AN INTEGRAL EQUATION FOR WHITTAKER'S CONFLUENT HYPERGEOMETRIC FUNCTION

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1. An integral equation

$$\phi(z) = \frac{\Gamma(1-2k)}{\Gamma(\frac{1}{2}-k-m)\,\Gamma(\frac{1}{2}-k+m)}\int_0^\infty e^{-\frac{1}{2}(t+z)}(t+z)^{2k-1}(tz)^{-k}\phi(t)\,dt, \tag{1·1}$$

for the function $\phi(z) = z^{-1}W_{k,m}(z)$, has been given by Sharma (3). In this note I find by operational calculus an integral equation for $z^{-1}W_{k,m}(z)$ which is more general than (1·1) given above.

2. We write $f(p) \doteqdot h(x)$ or $h(x) \doteqdot f(p)$, when

$$f(p) = p\int_0^\infty e^{-px}h(x)\,dx;$$

we then have, from Parseval's formula,

$$\int_0^\infty \frac{f(p+t)\,g(t)}{p+t}\,dt = \int_0^\infty e^{-pt}h(t)\,\phi(t)\,t^{-1}\,dt, \tag{2·1}$$

where $\phi(p) \doteqdot g(x)$. This shows that

$$t^{-1}h(t)\,\phi(t) \doteqdot p\int_0^\infty \frac{f(p+u)\,g(u)}{p+u}\,du. \tag{2·2}$$

Thus, taking (4)
$$f(p) = e^{\frac{1}{2}p}p^{\frac{1}{2}-\nu}W_{k,\nu}(p)$$
$$\doteqdot \frac{x^{-\frac{1}{2}-k+\nu}(1+x)^{-\frac{1}{2}+k+\nu}}{\Gamma(\frac{1}{2}-k+\nu)} = h(x),$$

and (1)
$$\phi(p) = \frac{\Gamma(\nu-m)\,\Gamma(\nu+m)}{\Gamma(\frac{1}{2}-k+\nu)}p\,{}_2F_1[\nu-m, \nu+m; \tfrac{1}{2}-k+\nu, -p]$$
$$\doteqdot e^{-\frac{1}{2}x}x^{\nu-\frac{1}{2}}W_{k,m}(x) = g(x),$$

we find that

$$x^{-1}\phi(x)\,h(x) = \frac{\Gamma(\nu-m)\,\Gamma(\nu+m)}{\Gamma(\frac{1}{2}-k+\nu)\,\Gamma(\frac{1}{2}-k+\nu)}x^{-\frac{1}{2}-k+\nu}(1+x)^{-\frac{1}{2}+k+\nu}\,{}_2F_1[\nu-m, \nu+m; \tfrac{1}{2}-k+\nu; -x]$$
$$= \frac{\Gamma(\nu-m)\,\Gamma(\nu+m)}{\Gamma(\frac{1}{2}-k+\nu)\,\Gamma(\frac{1}{2}-k+\nu)}x^{-\frac{1}{2}-k+\nu}\,{}_2F_1[\tfrac{1}{2}-k-m, \tfrac{1}{2}-k+m; \tfrac{1}{2}-k+\nu; -x],$$

and accordingly we have (2)

$$x^{-1}\phi(x)\,h(x) \doteqdot \frac{\Gamma(\nu-m)\,\Gamma(\nu+m)}{\Gamma(\frac{1}{2}-k+\nu)}e^{\frac{1}{2}p}p^{\frac{1}{2}-\nu}W_{k,m}(p). \tag{2·3}$$

Thus, substituting from (2·3) in (2·2), we have

$$p^{-1}W_{k,m}(p) = \frac{\Gamma(\frac{1}{2}-k+\nu)}{\Gamma(\nu-m)\,\Gamma(\nu+m)}\int_0^\infty (p+t)^{-\frac{1}{2}-\nu}(tp)^{\nu-\frac{1}{2}}W_{k,\nu}(p+t)\,t^{-1}W_{k,m}(t)\,dt, \tag{2·4}$$

where $\Re(\nu \pm m) > 0$, $\Re(\frac{1}{2}-k+\nu) > 0$.

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This shows that $\phi(z) = z^{-1}W_{k,m}(z)$ satisfies the integral equation

$$\phi(p) = \frac{\Gamma(\frac{1}{2}-k+\nu)}{\Gamma(\nu-m)\,\Gamma(\nu+m)}\int_0^\infty (p+t)^{-\frac{1}{2}-\nu}(tp)^{\nu-\frac{1}{2}}W_{k,\nu}(p+t)\,\phi(t)\,dt, \tag{2·5}$$

where $\Re(\nu \pm m) > 0$, $\Re(\frac{1}{2}-k+\nu) > 0$.

The restriction $\Re(\frac{1}{2}-k+\nu) > 0$ may be removed by the theory of analytic continuation, although the right-hand side of equations (2·4) and (2·5) will appear in an indefinite form whenever $\frac{1}{2}-k+\nu$ is zero or a negative integer.

The integral equation (1·1) is the particular case of (2·5) when $\nu = \frac{1}{2}-k$.

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APPROXIMATE TESTS OF CORRELATION IN TIME-SERIES 3

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An approximate distribution of the first serial correlation coefficient defined by

$$r_1 = \left(\sum_{i=1}^{n} x_i x_{i+1}\right) \Big/ \left(\sum_{i=1}^{n} x_i^2\right) \quad (x_{n+1} = x_1), \tag{1}$$

where the x_i are assumed to be normally distributed has been given by Dixon (1), and Madow (2) has shown how this approximation may be extended to cover the case where the x_i are connected by a linear Markoff process $x_{i+1} = \rho x_i + \epsilon_{i+1}$ to give the frequency function

$$\frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n+\frac{1}{2})\,\Gamma(\frac{1}{2})}\,\frac{(1-r_1^2)^{\frac{1}{2}(n-1)}}{(1-2\rho r_1+\rho^2)^{\frac{1}{2}n}}. \tag{2}$$

Since this is in any case an approximate distribution it is worth while investigating whether to the same order of approximation any of the usual tests of significance can be applied.

It has already been shown (3) that if the usual transformation of the correlation coefficient is applied, i.e. $r_1 = \tanh z$, $\rho = \tanh \zeta$, then $z-\zeta$ is approximately normally distributed with variance $(1-\rho^2)/n$. However, since this variance depends upon ρ the

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