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Mixed Ramsey Numbers Revisited

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Given a graph H with no isolates, the (generalized) mixed Ramsey number $r(H,\overline{I}_m)$ is the smallest integer r such that every H-free graph of order r contains an m-element irredundant set. We consider some questions concerning the asymptotic behaviour of this number (i) with H fixed and $m \to \infty$, (ii) with m fixed and (H_n) a sequence of dense graphs, in particular for the sequence (K_n), $n \to \infty$. Open problems are mentioned throughout the paper.

1. Definitions

Let G be an ordinary graph. A set of vertices $S \subseteq V(G)$ is *irredundant* in G if each vertex $x \in S$ is either isolated in $\langle S \rangle_G$ or else has a *private neighbour* $y \in V \backslash S$ that is adjacent to x alone among the vertices of S. The *irredundance number* $\mathrm{IR}(G)$ is the largest cardinality among the irredundant sets in G. Every independent set is irredundant, so $IR(G) \geqslant \alpha(G)$. Irredundant sets play a role in the theory of domination. For example, the minimal dominating sets in a graph are those that are both dominating and irredundant. See [13, pp. 71–75].

For notation not explicitly defined in this paper, see [2]. Let H and K be graphs without isolates. The Ramsey number $r(H,K)$ is the smallest integer r such that, for every graph G with r vertices, either $H \subset G$ or $K \subset \overline{G}$. For the case $K = K_m$ (the complete graph of order m) the defining dichotomy becomes $H \subset G$ or $\alpha(G) \geqslant m$. Replacing K by a finite class of graphs $\mathscr{G} = \{G_1, G_2, \ldots, G_k\}$, we obtain the *class Ramsey number* $r(H,\mathscr{G})$. In the defining statement, replace the condition $K \subset \overline{G}$ by $G_i \subset \overline{G}$ for at least one $i \in [k]$. To define the (generalized) *mixed Ramsey number* $r(H,\overline{I}_m) = r(\overline{I}_m, H)$, replace 'independent' by 'irredundant' in the definition of $r(H,K_m)$. It is evident that such a number exists and that it satisfies

$$r(H,\overline{I}_m) \leqslant r(H,K_m).$$

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We view $r(H,\overline{I}_m)$ as a class Ramsey number, with $\overline{I}_m$ standing for the collection of complements of (induced) subgraphs spanned by the irredundant sets and their corresponding private neighbours. Specifically, if G contains an m-element irredundant set, then $\overline{G}$ contains one or more graphs from the class $\{G_1, G_2, \ldots, G_{m-1}\}$ where $G_1 = K_m$, $G_{m-1} = K(m,m) - mK_2$, and $G_k = K_{m-k-1} + (K(k+1,k+1) - (k+1)K_2)$ for $2 \leqslant k \leqslant m-2$. Conversely, if $\overline{G}$ contains either K_m or $K(m,m) - mK_2$ (as an induced bipartite graph), then G contains an irredundant set. Thus, in all cases, $\overline{I}_m$ includes K_m, corresponding to the case where the irredundant m-element subset is an independent set, and $K_{m,m} - mK_2$ (induced), corresponding to the case where every vertex of the irredundant set has a private neighbour. For $m = 3$, the corresponding class is

$$\overline{I}_3 = \{K_3, \langle C_6 \rangle\},$$

where $\langle C_6 \rangle$ denotes an induced C_6 [12]. The mixed Ramsey number $t(m,n) = r(\overline{I}_m, K_n)$ was introduced in [7] and subsequently studied in [5] and elsewhere. Also, the *irredundant Ramsey number* $s(m,n) = r(\overline{I}_m, \overline{I}_n)$ has been studied extensively. See [17] for a survey of results on mixed and irredundant Ramsey numbers. In this paper, we return to some questions raised in [5] and study other problems related to $r(H,\overline{I}_m)$.

2. A simple observation

A connected graph H of order n is *m-good* if $r(H,K_m) = (m-1)(n-1)+1$. See [11] for a survey. It is easy to see that if H is m-good then $r(H,\overline{I}_m) = (m-1)(n-1)+1$. Indeed, since $r(H,\overline{I}_m) \leqslant r(H,K_m) = (m-1)(n-1)+1$, it is enough to verify that $r(H,\overline{I}_m) \geqslant (m-1)(n-1)+1$. Let $G = (m-1)K_{n-1}$. Then $H \not\subset G$, since no component of G has n or more vertices. Also G has no m-element irredundant set. Just note that any set of m vertices must contain two, say x and y, from the same component of G. Since x and y are adjacent, they must have private neighbours. But any vertex in the same component is adjacent to both x and y, and any vertex in another component is adjacent to neither x nor y.

3. Asymptotic bounds involving fixed H and large irredundant sets

For the case where H is a graph with $m \geqslant 3$ vertices and $q \geqslant m$ edges and n is large, the known bounds for $r(H,\overline{I}_n)$ are basically the same as for $r(H,K_n)$. To simplify the notation in the following theorem, let $K(\frac{n}{2},\frac{n}{2})$ denote the balanced bipartite graph of order n, with the understanding that this is $K(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ when n is odd.

Theorem 3.1. *If H is a fixed graph with $m \geqslant 3$ vertices and $q \geqslant m$ edges, then*

$$c\left(\frac{n}{\log n}\right)^{\frac{q-1}{m-2}} \leqslant r\left(H, K\left(\tfrac{n}{2},\tfrac{n}{2}\right)\right) \leqslant r(H,\overline{I}_n) \leqslant r(H,K_n).$$

Proof. With $N = r(H,\overline{I}_n)$, let G be a graph of order N such that $H \not\subset G$. Then G contains an n-element irredundant set S. We then claim that $\overline{G}$ contains $K(\frac{n}{2},\frac{n}{2})$. There are two cases.

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(i) If the non-isolated vertices of S can be covered by a set A of $\lceil n/2 \rceil$ vertices, then A and $S \backslash A$ are the colour classes of the desired complete bipartite graph in $\overline{G}$.

(ii) Otherwise, take a set A consisting of $\lceil n/2 \rceil$ non-isolated vertices, and let P denote the corresponding set of private neighbours. Then $S \backslash A$ and P are the colour classes of the desired bipartite graph.

The proof that

$$r\left(H, K\left(\tfrac{n}{2}, \tfrac{n}{2}\right)\right) \geqslant c\left(\frac{n}{\log n}\right)^{\frac{q-1}{m-2}}$$

uses the probabilistic method. Note: The proof is very close to that of a similar result for $r(H, K_n)$ given in [10]. We use the Lovász Local Lemma [9], as implemented by Spencer in [19]. (See [14] for an alternative technique.) Let $C_1, C_2, \ldots, C_k$ be events in a probability space $(\Omega, \mathscr{F}, P)$, and suppose that there exists corresponding positive numbers $x_1, x_2, \ldots, x_k$ such that $x_i P(C_i) < 1$ and

$$\log x_i > \sum_{\text{dep}} x_j P(C_j), \qquad i = 1, 2, \ldots, k,$$

where the sum is taken over all $j \neq i$ such that C_i and C_j are dependent. Then

$$P\left(\bigcap_{i=1}^{k} \overline{C_i}\right) > 0.$$

To apply this result to the Ramsey problem at hand, consider a random graph $G_{N,p}$. We distinguish two kinds of events. First, for $A \subset [N]$ and $|A| = m$, let

$$A_S = \{G |\ H \subseteq \langle S \rangle_G\}.$$

Secondly, for $B \subset [N]$ and $|B| = n$, let

$$B_T = \left\{G \middle|\ K\left(\tfrac{n}{2}, \tfrac{n}{2}\right) \subseteq \langle T \rangle_{\overline{G}}\right\}.$$

Each A_S has the same probability, which we denote by $P(A)$. Similarly, the probability of each B_T is denoted by $P(B)$. In view of the Local Lemma, it suffices to show that there are positive numbers a and b such that $aP(A) < 1$, $bP(b) < 1$, and the pair of inequalities

$$\log a > aN_{AA}P(A) + bN_{AB}P(B), \tag{3.1}$$

$$\log b > aN_{BA}P(A) + bN_{BB}P(B) \tag{3.2}$$

hold, where, for example, N_{AB} denotes (for fixed S) the number of T such that the events A_S and B_T are dependent. Clearly A_S and B_T are dependent only if S and T have at least two vertices in common. We have the following simple bounds for the quantities $P(A), P(B), N_{AA}, N_{AB}, N_{BA}, N_{BB}$:

$$P(A) \leqslant m!\, p^q,$$

$$P(B) \leqslant \binom{n}{\lfloor n/2 \rfloor}(1-p)^{(n^2-1)/4} < 2^n e^{-n^2 p/4},$$

$$N_{AA} \leqslant \binom{m}{2}\binom{N-2}{m-2} = O(N^{m-2}),$$

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$$N_{AB}, N_{BB} \leqslant \binom{N}{n} < \frac{N^n}{n!} < \left(\frac{N}{2}\right)^n,$$

$$N_{BA}, N_{BB} \leqslant \binom{n}{2}\binom{N-2}{m-2} < \frac{n^2 N^{m-2}}{2(m-2)!}.$$

With $s = (m-2)/(q-1)$, set

$$\begin{aligned} p &= C_1 N^{-s}, \\ n &= C_2 N^s \log N, \\ a &= C_3 > 1, \\ b &= \exp(C_4 N^s (\log N)^2), \end{aligned}$$

where C_1 to C_4 are positive constants to be chosen. Then $\log a > 0$,

$$aN_{AA}P(A) = O(N^{m-2}N^{-sq}) = O(N^{-s}) = o(1)$$

and

$$bN_{AB}P(B) < bN^n e^{-n^2 p/4} < \exp\left(N^s (\log N)^2 \left(C_4 + C_2 - C_1 C_2^2/4\right)\right).$$

It follows that (3.1) holds provided $C_1 C_2^2 > 4(C_2 + C_4)$. With such a choice, $bN_{BB}P(B) = o(1)$, so substitution of $b = \exp(C_4 N^s (\log N)^2)$ and the appropriate bounds for N_{BA} and $P(A)$ shows that (3.2) holds provided $C_4 > \binom{m}{2} C_3 C_2^2 C_1^q$. To see that there are constants C_1 to C_4 that satisfy the required inequalities, let $C_3 = 1.1$ and then pick C_1 so that $5\binom{m}{2} C_3 C_1^{q-1} = 1$. Then choose C_2 sufficiently large that $C_1 C_2 > 20$. With these choices, $5\binom{m}{2} C_3 C_2^2 C_1^q = C_1 C_2^2$ and $5C_1 C_2^2 - 20C_2 > 4C_1 C_2^2$. Finally, choose C_4 so that

$$5C_1 C_2^2 - 20C_2 > 20C_4 > 4C_1 C_2^2.$$

Then we have both $C_1 C_2^2 > 4(C_2 + C_4)$ and $C_4 > \frac{1}{5} C_1 C_2^2 = \binom{m}{2} C_3 C_2^2 C_1^q$. It is easy to see that the conditions $aP(A) < 1$ and $bP(B) < 1$ are satisfied. Thus, for sufficiently large N, there exists a graph G such that $H \not\subset G$ and $K(\frac{n}{2}, \frac{n}{2}) \not\subset \overline{G}$, where $n = C_2 N^s \log N$. Solving for N in terms of n, we obtain the stated result. □

Corollary 3.2. *For some positive constant c,*

$$c\left(\frac{n}{\log n}\right)^2 < r(K_3, \overline{I}_n) < (1 + o(1))\frac{n^2}{\log n}, \qquad n \to \infty.$$ □

Problem 3.3. *Determine (up to a constant factor) the asymptotic behaviour of $r(K_3, \overline{I}_n)$.*

At this point, we do not have an example H where it is known that $r(H, \overline{I}_n) = o(r(H, K_n))$, nor do we know that $r(H, \overline{I}_n) = \Theta(r(H, K_n))$ holds in general.

Problem 3.4. *For which graphs H, if any, is it true that*

$$\lim_{n\to\infty} \frac{r(H, \overline{I}_n)}{r(H, K_n)} = 0?$$

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4. Asymptotic bounds involving irredundant sets of fixed size and large independent sets

The bound

$$r(\overline{I}_3, K_n) \leqslant \sqrt{\frac{5}{2}}\, n^{3/2} \tag{4.1}$$

is proved in [5]. Thus

$$\lim_{n\to\infty} \frac{r(\overline{I}_3, K_n)}{r(K_3, K_n)} = 0.$$

The following problem is thus suggested.

Problem 4.1. *Is it true that, for every* $m \geqslant 3$,

$$\lim_{n\to\infty} \frac{r(\overline{I}_m, K_n)}{r(K_m, K_n)} = 0?$$

So far, even the case of $m = 4$ is open. In this section, we prove the following modest improvement of (4.1):

$$r(\overline{I}_3, K_n) \leqslant \frac{5\, n^{3/2}}{\sqrt{\log n}}. \tag{4.2}$$

In view of the characterization of $\overline{I}_3$ mentioned earlier, we may translate this result as

$$r(K_3 \text{ or } \langle C_6\rangle, K_n) \leqslant \frac{5\, n^{3/2}}{\sqrt{\log n}}.$$

The proof uses two key results, one due to Hattingh and the other due to Shearer. Let $u \in V(G)$. A vertex w is a *second neighbour* of u if $uw \notin E(G)$ but $uv \in E(G)$ and $vw \in E(G)$ for some $v \in V(G)$. The following result is a rewording of one found in [12].

Lemma 4.2. (Hattingh) *Suppose* G *contains neither a* K_3 *nor an induced* C_6. *With* $v \in V(G)$ *arbitrary, let* X *denote the set of second neighbours of* v. *Then* $\langle X\rangle_G$ *is bipartite.*

Actually, Hattingh proved a slightly stronger result, but the version quoted is sufficient for obtaining our asymptotic bound. The following result appeared initially in [18]. It improves and provides a simpler proof of an earlier result of Ajtai, Komlós and Szemerédi [1].

Lemma 4.3. (Shearer) *Let*

$$f(x) = \frac{x\log x - (x-1)}{(x-1)^2}, \qquad f(1) = \tfrac{1}{2}.$$

If G *is any triangle-free graph of order* N *and average degree* $\overline{d}$, *then* $\alpha(G) \geqslant Nf(\overline{d})$.

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Theorem 4.4. *For all* $n \geqslant 2$,

$$r(\overline{I}_3, K_n) \leqslant \frac{5\, n^{3/2}}{\sqrt{\log n}}.$$

Proof. The constant 5 has been chosen for convenience in checking the required inequalities. There is no doubt that numerical calculation would yield an improvement. In one of the two cases, the proof will be by induction. From [5] we know that $r(\overline{I}_3, K_n) \leqslant \sqrt{5/2}\, n^{3/2}$. Hence the bound to be proved holds for all $n \leqslant e^{10}$. In this way, we can restrict our attention to 'suitably large' values of n from the beginning. Let G be a graph of order $N \geqslant 5n^{3/2}/\sqrt{\log n}$, and let $\overline{d}$ denote its average degree. There are two cases.

Case (i): $\overline{d} < 2\sqrt{n \log n}$.
Assume that G contains no K_3. By Lemma 2, it suffices to show that $Nf(\overline{d}) > n$. We use the fact that f is decreasing on $(0, \infty)$. In view of this fact, it suffices to show that

$$5\sqrt{\frac{n}{\log n}}\, f(2\sqrt{n \log n}) > 1.$$

Since $n > e^{10}$, we have $2\sqrt{n \log n} > e^5$. A simple calculation shows that

$$f(x) > \frac{4}{5}\,\frac{\log x}{x} \quad \text{for all} \quad x > e^5.$$

(The stated inequality is equivalent to $(5x^2 - 4(x-1)^2)\log x > 5x(x-1)$, which is clearly true for $x > e^5$.) Thus, for all $n > e^{10}$,

$$5\sqrt{\frac{n}{\log n}}\, f(2\sqrt{n \log n}) > 5\sqrt{\frac{n}{\log n}}\,\frac{4}{5}\,\frac{\frac{1}{2}\log n}{2\sqrt{n \log n}} = 1,$$

as required.

Case (ii): $\overline{d} \geqslant 2\sqrt{n \log n}$.
Let v be any vertex with degree $d(v) \geqslant 2\sqrt{n \log n}$. Assume that G contains neither a K_3 nor an induced C_6 and that $\alpha(G) < n$. It follows that $d(v) \leqslant n - 1$. By Lemma 1, the subgraph spanned by the second neighbours of v is bipartite. Hence the set of second neighbours has at most $2n - 2$ vertices. Thus we are left to consider what happens if v together with its neighbours and second neighbours comprise fewer than $3n$ vertices. Let W denote the remaining vertices of G after v, its neighbours and second neighbours have been deleted, and let $H = \langle W \rangle_G$. Note that $\alpha(G) \geqslant \alpha(H) + d(v) \geqslant \alpha(H) + 2\sqrt{n \log n}$, since the neighbourhood of v is an independent set and no vertex in W is adjacent to a vertex of $\Gamma(v)$. Hence $N < 3n + r(\overline{I}_3, K_{n-k})$, where $k = k(n) = 2\sqrt{n \log n}$. By induction,

$$r(\overline{I}_3, K_{n-k}) \leqslant \frac{5(n-k)^{3/2}}{\sqrt{\log(n-k)}}.$$

Thus, to obtain the desired contradiction, it suffices to verify that

$$\frac{5(n-k)^{3/2}}{\sqrt{\log(n-k)}} + 3n < \frac{5n^{3/2}}{\sqrt{\log n}}.$$

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This inequality can be written as

$$\sqrt{\frac{n}{\log n}}\left(1-\frac{(1-k/n)^{3/2}}{\sqrt{\log(n-k)/\log n}}\right)>\frac{3}{5}.$$

Let $x=k/n=2\sqrt{(\log n)/n}$. Then the desired result becomes

$$\left(1-\frac{3x}{10}\right)\sqrt{1+\frac{\log(1-x)}{\log n}}>(1-x)^{3/2}. \tag{4.3}$$

For $0<u<\frac{1}{2}$,

$$\log(1-u)=-u-\frac{u^2}{2}-\frac{u^3}{3}-\cdots>-\frac{u}{1-u}>-2u.$$

Since $n>e^{10}$, we have $\log n>10$ and $x<2\sqrt{10/e^{10}}<1/20$, so

$$\sqrt{1+\frac{\log(1-x)}{\log n}}>1+\frac{\log(1-x)}{\log n}\geqslant 1+\frac{\log(1-x)}{10}>1-\frac{x}{5}.$$

Hence the left-hand side of (4.3) exceeds

$$\left(1-\frac{3x}{10}\right)\left(1-\frac{x}{5}\right)>1-\frac{x}{2}>1-x>(1-x)^{3/2},$$

and the truth of (4.3) is evident. Since cases (i) and (ii) are exhaustive, the proof is complete. □

The lower bound

$$r(\overline{I}_m,K_n)>c_m\left(\frac{n}{\log n}\right)^{\frac{m^2-m-1}{2(m-1)}} \tag{4.4}$$

is proved in [5]. At this time, we have no lower bound for $r(\overline{I}_3,K_n)$ that is any stronger than that given by the $m=3$ case of (4.4). Thus there is much room for improvement in our knowledge of the asymptotic behaviour of $r(\overline{I}_3,K_n)$.

Problem 4.5. *Improve the upper and/or lower bound in*

$$c\left(\frac{n}{\log n}\right)^{5/4}<r(\overline{I}_3,n)\leqslant\frac{5n^{3/2}}{\sqrt{\log n}}.$$

Note. The same lower bound as in (4.4) has been given for $s(m,n)$, independently by Erdős and Hattingh in [8] and by Krivelevich in [15]. From Theorem 4.4 we easily obtain

$$r(\overline{I}_4,K_n)\leqslant C\frac{n^{5/2}}{\sqrt{\log n}},$$

and at present, we know that

$$c_1\left(\frac{n}{\log n}\right)^{5/2}<r(K_4,K_n)<(1+o(1))\frac{n^3}{(\log n)^2}.$$

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The lower bound is that of Spencer [19] and has not been improved in 25 years. The upper bound is due to Li, Rousseau and Zang [16]. This merely improves the constant factor in a result of Ajtai, Komlós and Szemerédi [1]. Erdős asked whether or not $r(K_4, K_n) > n^3/(\log n)^c$ for some c. (See [6].) If so, then $r(\overline{I}_4, K_n)/r(K_4, K_n) \to 0$, but this seems well beyond our reach.

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