_1, z_2, z_3) \in (S^1)^3$.

Consider now a homeomorphism f of S^1 with lift $\widehat{f}$. We define the *cross-ratio distortion* of the triple (z_1, z_2, z_3), $z_1, z_2, z_3 \in S^1$ with respect to f by

$$\mathrm{Dcr}_f(z_1, z_2, z_3) := \mathrm{Dcr}_{\widehat{f}}(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3}),$$

where $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3})$ is the lifted vector of (z_1, z_2, z_3). This definition is independent of the lift of f since the cross-ratio is invariant by translation.

Definition 2.10. Let $R > 1$ be a real number and $x_0 \in S^1$. We say that a triple (z_1, z_2, z_3) of S^1 $(z_1 \prec z_2 \prec z_3)$ with lifted vector $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3})$ satisfies the conditions (a) and (b) for the point x_0 and the constant R if:

(a) $R^{-1} \le \dfrac{\widetilde{z_3} - \widetilde{z_2}}{\widetilde{z_2} - \widetilde{z_1}} \le R$;

(b) $\max\limits_{1 \le i \le 3} l([x_0, z_i]) \le R l([z_1, z_2])$.

3. *Cells*

Let $x_0 \in S^1$, $\delta > 0$ and $U_\delta(x_0)$ a δ-neighborhood of x_0. Let $f \in \mathcal{P}(S^1)$ with irrational rotation number $\alpha = \rho(f)$. Let $c \in C(f)$ and n an integer. By Lemma 2.3, either $c \in \Delta^{(n-1)}_{i_n(c)}(x_0)$ for some $0 \le i_n(c) < q_n$ or $c \in \Delta^{(n)}_{i_n(c)}(x_0)$ for some $0 \le i_n(c) < q_{n-1}$. Set

$$y_2 = f^{-i_n(c)}(c), \quad y_1 = f^{-q_{n-1}}(y_2), \quad y_3 = f^{q_{n-1}}(y_2).$$

PROPOSITION 3.1. *Under the notation above, there exists $N = N(x_0, \delta) \in \mathbb{N}$ such that for all $n \ge N$, there is a triple $(y_1, y_2, y_3) = (y_1(n), y_2(n), y_3(n))_{n \ge N}$ with the following properties:*

(c-0) (y_1, y_2, y_3) (respectively $f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))_{n \ge N} \subset U_\delta(x_0)$;

(c-1) $y_2 \in \Delta^{(n-1)}_0(x_0)$ *or* $y_2 \in \Delta^{(n)}_0(x_0)$;

(c-2) $l(f^j([y_1, y_3])) \le K\lambda^n$, *for every* $0 \le j < q_n$, *where* $\lambda = (1 + e^{-V})^{-\frac{1}{2}}$ *and* K *a constant independent of* n;

(c-3) $\sum_{j=0}^{q_n - 1} l(f^j([y_1, y_3])) \le 2$;

(c-4) *the triples* (y_1, y_2, y_3) *and* $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$ *satisfy conditions* (a) *and* (b) *of Definition 2.10 for* x_0 *and the constant* $R = e^{3V} + e^V + 1$;

(c-5)
- $f^i([y_1, y_3])$ *does not contain* c *for every* $0 \le i \ne i_n(c) < q_n$;
- $f^{i_n(c)}([y_1, y_3])$ *does not contain any break point of* f *other than* c.

Proof. The property (c-1) is obvious by construction. The property (c-2) holds by Corollary 2.6. Since $[y_1, y_2] = \Delta^{(n-1)}_0(y_1)$ and $[y_2, y_3] = \Delta^{(n-1)}_0(y_2)$, one has that $f^i([y_1, y_2])$ and $f^i([y_2, y_3])$ are elements of the partition $\xi_n(y_1)$ and $\xi_n(y_2)$ respectively,

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hence the first part of (c-5) is satisfied. For the second part of (c-5), choose an integer

$$n_0 \geq \frac{\log(K^{-1}L_0)}{\log \lambda} \quad \text{where } L_0 = \min_{x,y\in C(f),x\neq y} l([x, y]).$$

Then for every $n \in \mathbb{N}_0$, $n > n_0$, if there is $d \in C(f)\backslash\{c\}$ such that $d \in f^{i_n(c)}([y_1, y_3])$, then by (c-2) one has $l(f^{i_n(c)}([y_1, y_3])) \leq K\lambda^n < K\lambda^{n_0} \leq L_0 \leq l([c, d])$, which is a contradiction. The property (c-3) is satisfied since

$$\sum_{j=0}^{q_n-1} l(f^j([y_1, y_3])) = \sum_{j=0}^{q_n-1} l(f^j([y_1, y_2])) + \sum_{j=0}^{q_n-1} l(f^j([y_2, y_3])) \leq 2.$$

Let us show the property (c-4). By Denjoy's inequality, one has

$$e^{-V} \leq l([y_2, y_3])/l([y_1, y_2]) \leq e^V.$$

On the other hand,

$$\max_{1\leq i\leq 3} l([x_0, y_i]) \leq l([y_1, y_3]) \leq (1 + e^V)l([y_1, y_2]).$$

Hence (y_1, y_2, y_3) satisfies conditions (a) and (b) for $R_1 = e^V + 1$. For the triple $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$, Lemma 2.8 yields

$$e^{-2V}\frac{l([y_2, y_3])}{l([y_1, y_2])} \leq \frac{l(f^{q_n}([y_2, y_3]))}{l(f^{q_n}([y_1, y_2]))} \leq e^{2V}\frac{l([y_2, y_3])}{l([y_1, y_2])}.$$

Thus

$$e^{-3V} \leq \frac{l(f^{q_n}([y_2, y_3]))}{l(f^{q_n}([y_1, y_2]))} \leq e^{3V}.$$

Hence $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$ satisfies condition (a) for $R_2 = e^{3V}$. We will check now condition (b). We consider two cases. If $y_2 \in \Delta_0^{(n-1)}(x_0)$ then $(f^{q_n}([y_1, y_3]))$ contains x_0 since $f^{q_n}(y_3)$ lies to the right of y_2. Therefore,

$$\max_{1\leq i\leq 3} l([x_0, f^{q_n}y_i]) \leq l(f^{q_n}([y_1, y_3])) \leq (1 + e^V)l(f^{q_n}([y_1, y_2])).$$

If $y_2 \in \Delta_0^{(n)}(x_0)$ then either x_0 belongs to $f^{q_n}([y_1, y_3])$ or not. In the former case, one has as above

$$\max_{1\leq i\leq 3} l([x_0, f^{q_n}(y_i)]) \leq (1 + e^V)l(f^{q_n}([y_1, y_2])).$$

In the latter case, $f^{q_n}(y_3)$ lies on the left of x_0. Thus,

$$\max_{1\leq i\leq 3} l([x_0, f^{q_n}(y_i)]) \leq l(f^{q_n}([y_1, y_3])) + l([y_2, y_3]).$$

By Denjoy's inequality, one has

$$e^{-V} \leq \frac{l(f^{q_n}([y_1, y_2]))}{l([y_1, y_2])} \leq e^V,$$

so

$$\begin{aligned}\max_{1\leq i\leq 3} l([x_0, f^{q_n}(y_i)]) &\leq (1 + e^V)l(f^{q_n}([y_1, y_2])) + e^V l([y_1, y_2]) \\ &\leq (1 + e^V + e^{2V})l(f^{q_n}([y_1, y_2])).\end{aligned}$$

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Therefore, condition (b) with the constant $R = 1 + e^V + e^{2V}$ holds for $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$. In conclusion, conditions (a) and (b) with the constant $R = 1 + e^V + e^{3V}$ hold for (y_1, y_2, y_3) and $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$. It remains to show the property (c-0): since $x_0 \in [y_1, y_3]$, it follows by (c-2) that there exists $N = N(x_0, \delta) \in \mathbb{N}$, $N > n_0$, such that

$$(y_1, y_2, y_3) = (y_1(n), y_2(n), y_3(n))_{n \geq N} \subset U_\delta(x_0)$$

and

$$(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))_{n \geq N} \subset U_\delta(x_0),$$

since by (c-4), we have

$$l([x_0, f^{q_n}(y_i)]) \leq Rl(f^{q_n}([y_1, y_2])) \leq Re^V l([y_1, y_2]).$$

The proof is complete. □

Definition (cell). The triple (y_1, y_2, y_3) given in Proposition 3.1 is called a *standard cell associated to* (f, x_0, c) for the constant $R = e^{3V} + e^V + 1$.

In the following, set $\mathbb{N}_0 = \{n \in \mathbb{N} : n \geq N\}$. Let $c \in C(f)$, $d \notin O_f(c)$ and $(y_1, y_2, y_3) := (y_1(n), y_2(n), y_3(n))_{n \in \mathbb{N}_0}$ be a standard cell associated to (f, x_0, c). Set for every $n \in \mathbb{N}_0$,

$$a_n(d) = \frac{l([f^{i_n(d)}(y_2), d])}{l([f^{i_n(d)}(y_2), f^{j_n(d)}(y_2)])}, \quad b_n(d) = \frac{l([f^{j_n(d)}(y_2), d])}{l([f^{i_n(d)}(y_2), f^{j_n(d)}(y_2)])},$$

where $j_n(d) = \varphi_n(i_n(d))$. Obviously, $b_n(d) = 1 - a_n(d)$.

Denote by $A(d)$ the set of adherence values of $(a_n(d))_{n \in \mathbb{N}_0}$. We have $A(d) \subset [0, 1]$.

Set

$$\begin{aligned} Q_1 &= \{d \in C(f)\backslash\{c\} : A(d) \cap]0, 1[\neq \emptyset\}, \\ Q_2 &= \{d \in C(f)\backslash\{c\} : A(d) = \{0\}\}, \\ Q_3 &= \{d \in C(f)\backslash\{c\} : A(d) = \{1\}\}, \\ Q_4 &= \{d \in C(f)\backslash\{c\} : A(d) = \{0, 1\}\}. \end{aligned}$$

Notice that Q_1, Q_2, Q_3 and Q_4 are defined *with respect to* f, x_0, c *and* $\mathbb{N}_0$. It is clear that Q_1, Q_2, Q_3, Q_4 form a partition of $C(f)\backslash\{c\}$.

LEMMA 3.2. *Let* $c \in C(f)$ *and* $(y_1, y_2, y_3) := (y_1(n), y_2(n), y_3(n))_{n \in \mathbb{N}_0}$ *be a standard cell associated to* (f, x_0, c). *Let* $d \in C(f)\backslash\{c\}$. *Then:*

(1) *if* $d \in Q_1$ *then there exist an infinite subset* $\mathbb{N}_{d,1} \subset \mathbb{N}_0$, *and a constant* $\gamma_1(d) > 0$ *such that* $a_n(d), b_n(d) \geq \gamma_1(d)$, *for all* $n \in \mathbb{N}_{d,1}$;

(2) *if* $d \in Q_2 \cup Q_4$ *then there exists an infinite subset* $\mathbb{N}_{d,2} \subset \mathbb{N}_0$ *such that* $(a_n(d))_{n \in \mathbb{N}_{d,2}}$ *is decreasing to* 0 *and* $b_n(d) \geq \frac{1}{2}$, *for all* $n \in \mathbb{N}_{d,2}$;

(3) *if* $d \in Q_3$ *then* $(b_n(d))_{n \in \mathbb{N}_0}$ *is decreasing to* 0 *and there exists* $N_3 \in \mathbb{N}$ *such that* $a_n(d) \geq \frac{1}{2}$, *for all* $n \in \mathbb{N}_0$, $n \geq N_3$.

Proof. (1) If $d \in Q_1$ then there exists an infinite subset $\mathbb{N}^0_{d,1} \subset \mathbb{N}_0$ such that $(a_n(d))_{n \in \mathbb{N}^0_{d,1}}$ converges to $\alpha \in]0, 1[$. Hence there exists $N_1 \in \mathbb{N}$ such that $a_n(d), b_n(d) \geq$

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$\gamma_1(d)$, for all $n \in \mathbb{N}^0_{d,1}$, $n \geq N_1$, where

$$\gamma_1(d) = \min\left(\frac{\alpha}{2}, \frac{1-\alpha}{2}\right) > 0.$$

Here

$$\mathbb{N}_{d,1} = \{n \in \mathbb{N}^0_{d,1} : n \geq N_1\}.$$

(2) If $d \in Q_2$ then $(a_n(d))_{n\in\mathbb{N}_0}$ is decreasing to 0 and hence $(b_n(d))_{n\in\mathbb{N}_0}$ converges to 1. So there exists $N_2 \in \mathbb{N}$ such that $b_n(d) \geq \frac{1}{2}$, for all $n \in \mathbb{N}_0$, $n \geq N_2$. Here $\mathbb{N}_{d,2} = \{n \in \mathbb{N}_0 : n \geq N_2\}$.

If $d \in Q_4$, then there exists an infinite subset $\mathbb{N}^0_{d,2}$ of $\mathbb{N}_0$ such that $(a_n(d))_{n\in\mathbb{N}^0_{d,2}}$ is decreasing to 0 and $(a_n(d))_{n\in\mathbb{N}_0\setminus\mathbb{N}^0_{d,2}}$ converges to 1. So there exists $N_4 \in \mathbb{N}$ such that $b_n(d) \geq \frac{1}{2}$, for all $n \in \mathbb{N}^0_{d,2}$, $n \geq N_4$. Here $\mathbb{N}_{d,2} = \{n \in \mathbb{N}^0_{d,2} : n \geq N_4\}$.

(3) If $d \in Q_3$ then $(a_n(d))_{n\in\mathbb{N}_0}$ converges to 1 or equivalently $(b_n(d))_{n\in\mathbb{N}_0}$ is decreasing to 0. So there exists $N_3 \in \mathbb{N}$ such that $a_n(d) \geq \frac{1}{2}$, for all $n \in \mathbb{N}_0$, $n \geq N_3$. □

PROPOSITION 3.3. *Let $c \in C(f)$ and $(y_1, y_2, y_3) := (y_1(n), y_2(n), y_3(n))_{n\in\mathbb{N}_0}$ be a standard cell associated to (f, x_0, c). Then there exist an infinite subset $\mathbb{N}_c \subset \mathbb{N}_0$ and a constant $\gamma > 0$ such that for any $d \in C(f)\setminus\{c\}$, we have one of the following properties.*

$S_1(d)$: $a_n(d), b_n(d) \geq \gamma$, for all $n \in \mathbb{N}_c$.

$S_2(d)$: $(a_n(d))_{n\in\mathbb{N}_c}$ is decreasing to 0 and $b_n(d) \geq \gamma$, for all $n \in \mathbb{N}_c$.

$S_3(d)$: $(b_n(d))_{n\in\mathbb{N}_c}$ is decreasing to 0 and $a_n(d) \geq \gamma$, for all $n \in \mathbb{N}_c$.

In Proposition 3.3, we say that $S_i(d)$ $(i = 1, 2, 3)$ is satisfied on $\mathbb{N}_c$ for the constant γ.

Proof. Recall that $c_0, c_1, \ldots, c_p$ are the break points of f and write $c_0 = c$. By Lemma 3.2, we get the following property.

S-property: for any infinite subset $\mathbb{M} \subset \mathbb{N}_0$ and any break point $c_i \in C(f)\setminus\{c_0\}$, there exist an infinite subset $\mathbb{N}_i \subset \mathbb{M}$, a positive constant $\gamma_i > 0$ and $j \in \{1, 2, 3\}$ such that $S_j(c_i)$ is satisfied on $\mathbb{N}_i$ for the constant γ_i.

Apply the S-property to $\mathbb{M} = \mathbb{N}_0$ and c_1: there exist an infinite subset $\mathbb{N}_1 \subset \mathbb{N}_0$, a positive constant $\gamma_1 > 0$ and $j_1 \in \{1, 2, 3\}$ such that $S_{j_1}(c_1)$ is satisfied on $\mathbb{N}_1$ for the constant γ_1.

Apply the S-property to $M = \mathbb{N}_1$ and c_2: there exist an infinite subset $\mathbb{N}_2 \subset \mathbb{N}_1$, a positive constant $0 < \gamma_2 < \gamma_1$ and $j_2 \in \{1, 2, 3\}$ such that $S_{j_2}(c_2)$ is satisfied on $\mathbb{N}_2$ for the constant γ_2.

We continue in this fashion obtaining a decreasing sequence $\mathbb{N}_p \subset \mathbb{N}_{p-1} \subset \cdots \subset \mathbb{N}_2 \subset \mathbb{N}_1 \subset \mathbb{N}_0$ of infinite subsets of $\mathbb{N}_0$, a decreasing sequence $0 < \gamma_p < \cdots \gamma_2 < \gamma_1$ of positive constants and a sequence $(j_1, j_2, \ldots, j_p) \in \{1, 2, 3\}^p$ such that for every $i \in \{1, 2, \ldots, p\}$ the property $S_{j_i}(c_i)$ is satisfied on $\mathbb{N}_i$ for the constant γ_i. Therefore, for every $i \in \{1, 2, \ldots, p\}$ the property $S_{j_i}(c_i)$ is satisfied on $\mathbb{N}_c = \mathbb{N}_p$ for the constant $\gamma = \gamma_p$. □

PROPOSITION 3.4. *Let $(y_1, y_2, y_3) := (y_1(n), y_2(n), y_3(n))_{n\in\mathbb{N}_0}$ be a standard cell associated to (f, x_0, c). Let h be the conjugating homeomorphism from f to R_α: $h \circ f = R_\alpha \circ h$. Assume that h admits at x_0 a positive derivative $Dh(x_0) > 0$ and that $\rho(f) = \alpha$ is of bounded type. Let $d \in C(f)\setminus O_f(c)$. Then:*

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(i) *$d \in Q_1$ and hence there exist an infinite subset $\mathbb{N}_{c,d} \subset \mathbb{N}_0$ and a constant $\gamma_{c,d} > 0$ such that $a_n(d), b_n(d) \geq \gamma_{c,d}$, for all $n \in \mathbb{N}_{c,d}$;*

(ii) *there exist an infinite subset $\mathbb{N}^0_{c,d} \subset \mathbb{N}_{c,d}$ and a constant $\gamma^0_{c,d} > 0$ such that for any $b \in C(f)\backslash\{c, d\}$, one of the properties $S_1(b)$, $S_2(b)$ or $S_3(b)$ is satisfied on $\mathbb{N}^0_{c,d}$ for the constant $\gamma^0_{c,d}$.*

Proof. (i) Suppose that $d \notin Q_1$, that is, $A(d) \subset \{0, 1\}$. We distinguish three cases.

Case 1: $A(d) = \{0\}$, i.e. $d \in Q_2$. In this case, $(a_n(d))_{n\in\mathbb{N}_0}$ is decreasing to 0. Write

$$t_n(d) := \frac{l([f^{-i_n(d)}(d), y_2])}{l([y_1, y_2])}, \quad n \in \mathbb{N}_0.$$

By Lemma 2.8, one has

$$e^{-2V} t_n(d) \leq \frac{l([f^{i_n(d)}(y_2), d])}{l(f^{i_n(d)}([y_1, y_2]))} \leq e^{2V} t_n(d). \tag{1}$$

We also have

$$1 \leq \frac{l([f^{i_n(d)}(y_2), f^{j_n(d)}(y_2)])}{l(f^{i_n(d)}([y_1, y_2]))} \leq 1 + e^V. \tag{2}$$

Indeed:

- if $q_{n-1} \leq i_n(d) < q_n$, then $j_n(d) = i_n(d) - q_{n-1}$ (Lemma 2.5), hence $f^{i_n(d)}([y_1, y_2]) = [f^{i_n(d)}(y_2), f^{j_n(d)}(y_2)]$ and so
$$\frac{l([f^{i_n(d)}(y_2), f^{j_n(d)}(y_2)])}{l(f^{i_n(d)}([y_1, y_2]))} = 1;$$
- if $0 \leq i_n(d) < q_{n-1}$, then $j_n(d) = i_n(d) + q_n - q_{n-1}$ (Lemma 2.5), hence $f^{j_n(d)}(y_2) = f^{q_n}(f^{i_n(d)}(y_1))$. As $f^{i_n(d)}(y_1) \in]f^{j_n(d)}(y_2), f^{j_n(d)}(y_3)[$, $f^{j_n(d)}(y_3) \in]f^{j_n(d)}(y_2), f^{i_n(d)}(y_2)[$, then by Denjoy's inequality, we get
$$\begin{aligned} 1 \leq \frac{l([f^{j_n(d)}(y_2), f^{i_n(d)}(y_2)])}{l(f^{i_n(d)}([y_1, y_2]))} &= 1 + \frac{l([f^{i_n(d)}(y_1), f^{j_n(d)}(y_2)])}{l(f^{i_n(d)}([y_1, y_2]))} \\ &\leq 1 + \frac{l([f^{j_n(d)}(y_2), f^{j_n(d)}(y_3)])}{l(f^{i_n(d)}([y_1, y_2]))} \\ &= 1 + \frac{l([f^{q_n}(f^{i_n(d)}([y_1, y_2])))}{l(f^{i_n(d)}([y_1, y_2]))} \\ &\leq 1 + e^V. \end{aligned}$$

Combining (1) and (2), we get

$$\frac{e^{-2V}}{1 + e^V} t_n(d) \leq a_n(d) \leq e^{2V} t_n(d) \quad \text{for every } n \in \mathbb{N}_0,$$

and therefore $\lim_{n\to+\infty} t_n(d) = 0$. Since $y_2 = f^{-i_n(c)}(c)$, we conclude that

$$l([f^{-i_n(d)}(d), f^{-i_n(c)}(c)]) = o(l([y_1, y_2])), \quad n \to +\infty. \tag{3}$$

Let us prove that

$$l([R_{-i_n(d)\alpha}(h(d)), R_{-i_n(c)\alpha}(h(c))]) = o(\|q_{n-1}\alpha\|), \quad n \to +\infty, \tag{4}$$

where $\|q_{n-1}\alpha\| = \inf_{k\in\mathbb{Z}} |q_{n-1}\alpha - k|$.

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First, for every $n \in \mathbb{N}_0$, one has

$$\frac{l([R_{-i_n(d)\alpha}(h(d)), h(x_0)])}{l([f^{-i_n(d)}(d), x_0])} = \frac{l([h(f^{-i_n(d)}(d)), h(x_0)])}{l([f^{-i_n(d)}(d), x_0])} \longrightarrow Dh(x_0), \quad n \to +\infty,$$

thus

$$l([R_{-i_n(d)\alpha}(h(d)), h(x_0)]) = Dh(x_0) l([f^{-i_n(d)}(d), x_0]) + o(l([f^{-i_n(d)}(d), x_0])). \tag{4-1}$$

Similarly, we also have

$$l([R_{-i_n(c)\alpha}(h(c)), h(x_0)]) = Dh(x_0) l([f^{-i_n(c)}(c), x_0]) + o(l([f^{-i_n(c)}(c), x_0])), \tag{4-2}$$

and

$$\begin{aligned} l([R_{-q_{n-1}\alpha}(h(x_0)), h(x_0)]) &= \|q_{n-1}\alpha\| \\ &= Dh(x_0) l([f^{-q_{n-1}}(x_0), x_0]) + o(l([f^{-q_{n-1}}(x_0), x_0])). \end{aligned} \tag{4-3}$$

By (4-3), we get

$$l([f^{-q_{n-1}}(x_0), x_0]) = o(\|q_{n-1}\alpha\|). \tag{5}$$

Moreover, we have

$$o(l([f^{-q_{n-1}}(x_0), x_0])) = o(l([y_1, y_2])). \tag{6}$$

Indeed:

- if $y_2 \in [x_0, f^{q_{n-1}}(x_0)]$ then

$$\begin{aligned} l([f^{-q_{n-1}}(x_0), x_0]) &\le l([f^{-q_{n-1}}(x_0), y_1]) + l([y_1, x_0]) \\ &\le e^V l([x_0, y_2]) + l([y_1, y_2]) \quad \text{(by Denjoy's inequality)} \\ &\le (e^V + 1) l([y_1, y_2]); \end{aligned}$$

- if $y_2 \in [f^{q_n}(x_0), x_0]$ then

$$\begin{aligned} l([f^{-q_{n-1}}(x_0), x_0]) &\le l([f^{-q_{n-1}}(x_0), y_2]) + l([y_2, x_0]) \\ &\le l([y_1, y_2]) + l([y_1, y_2]) \\ &= 2l([y_1, y_2]). \end{aligned}$$

Second, we distinguish two cases.

(1) The case where $x_0 \in [f^{-i_n(c)}(c), f^{-i_n(d)}(d)]$. Since h preserves the orientation,

$$h(x_0) \in [R_{-i_n(c)\alpha}(h(c)), R_{-i_n(d)\alpha}(h(d))].$$

Therefore

$$l([f^{-i_n(c)}(c), f^{-i_n(d)}(d)]) = l([f^{-i_n(c)}(c), x_0]) + l([f^{-i_n(d)}(d), x_0])$$

and

$$\begin{aligned} &l([R_{-i_n(d)\alpha}(h(d)), R_{-i_n(c)\alpha}(h(c))]) \\ &\quad = l([R_{-i_n(d)\alpha}(h(d)), h(x_0)]) + l([h(x_0), R_{-i_n(c)\alpha}(h(c))]) \\ &\quad = Dh(x_0) l([f^{-i_n(d)}(d), x_0]) + o(l([f^{-i_n(d)}(d), x_0])) \\ &\qquad + Dh(x_0) l([f^{-i_n(c)}(c), x_0]) + o(l([f^{-i_n(c)}(c), x_0])) \quad \text{(by (4-1), (4-2))} \\ &\quad = Dh(x_0)[l([f^{-i_n(d)}(d), x_0]) + l([f^{-i_n(c)}(c), x_0])] \\ &\qquad + o(l([f^{-i_n(d)}(d), x_0])) + o(l([f^{-i_n(c)}(c), x_0])) \end{aligned}$$

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$$
\begin{aligned}
&= Dh(x_0)l([f^{-i_n(c)}(c),\, f^{-i_n(d)}(d)]) + \mathrm{o}(l([f^{-i_n(c)}(c),\, x_0])) + \mathrm{o}(l([f^{-i_n(d)}(d), x_0]))\\
&= \mathrm{o}(l([y_1,\, y_2])) \quad \text{(by (3))}\\
&= \mathrm{o}(l([f^{-q_{n-1}}(x_0),\, x_0])) \quad \text{(by (6))}\\
&= \mathrm{o}(\|q_{n-1}\alpha\|) \quad \text{(by (5))},
\end{aligned}
$$

which is our claim (4).

(2) The case where $x_0 \notin [f^{-i_n(c)}(c),\, f^{-i_n(d)}(d)]$. Since h preserves the orientation,

$$h(x_0) \notin [R_{-i_n(c)\alpha}(h(c)),\, R_{-i_n(d)\alpha}(h(d))].$$

Then

$$l([f^{-i_n(c)}(c),\, f^{-i_n(d)}(d)]) = s[l([f^{-i_n(d)}(d),\, x_0]) - l([f^{-i_n(c)}(c),\, x_0])],$$

where $s \in \{-1,\, 1\}$, and

$$
\begin{aligned}
&l([R_{-i_n(d)\alpha}(h(d)),\, R_{-i_n(c)\alpha}(h(c))])\\
&\quad = s[l([R_{-i_n(d)\alpha}(h(d)),\, h(x_0)]) - l([h(x_0),\, R_{-i_n(c)\alpha}(h(c))])]\\
&\quad = s[Dh(x_0)l([f^{-i_n(d)}(d),\, x_0]) + \mathrm{o}(l([f^{-i_n(d)}(d),\, x_0]))]\\
&\qquad - s[Dh(x_0)l([f^{-i_n(c)}(c),\, x_0]) + \mathrm{o}(l([f^{-i_n(c)}(c),\, x_0]))] \quad \text{(by (4-1), (4-2))}\\
&\quad = Dh(x_0)s[l([f^{-i_n(d)}(d),\, x_0]) - l([f^{-i_n(c)}(c),\, x_0])]\\
&\qquad + s[\mathrm{o}(l([f^{-i_n(d)}(d),\, x_0])) - \mathrm{o}(l([f^{-i_n(c)}(c),\, x_0]))]\\
&\quad = Dh(x_0)l([f^{-i_n(c)}(c),\, f^{-i_n(d)}(d)]) + \mathrm{o}(l([f^{-i_n(c)}(c), x_0])) + \mathrm{o}(l([f^{-i_n(d)}(d), x_0]))\\
&\quad = \mathrm{o}(l([y_1,\, y_2])) \quad \text{(by (3))}\\
&\quad = \mathrm{o}(l([f^{-q_{n-1}}(x_0),\, x_0])) \quad \text{(by (6))}\\
&\quad = \mathrm{o}(\|q_{n-1}\alpha\|) \quad \text{(by (5))},
\end{aligned}
$$

which is our claim (4). We see by (4) that:

$$
\begin{aligned}
l([R_{-i_n(d)\alpha}(h(d)),\, R_{-i_n(c)\alpha}(h(c))]) &= \|(i_n(c) - i_n(d))\alpha - h(c) - h(d)\|\\
&= \mathrm{o}(\|q_{n-1}\alpha\|) \quad \text{as } n \to +\infty,\, n \in \mathbb{N}_0.
\end{aligned}
$$

Set $k_n = i_n(c) - i_n(d)$, $n \in \mathbb{N}_0$ and $\beta = h(c) - h(d)$. Then

$$|k_n| < q_n \quad \text{and} \quad \|k_n\alpha - \beta\| = \mathrm{o}(\|q_{n-1}\alpha\|), \quad n \to +\infty.$$

Since

$$\|(k_{n+1} - k_n)\alpha\| \le \|k_{n+1}\alpha - \beta\| + \|k_n\alpha - \beta\|, \quad \text{and} \quad \|q_n\alpha\| < \|q_{n-1}\alpha\|,$$

we see that

$$
\begin{aligned}
\|(k_{n+1} - k_n)\alpha\| &= \mathrm{o}(\|q_{n-1}\alpha\| + \|q_n\alpha\|)\\
&= \mathrm{o}(\|q_{n-1}\alpha\|), \quad n \to +\infty.
\end{aligned}
$$

Since α is of bounded type, there exists a constant $C > 0$ such that for every $n \ge 1$, $n \in \mathbb{N}_0$, $\|q_{n-1}\alpha\| \le C\|q_{n+2}\alpha\|$. Therefore, there exists $n_0 \ge 1$ such that for every $n \ge n_0$,

$$\|(k_{n+1} - k_n)\alpha\| < \|q_{n+2}\alpha\|,$$

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where

$$|k_{n+1} - k_n| < q_{n+1} + q_n \leq q_{n+2}.$$

Recall that

$$q_n = \min\{k \in \mathbb{N}^* : \|k\alpha\| < \|q_{n-1}\alpha\|\}.$$

Therefore, $|k_{n+1} - k_n| \geq q_{n+3}$; this is impossible since $q_{n+2} < q_{n+3}$. It follows that for every $n \geq n_0$, $k_{n+1} - k_n = 0$. So, for every $n \geq n_0$, $k_n = k_{n_0}$, hence

$$\|k_{n_0}\alpha - \beta\| = \|k_n\alpha - \beta\| = o(\|q_{n-1}\alpha\|) \quad \text{as } n \to +\infty.$$

This implies that $\beta = k_{n_0}\alpha$ (since $\|q_{n-1}\alpha\| \to 0$) and therefore $h(d) \in O_{R_\alpha}(h(c))$, so $d \in O_f(c)$, which contradicts the hypothesis.

Case 2: $A(d) = \{1\}$, i.e. $d \in Q_3$. In this case, $(a_n(d))_{n\in\mathbb{N}_0}$ converges to 1 or equivalently $(b_n(d))_{n\in\mathbb{N}_0}$ is decreasing to 0. Then, similarly to case 1, we get

$$l([R_{-j_n(d)\alpha}(h(d)),\, R_{-i_n(c)\alpha}(h(c))]) = o(\|q_{n-1}\alpha\|), \quad n \to +\infty.$$

By taking

$$k_n = i_n(c) - j_n(d), \quad n \in \mathbb{N}_0,$$
$$\beta = h(c) - h(d)$$

and running the proof as in case 1, we arrive at a contradiction.

Case 3: $A(d) = \{0, 1\}$, i.e. $d \in Q_4$. In this case, there exists an infinite subset $\mathbb{N}_1$ of $\mathbb{N}_0$ such that $(a_n(d))_{n\in\mathbb{N}_1}$ converges to 0 and $(a_n(d))_{n\in\mathbb{N}_0\setminus\mathbb{N}_1}$ converges to 1. As in case 1, we have the property (4) for $n \in \mathbb{N}_1$:

$$\begin{aligned} l([R_{-i_n(d)\alpha}(h(d)),\, R_{-i_n(c)\alpha}(h(c))]) &= \|(i_n(c) - i_n(d))\alpha - h(c) - h(d)\| \\ &= o(\|q_{n-1}\alpha\|) \quad \text{as } n \to +\infty,\ n \in \mathbb{N}_1. \end{aligned}$$

Similarly,

$$\begin{aligned} l([R_{-j_n(d)\alpha}(h(d)),\, R_{-i_n(c)\alpha}(h(c))]) &= \|(i_n(c) - i_n(d))\alpha - h(c) - h(d)\| \\ &= o(\|q_{n-1}\alpha\|) \quad \text{as } n \to +\infty,\ n \in \mathbb{N}_0\setminus\mathbb{N}_1. \end{aligned}$$

Set

$$k_n = \begin{cases} i_n(c) - i_n(d) & \text{if } n \in \mathbb{N}_0, \\ i_n(c) - j_n(d) & \text{if } n \in \mathbb{N}_0\setminus\mathbb{N}_1 \end{cases}$$

and

$$\beta = h(c) - h(d).$$

Then,

$$\|k_n\alpha - \beta\| = o(\|q_{n-1}\alpha\|), \quad n \to +\infty \text{ with } |k_n| < q_n.$$

Therefore, there exists $n_0 \geq 1$ such that for every $n \geq n_0$, $k_n = k_{n_0}$.

Running the proof as in case 1, we arrive at a contradiction. We conclude by Lemma 3.2(1), the assertion (i).

Proof of (ii): this follows directly from Proposition 3.3. □

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4. *Derived cells*

We introduce in this section a new triple (z_1, z_2, z_3) in $U_\delta(x_0)$ depending on two real parameters r and u. Let $r, u \in \mathbb{R}$ be such that $r \geq 1$, $u > 0$ and $ru \leq 1$. Let $(y_1, y_2, y_3) \subset U_\delta(x_0)$ be the standard cell for the constant $R = e^{3V} + e^V + 1$ associated to (f, x_0, c). We define the points $z_1, z_2, z_3 \in S^1$ as follows:

$$z_2 = y_2, \quad z_1 \in [y_1, y_2], \quad z_3 \in [y_2, y_3]$$

such that

$$r^{-1}u \leq \frac{l([z_1, z_2])}{l([y_1, y_2])} \leq ru \quad \text{and} \quad r^{-1}u \leq \frac{l([z_2, z_3])}{l([y_2, y_3])} \leq ru.$$

Definition (derived cell). The triple (z_1, z_2, z_3) is called an (r, u)-*derived cell associated to* (f, x_0, c). The $(1, 1)$-derived cell corresponds to the standard cell (y_1, y_2, y_3) associated to (f, x_0, c).

PROPOSITION 4.1. *Under the notation above, let $u \in]0, e^{-V}/r]$. Then we have:*

(c-0) $[z_1, z_3] \subset [y_1, y_3]$;

(c-1) *there exists a unique $0 \leq i_n(c) < q_n$ such that $f^{i_n(c)}(z_2) = c$;*

(c-2) $l(f^j([z_1, z_3])) \leq K\lambda^n$, *for every $0 \leq j < q_n$, where $\lambda = (1 + e^{-V})^{-\frac{1}{2}}$ and K is a constant independent of n;*

(c-3) $\sum_{j=0}^{q_n-1} l(f^j([z_1, z_3])) \leq 2$;

(c-4) *the triples (z_1, z_2, z_3) and $(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))$ satisfy conditions (a) and (b) of Definition 2.10 for x_0 and the constant $R(u) = rRe^{2V}/u$;*

(c-5)
- *$f^i([z_1, z_3])$ does not contain c for every $0 \leq i \neq i_n(c) < q_n$;*
- *$f^{i_n(c)}([z_1, z_3])$ does not contain any break point of f other than c.*

Proof. The properties (c-0) and (c-1) are clear since $ru \leq 1$ and $z_2 = y_2 = f^{-i_n(c)}(c)$, $0 \leq i_n(c) < q_n$. Also the properties (c-2), (c-3) and (c-5) hold since $[z_1, z_3] \subset [y_1, y_3]$ and (y_1, y_2, y_3) satisfies (c-2), (c-3) and (c-5) by Proposition 3.1. It remains to show the property (c-4). We have

$$r^{-2}e^{-V} \leq \frac{r^{-1}ul([y_2, y_3])}{rul([y_1, y_2])} \leq \frac{l([z_2, z_3])}{l([z_1, z_2])} \leq \frac{rul([y_2, y_3])}{r^{-1}ul([y_1, y_2])} \leq r^2e^V,$$

then by Lemma 2.8, we get

$$r^{-2}e^{-3V} \leq e^{-2V}\frac{l([z_2, z_3])}{l([z_1, z_2])} \leq \frac{l(f^{q_n}([z_2, z_3]))}{l(f^{q_n}([z_1, z_2]))} \leq e^{2V}\frac{l([z_2, z_3])}{l([z_1, z_2])} \leq r^2e^{3V}.$$

On the other hand, since $[z_1, z_3] \subset [y_1, y_3]$ and (y_1, y_2, y_3) satisfies condition (b) for the constant $R = e^{3V} + e^V + 1$, it follows that

$$\max_{1\leq i\leq 3} l([x_0, z_i]) \leq \max_{1\leq i\leq 3} l([x_0, y_i]) \leq Rl([y_1, y_2]) \leq \frac{rR}{u}l([z_1, z_2]).$$

Thus (z_1, z_2, z_3) satisfies conditions (a) and (b) for $R_1(u) = rR/u$ (because $r^2e^{3V} < rR/u$ since $u \in]0, e^{-V}/r]$). On the other hand, since $f^{q_n}([z_1, z_3]) \subset f^{q_n}([y_1, y_3])$ and $f^{q_n}(y_1, y_2, y_3)$ satisfy condition (b) for $R = e^{3V} + e^V + 1$, we get:

$$\max_{1\leq i\leq 3} l([x_0, f^{q_n}z_i]) \leq \max_{1\leq i\leq 3} l([x_0, f^{q_n}y_i]) \leq Rl(f^{q_n}([y_1, y_2])).$$

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By Lemma 2.8, we have

$$\frac{l(f^{q_n}([y_1, y_2]))}{l(f^{q_n}([z_1, z_2]))} \le e^{2V} \frac{l([y_1, y_2])}{l([z_1, z_2])} \le \frac{re^{2V}}{u}.$$

Hence

$$\max_{1\le i\le 3} l([x_0, f^{q_n} z_i]) \le \frac{rRe^{2V}}{u} l(f^{q_n}([z_1, z_2])).$$

It follows that conditions (a) and (b) hold for $(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))$ (since $u \in]0, e^{-V}/r]$) for constant $R(u) := rRe^{2V}/u$. □

Set

$$u_0 = e^{-2V} r^{-1} \tau^{-1} \min(1, \gamma),$$

where:

- $\tau = (1 + e^V)e^{3V} r$;
- γ is the constant given by Proposition 3.3.

PROPOSITION 4.2. *Suppose that $c \in C(f)$ is the unique break point in its orbit. Let $(y_1, y_2, y_3) = (y_1(n), y_2(n), y_3(n))_{n\in\mathbb{N}_0}$ be a standard cell associated to (f, x_0, c). Then there exists an infinite subset $\mathbb{N}_c \subset \mathbb{N}_0$ such that for any $u \in]0, u_0[$, there exists $n_{u,r} \in \mathbb{N}_c$ such that for every $n \in \mathbb{N}_c$, $n \ge n_{u,r}$: the (r, u)-derived cell $(z_1, z_2, z_3) := (z_1(n), z_2(n), z_3(n))_{n\in\mathbb{N}_c}$ associated to (f, x_0, c) has the following properties.*

(a) *For every $0 \le i < q_n$:*
  (i) $f^i(]z_1, z_3[)$ *does not contain any point of Q_1;*
  (ii) $f^i(]z_1, z_2[)$ *does not contain any point of Q_3;*
  (iii) $f^i(]z_2, z_3[)$ *does not contain any point of $Q_2 \cup Q_4$.*

(b) *For every $d \in Q_2 \cup Q_4$, $f^{i_n(d)}(]z_1, z_2[)$ does not contain any break point of f other than d.*

(c) *For every $d \in Q_3$, $f^{j_n(d)}(]z_2, z_3[)$ does not contain any break point of f other than d where $j_n(d) = \varphi_n(i_n(d))$.*

We need the following lemma.

LEMMA 4.3. *Under the hypotheses of Proposition 4.2, let $(z_1, z_2, z_3) := (z_1(n), z_2(n), z_3(n))_{n\in\mathbb{N}_0}$ be an (r, u)-derived cell associated to (f, x_0, c). Then for every $n \in \mathbb{N}_0$*

(1)
$$\frac{\tau^{-1}}{u} a_n(d) \le \frac{l([f^{i_n(d)}(z_2), d])}{l(f^{i_n(d)}([z_1, z_2]))} \le \frac{\tau}{u} a_n(d),$$
$$\frac{\tau^{-1}}{u} b_n(d) \le \frac{l([f^{j_n(d)}(z_2), d])}{l(f^{j_n(d)}([z_1, z_2]))} \le \frac{\tau}{u} b_n(d),$$

(2)
$$\frac{\tau^{-1}}{u} a_n(d) \le \frac{l([f^{i_n(d)}(z_2), d])}{l(f^{i_n(d)}([z_2, z_3]))} \le \frac{\tau}{u} a_n(d),$$
$$\frac{\tau^{-1}}{u} b_n(d) \le \frac{l([f^{j_n(d)}(z_2), d])}{l(f^{j_n(d)}([z_2, z_3]))} \le \frac{\tau}{u} b_n(d),$$

where $j_n(d) = \varphi_n(i_n(d))$.

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Proof. Set

$$\beta_n = \frac{l(f^{i_n(d)}([y_1, y_2]))}{l([f^{j_n(d)}(y_2),\ f^{i_n(d)}(y_2)])}, \quad \gamma_n = \frac{l(f^{j_n(d)}([y_2, y_3]))}{l([f^{j_n(d)}(y_2),\ f^{i_n(d)}(y_2)])}.$$

Since $1 \le \beta_n + \gamma_n \le 2$ and by Denjoy's inequality $e^{-V} \le \beta_n/\gamma_n \le e^V$, we obtain the inequalities $1 \le \beta_n^{-1} \le 1 + e^V$, $1 \le \gamma_n^{-1} \le 1 + e^V$. Since $z_2 = y_2$, we have

$$\frac{l([f^{i_n(d)}(z_2), d])}{l(f^{i_n(d)}([z_1, z_2]))} = \beta_n^{-1} \frac{l(f^{i_n(d)}([y_1, y_2]))}{l(f^{i_n(d)}([z_1, z_2]))} a_n(d)$$

and

$$\frac{l([f^{j_n(d)}(z_2), d])}{l(f^{j_n(d)}([z_1, z_2]))} = \gamma_n^{-1} \frac{l(f^{j_n(d)}([y_2, y_3]))}{l(f^{j_n(d)}([z_1, z_2]))} b_n(d).$$

Then Lemma 2.8 implies that

$$\begin{aligned} \frac{e^{-2V}}{ru} &\le e^{-2V} \frac{l([y_1, y_2])}{l([z_1, z_2])} \\ &\le \frac{l(f^{i_n(d)}([y_1, y_2]))}{l(f^{i_n(d)}([z_1, z_2]))} \\ &\le e^{2V} \frac{l([y_1, y_2])}{l([z_1, z_2])} \\ &\le \frac{e^{2V} r}{u} \end{aligned}$$

and

$$\frac{e^{-3V}}{ru} \le \frac{l(f^{j_n(d)}([y_2, y_3]))}{l(f^{j_n(d)}([z_1, z_2]))} \le \frac{e^{3V} r}{u};$$

this implies the assertion (1). By the same method, we obtain the assertion (2). □

Proof of Proposition 4.2. If $C(f) = \{c\}$ the Proposition 4.2 is clear. Now, suppose that $C(f) \neq \{c\}$. By Proposition 3.3, there exist an infinite subset $\mathbb{N}_c \subset \mathbb{N}_0$ and a constant $\gamma > 0$ such that for every $d \in C(f)\backslash\{c\}$, one of the properties $S_1(d)$, $S_2(d)$ or $S_3(d)$ is satisfied on $\mathbb{N}_c$ for the constant γ. Let $u \in]0, u_0[$, $n \in \mathbb{N}_c$ and $d \in C(f)\backslash\{c\}$. Then from Lemma 2.5, there exists a unique $0 \le i_n(d) < q_n$ such that $d \in f^{i_n(d)}(]y_1, y_2[) \cup f^{j_n(d)}(]y_2, y_3[)$ with $j_n(d) = \varphi_n(i_n(d))$. Therefore:

- $d \notin f^i([z_1, z_3])$ for every $0 \le i \ne j_n(d)$, $i \ne i_n(d) < q_n$ (as $d \notin f^i([y_1, y_3])$);
- $d \notin f^{i_n(d)}([z_2, z_3]) \cup f^{j_n(d)}([z_1, z_2])$ (as $d \notin f^{i_n(d)}([y_2, y_3]) \cup f^{j_n(d)}([y_1, y_2])$).

Assertion (a). (i) If $d \in Q_1$ then by Proposition 3.3, $S_1(d)$ is satisfied on $\mathbb{N}_c$ for the constant γ: $a_n(d), b_n(d) \ge \gamma$, for every $n \in \mathbb{N}_c$, so by the inequalities (1) and (2) of Lemma 4.3, we get

$$\frac{l([f^{i_n(d)}(z_2), d])}{l(f^{i_n(d)}([z_1, z_2]))} \ge a_n(d) \frac{\tau^{-1}}{u} > \gamma \frac{\tau^{-1}}{u_0} \ge 1,$$

and

$$\frac{l([f^{j_n(d)}(z_2), d])}{l(f^{j_n(d)}([z_2, z_3]))} \ge b_n(d) \frac{\tau^{-1}}{u} > \gamma \frac{\tau^{-1}}{u_0} \ge 1,$$

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for every $n \in \mathbb{N}_c$. Therefore

$$d \notin f^{i_n(d)}([z_1, z_2]) \cup f^{j_n(d)}([z_2, z_3]).$$

Thus, assertion (a), (i) is proven.

(ii) If $d \in Q_3$ then by Proposition 3.3, $S_3(d)$ is satisfied on $\mathbb{N}_c$ for the constant $\gamma : a_n(d) \geq \gamma$, for every $n \in \mathbb{N}_c$, so by the inequality (1) of Lemma 4.3, we get

$$\frac{l([f^{i_n(d)}(z_2), d])}{l(f^{i_n(d)}([z_1, z_2]))} \geq a_n(d)\frac{\tau^{-1}}{u} > \gamma\frac{\tau^{-1}}{u_0} \geq 1.$$

Therefore $d \notin f^{i_n(d)}([z_1, z_2])$, and thus assertion (a), (ii) is proven.

(iii) If $d \in Q_2 \cup Q_4$ then $S_2(d)$ is satisfied on $\mathbb{N}_c$ for the constant $\gamma : b_n(d) \geq \gamma$, for every $n \in \mathbb{N}_c$, so by the inequality (2) of Lemma 4.3, we get

$$\frac{l([f^{j_n(d)}(z_2), d])}{l(f^{j_n(d)}([z_2, z_3]))} \geq b_n(d)\frac{\tau^{-1}}{u} > \gamma\frac{\tau^{-1}}{u_0} \geq 1.$$

Therefore $d \notin f^{j_n(d)}([z_2, z_3])$, this implies the assertion (a), (iii).

Now choose an integer

$$n_0 \geq \frac{\log(K^{-1}L_0)}{\log \lambda},$$

where $L_0 = \min_{x,y \in C(f), x \neq y} l([x, y])$.

Assertion (b). Let $d \in Q_2 \cup Q_4$, then by Proposition 3.3, $S_2(d)$ is satisfied on $\mathbb{N}_c$ for the constant γ: $(a_n(d))_{n \in \mathbb{N}_c}$ is decreasing to 0, so there exists an integer $n_{u,r} = n_c > n_0$ such that for every $n \in \mathbb{N}_c$, $n > n_c$, one has $d \in f^{i_n(d)}([z_1, z_2])$ by Lemma 4.3(1). By Proposition 4.1(c-2) and for every $b \in C(f)\backslash\{d\}$, one has:

$$l(f^{i_n(d)}([z_1, z_2])) \leq K\lambda^n < K\lambda^{n_0} \leq L_0 \leq l([b, d]).$$

It follows that $b \notin f^{i_n(d)}([z_1, z_2])$ for every $b \in C(f)\backslash\{d\}$. Assertion (b) is proven.

Assertion (c). Let $d \in Q_3$, then by Proposition 3.3, $(b_n(d))_{n \in \mathbb{N}_c}$ is decreasing to 0, so there exists an integer $n'_{u,r} = n'_c > n_0$ such that for every $n \in \mathbb{N}_c$, $n > n'_c$, one has $d \in f^{j_n(d)}([z_2, z_3])$ by Lemma 4.3(2). By Proposition 4.1(c-2) and for every $b \in C(f)\backslash\{d\}$, one has:

$$l(f^{j_n(d)}([z_2, z_3])) \leq K\lambda^n < K\lambda^{n_0} \leq L_0 \leq l([b, d]).$$

It follows that $b \notin f^{j_n(d)}([z_2, z_3])$ for every $b \in C(f)\backslash\{d\}$. Assertion (c) is proven. □

5. *Control of distortions*

From now on, $f \in \mathcal{P}(S^1)$ have irrational rotation number α such that Df is absolutely continuous on every connected interval of $S^1\backslash C(f)$.

PROPOSITION 5.1. *Let u_0 and $\mathbb{N}_c$ be as in Proposition 4.2. Assume that the conjugating homeomorphism h from f to R_α admits at x_0 a positive derivative $Dh(x_0) = \omega_0 > 0$. Then for any $u \in]0, u_0[$, there exists an integer $n_u \in \mathbb{N}_c$ such that for every $n \in \mathbb{N}_c$, $n \geq n_u$,*

$$|\mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3) - 1| \leq \frac{2}{1 - u_0}u.$$

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We need the following lemma.

LEMMA 5.2. [**6**, Lemma 3] *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number α. Assume that the conjugating homeomorphism h of f to R_α admits at x_0 a positive derivative $Dh(x_0) = \omega_0 > 0$. Then for any $x_0 \in S^1$ and $R > 1$, there exists a constant $C > 0$ (dependent only on R and ω_0) such that for any $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that for any triple $(z_1, z_2, z_3) \subset U_\delta(x_0)$ $(z_1 \prec z_2 \prec z_3)$ satisfying the conditions (a) and (b) for the point x_0 and the constant R, one has*

$$|Dcr_h(z_1, z_2, z_3) - 1| \leq C\varepsilon.$$

Remark 3. One can show precisely that $C = 8R^2/\omega_0$.

Proof of Proposition 5.1. Let $u \in]0, u_0[$. Applying Lemma 5.2 to x_0 and $R(u) = Rre^{2V}/u > 1$ there exists a constant $C_u = 8R(u)^2/\omega_0$ (see Remark 3) such that for

$$\varepsilon(u) = \frac{\omega_0}{8r^2R^2e^{4V}}u^3,$$

there exists $\delta(u) > 0$ (dependent on u and x_0) such that for any triple $(t_1, t_2, t_3) \subset U_{\delta(u)}(x_0)$, $(t_1 \prec t_2 \prec t_3)$ satisfying conditions (a) and (b) for the point x_0 and the constant $R(u)$, we have

$$|\mathrm{Dcr}_h(t_1, t_2, t_3) - 1| \leq C_u\varepsilon(u) = u.$$

Furthermore, by Proposition 4.1(c-4), the triples (z_1, z_2, z_3) and $(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))$ satisfy conditions (a) and (b) for the point x_0 and the constant $R(u)$. For every $n \in \mathbb{N}_c$, $u \in]0, u_0[$, we have

$$\max_{1\leq i\leq 3} l([x_0, z_i]) \leq R(u)l([z_1, z_2]) \leq R(u)K\lambda^n$$

and

$$\max_{1\leq i\leq 3} l(f^{q_n}([x_0, z_i])) \leq R(u)l(f^{q_n}([z_1, z_2])) \leq R(u)e^VK\lambda^n.$$

As $\lim_{n\to+\infty} \lambda^n = 0$, there exists $n_u \in \mathbb{N}_c$ (dependent on u) such that for every $n \in \mathbb{N}_c$, $n \geq n_u$, one has $[z_1, z_3]$(respectively $f^{q_n}([z_1, z_3])) \subset U_{\delta(u)}(x_0)$. In particular, for every $n \in \mathbb{N}_c$, $n \geq n_u$, $|\mathrm{Dcr}_h(z_1, z_2, z_3) - 1| < u$ and $|\mathrm{Dcr}_h(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3)) - 1| < u$. Since $h \circ f^{q_n} = R_\alpha^{q_n} \circ h$, for every $n \in \mathbb{N}_c$, $n \geq n_u$, we get

$$\begin{aligned}\mathrm{Cr}(h(f^{q_n}(z_1)), h(f^{q_n}(z_2)), h(f^{q_n}(z_3))) &= \mathrm{Cr}(R_\alpha^{q_n}(h(z_1)), R_\alpha^{q_n}(h(z_2)), R_\alpha^{q_n}(h(z_3)))\\ &= \mathrm{Cr}(h(z_1), h(z_2), h(z_3))\end{aligned}$$

and

$$\mathrm{Dcr}_h(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3)) = \frac{\mathrm{Cr}(h(z_1), h(z_2), h(z_3))}{\mathrm{Cr}(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))}.$$

Therefore

$$\mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3) = \frac{\mathrm{Dcr}_h(z_1, z_2, z_3)}{\mathrm{Dcr}_h(f^{q_n}(z_1), f^{q_n}(z_2), f^{q_n}(z_3))} \in \left[\frac{1-u}{1+u}, \frac{1+u}{1-u}\right].$$

As for every $u \in]0, u_0[$,

$$\left[\frac{1-u}{1+u}, \frac{1+u}{1-u}\right] = \left[1 - \frac{2u}{1+u}, 1 + \frac{2u}{1-u}\right] \subset \left[1 - \frac{2u}{1-u_0}, 1 + \frac{2u}{1-u_0}\right],$$

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then for every $n \in \mathbb{N}_c$, $n \geq n_u$, we get

$$|\mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3) - 1| \leq \frac{2}{1-u_0}u. \qquad \square$$

The next proposition will show another distortion control which is opposite to that of Proposition 5.1. This allows us to prove that the invariant measure μ_f of f is singular with respect to the Haar measure m.

PROPOSITION 5.3. *Let u_0 and $\mathbb{N}_c$ be as in Proposition 4.2. Then for any $\varepsilon > 0$, there exists $\eta > 0$, such that for any $u \in]0, \eta[$ there exists $n_{u,r} \in \mathbb{N}_c$ such that for any $n \in \mathbb{N}_c$, $n \geq n_{u,r}$, the (r, u)-derived cell (z_1, z_2, z_3) associated to (f, x_0, c) satisfies*

$$|\mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3) - \Pi(c)| \leq \theta(\varepsilon),$$

where

$$\Pi(c) = \sigma_f(c) \prod_{d \in Q_2 \cup Q_3 \cup Q_4} \sigma_f(d),$$

$$\theta(\varepsilon) = \Pi(c)\left(e^{C\varepsilon} + \frac{C_1}{\sigma_f(c)}e^{C\varepsilon}\varepsilon - 1\right),$$

for some constants $C, C_1 > 0$ and where Q_2, Q_3 and Q_4 are defined with respect to f, x_0, c and $\mathbb{N}_c$.

Proof. Let $\varepsilon > 0$. Since Df is absolutely continuous on every interval $[c_i, c_{i+1}]$ $(0 \leq i \leq p)$, there exists $\eta_0 > 0$ such that for any disjoint intervals $([a_j, b_j])_{j \in J}$ of continuity intervals of Df (these latter are the $[c_i, c_{i+1}]$, $0 \leq i \leq p$) such that $\sum_{j \in J} |b_j - a_j| < \eta_0$ then $\sum_{j \in J} |Df(b_j) - Df(a_j)| < \varepsilon$. Set $\eta = \eta_0 e^{-2V}/r$. Let $u \in]0, \eta[$ and (z_1, z_2, z_3) be the (r, u)-derived cell associated to (f, x_0, c). Denote the following:

$$I = \{0, \ldots, q_n - 1\} \backslash (\{i_n(d) : d \in Q_2 \cup Q_4\} \cup \{j_n(d) : d \in Q_3\} \cup \{i_n(c)\});$$

$$D_i(f) := \mathrm{Dcr}_f(f^i(z_1), f^i(z_2), f^i(z_3)), \quad 0 \leq i < q_n;$$

$$P = \prod_{i \in I} D_i(f); \quad P_2 = \prod_{d \in Q_2 \cup Q_4} D_{i_n(d)}(f); \quad P_3 = \prod_{d \in Q_3} D_{j_n(d)}(f);$$

$$D_n(u)(f) := \mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3), \quad n \in \mathbb{N}_c.$$

One has $D_n(u) = P P_2 P_3 D_{i_n(c)}(f)$.

First we consider $D_i(f)$, $i \in I$. By Propositions 4.2 and 4.1(c-5), Df is continuous on the interval $[f^i(z_1), f^i(z_3)]$. Then by the mean value theorem, there exist $a_i \in]f^i(z_1), f^i(z_2)[$, and $b_i \in]f^i(z_2), f^i(z_3)[$ such that $D_i(f) = Df(a_i)/Df(b_i)$.

Since $([a_i, f^i(z_2)])_{i \in I}$ are disjoint intervals of continuity of Df, by Lemma 2.8, they satisfy the inequality

$$\begin{aligned}
\sum_{i \in I} l([a_i, f^i(z_2)]) &\leq \sum_{i \in I} l([f^i(z_1), f^i(z_2)]) \\
&\leq \sum_{i \in I} e^{2V} \frac{l([z_1, z_2])}{l([y_1, y_2])} l([f^i(y_1), f^i(y_2)]) \\
&\leq rue^{2V} \sum_{i \in I} l([f^i(y_1), f^i(y_2)]) \\
&\leq rue^{2V} < \eta_0.
\end{aligned}$$

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Consequently

(i) $$\sum_{i\in I} |Df(f^i(z_2)) - Df(a_i)| \le \varepsilon.$$

Also, $([f^i(z_2), b_i])_{i\in I}$ are disjoint intervals of continuity of Df. So, by Lemma 2.8, they satisfy

$$\begin{aligned}\sum_{i\in I} l([f^i(z_2), b_i]) &\le \sum_{i\in I} l([f^i(z_2),\ f^i(z_3)]) \\ &\le e^{2V} \sum_{i\in I} \frac{l([z_2, z_3])}{l([y_1, y_2])} l([f^i(y_1),\ f^i(y_2)]) \\ &\le rue^{2V} \sum_{i\in I} l([f^i(y_1),\ f^i(y_2)]) \\ &\le rue^{2V} < \eta_0.\end{aligned}$$

Consequently

(ii) $$\sum_{i\in I} |Df(f^i(z_2)) - Df(b_i)| \le \varepsilon.$$

Combining (i) and (ii) we get

(1) $$\sum_{i\in I} |Df(b_i) - Df(a_i)| \le 2\varepsilon.$$

Set

$$D_i^*(f) := \max\left(D_i(f), \frac{1}{D_i(f)}\right) \quad \text{and} \quad P^* = \prod_{i\in I} D_i^*(f).$$

Since Df takes its value in $[m_1, M_1]$, $m_1 > 0$, the following properties hold:

$$\begin{aligned}&0 \le D_i^*(f) - 1 \le \frac{1}{m_1} |Df(b_i) - Df(a_i)|, \\ &\sum_{i\in I} |D_i^*(f) - 1| \le \frac{2}{m_1}\varepsilon, \\ &|\log D_i(f)| = \log D_i^*(f) \le D_i^*(f) - 1, \\ &\frac{1}{P^*} \le P \le P^*.\end{aligned}$$

It follows that

(2) $$e^{-C_0\varepsilon} \le P \le e^{C_0\varepsilon} \quad \text{where } C_0 = \frac{2}{m_1}.$$

Next we consider $D_{i_n(c)}(f)$. Recall that $f^{i_n(c)}(z_2) = c$. By Proposition 4.1(c-5), Df is continuous on each of the intervals $[f^{i_n(c)}(z_1), c]$ and $[c,\ f^{i_n(c)}(z_3)]$. By the mean value theorem, there exist $a_{i_n(c)} \in]f^{i_n(c)}(z_1), c[$ and $b_{i_n(c)} \in]c,\ f^{i_n(c)}(z_3)[$ such that

$$D_{i_n(c)}(f) = \frac{Df(a_{i_n(c)})}{Df(b_{i_n(c)})}.$$

One has

$$\begin{aligned}|D_{i_n(c)}(f) - \sigma_f(c)| = \frac{1}{Df_+(c)Df(b_{i_n(c)})} &|Df_+(c)(Df(a_{i_n(c)}) - Df_-(c)) \\ &+ Df_-(c)(Df_+(c) - Df(b_{i_n(c)}))|.\end{aligned}$$

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Hence

$$|D_{i_n(c)}(f) - \sigma_f(c)| \le \frac{1}{Df_+(c)Df(b_{i_n(c)})}(Df_+(c)|Df(a_{i_n(c)}) - Df_-(c)| + Df_-(c)|Df_+(c) - Df(b_{i_n(c)})|).$$

Since $[a_{i_n(c)}, c]$ and $[c, b_{i_n(c)}]$ are intervals of continuity of Df, using Lemma 2.8 they satisfy

$$l([a_{i_n(c)}, c]) \le l([f^{i_n(c)}(z_1), f^{i_n(c)}(z_2)]) \le rue^{2V} < \eta_0$$

and

$$l([c, b_{i_n(c)}]) \le l([f^{i_n(c)}(z_2), f^{i_n(c)}(z_3)]) \le rue^{2V} < \eta_0.$$

It follows that

$$|Df(a_{i_n(c)}) - Df_-(c)| < \varepsilon \quad \text{and} \quad |Df_+(c) - Df(b_{i_n(c)})| < \varepsilon.$$

Therefore

$$|D_{i_n(c)}(f) - \sigma_f(c)| \le C_1\varepsilon \quad \text{where } C_1 = \frac{2M_1}{m_1^2}. \tag{3}$$

Consider now $D_{i_n(d)}(f)$, $d \in Q_2 \cup Q_4$. By Proposition 4.2, Df is continuous on the intervals

$$[f^{i_n(d)}(z_1), d], [d, f^{i_n(d)}(z_2)] \quad \text{and} \quad [f^{i_n(d)}(z_2), f^{i_n(d)}(z_3)].$$

By the mean value theorem, there exist

$$b_1 \in]f^{i_n(d)}(z_1), d[, b_2 \in]d, f^{i_n(d)}(z_2)[, b_3 \in]f^{i_n(d)}(z_2), f^{i_n(d)}(z_3)[,$$

such that

$$D_{i_n(d)}(f) = r_n(d)\frac{Df(b_2)}{Df(b_3)} + (1 - r_n(d))\frac{Df(b_1)}{Df(b_3)},$$

where

$$r_n(d) = \frac{l([f^{i_n(d)}(z_2), d])}{l([f^{i_n(d)}(z_1), f^{i_n(d)}(z_2)])}.$$

We have

$$|D_{i_n(d)}(f) - \sigma_f(d)| \le \left|\frac{Df(b_1)}{Df(b_3)} - \frac{Df_-(d)}{Df_+(d)}\right| + r_n(d)\frac{|Df(b_1) - Df(b_2)|}{Df(b_3)}.$$

Since

$$\left|\frac{Df(b_1)}{Df(b_3)} - \frac{Df_-(d)}{Df_+(d)}\right| \le \frac{M_1}{m_1^2}(|Df(b_1) - Df_-(d)| + |Df(b_3) - Df_+(d)|)$$

and by Lemma 4.3, $r_n(d) \le a_n(d)(\tau/u)$, it follows that

$$|D_{i_n(d)}(f) - \sigma_f(d)| \le K_0\Bigg(\frac{a_n(d)}{u} + |Df(b_1) - Df_-(d)| + |Df(f^{i_n(d)}(z_2)) - Df_+(d)| + |Df(b_3) - Df(f^{i_n(d)}(z_2))|\Bigg),$$

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where $K_0 = \max(M_1/m_1^2, 2\tau M_1/m_1)$. The intervals $(]b_1, d[)_{d\in Q_2\cup Q_4}$ and $(]d, f^{i_n(d)}(z_2)[)_{d\in Q_2\cup Q_4}$ and $(]f^{i_n(d)}(z_2), b_3[)_{d\in Q_2\cup Q_4}$ are disjoint intervals of continuity of Df so by Lemma 2.8, they satisfy

$$\sum_{d\in Q_2\cup Q_4} l([b_1, d]) \le \sum_{d\in Q_2\cup Q_4} l([f^{i_n(d)}(z_1), f^{i_n(d)}(z_2)]) < \eta_0,$$
$$\sum_{d\in Q_2\cup Q_4} l([d, f^{i_n(d)}(z_2)]) \le \sum_{d\in Q_2\cup Q_4} l([f^{i_n(d)}(z_1), f^{i_n(d)}(z_2)]) < \eta_0,$$
$$\sum_{d\in Q_2\cup Q_4} l([f^{i_n(d)}(z_2), b_3]) \le \sum_{d\in Q_2\cup Q_4} l([f^{i_n(d)}(z_2), f^{i_n(d)}(z_3)]) < \eta_0.$$

So

$$\sum_{d\in Q_2\cup Q_4} (|Df(b_1) - Df_-(d)| + |Df(f^{i_n(d)}(z_2)) - Df_+(d)| + |Df(b_3) - Df(f^{i_n(d)}(z_2))|) < 3\varepsilon.$$

It follows that

$$\sum_{d\in Q_2\cup Q_4} |D_{i_n(d)}(f) - \sigma_f(d)| \le K_0\left(3\varepsilon + \frac{N_2}{u}\max_{d\in Q_2\cup Q_4} a_n(d)\right),$$

where $N_2 = \mathrm{card}(Q_2 \cup Q_4)$. Since $\max_{d\in Q_2\cup Q_4} a_n(d)$ converges to 0, there exists $n_u \in \mathbb{N}_c$ such that for every $n \in \mathbb{N}_c$, $n \ge n_u$, $N_2 \max_{d\in Q_2\cup Q_4} a_n(d) \le u\varepsilon$.

Therefore, for every $n \in \mathbb{N}_c$, $n \ge n_u$,

$$\sum_{d\in Q_2\cup Q_4} |D_{i_n(d)}(f) - \sigma_f(d)| \le 4K_0\varepsilon.$$

In particular, $|D_{i_n(d)}(f) - \sigma_f(d)| \le 4K_0\varepsilon$.

Set

$$D^*_{i_n(d)}(f) := \max\left(\frac{D_{i_n(d)}(f)}{\sigma_f(d)}, \frac{\sigma_f(d)}{D_{i_n(d)}(f)}\right) \ge 1,$$
$$P_2^* = \prod_{d\in Q_2\cup Q_4} D^*_{i_n(d)}(f).$$

Then we have the following properties:

$$\left|\log \frac{D_{i_n(d)}(f)}{\sigma_f(d)}\right| = \log D^*_{i_n(d)}(f) \le D^*_{i_n(d)}(f) - 1,$$
$$\frac{1}{P_2^*} \le \frac{P_2}{\prod_{d\in Q_2\cup Q_4} \sigma_f(d)} \le P_2^*.$$

For $0 < \varepsilon < \min_{d\in C(f)}(\sigma_f(d)/5K_0)$, we get

$$\sum_{d\in Q_2\cup Q_4} |D^*_{i_n(d)}(f) - 1| \le \sum_{d\in Q_2\cup Q_4} \frac{|D_{i_n(d)}(f) - \sigma_f(d)|}{\sigma_f(d) - 4K_0\varepsilon} \le \sum_{d\in Q_2\cup Q_4} \frac{|D_{i_n(d)}(f) - \sigma_f(d)|}{\frac{1}{5}\sigma_f(d)} \le C_2\varepsilon,$$

where $C_2 = (20/m_1)M_1K_0$ is a constant independent of n, u, ε.

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Hence

$$P_2^* = \prod_{d\in Q_2\cup Q_4} D^*_{i_n(d)}(f) \le e^{(\sum_{d\in Q_2\cup Q_4} |D^*_{i_n(d)}(f)-1|)} \le e^{C_2\varepsilon}.$$

It follows that for every $n \in \mathbb{N}_c$, $n \ge n_u$,

$$e^{-C_2\varepsilon} \prod_{d\in Q_2\cup Q_4} \sigma_f(d) \le P_2 \le e^{C_2\varepsilon} \prod_{d\in Q_2\cup Q_4} \sigma_f(d). \tag{4}$$

Consider $D_{j_n(d)}(f)$, $d \in Q_3$. Similarly to before, there exists $N_u \ge n_u$ such that for every $n \in \mathbb{N}_c$, $n \ge N_u$,

$$e^{-C_3\varepsilon} \prod_{d\in Q_3} \sigma_f(d) \le P_3 \le e^{C_3\varepsilon} \prod_{d\in Q_3} \sigma_f(d), \tag{5}$$

where C_3 is a constant independent of n, u, ε.

Finally, set

$$\theta_f(\varepsilon) = \nu(c)\left(e^{C\varepsilon}\left(1 + \frac{C_1}{\sigma_f(c)}\varepsilon\right) - 1\right) \quad \text{where } C = C_0 + C_2 + C_3.$$

The function $x \to \theta_f(x)$ is continuous on $\mathbb{R}$; moreover:

(i) $\theta_f(0) = 0$;

(ii) $\theta_f(x) > -\theta_f(-x)$ and $\theta_f(x) > 0$, for every $x > 0$.

Combining (2), (3), (4) and (5), we obtain the following.

For any $\varepsilon \in]0, \min_{d\in C(f)} \sigma_f(d)/5K_0[$, there exists $\eta > 0$ such that for any $u \in]0, \eta[$ there exists $N_{u,r} \in \mathbb{N}_c$ such that for every $n \in \mathbb{N}_c$, $n \ge N_{u,r}$, we have

$$-\theta_f(\varepsilon) < \theta_f(-\varepsilon) \le D_n(u)(f) - \nu(c) \le \theta_f(\varepsilon).$$

This concludes the proof of Proposition 5.3. □

COROLLARY 5.4. *Let $d \in C(f)\backslash\{c\}$, u_0, $\gamma_{c,d}$ and $\mathbb{N}_{c,d}$ be as in Proposition 3.4. Let h be the conjugating homeomorphism from f to R_α: $h \circ f = R_\alpha \circ h$. Assume that $Dh(x_0) > 0$ and that the rotation number α of f is of bounded type. Then for any $\varepsilon > 0$, there exists $\eta > 0$, such that for any $u \in]0, \eta[$ there exists $n_{u,r} \in \mathbb{N}_{c,d}$ such that for any $n \in \mathbb{N}_{c,d}$, $n \ge n_{u,r}$, the (r, u)-derived cell (z_1, z_2, z_3) associated to (f, x_0, c) satisfies*

$$|\mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3) - \Pi(c, d)| \le \theta(\varepsilon),$$

where

$$\Pi(c, d) = \sigma_f(c) \prod_{b\in Q_2\cup Q_3\cup Q_4} \sigma_f(b)$$

and

$$\theta(\varepsilon) = \nu(c, d)\left(e^{C\varepsilon}\left(1 + \frac{C_1}{\sigma_f(c)}\varepsilon\right) - 1\right),$$

for some constants $C, C_1 > 0$ and where Q_2, Q_3 and Q_4 are defined with respect to f, x_0, c, d and $\mathbb{N}_{c,d}$.

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Proof. By Proposition 3.4(i), $d \in Q_1$. So the proof follows similarly to that of Proposition 5.3. □

6. *Proof of Theorems A and B*

To show that the measure μ_f is singular with respect to the Haar measure m, it suffices to prove that the derivative Dh of the conjugating homeomorphism h from f to R_α is zero on a set of total Haar measure. On the contrary, h admits at a point x_0 a positive derivative: $\mathrm{Dh}(x_0) = \omega_0 > 0$. Let u_0 and $\mathbb{N}_c$ be as in Proposition 3.3. Write

$$D_n(u) = \mathrm{Dcr}_{f^{q_n}}(z_1, z_2, z_3).$$

By Proposition 5.1, we have:

(∗) for every $u \in]0, u_0[$, there exists an integer $m_{u,r} \in \mathbb{N}_c$, such that for every $n \in \mathbb{N}_c$, $n \geq m_{u,r}$,

$$|D_n(u) - 1| \leq \frac{2}{1 - u_0} u.$$

Proof of Theorem A. The function

$$\theta(\varepsilon) = \Pi(c)\left(e^{C\varepsilon}\left(1 + \frac{C_1}{\sigma_f(c)}\varepsilon\right) - 1\right)$$

is continuous at 0, then there exists $\varepsilon > 0$ such that $\theta(\varepsilon) < |\nu(c) - 1|/4$. Since by hypothesis $\Pi(c) \neq 1$, we can chose $0 < \varepsilon < |\Pi(c) - 1|/4A$. By Proposition 5.3, we have:

(∗∗) there exists $\eta \in]0, u_0[$ such that for every $u \in]0, \eta[$, there exists $n_{u,r} \geq m_{u,r}$, such that for any $n \in \mathbb{N}_c$, $n \geq n_{u,r}$,

$$|D_n(u) - \Pi(c)| \leq \frac{|\Pi(c) - 1|}{4}.$$

Set

$$u_\varepsilon := \min\left(\eta, \frac{1}{8}|\Pi(c) - 1|(1 - u_0), u_0\right).$$

By the property (∗), we conclude that for every $u \in]0, u_\varepsilon[$, there exists $n_{u,r} \in \mathbb{N}_c$ such that for any $n \in \mathbb{N}_c$, $n \geq n_{u,r}$, one has

$$|D_n(u) - 1| \leq \frac{|\Pi(c) - 1|}{4}.$$

It follows by the property (∗∗) that for any $u \in]0, u_\varepsilon[$, there exists $n_{u,r} \in \mathbb{N}_c$ such that for any $n \in \mathbb{N}_c$ and $n \geq n_{u,r}$,

$$|\Pi(c) - 1| \leq |D_n(u) - \Pi(c)| + |D_n(u) - 1| \leq \frac{|\Pi(c) - 1|}{4} + \frac{|\Pi(c) - 1|}{4},$$

so

$$|\Pi(c) - 1| \leq \frac{|\Pi(c) - 1|}{2},$$

a contradiction. □

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Proof of Theorem B. Proof of (*b*). By Corollary 2.2, f is conjugate to a C^1-diffeomorphism $F = K \circ f \circ K^{-1}$ through a piecewise polynomial class P-homeomorphism K of S^1. Moreover, DF is absolutely continuous on S^1 and $D^2F \in L^p(S^1)$ for some $p > 1$. By Katznelson–Ornstein's theorem, μ_F is equivalent to the total Haar measure m and so is μ_f.

Proof of (*a*). Following the proof of Theorem A and replacing $\Pi(c)$ by $\Pi(c, d)$, we obtain $|\Pi(c, d) - 1| \leq |\Pi(c, d) - 1|/2$; this implies that $\Pi(c, d) = 1$. Let $(i, k) \in \{0, \ldots, p\}^2$, $i \neq k$. For $c = c_i$ and $d = c_k$, one has $\Pi(c_i, c_k) = 1$. Therefore

$$\sigma_f(c_i) \prod_{j=0, j\neq i,k}^{p} \sigma_f(c_j)^{e(i,k,j)} = 1 \quad \text{where } e(i, k, j) \in \{0, 1\}.$$

Set for $(i, k, j) \in \{0, \ldots, p\}^3$,

$$\varepsilon_{k+i(p+1),j} = \begin{cases} e(i, k, j) & \text{if } k \neq i,\ j \neq i \text{ and } k \neq j, \\ 1 & \text{if } k \neq i \text{ and } j = i, \\ 0 & \text{if } k = j \text{ and } k \neq i, \\ 0 & \text{if } k = i. \end{cases}$$

It follows that for every $(i, k) \in \{0, \ldots, p\}^2$, we have

$$(*) \qquad \sum_{j=0}^{p} \varepsilon_{k+i(p+1),j} \log(\sigma_f(c_j)) = 0.$$

Set the matrix

$$A_{p+1}(f) = (\varepsilon_{l,j})_{0\leq l<(p+1)^2, 0\leq j\leq p} \in \mathcal{M}_{(p+1)^2, p+1}(\{0, 1\}).$$

Then $A_{p+1}(f)$ has the following obvious properties.

- For every $(i, j) \in \{0, \ldots, p\}^2$, $\varepsilon_{i+i(p+1),j} = 0$ and $\varepsilon_{j+i(p+1),j} = 0$.
- For every $(i, k) \in \{0, \ldots, p\}^2$ with $k \neq i$, $\varepsilon_{k+i(p+1),i} = 1$.

Moreover, we have the following lemma.

LEMMA 6.1. *We have* $\operatorname{rank}(A_{p+1}(f)) = p + 1$.

The proof of this lemma will be given in the Appendix.

We complete the proof of (a) as follows. Since by hypothesis, f does not have the (D)-property, so there exists a break point c_i which is the unique break point in its orbit, $\sigma_f(c_i) \neq 1$. Now using Lemma 6.1, we conclude by (*) that for every $j = 0, 1, \ldots, p$, $\log(\sigma_f(c_j)) = 0$. So, f has no break points, which is a contradiction. The proof of Theorem B is complete. □

Proof of Corollary 1.3. The proof follows (as a particular case) from Theorem A. □

Proof of Corollary 1.4. The proof follows from Theorem A for the case (i) and from Theorem B for the case (ii). □

Proof of Corollary 1.5. The proof follows easily from Theorem B. □

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A. *Appendix. The rank of $A_{p+1}(f)$*

Proof of Lemma 6.1. We proceed by induction on p.

For $p = 1$, $A_2(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and it is plain that $\operatorname{rank}(A_2(f)) = 2$.

For $p = 2$, write $A_3(f) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \varepsilon_{1,2} \\ 1 & \varepsilon_{2,1} & 0 \\ 0 & 1 & \varepsilon_{3,2} \\ 0 & 0 & 0 \\ \varepsilon_{5,0} & 1 & 0 \\ 0 & \varepsilon_{6,1} & 1 \\ \varepsilon_{7,0} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ where $\varepsilon_{i,j} \in \{0, 1\}$.

The 3×3 submatrix $\begin{pmatrix} 1 & 0 & \varepsilon_{1,2} \\ \varepsilon_{5,0} & 1 & 0 \\ 0 & \varepsilon_{6,1} & 1 \end{pmatrix}$ of $A_3(f)$ has determinant $1 + \varepsilon_{1,2}\varepsilon_{5,0}\varepsilon_{6,1} \geq 1$, hence $\operatorname{rank}(A_3(f)) = 3$. Assume that $\operatorname{rank}(A_p(f)) = p$, for $p \in \mathbb{N}^*$. Note that the $p^2 \times p$ submatrix of $A_{p+1}(f)$ obtained by removing the last column, the last $p + 1$ rows, and the rows numbers $p + 1, 2(p + 1), \ldots, p(p + 1)$ is $A_p(f)$, hence $\operatorname{rank}(A_{p+1}(f)) \in \{p, p + 1\}$. Suppose that $\operatorname{rank}(A_{p+1}(f)) = p$. Let $B_{p+1}(f)$ denote the $p(p + 1) \times (p + 1)$ submatrix of $A_{p+1}(f)$ obtained by removing the last $(p + 1)$ rows. As $A_p(f)$ is a submatrix of $B_{p+1}(f)$ then $\operatorname{rank}(B_{p+1}(f)) = p$. The Gauss–Jordan rows operation converts the matrix $B_{p+1}(f)$ to the $p(p + 1) \times (p + 1)$ matrix $J_{p+1}(f) = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$ where I_p is the identity $p \times p$ matrix. Let $E_{p+1}(f)$ be the matrix obtained by adding to $J_{p+1}(f)$ the last $(p + 1)$ rows of $A_{p+1}(f)$, then $\operatorname{rank}(E_{p+1}(f)) = p + 1$ and $\operatorname{rank}(A_{p+1}(f)) = \operatorname{rank}(E_{p+1}(f))$, so $p = p + 1$, a contradiction. □

Example. Consider the case $p = 3$.

$$A_4(f) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \varepsilon_{1,2} & \varepsilon_{1,3} \\ 1 & \varepsilon_{2,1} & 0 & \varepsilon_{2,3} \\ 1 & \varepsilon_{3,1} & \varepsilon_{3,2} & 0 \\ 0 & 1 & \varepsilon_{4,2} & \varepsilon_{4,3} \\ 0 & 0 & 0 & 0 \\ \varepsilon_{6,0} & 1 & 0 & \varepsilon_{6,3} \\ \varepsilon_{7,0} & 1 & \varepsilon_{7,2} & 0 \\ 0 & \varepsilon_{8,1} & 1 & \varepsilon_{8,3} \\ \varepsilon_{9,0} & 0 & 1 & \varepsilon_{9,3} \\ 0 & 0 & 0 & 0 \\ \varepsilon_{11,0} & \varepsilon_{11,1} & 1 & 0 \\ 0 & \varepsilon_{12,1} & \varepsilon_{12,2} & 1 \\ \varepsilon_{13,0} & 0 & \varepsilon_{13,2} & 1 \\ \varepsilon_{14,0} & \varepsilon_{14,1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{where } \varepsilon_{i,j} \in \{0, 1\}.$$

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One has

$$J_4(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_4(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{12,1} & \varepsilon_{12,2} & 1 \\ \varepsilon_{13,0} & 0 & \varepsilon_{13,2} & 1 \\ \varepsilon_{14,0} & \varepsilon_{14,1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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