COMPOSITIO MATHEMATICA

A counterexample to Lagrangian Poincaré recurrence in dimension four

Joel Schmitz

Compositio Math. **161** (2025), 2373–2379.

doi: 10.1112/S0010437X2510239X

FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
EST. 1865



https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

Compositio Math. 161 (2025) 2373–2379
doi:10.1112/S0010437X2510239X

A counterexample to Lagrangian Poincaré recurrence in dimension four

Joel Schmitz

Abstract

Counterexamples to Lagrangian Poincaré recurrence were recently found in dimensions greater than six by Brocić and Shelukhin. We construct counterexamples in dimension four using almost toric fibrations.

1. Introduction

Guided by the classical Poincaré recurrence for sets of positive measure and by the many Lagrangian intersection results in symplectic topology, Viterbo and Ginzburg around 2010 independently formulated the Lagrangian Poincaré recurrence (LPR) conjecture:

> For every symplectic manifold (M, ω) it holds true that for every closed Lagrangian submanifold $L \subset M$ and every compactly supported Hamiltonian diffeomorphism φ of M, there exists a sequence of integers $k_i \to \infty$ such that $\varphi^{k_i}(L) \bigcap L \neq \emptyset$.

We refer to [GG19] for a detailed discussion of this conjecture. In dimensions ≥ 6, counterexamples to LPR were recently constructed by Brocić and Shelukhin [BS24] for certain product tori in the ball, and hence in any symplectic manifold. They applied Chekanov's construction from [Che96]. In this note, we construct the first counterexamples in dimension four. Our ambient manifolds are 1-point blow ups of $S^2 \times S^2$, and the Lagrangians are tori. The Hamiltonian diffeomorphisms without recurrence for these tori will be obtained with the help of Symington's structure theorem for almost toric symplectic 4-manifolds. The construction is similar to [HK21, Section 6], in that it is given by a loop of nodal slides. Our arguments also work in all closed symplectic toric 4-manifolds except for a short explicit list. See Remarks 3.3 and 3.2 for a more detailed discussion. We also get counterexamples in dimensions greater than four, with quite a different character to the counterexamples in [BS24]. See Remark 3.5.

The counterexamples in [BS24] and here mean that LPR does not hold unconditionally. On the other hand, LPR was established in [GG18] for a certain class of Hamiltonian diffeomorphisms of $\mathbb{C}P^n$ called pseudo-rotations. This was generalized in [JS24]. Furthermore, the Lagrangian packing number of certain Lagrangian tori in non-monotone $S^2 \times S^2$ is finite, which in particular implies LPR, as shown by Polterovich and Shelukhin [PS23]. See also the previous work by Mak and Smith [MS21].

Received 19 November 2024, accepted in final form 3 March 2025, published online 3 November 2025.
2020 Mathematics Subject Classification 37J11 (primary), 53D12, 53D20 (secondary).
Keywords: Lagrangian Poincaré recurrence; dynamical systems; almost toric fibrations.
© The Author(s), 2025. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited. *Compositio Mathematica* is © Foundation Compositio Mathematica.

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

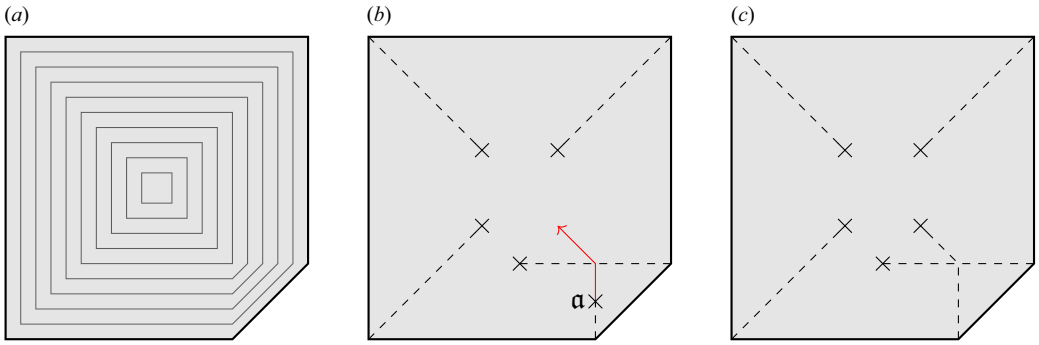


Figure 1 (colour online). Constructing the almost toric fibration π_0.

2. The construction

Let M be the one point blow up of $(S^2 \times S^2, \omega_{a,b})$ by weight c, where $\omega_{a,b}$ is the product symplectic form giving areas a and b to the factors with $a \geq b$ and $c < (b/2)$.

We will first construct a symplectomorphism φ and a Lagrangian L such that the Lagrangians $\{\varphi^n(L)\}_{n\in\mathbb{N}}$ are pairwise disjoint. Then, we conclude using Lemma 2.2 that φ (if $a \neq b$) or φ^2 (if $a = b$) must be a Hamiltonian symplectomorphism.

Let $\mu\colon M \to \Delta$ be the toric moment map where Δ is the Delzant polygon in Figure 1(a); Δ can be written as the non-negative locus of the function

$$\mathcal{F}_\Delta(x_1, x_2) = \min\left\{\frac{a}{2} - |x_1|\,, \frac{b}{2} - |x_2|\,, x_2 - x_1 + \frac{a+b}{2} - c\right\}.$$

The parameter $\mathcal{F}_\Delta$ measures the integral affine distance to the boundary of Δ. The level sets of $\mathcal{F}_\Delta$ are also depicted in Figure 1(a). By making a nodal trade [Sym03, Theorem 6.5] in each corner we obtain an almost toric fibration $\pi_0'\colon X \to B_0'$ whose base diagram is depicted in Figure 1(b). The solid broken line segment emanating from node $\mathfrak{a}$ in Figure 1(b) is a straight line segment in the integral affine structure on B_0' inherited from the almost toric fibration and is contained in the eigenline of node $\mathfrak{a}$. Modifying π_0' by a nodal slide [Sym03, Proposition 6.2] along this line segment, we obtain the almost toric fibration $\pi_0\colon M \to B_0$ depicted in Figure 1(c).

Note that $\max_{x\in\Delta}\{\mathcal{F}_\Delta(x)\} = (b/2)$. Define the $\mathbb{R}$-action $t\cdot$ on $\Delta \setminus \mathcal{F}_\Delta^{-1}(b/2)$ such that $t \cdot x$ is the translation of x by integral affine distance t along the level set $\mathcal{F}_\Delta^{-1}(\mathcal{F}_\Delta(x))$ in the counter-clockwise direction.

Given $\varepsilon \in (0, \min\{c, (b/2) - c\})$, we will construct $\varphi \in \mathrm{Symp}(M)$ such that φ acts on the fibres of π_0 as follows: let $h = \mathcal{F}_\Delta(x)$. If $h > c + \varepsilon$, then $\varphi(\pi_0^{-1}(x)) = \pi_0^{-1}(x)$. If $h < c - \varepsilon$, then $\varphi(\pi_0^{-1}(x)) = \pi_0^{-1}((c-h)\cdot x)$. The integral affine length of a level set $\mathcal{F}_\Delta^{-1}(h)$ with $0 \leq h < c$ is $2(a+b) - c - 7h$. So for any $h \in (0, c-\varepsilon)$ with $\frac{c-h}{2(a+b)-c-7h}$ irrational and $x \in \mathcal{F}_\Delta^{-1}(h)$, the orbit of $\pi_0^{-1}(x)$ under φ is dense in the level set $\mathcal{F}_\Delta^{-1}(h)$ and the Lagrangians $\{\varphi^n(\pi_0^{-1}(x))\}_{n\in\mathbb{N}}$ are pairwise disjoint.

2.1.1 *Constructing φ.* Note that the node $\mathfrak{n}$ in Figure 2(a) has $\mathcal{F}_\Delta(\mathfrak{n}) = c$. We modify the fibration π_0 by a sequence of four nodal slides, illustrated in Figure 2. First, slide the node $\mathfrak{n}$ along the marked path in Figure 2(a) to obtain an almost toric fibration with base diagram Figure 2(b). During the nodal slide, using [Eva23, Proof of Theorem 8.10], we may assume that we do not modify the fibration outside of $\mathcal{F}_\Delta^{-1}(c-\varepsilon, c+\varepsilon)$. Note that the branch cuts in the base diagram Figure 2(b) cut the base B_0 into three connected components. Removing the horizontal branch cut drawn with a thick dashed line in Figure 2(b), we get the base diagram Figure 2(c), which

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

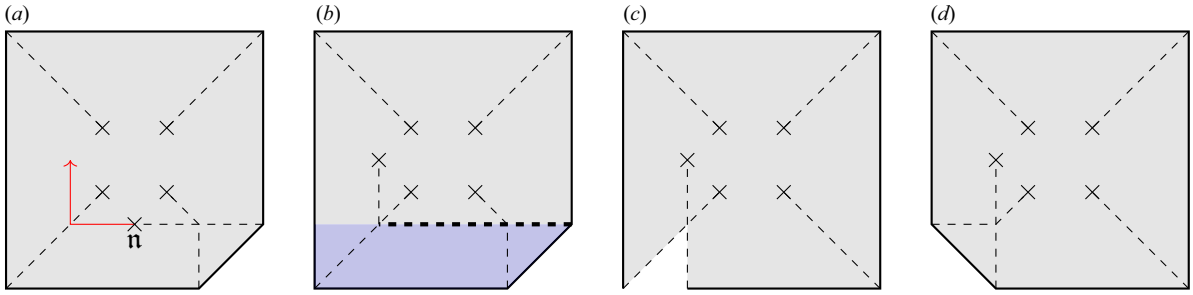


Figure 2 (colour online). Nodal slide and change of branch cut.

has only one connected component. This diagram has a discontinuity in the lower left corner, which can be removed by introducing a branch cut and applying a suitable shear map (given below), giving Figure 2(d). The effect of the two steps (b)→ (c)→ (d) is to apply the shear map

$$(x_1, x_2) \mapsto \left(x_1 + \left(c - \frac{b}{2} - x_2\right), x_2\right), \tag{2.1}$$

to points with $x_2 < c - (b/2)$ (the dark shaded region in Figure 2(b)), and the identity everywhere else, giving the base diagram Figure 2(d). Note that the diagram (d) is made identical to diagram (a) by a 90°-rotation. Repeat this process three more times as illustrated in Figure 3 until $\mathfrak{n}$ has returned to its original position. By this process we obtain an almost toric fibration $\pi_1 \colon M \to B_1$ such that, for $x \in \Delta$, if $h > c + \varepsilon$ the fibres $\pi_1^{-1}(x)$ and $\pi_0^{-1}(x)$ are equal and if $h < c - \varepsilon$, by combining the effect of the four shear maps (2.1), we have

$$\pi_1^{-1}(x) = \pi_0^{-1}((c-h)\cdot x). \tag{2.2}$$

Now [Sym03, Corollary 5.4] and [Eva23, Proof of Theorem 8.5] give a symplectomorphism $\varphi \in \mathrm{Symp}(M)$ such that, outside a small neighbourhood of the nodes,

$$\varphi(\pi_0^{-1}(x)) = \pi_1^{-1}(x)\,.$$

From (2.2) we conclude that, for $h = \mathcal{F}_\Delta(x) < c - \varepsilon$,

$$\varphi(\pi_0^{-1}(x)) = \pi_1^{-1}(x) = \pi_0^{-1}((c-h)\cdot x).$$

Remark 2.1. As pointed out by the referee, here is an equivalent manner of seeing the effect of the nodal slide in Figure 3: Figure 4(a) gives an alternative base diagram of the almost toric base B_0 pictured in Figure 1(c), which can also be obtained from $(S^2 \times S^2, \omega_{a,b})$ by a non-toric blow up as in [Eva23, Section 9.1]. Similarly to [Eva23, Section 10.1], we may replace the four branch cuts emanating from the nodes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ by the tetrapod branch cut T highlighted in Figure 4(a). Denote the branch cut emanating from the node $\mathfrak{n}$ by ℓ. Then, $B_0 \setminus (T \cup \ell)$ is homotopy equivalent to S^1, and a base diagram of its universal cover is given by Figure 4(b). The concatenation of all the nodal slides in Figure 3 can be written as one nodal slide marked in Figure 4(a,b) by the solid line emanating from $\mathfrak{n}$. Performing the nodal slide, we get the base diagram Figure 4(c) and the almost toric fibration $\pi_1 \colon M \to B_1$ as above. As described in [Eva23, Section 7.2], we can rotate the branch cut ℓ emanating from $\mathfrak{n}$ until it matches the branch cut in Figure 4(b) to obtain the base diagram of Figure 4(d) for the universal cover of $B_1 \setminus (T \cup \ell)$. The combined effect of steps (b)→ (c)→ (d) is to apply the shear matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ to points below the eigenline of $\mathfrak{n}$: the 'missing' triangle below $\mathfrak{n}$ is spanned by vectors in directions $(a, -1), (a+1, -1)$ for some value of $a \in \mathbb{R}$, so when a point x touches the sliding or rotating branch cut, in the base diagram the point jumps over the gap given by the missing triangle according the shear matrix A. As above we use [Eva23, Proof of Theorem 8.10] to ensure the nodal slide only modifies fibres

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

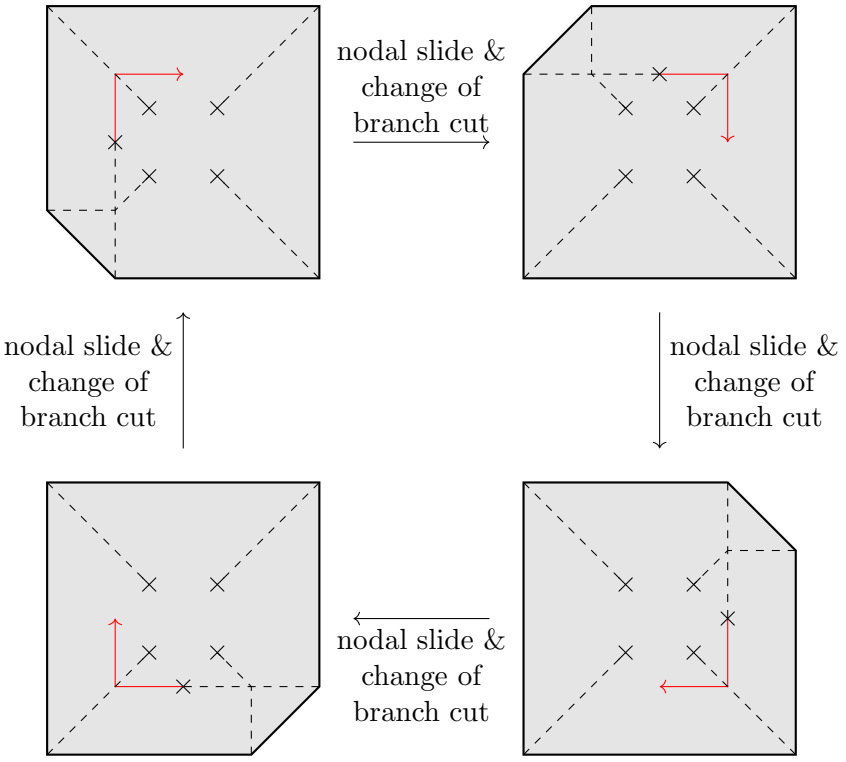


Figure 3 (colour online). Constructing the almost toric fibration π_1.

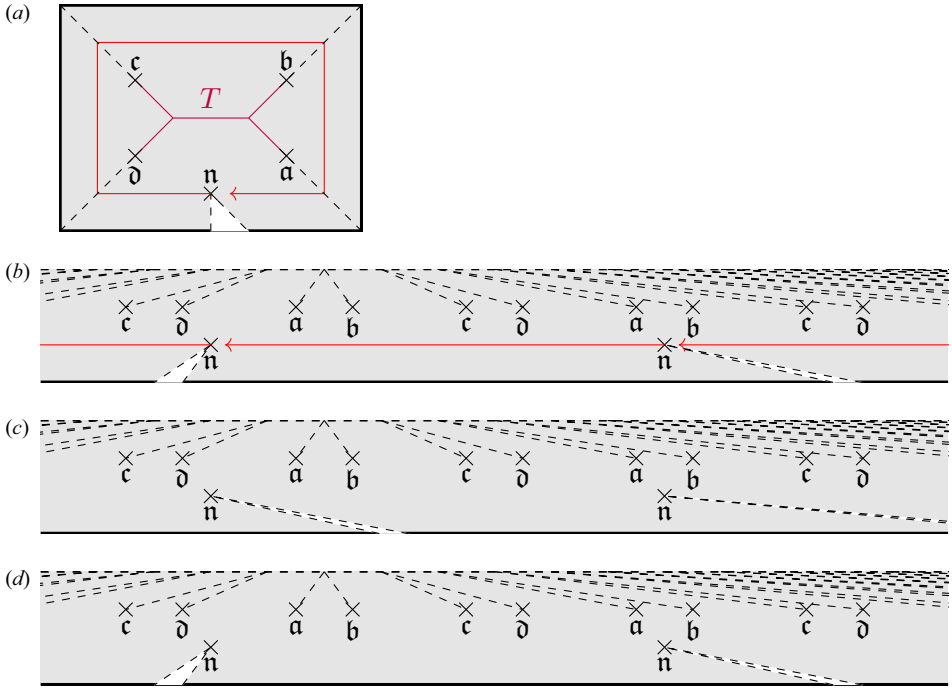


Figure 4 (colour online). An alternative picture of the nodal slides used in the construction of the symplectomorphism φ.

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

near the eigenline of $\mathfrak{n}$, and [Sym03, Corollary 5.4] and [Eva23, Proof of Theorem 8.5] to get the desired symplectomorphism.

We conclude the construction by showing that φ or φ^2 is a Hamiltonian diffeomorphism.

Lemma 2.2. *Let ψ be a symplectomorphism of M.*
(1) *If $a \neq b$, then ψ is a Hamiltonian diffeomorphism.*
(2) *If $a = b$, then ψ^2 is a Hamiltonian diffeomorphism.*

Proof. It is shown in [LLW15, Corollary 1.3] that the symplectic mapping class group of M is a finite reflection group generated by Dehn twists along embedded Lagrangian spheres, and that a generating set is given by the classes $\xi \in H_2(M;\mathbb{Z})$ which can be represented by smoothly embedded spheres and satisfy

$$\xi \cdot \xi = -2, \quad c_1(M)(\xi) = 0, \quad [\omega](\xi) = 0.$$

Already, the conditions $\xi \cdot \xi = -2$ and $c_1(M)(\xi) = 0$ imply that ξ is the anti-diagonal class $-D$, see e.g. [Eva10, Lemma 4.1.4]. Since the symplectic area of $-D$ vanishes only for $a = b$, and since in this case the Dehn twist along the anti-diagonal has order of at most 2, the symplectic mapping class group is trivial in case $a \neq b$ and has at most two elements in case $a = b$. Finally, note that the symplectomorphisms in the component of the identity are Hamiltonian, since the first homology of M vanishes. □

Alternatively [LLW22, Corollary 4.6] yields a shorter but slightly weaker argument. See Remark 3.3.

3. Discussion

Remark 3.1. The set of $h \in (0, c-\varepsilon)$ such that, for $x \in \mathcal{F}_\Delta^{-1}(h)$, the Lagrangians $\{\varphi^n(\pi_0^{-1}(x))\}_{n\in\mathbb{N}}$ are pairwise disjoint is dense and has full measure in $(0, c-\varepsilon)$.

Remark 3.2. The construction in § 2 of a symplectomorphism φ can be generalized to more toric symplectic 4-manifolds: Let Δ be a compact Delzant polygon and $\mu\colon X \to \Delta$ the toric symplectic manifold corresponding to Δ. Denote by $\mathcal{F}_\Delta\colon \Delta \to \mathbb{R}$ the integral affine distance to the boundary of Δ. Then we can repeat the construction in § 2 if Δ has an edge of length $c < \max_{x\in\Delta}\{\mathcal{F}_\Delta(x)\}$ corresponding to a symplectic sphere of self-intersection -1. By [MS23], these are all closed symplectic toric 4-manifolds except:

(1) the five monotone symplectic toric 4-manifolds;
(2) $(S^2 \times S^2, \omega_{a,b})$ with $a \neq b$;
(3) the one- or two-fold blow up of $(S^2 \times S^2, \omega_{a,b})$ of size $c = \frac{\min\{a,b\}}{2}$.

Remark 3.3. Let (M,ω) be a closed toric symplectic 4-manifold. By [LLW22, Corollary 4.6], the symplectic mapping class group of (M,ω) is finite.

Thus the generalized construction in Remark 3.2 yields a symplectomorphism φ such that, for many fibre tori L, the tori $\{\varphi^n(L)\}_{n\in\mathbb{N}}$ are pairwise disjoint. Since the mapping class group is finite, $\psi = \varphi^k$ is Hamiltonian for some $k \geq 1$ and the tori $\{\psi^n(L)\}_{n\in\mathbb{N}}$ are pairwise disjoint. In particular, LPR does not hold for these toric symplectic 4-manifolds.

Remark 3.4. While our construction does not yield a result for $S^2 \times S^2$, in [BK25] Brendel and Kim show that most toric fibres of non-monotone $S^2 \times S^2$ have infinite Lagrangian packing number.

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

Remark 3.5. Taking the product of our counterexamples (M, L, φ) to LPR with any pair (M', L') and the identity of M', we obtain many counterexamples to LPR in dimensions ≥ 6. By our construction, every Lagrangian $\varphi^k(L) \times L'$ is an accumulation point of the orbit $\varphi^n(L) \times L'$. These counterexamples are therefore very different from the ones in [BS24], where the sequence of tori $\phi^n(L)$ converges to one Lagrangian torus.

Using Delzant's construction [Del88], we can obtain any closed symplectic toric manifold by symplectic reduction from $\mathbb{C}^N$, where N is the number of edges of the moment polytope. Lifting the Hamiltonians constructed above, we also get counterexamples to LPR in symplectic balls in dimensions ≥ 8 (e.g. a non-monotone one point blow-up of $\mathbb{C}P^2$ can be obtained from $\mathbb{C}^4$ by symplectic reduction), and thus any symplectic manifold of dimensions ≥ 8.

Acknowledgement

We thank Felix Schlenk for fruitful inputs and discussions and Joé Brendel for his critical remarks.

Conflicts of interest

None.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

BK25 J. Brendel and J. Kim, *Lagrangian split tori in $S^2 \times S^2$ and billiards*, Preprint (2025), arXiv:2502.03324 [math.SG].

BS24 F. Broi and E. Shelukhin, *A counterexample to Lagrangian Poincaré recurrence*, Preprint (2024), arXiv:2409.14225.

Che96 Y. V. Chekanov, *Lagrangian tori in a symplectic vector space and global symplectomorphisms*, Math. Z. **223** (1996), 547–559.

Del88 T. Delzant, *Hamiltoniens priodiques et images convexes de l'application moment. [Periodic Hamiltonians and convex images of moment mappings]*, Bull. Soc. Math. Fr. **116** (1988), 315–339.

Eva10 J. Evans, *Symplectic topology of some Stein and rational surfaces*, PhD thesis, Christs College (2010).

Eva23 J. Evans, *Lectures on Lagrangian torus fibrations*, London Mathematical Society Student Texts, vol. 105 (Cambridge University Press, Cambridge, 2023).

GG18 V. L. Ginzburg and B. Z. Grel, *Hamiltonian pseudo-rotations of projective spaces*, Invent. Math. **214** (2018), 1081–1130.

GG19 V. L. Ginzburg and B. Z. Grel, *Approximate identities and Lagrangian Poincaré recurrence*, Arnold Math. J. **5** (2019), 5–14.

HK21 P. Hacking and A. Keating, *Symplectomorphisms of some Weinstein 4-manifolds*, Preprint (2021), arXiv:2112.06797 [math.SG].

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press

JS24 D. Joksimovi and S. Seyfaddini, *A Hlder-type inequality for the C^0 distance and Anosov-Katok pseudo-rotations*, Int. Math. Res. Not. IMRN **2024** (2024), 6303–6324.

LLW15 J. Li, T.-J. Li and W. Wu, *The symplectic mapping class group of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with $n \leq 4$*, Mich. Math. J. **64** (2015), 319–333.

LLW22 J. Li, T.-J. Li and W. Wu, *Symplectic Torelli groups of rational surfaces*, Preprint (2022), arXiv:2212.01873 [math.SG].

MS21 C. Y. Mak and I. Smith, *Non-displaceable Lagrangian links in four-manifolds*, Geom. Funct. Anal. **31** (2021), 438–481.

MS23 G. Mikhalkin and M. Shkolnikov, *Wave fronts and caustics in the tropical plane*, in *Proceedings of 28th Gökova Geometry-Topology Conference (2023)* (International Press, 2023), 11–48.

PS23 L. Polterovich and E. Shelukhin, *Lagrangian configurations and Hamiltonian maps*, Compos. Math. **159** (2023), 2483–2520.

Sym03 M. Symington, *Four dimensions from two in symplectic topology*, in *Topology and geometry of manifolds, Proceedings of the 2001 Georgia topology conference, University of Georgia, Athens, GA, USA, May 21–June 2, 2001* (American Mathematical Society, Providence, RI, 2003), 153–208.

Joel Schmitz joel.schmitz@unine.ch
Institut de mathématiques, Rue Emile-Argand 11, 2000 Neuchâtel, Switzerland

https://doi.org/10.1112/S0010437X2510239X Published online by Cambridge University Press