A NEW MODEL FOR ALL C-SEQUENCES ARE TRIVIAL

ASSAF RINOT, ZHIXING YOU, AND JIACHEN YUAN

Abstract. We construct a model in which all C-sequences are trivial, yet there exists a κ-Souslin tree all of whose limit levels are vanishing levels. This provides an optimal combination of compactness and incompactness. It is obtained by incorporating a so-called *mutually exclusive ascent path* to Kunen's 1978 forcing construction, and by analyzing a gallery of κ-cc forcing extensions of the outcome model.

§1. Introduction. Weakly compact cardinals were introduced by Erdős and Tarski in the 1960s. There are many equivalent ways to define these cardinals, and this variety testifies to its significance: via model theory of infinitary languages, compactness of topological spaces, basis problems for linear orders, tree properties, Ramsey theory, elementary embeddings, indescribable cardinals, and more.

Motivated by a characterization of weak compactness in terms of C-sequences due to Todorčević [15, Theorem 1.8], Lambie-Hanson and Rinot [9, Definition 1.6] introduced a new cardinal characteristic, *the C-sequence number*, to measure the compactness of a regular uncountable cardinal κ, where the best case $\chi(\kappa) = 0$ amounts to saying that κ is weakly compact (see Definition 2.18 below). Once the C-sequence number of κ is bigger than 1 (i.e., $\chi(\kappa) > 1$), the method of walks on ordinals becomes active and enables the construction of interesting incompact objects. To mention two such applications at the level of a strongly inaccessible cardinal κ, if $\chi(\kappa) > 1$, then the κ-chain condition is not infinitely productive (see [8, Lemma 3.3] and [9, Lemma 5.8]), and there is a κ-Aronszajn tree with no ascent path of width less than $\chi(\kappa)$ (see [5, Lemma 7.9]).[1]

The strong assertion $\chi(\kappa) = 1$ indeed implies that κ shares features of weakly compact cardinals such as satisfying a diagonal form of simultaneous reflection of stationary subsets of κ [9, Lemma 2.12], and as being weakly compact in Gödel's constructible universe [15, Theorem 1.10].

In the other direction, very little is known about the compatibility of $\chi(\kappa) = 1$ with incompact postulates. Not only because walks on ordinals is unavailable in this context, but because Kunen's proof [7, Section 3] is currently the only known method to construct models for $\chi(\kappa) = 1$. We defer the discussion on Kunen's method to Section 1.1, and now turn to describe and motivate the kind of incompactness we would like to show compatible with $\chi(\kappa) = 1$.

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[1] Undefined terminology can be found in Section 2.

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The axiom $\clubsuit_{\mathrm{AD}}$ was introduced by Rinot and Shalev in [12] as a guessing principle useful in solving problems in set-theoretic topology. Roughly speaking, $\clubsuit_{\mathrm{AD}}$ holds over a set of ordinals S iff there is a ladder system over S consisting of pairwise almost-disjoint sets (in a sense sufficient to imply that the set S is not *subtle*), and this system has a guessing feature akin to Ostaszewski's principle $\clubsuit$ [11]. By [12, Theorem 2.30], $\clubsuit_{\mathrm{AD}}$ fails over every club of a weakly compact cardinal. By [12, Theorem 2.23], if there exists a κ-Souslin tree T whose set of vanishing levels $V(T)$ is stationary (see Definition 2.5 below), then $\clubsuit_{\mathrm{AD}}$ holds over $V(T) \cap C$ for some club $C \subseteq \kappa$.

As $\clubsuit_{\mathrm{AD}}$ holding over a club in κ implies that κ is neither subtle nor weakly compact, this raises the question of the degree of compactness that is compatible with it. The best known result in this vein is [13, Theorem E] asserting that assuming the consistency of a weakly compact cardinal, it is consistent that for some strongly inaccessible cardinal κ satisfying $\chi(\kappa) = \omega$, $\clubsuit_{\mathrm{AD}}$ holds over a club in κ, furthermore, there is a κ-Souslin tree T such that $V(T) = \mathrm{acc}(\kappa)$.[2] Whether this result may be improved to $\chi(\kappa) < \omega$ remained an evasive open problem. Here, this remaining case is resolved.

Main Theorem. *Suppose that κ is a weakly compact cardinal. Then there exists a forcing extension in which κ is strongly inaccessible, $\chi(\kappa) = 1$ and there exists a κ-Souslin tree T such that $V(T) = \mathrm{acc}(\kappa)$.*

To crystallize two concise compactness principles that our model manages to separate, consider the following two threading-type postulates:

(I) For every C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exists a cofinal $\Delta \subseteq \kappa$ such that, for every $\alpha < \kappa$, there exists a $\beta < \kappa$ with $\Delta \cap \alpha \subseteq \mathrm{nacc}(C_\beta)$.[3]

(II) For every C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exists a cofinal $\Delta \subseteq \kappa$ such that, for every $\alpha < \kappa$, there exists a $\beta < \kappa$ with $\Delta \cap \alpha \subseteq C_\beta$.

By Todorčević's theorem [15, Theorem 1.8], if κ is weakly compact, then (I) holds. The postulate (II) is equivalent to asserting that $\chi(\kappa) \leq 1$. Our model demonstrates that (II) is strictly weaker than (I) (see Figure 1).

1.1. About the proof. The standard way to argue that $\chi(\kappa) = 1$ holds in a given model is to prove that κ is not weakly compact, but can become one via some κ-cc notion of forcing. The prototype example is Kunen's model [7, Section 3] in which there is a κ-Souslin tree T such that forcing with it makes κ a measurable cardinal (in particular, weakly compact and subtle). Modifying Kunen's proof to make the generic κ-Souslin tree T satisfy that $V(T)$ covers a club in κ brings up two obstructions. First, as explained earlier, the existence of such a T implies that κ is not subtle, so that it cannot become measurable in some κ-cc forcing extension. The second is that in any extension in which κ becomes weakly compact, our tree would have to admit at least κ^+-many cofinal branches (see Proposition 2.7). So if, as in Kunen's case, T is slim[4] and the resurrection of weak compactness is obtained by

[2] For a set A of ordinals, $\mathrm{acc}(A) := \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha > 0\}$.

[3] The focus on $\mathrm{nacc}(C_\beta) := C_\beta \setminus \mathrm{acc}(C_\beta)$ amounts to replacing the quantification from ranging over C-sequences to ranging over ladder systems.

[4] The definition of *slim* trees may be found at [1, Definition 6.3]. An anti-slim variation of Kunen's model may be found at [10, Theorem 2.6]

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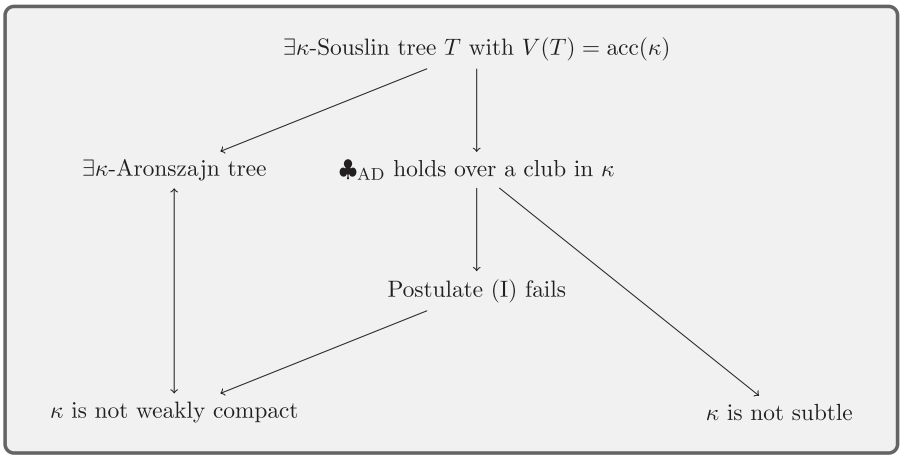


FIGURE 1. Diagram of implications at the level of a strongly inaccessible cardinal κ.

forcing with T, then such a T must have been a so-called *almost-Kurepa* κ-Souslin tree.

While we do not know it is impossible to prove the Main Theorem using Kunen's original approach, we ended up taking a different path. Instead of resurrection of weak compactness in a single forcing extension, our approach is to produce a model admitting a family of generic elementary embeddings each living in its own κ-cc forcing extension, as follows. We will start with κ a weakly compact cardinal, carry out some preparatory forcing below κ, and then force to add a uniformly homogeneous κ-Souslin subtree T of ${}^{<\kappa}\kappa$ such that $V(T)$ is a club in κ and such that T admits an $\mathcal{F}$-ascent path for an educated choice of a filter $\mathcal{F}$ over ω. Our plan is to argue that $\chi(\kappa) = 1$ holds in the final model by showing that for every C-sequence $\vec{C}$ over κ, a nontrivial elementary embedding $j : M \to N$ between two κ-models with $\mathrm{crit}(j) = \kappa$ and $\vec{C} \in M$ exists in some further κ-cc forcing extension. In the inevitable case that our tree T belongs to M, it would admit a branch $b : \kappa \to \kappa$, as witnessed by any element of the κ^{th}-level of $j(T)$. Meanwhile, since $V(T)$ covers a club, $\kappa \in j(V(T)) = V(j(T))$, and hence the tree $j(T)$ would have a vanishing κ-branch $b' : \kappa \to \kappa$. As b is non-vanishing and b' is vanishing, the fact that T is uniformly homogeneous implies that b and b' must disagree cofinally often. By iterating such an argument, we infer that the tree must have at least ω-many cofinal branches that disagree with each other cofinally often. Thus, we shall need a notion of forcing that introduces such cofinal branches in a κ-cc fashion. This is exactly where the definition of a *mutually exclusive $\mathcal{F}$-ascent path* arises. To connect on a previous remark, instead of adding κ^+-many branches, in our approach only countably many (mutually exclusive ones) are added.

1.2. Organization of this article. In Section 2, we provide some necessary background on trees, ascent paths, and C-sequences. We motivate the new notion

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of a *mutually exclusive $\mathcal{F}$-ascent path* by showing that variations of Kunen's forcing that do not make use of this kind of ascent path are not well behaved.

In Section 3, we present our main notions of forcing. Large cardinals play no role in this section, and the results are applicable as low as at $\aleph_2$.

In Section 4, we prove the main theorem by iterating the posets of Section 3 below and at a weakly compact cardinal.

§2. Preliminaries.

2.1. Trees and vanishing levels. For simplicity, instead of working with abstract trees, we opt to work with the following more concrete implementation. A *streamlined tree* is a subset $T \subseteq {}^{<\kappa}H_\kappa$ for some cardinal κ that is downward-closed, i.e., for every $t \in T$, $\{t \restriction \alpha \mid \alpha < \kappa\} \subseteq T$. The height of a node $x \in T$ is $\mathrm{dom}(x)$. The height of T, denoted $\mathrm{ht}(T)$, is the least ordinal β such that $T_\beta := \{x \in T \mid \mathrm{dom}(x) = \beta\}$ is empty. Note that for every ordinal β, $T \restriction \beta := \{x \in T \mid \mathrm{dom}(x) < \beta\}$ is as well a streamlined tree. A streamlined tree T is said to be *normal* iff for every $x \in T$ and every $\alpha < \mathrm{ht}(T)$, there exists some $y \in T_\alpha$ with either $x \subseteq y$ or $y \subseteq x$. A *streamlined κ-tree* [2, Definition 2.3] is a downward-closed subset T of ${}^{<\kappa}H_\kappa$ satisfying that $0 < |T_\alpha| < \kappa$ for every $\alpha < \kappa$.[5] A *streamlined κ-Souslin tree* is a streamlined κ-tree having no κ-branches and no κ-sized antichains with respect to the ordering $\subsetneq$.

Convention 2.1. *Hereafter, by a tree, we mean a streamlined tree.*

Definition 2.2. For all $s, t \in {}^{<\kappa}H_\kappa$, let

$$\Delta(s,t) := \min(\{\mathrm{dom}(s), \mathrm{dom}(t)\} \cup \{\delta \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\delta) \neq t(\delta)\}).$$

In addition, we define $(s * t) : \mathrm{dom}(t) \to H_\kappa$ via:

$$(s*t)(\varepsilon) := \begin{cases} s(\varepsilon), & \text{if } \varepsilon \in \mathrm{dom}(s);\\ t(\varepsilon), & \text{otherwise.}\end{cases}$$

Definition 2.3. A κ-tree T is *uniformly homogeneous* iff for all $s, t \in T$, $s * t$ is in T.

Definition 2.4. Suppose that T is a tree. An *α-branch* is a subset $B \subseteq T$ linearly ordered by $\subsetneq$ and satisfying that $\{\mathrm{dom}(x) \mid x \in B\} = \alpha$. For $\alpha \in \mathrm{acc}(\kappa)$, an α-branch B is *vanishing* iff $\bigcup B \notin T$.

Definition 2.5 (The levels of vanishing branches, [12, Definition 2.18]). For a κ-tree T, $V(T)$ stands for the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that for every node $x \in T$ of height less than α there exists a vanishing α-branch containing x.

In the context of uniformly homogeneous trees, the preceding admits a simpler characterization (see [13, Propositions 2.6 and 2.19]).

Fact 2.6. *For a uniformly homogeneous κ-tree T, $V(T)$ coincides with the set of all $\alpha \in \mathrm{acc}(\kappa)$ for which there exists a vanishing α-branch.*

[5] In this case, the poset $(T, \subsetneq)$ is a set-theoretic κ-tree in the usual abstract sense.

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PROPOSITION 2.7. *Suppose that κ is a weakly compact cardinal, and T is a κ-tree such that $V(T)$ covers a club in κ. Then T has at least κ^+-many cofinal branches.*

PROOF. Towards a contradiction, suppose that the set of cofinal branches of T may be enumerated (possibly with repetitions) as $\langle b_\alpha \mid \alpha < \kappa\rangle$. It follows that we may take a κ-model M containing T and each of the b_α's. Using the weak compactness of κ, fix an elementary embedding $j : M \to N$ with critical point κ, where N is as well a κ-model. By elementarity, $\kappa \in j(V(T))$, so let b be a vanishing κ-branch of $j(T)$. Since the restriction of $j(T)$ to all levels below κ coincides with T, it follows that b is a cofinal branch of T, so that $b = b_\alpha$ for some $\alpha < \kappa$. By elementarity, $j(b_\alpha)$ is a cofinal branch of $j(T)$ having b_α as an initial segment, contradicting the fact $b_\alpha = b$ is a vanishing κ-branch of $j(T)$. ⊣

2.2. Ascent paths. The notions of forcing in Section 3 below will make use of the upcoming Definitions 2.8, 2.14, and 2.15. To motivate them, we shall inspect here two earlier attempts to define these notions of forcing, demonstrating the problem with these attempts.

Let $\theta < \kappa$ be a pair of infinite regular cardinals. We start by recalling the vanilla definition of an ascent path.

DEFINITION 2.8. Suppose that T is a tree, and θ is some cardinal. For all $f, g \in {}^{\theta}T$, denote

$$\circledcirc(f,g) := \{\tau < \theta \mid f(\tau) \subseteq g(\tau) \text{ or } f(\tau) \supseteq g(\tau)\}.$$

DEFINITION 2.9 (Laver). Suppose that T is a tree of some height γ. A sequence $\vec{f} = \langle f_\alpha \mid \alpha < \gamma\rangle$ is a *θ-ascent path* through T iff the following two hold:

- for every $\alpha < \gamma$, $f_\alpha : \theta \to T_\alpha$ is a function;
- for all $\alpha < \beta < \gamma$, $\circledcirc(f_\alpha, f_\beta)$ is co-bounded in θ.

REMARK 2.10. We often write $\vec{f}(\alpha)$ for the α^{th}-element of $\vec{f}$, namely, f_α.

A natural notion of forcing for adding a κ-Souslin tree with a θ-ascent path may be found in the proof of [4, Theorem 3.1]. It reads as follows.

DEFINITION 2.11. $\mathbb{S}^\kappa_\theta$ is defined to be the notion of forcing consisting of all pairs $\langle T, \vec{f}\rangle$ for which the following two hold:

(1) $T \subseteq {}^{<\kappa}2$ is a normal uniformly homogeneous tree of a successor height all of whose levels have size less than κ;
(2) $\vec{f}$ is a θ-ascent path through T.

The order on $\mathbb{S}^\kappa_\theta$ is defined by taking end-extension on both coordinates.

Work in $V[G]$ for G an $\mathbb{S}^\kappa_\theta$-generic filter over V. It is the case that $T(G) := \bigcup\{T \mid \langle T, \vec{f}\rangle \in G\}$ is a uniformly homogeneous κ-Souslin tree, and $\vec{f}^G := \bigcup\{\vec{f} \mid \langle T, \vec{f}\rangle \in G\}$ is a θ-ascent path through $T(G)$. Next, consider the following further forcing.

DEFINITION 2.12. $\mathbb{A}^\kappa_\theta$ has underlying set κ, and ordering

$$\beta \leq_{\mathbb{A}^\kappa_\theta} \alpha \text{ iff } (\alpha \leq \beta \ \&\ \circledcirc(\vec{f}^G(\alpha), \vec{f}^G(\beta)) = \theta).$$

PROPOSITION 2.13. *$\mathbb{A}^\kappa_\theta$ does not satisfy the κ-chain condition.*

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PROOF. Let us say that $\beta < \kappa$ is *bad* iff $\beta = \alpha + 1$ is a successor ordinal and $\Delta(\vec{f}^G(\beta)(0), \vec{f}^G(\beta)(1)) = \alpha$.

CLAIM 2.13.1. *There are cofinally many bad $\beta < \kappa$.*

PROOF. Back in V, given a condition $\langle T, \vec{f}\rangle$ in $\mathbb{S}^\kappa_\theta$, fix α such that $\mathrm{ht}(T) = \alpha + 1$. Let $T' := T \cup \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle \mid t \in T_\alpha\}$, and define $\vec{f}' : \alpha + 2 \to {}^\theta T'$ via

$$\vec{f}'(\beta)(\tau) := \begin{cases} \vec{f}(\beta)(\tau), & \text{if } \beta \leq \alpha; \\ \vec{f}(\alpha)(\tau)^\frown\langle 0\rangle, & \text{if } \beta = \alpha + 1 \ \&\ \tau \neq 1; \\ \vec{f}(\alpha)(0)^\frown\langle 1\rangle, & \text{otherwise.} \end{cases}$$

Then $\langle T', \vec{f}'\rangle$ extends $\langle T, \vec{f}\rangle$ and it forces that $\mathrm{dom}(\vec{f}')$ is bad. ⊣

As the collection of all bad $\beta < \kappa$ is an antichain in $\mathbb{A}^\kappa_\theta$, the latter fails to satisfy the κ-cc. ⊣

To mitigate the problem arising from the preceding proposition, and in view of the goal of securing a good chain condition, we introduce the next two definitions.

DEFINITION 2.14. Two elements $s, t \in {}^{<\kappa}H_\kappa$ are *mutually exclusive* iff $s(\varepsilon) \neq t(\varepsilon)$ for every $\varepsilon \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$.

DEFINITION 2.15. Suppose that T is a tree of some height γ, and $\mathcal{F}$ is a filter over θ. A sequence $\vec{f} = \langle f_\alpha \mid \alpha < \gamma\rangle$ is a *mutually exclusive $\mathcal{F}$-ascent path* through T iff the following three hold:

- for every $\alpha < \gamma$, $f_\alpha : \theta \to T_\alpha$ is a function;
- for all $\alpha < \beta < \gamma$, $\circledcirc(f_\alpha, f_\beta) \in \mathcal{F}$;
- for every nonzero $\alpha < \gamma$, $\langle f_\alpha(\tau) \mid \tau < \theta\rangle$ consists of mutually exclusive nodes.

Fix some uniform filter $\mathcal{F}$ over θ, and revise Definition 2.11, as follows.

DEFINITION 2.16. $\mathbb{S}^\kappa_\mathcal{F}$ is defined to be the notion of forcing consisting of all pairs $\langle T, \vec{f}\rangle$ for which the following two hold:

(1) $T \subseteq {}^{<\kappa}\kappa$ is a normal uniformly homogeneous tree of a successor height all of whose levels have size less than κ;
(2) $\vec{f}$ is a mutually exclusive $\mathcal{F}$-ascent path through T.

The order on $\mathbb{S}^\kappa_\mathcal{F}$ is defined by taking end-extension on both coordinates.

Work in $V[G]$ for G an $\mathbb{S}^\kappa_\mathcal{F}$-generic filter over V. Define $\mathbb{A}^\kappa_\mathcal{F}$ to have underlying set κ, and ordering

$$\beta \leq_{\mathbb{A}^\kappa_\mathcal{F}} \alpha \text{ iff } (\alpha \leq \beta \ \&\ \circledcirc(\vec{f}^G(\alpha), \vec{f}^G(\beta)) \in \mathcal{F}).$$

This time—for an educated choice of a filter $\mathcal{F}$—the corresponding notion of forcing $\mathbb{A}^\kappa_\mathcal{F}$ does satisfy the κ-chain condition. In particular, the tree generated by the ascent path is a κ-Souslin tree. Curiously, the whole tree $T(G)$ is not Souslin. Indeed, during a talk given by the second author at the Chinese Academy of Sciences, Yinhe Peng pointed out the following.

PROPOSITION 2.17 (Peng). *Let G be an $\mathbb{S}^\kappa_\mathcal{F}$-generic filter. Then $T(G)$ has an antichain of size κ.*

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PROOF. By a density argument, there are cofinally many $\alpha < \kappa$, for which there is a node $t_\alpha : \alpha + \theta \to \kappa$ such that for every $\varepsilon < \alpha + \theta$,

$$t_\alpha(\varepsilon) = \begin{cases} \vec{f}^G_\alpha(0)(\varepsilon), & \text{if } \varepsilon < \alpha; \\ \vec{f}^G_{\alpha+\theta}(\tau)(\alpha+\tau), & \text{if } \varepsilon = \alpha + \tau. \end{cases}$$

Now, if $\alpha < \alpha + \theta \leq \beta$ and t_α is extended by t_β, then there are cofinally many $\tau < \theta$ such that $f_{\alpha+\theta}(\tau) \subseteq f_\beta(\tau)$. In particular, we may find a nonzero τ such that $f_{\alpha+\theta}(\tau)(\alpha+\tau) = f_\beta(\tau)(\alpha+\tau)$. Since $t_\alpha \subseteq t_\beta$, we have

$$f_\beta(\tau)(\alpha+\tau) = f_{\alpha+\theta}(\tau)(\alpha+\tau) = t_\alpha(\alpha+\tau) = t_\beta(\alpha+\tau) = f_\beta(0)(\alpha+\tau),$$

contradicting the fact that $f_\beta(\tau)$ and $f_\beta(0)$ are mutually exclusive. ⊣

In Section 3, we shall present a pair $(\mathbb{S}^\kappa_{\mathbf{X}}, \mathbb{A}^\kappa_{\mathbf{X}})$ of notions of forcing that overcome the problems detected in the two earlier attempts $(\mathbb{S}^\kappa_\theta, \mathbb{A}^\kappa_\theta)$ and $(\mathbb{S}^\kappa_{\mathcal{F}}, \mathbb{A}^\kappa_{\mathcal{F}})$ considered here. Specifically, $\mathbb{S}^\kappa_{\mathbf{X}}$ will be a $<\kappa$-strategically closed forcing that adds a uniformly homogeneous κ-Souslin tree with a mutually exclusive $\mathcal{F}$-ascent path and whose set of vanishing levels is a club in κ. As well, $\mathbb{S}^\kappa_{\mathbf{X}} * \dot{\mathbb{A}}^\kappa_{\mathbf{X}}$ will be a $<\kappa$-strategically closed forcing, and $\mathbb{A}^\kappa_{\mathbf{X}}$ will satisfy the κ-chain condition.

2.3. C-sequences. A *C-sequence* over a regular uncountable cardinal κ is a sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ such that for every $\beta < \kappa$, C_β is a closed subset of β with $\sup(C_\beta) = \sup(\beta)$.

DEFINITION 2.18 (The C-sequence number of κ, [9]). If κ is weakly compact, then let $\chi(\kappa) := 0$. Otherwise, let $\chi(\kappa)$ denote the least cardinal $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

REMARK 2.19. The special case of $\chi(\kappa) \leq 1$ is better known as "all C-sequences over κ are trivial" [16, Definition 6.3.1].

To motivate one ingredient of the proof of the main result in Section 4, we present here an abstract approach to producing models satisfying $\chi(\kappa) = 1$. It may be phrased as a *suspended form of the tree property*, as follows.

PROPOSITION 2.20. *Suppose that κ is a strongly inaccessible cardinal, and for every κ-tree T, there is a κ-cc forcing extension in which T has a κ-branch. Then $\chi(\kappa) \leq 1$.*

PROOF. Let $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ be a C-sequence over κ. Let T be the corresponding tree $T(\rho_0^{\vec{C}})$ from the theory of walks on ordinals. Since κ is strongly inaccessible, T is narrow, i.e., it is a κ-tree. Let V' be some κ-cc forcing extension of the universe in which T admits a κ-branch. By the proof of the forward implication of [16, Theorem 6.3.5], there must exists a club $C \subseteq \kappa$ such that, for every $\alpha < \kappa$, there is a $\beta \geq \alpha$ with $C \cap \alpha = C_\beta \cap \alpha$. Since V' is a κ-cc forcing extension of V, we may find a subclub $D \subseteq C$ residing in V. It follows that for every $\alpha < \kappa$, there is a $\beta \geq \kappa$ with $D \cap \alpha \subseteq C_\beta$. ⊣

REMARK 2.21. An analogous statement holds for $\chi(\kappa) = 0$. Indeed, by [17, Lemma 2.2], if κ is a strongly inaccessible cardinal and for every κ-tree T there is a forcing extension by a poset whose square has the κ-cc in which T has a κ-branch, then $\chi(\kappa) = 0$.

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§3. An ascent path variant of Kunen's construction. Throughout this section, θ, κ stand for infinite regular cardinals and we assume that $\lambda^\theta < \kappa$ for all $\lambda < \kappa$. In particular, $\theta^+ < \kappa$. We also fix a sequence $\mathbf{X} = \langle X_\xi \mid \xi < \zeta \rangle$ satisfying the following requirements:

- $\langle X_\xi \mid \xi < \zeta \rangle$ is a $\subseteq$-decreasing sequence of nonempty subsets of θ;
- $|\theta \setminus X_0| = \theta$;
- $\bigcap_{\xi<\zeta} X_\xi = \emptyset$.

We denote by $\mathcal{F}$ the filter generated by the elements of $\mathbf{X}$, i.e., $\mathcal{F} = \{ Y \subseteq \theta \mid \exists \xi < \zeta \, (X_\xi \subseteq Y)\}$. Note that $\mathcal{F}$ is proper and nonprincipal. Also, let $\mathcal{I}$ denote the dual ideal of $\mathcal{F}$.

Definition 3.1. Define an equivalence relation $=^*$ on ${}^{<\kappa}H_\kappa$ by letting $s =^* t$ iff $\mathrm{dom}(s) = \mathrm{dom}(t)$ and there exists an $\alpha < \mathrm{dom}(s)$ such that $s(\beta) = t(\beta)$ whenever $\alpha \leq \beta < \mathrm{dom}(s)$.

We are now ready to present the notions of forcing we will be using.

Definition 3.2. $\mathbb{S}^\kappa_{\mathbf{X}}$ is defined to be the notion of forcing consisting of all pairs $\langle T, \vec{f} \rangle$ that satisfy the following list of requirements:

(1) $T \subseteq {}^{<\kappa}\kappa$ is a normal uniformly homogeneous tree of a successor height, say $\eta_T + 1$, all of whose levels have size less than κ;
(2) $\vec{f}$ is a mutually exclusive $\mathcal{F}$-ascent path through T;
(3) $V(T)$ is a closed set, and if $\eta \in \mathrm{acc}(\kappa)$, then $\eta \in V(T)$;
(4) for every nonzero $\alpha \leq \eta$, for every $t \in T_\alpha$, the set $\{\tau < \theta \mid t \text{ and } \vec{f}(\alpha)(\tau) \text{ are mutually exclusive}\}$ is in $\mathcal{F}$.[6]

The order on $\mathbb{S}^\kappa_{\mathbf{X}}$ is defined by taking end-extension on both coordinates.

Notation 3.3. *For $\langle T, \vec{f} \rangle \in \mathbb{S}^\kappa_{\mathbf{X}}$:*

- *η_T denotes the unique ordinal to satisfy $\mathrm{ht}(T) = \eta_T + 1$;*
- *ν_T denotes the cardinality of the quotient ${}^{T_{\eta_T}}\!/_{=^*}$.*

Remark 3.4. If $\eta_T = \alpha + 1$ is a successor ordinal, then $\nu_T = |\{t(\alpha) \mid t \in T_{\eta_T}\}|$.

Lemma 3.5 (One-step β-extension). *Let $\langle T, \vec{f} \rangle \in \mathbb{S}^\kappa_{\mathbf{X}}$, $\beta \leq \eta_T$, and $\nu < \kappa$. Then there is a $\langle T', \vec{f}' \rangle \leq \langle T, \vec{f} \rangle$ with $\eta_{T'} = \eta_T + 1$ and $\nu_{T'} \geq \nu$ such that:*

- *$\circledcirc(\vec{f}(\beta), \vec{f}(\eta_T)) \subseteq \circledcirc(\vec{f}(\eta_T), \vec{f}'(\eta_{T'}))$, and*
- *$\circledcirc(\vec{f}(\beta), \vec{f}'(\eta_{T'})) = \theta$.*

Proof. Set $T' := T \cup \{t^\frown\langle \iota \rangle \mid t \in T_{\eta_T}, \ \iota < |\nu \cup \theta|\}$, and then define $\vec{f}' : \eta_{T'} + 1 \to {}^\theta({}^{<\kappa}\kappa)$ by letting for all $\alpha \leq \eta_{T'}$ and $\tau < \theta$,

$$\vec{f}'(\alpha)(\tau) := \begin{cases} \vec{f}(\alpha)(\tau), & \text{if } \alpha \leq \eta_T; \\ (\vec{f}(\beta)(\tau) * \vec{f}(\eta_T)(\tau))^\frown\langle \tau \rangle, & \text{otherwise.} \end{cases}$$

It is clear that $\langle T', \vec{f}' \rangle$ is an element of $\mathbb{S}^\kappa_{\mathbf{X}}$ as sought. ⊣

[6] This requirement ensures that the generic tree coincides with the one generated by its ascent path, thus eliminating the discrepancy demonstrated in Proposition 2.17.

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The next lemma identifies a feature of decreasing sequences of conditions sufficient to ensure the existence of a lower bound.

LEMMA 3.6. *Suppose* Γ *is a cofinal subset of some* $\gamma \in \mathrm{acc}(\kappa)$, $\delta \in (\gamma, \kappa)$, *and we are given a sequence* $\langle (T^\beta, \vec{f}^\beta, \vec{z}^\beta) \mid \beta \in \Gamma \rangle$ *such that*:

- $\langle \langle T^\beta, \vec{f}^\beta \rangle \mid \beta \in \Gamma \rangle$ *is a decreasing sequence of conditions in* $\mathbb{S}^\kappa_{\mathbf{X}}$;
- *for every* $\beta \in \Gamma$, $\vec{z}^\beta : (\beta, \delta) \to T^\beta$, *where* $\beta < i < \delta$ *implies*:
  - $\vec{z}^\beta(i)$ *belongs to the top level of* T^β;
  - $\vec{z}^\beta(i) \neq^* \vec{z}^\beta(i')$ *for every* $i' \in (\beta, \delta) \setminus \{i\}$;
  - $\vec{z}^\beta(i) \neq^* \vec{f}^\beta(\eta_{T^\beta})(0)$;
  - $\vec{z}^\beta(i)$ *and* $\vec{f}^\beta(\eta_{T^\beta})(\tau)$ *are mutually exclusive whenever* $0 < \tau < \theta$.[7]
- *for all* $\alpha, \beta \in \Gamma$:
  - $\circledcirc(\vec{f}^\alpha(\eta_{T^\alpha}), \vec{f}^\beta(\eta_{T^\beta})) = \theta$;
  - $\vec{z}^\alpha(i) \subseteq \vec{z}^\beta(i)$ *provided* $\alpha < \beta < i < \delta$.

Then there are a condition $\langle T^\gamma, \vec{f}^\gamma \rangle$ *in* $\mathbb{S}^\kappa_{\mathbf{X}}$ *and* $\vec{z}^\gamma : (\gamma, \delta) \to T^\gamma$ *such that*:

- $\eta_{T^\gamma} = \sup_{\beta \in \Gamma} \eta_{T^\beta}$;
- $\vec{z}^\gamma(i) = \bigcup_{\beta \in \Gamma} \vec{z}^\beta(i)$ *whenever* $\gamma < i < \delta$;
- *for every* $\beta \in \Gamma$, $\langle T^\gamma, \vec{f}^\gamma \rangle \leq \langle T^\beta, \vec{f}^\beta \rangle$ *and* $\circledcirc(\vec{f}^\beta(\eta_{T^\beta}), \vec{f}^\gamma(\eta_{T^\gamma})) = \theta$;
- *a function* $y : \eta_{T^\gamma} \to \kappa$ *is in* T^γ *iff one of the following holds*:
  - *there are* $x \in \bigcup_{\beta \in \Gamma} T^\beta$ *and* $\tau < \theta$ *such that* $y = x * \vec{f}^\gamma(\eta_{T^\gamma})(\tau)$, *or*
  - *there are* $x \in \bigcup_{\beta \in \Gamma} T^\beta$ *and* $i \in (\gamma, \delta)$ *such that* $y = x * \vec{z}^\gamma(i)$.

PROOF. Set $\eta := \sup_{\beta \in \Gamma} \eta_{T^\beta}$ and $\vec{f} := \bigcup_{\beta \in \Gamma} \vec{f}^\beta$. Define $\vec{f}^\gamma : \eta + 1 \to {}^\theta({}^{<\kappa}\kappa)$ by letting for all $\alpha \leq \eta$ and $\tau < \theta$,

$$\vec{f}^\gamma(\alpha)(\tau) := \begin{cases} \vec{f}(\alpha)(\tau), & \text{if } \alpha < \eta; \\ \bigcup_{\beta \in \Gamma} \vec{f}(\eta_{T^\beta})(\tau), & \text{if } \alpha = \eta. \end{cases}$$

Define $\vec{z}^\gamma : (\gamma, \delta) \to T^\gamma$ via

$$\vec{z}^\gamma(i) := \bigcup_{\beta \in \Gamma} \vec{z}^\beta(i).$$

Finally, consider $T := \bigcup_{\beta \in \Gamma} T^\beta$, and then let

$$T^\gamma := T \cup \bigcup \{ x * \vec{f}^\gamma(\eta)(\tau), x * \vec{z}^\gamma(i) \mid x \in T, \tau < \theta, \gamma < i < \delta \}.$$

To see that $\langle T^\gamma, \vec{f}^\gamma \rangle$ is as sought note that

$$y := \bigcup_{\beta \in \Gamma} \vec{z}^\beta(\gamma)$$

determines a vanishing η-branch of T^γ and hence $\eta \in V^-(T) = V(T)$. Also, for every $i \in (\gamma, \delta)$, $\vec{z}^\gamma(i)$ and $\vec{f}^\gamma(\eta)(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$.

[7] This clause together with the previous one provide better control on how Clause (4) of Definition 3.2 is implemented with respect to the nodes enumerated by $\vec{z}^\beta$.

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So, for every $t \in (T^\gamma)_\eta$, an intersection of two filter sets demonstrates that $\{\tau < \theta \mid t \text{ and } \vec{f}^\gamma(\eta)(\tau) \text{ are mutually exclusive}\}$ is in $\mathcal{F}$. ⊣

LEMMA 3.7. *$\mathbb{S}^\kappa_{\mathbf{X}}$ is $<\kappa$-strategically closed.*

PROOF. By Lemma 3.6. See the proof of the upcoming Lemma 3.10 for a stronger result. ⊣

REMARK 3.8. By [6], for a successor cardinal $\kappa = \lambda^+$, if $\square_\lambda$ holds, then every $(\lambda+1)$-strategically closed forcing is κ-strategically closed. So, in this case, $\mathbb{S}^\kappa_{\mathbf{X}}$ is moreover κ-strategically closed. On the other hand, if κ is subtle, then $\mathbb{S}^\kappa_{\mathbf{X}}$ is not κ-strategically closed; this is because a κ-strategically closed forcing preserves the subtleness of κ, whereas the existence of a κ-tree T for which $V(T)$ covers a club implies that κ is non-subtle [13, Corollary 2.22].

Given an $\mathbb{S}^\kappa_{\mathbf{X}}$-generic filter G, we let $T(G) := \bigcup\{T \mid \langle T, \vec{f}\rangle \in G\}$, and $\vec{f}^G := \bigcup\{\vec{f} \mid \langle T, \vec{f}\rangle \in G\}$.

DEFINITION 3.9. Suppose G is an $\mathbb{S}^\kappa_{\mathbf{X}}$-generic filter over V. In $V[G]$, for every $\xi < \zeta$, define the forcing notion $\mathbb{A}^\kappa_{\mathbf{X},\xi}$ associated with G to have underlying set κ, and ordering

$$\beta \leq_{\mathbb{A}^\kappa_{\mathbf{X},\xi}} \alpha \text{ iff } (\alpha \leq \beta \;\&\; X_\xi \subseteq \circledcirc(\vec{f}^G(\alpha), \vec{f}^G(\beta))).$$

Let $\mathbb{A}^\kappa_{\mathbf{X}}$ be the lottery sum of these $\mathbb{A}^\kappa_{\mathbf{X},\xi}$'s over all $\xi < \zeta$.

LEMMA 3.10. *$\mathbb{S}^\kappa_{\mathbf{X}} * \dot{\mathbb{A}}^\kappa_{\mathbf{X}}$ has a dense subset that is $<\kappa$-strategically closed.*

PROOF. Let $\xi < \zeta$, and we shall prove that $\mathbb{S}^\kappa_{\mathbf{X}} * \dot{\mathbb{A}}^\kappa_{\mathbf{X},\xi}$ has a dense subset that is $<\kappa$-strategically closed.

Let D be the collection of all pairs $(p, \dot{q})$ in $\mathbb{S}^\kappa_{\mathbf{X}} * \dot{\mathbb{A}}^\kappa_{\mathbf{X},\xi}$ such that if $p = \langle T, \vec{f}\rangle$, then $\dot{q} = \check{\eta}_T$. By Lemma 3.5, D is dense in $\mathbb{S}^\kappa_{\mathbf{X}} * \dot{\mathbb{A}}^\kappa_{\mathbf{X},\xi}$. For simplicity, we shall identify D with the collection of all triples $(T, \vec{f}, \eta)$ such that $\langle T, \vec{f}\rangle \in \mathbb{S}^\kappa_{\mathbf{X}}$ and $\eta = \eta_T$, where a triple $(T', \vec{f}', \eta')$ extends $(T, \vec{f}, \eta)$ iff all of the following hold:

- $T' \cap {}^{\leq\eta}\kappa = T$,
- $\vec{f}' \restriction (\eta+1) = \vec{f}$, and
- $X_\xi \subseteq \circledcirc(\vec{f}(\eta), \vec{f}'(\eta'))$.

To prove that D is $<\kappa$-strategically closed, let $\mu < \kappa$ be arbitrary, and we shall describe a winning strategy for **II** in the game $\Game_\mu(D)$. Playing the game will yield a decreasing sequence of conditions in D, $\langle (T^\beta, \vec{f}^\beta, \eta_\beta) \mid \beta < \mu\rangle$, along with an auxiliary sequence $\langle \vec{z}^\beta \mid \beta < \mu \text{ nonzero even ordinal}\rangle$.

We start by letting $(T^0, \vec{f}^0, \eta_0)$ be the maximal element of D, that is, $T^0 := \{\emptyset\}$, $\vec{f}^0 : 1 \to {}^\theta\{\emptyset\}$, and $\eta_0 := 0$. Next, suppose that $\gamma < \mu$ is a nonzero even ordinal and we have already obtained a decreasing sequence of conditions $\langle (T^\beta, \vec{f}^\beta, \eta_\beta) \mid \beta < \gamma\rangle$ in D, and a sequence $\langle \vec{z}^\beta \mid \beta < \gamma \text{ nonzero even ordinal}\rangle$ in such a way that all of the following hold:

(i) for all $\alpha < \beta < \gamma$, $X_\xi \subseteq \circledcirc(\vec{f}^\alpha(\eta_\alpha), \vec{f}^\beta(\eta_\beta))$;
(ii) for every nonzero even $\beta < \gamma$, $\vec{z}^\beta : (\beta, \mu) \to T^\beta$, where $\beta < i < \mu$ implies:

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- $\vec{z}^{\beta}(i)$ belongs to the top level of T^{β};
- $\vec{z}^{\beta}(i) \neq^{*} \vec{z}^{\beta}(i')$ for every $i' \in (\beta, \mu) \setminus \{i\}$;
- $\vec{z}^{\beta}(i) \neq^{*} \vec{f}^{\beta}(\eta_{\beta})(0)$;
- $\vec{z}^{\beta}(i)$ and $\vec{f}^{\beta}(\eta_{\beta})(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$;

(iii) for every pair $\alpha < \beta$ of even ordinals below γ:
- $\circledcirc(\vec{f}^{\alpha}(\eta_{\alpha}), \vec{f}^{\beta}(\eta_{\beta})) = \theta$;
- $\vec{z}^{\alpha}(i) \subseteq \vec{z}^{\beta}(i)$ provided $0 < \alpha < \beta < i < \mu$.

Our description of the γ^{th}-move is divided into three cases, as follows.

▶ If $\gamma = 2$, then using Lemma 3.5, fix a one-step η_1-extension $\langle T^2, \vec{f}^2 \rangle$ of $\langle T^1, \vec{f}^1 \rangle$ such that $v_{T^2} \geq \max\{\mu, \theta^+\}$. Letting $\eta_2 := \eta_{T^2}$, it is clear that $(T^2, \vec{f}^2, \eta_2)$ belongs to D and extends $(T^1, \vec{f}^1, \eta_1)$. Now, as $v_{T^2} \geq \max\{\mu, \theta^+\}$, we may fix a map $\vec{z}^2 : (2, \mu) \to T^2$, where $2 < i < \mu$ implies:

- $\vec{z}^2(i)$ belongs to the top level of T^2;
- $\vec{z}^2(i) \neq^{*} \vec{z}^2(i')$ for every $i' \in (2, \mu) \setminus \{i\}$;
- $\vec{z}^2(i) \neq^{*} \vec{f}^2(\eta_2)(0)$;
- $\vec{z}^2(i)$ and $\vec{f}^2(\eta_2)(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$.

Indeed, as $\eta_2 = \eta_1 + 1$, this can be achieved by ensuring that $\langle \vec{z}^2(i)(\eta_1) \mid 2 < i < \mu \rangle^{\frown} \langle \vec{f}^2(\tau)(\eta_1) \mid \tau < \theta \rangle$ be an injective sequence, whereas $\langle \vec{z}^2(i) \restriction \eta_1 \mid 2 < i < \mu \rangle$ be a constant sequence whose sole element is $\vec{f}^2(\eta_2)(0) \restriction \eta_1$.

▶ If γ is a successor ordinal, say, $\gamma = \alpha + 2$, then using Lemma 3.5, fix a one-step η_{α}-extension $\langle T^{\gamma}, \vec{f}^{\gamma} \rangle$ of $\langle T^{\alpha+1}, \vec{f}^{\alpha+1} \rangle$ such that $v_{T^{\gamma}} \geq \max\{\mu, \theta^+\}$. Letting $\eta_{\gamma} := \eta_{T^{\gamma}}$, it is clear that $(T^{\gamma}, \vec{f}^{\gamma}, \eta_{\gamma})$ belongs to D, and

$$X_{\xi} \subseteq \circledcirc(\vec{f}^{\alpha}(\eta_{\alpha}), \vec{f}^{\alpha+1}(\eta_{\alpha+1})) \subseteq \circledcirc(\vec{f}^{\alpha+1}(\eta_{\alpha+1}), \vec{f}^{\gamma}(\eta_{\gamma})).$$

Next, we fix a map $\vec{z}^{\gamma} : (\gamma, \mu) \to T^{\gamma}$, where $\gamma < i < \mu$ implies:

- $\vec{z}^{\gamma}(i)$ belongs to the top level of T^{γ};
- $\vec{z}^{\gamma}(i) \neq^{*} \vec{z}^{\gamma}(i')$ for every $i' \in (\gamma, \mu) \setminus \{i\}$;
- $\vec{z}^{\alpha}(i) \subseteq \vec{z}^{\gamma}(i)$ and $\vec{z}^{\gamma}(i) \neq^{*} \vec{f}^{\gamma}(\eta_{\gamma})(0)$;
- $\vec{z}^{\gamma}(i)$ and $\vec{f}^{\gamma}(\eta_{\gamma})(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$.

This is possible because the combination of $v_{T^{\gamma}} \geq \max\{\mu, \theta^+\}$ with $\eta_{\gamma} = \eta_{\alpha+1} + 1$ and $\circledcirc(\vec{f}^{\alpha}(\eta_{\alpha}), \vec{f}^{\gamma}(\eta_{\gamma})) = \theta$ enables us to ensure that $\langle \vec{z}^{\gamma}(i)(\eta_{\alpha+1}) \mid \gamma < i < \mu \rangle^{\frown} \langle \vec{f}^{\gamma}(\tau)(\eta_{\alpha+1}) \mid \tau < \theta \rangle$ be an injective sequence, and $\langle \vec{z}^{\gamma}(i) \restriction \eta_{\alpha+1} \mid \gamma < i < \mu \rangle$ be equal to $\langle \vec{z}^{\alpha}(i) * (\vec{f}^{\gamma}(\eta_{\gamma})(0) \restriction \eta_{\alpha+1}) \mid \gamma < i < \mu \rangle$.

▶ If γ is a limit ordinal, then obtain $\langle T^{\gamma}, \vec{f}^{\gamma} \rangle$ and $\vec{z}^{\gamma}$ by appealing to Lemma 3.6 with the sequence $\langle (T^{\beta}, \vec{f}^{\beta}, \vec{z}^{\beta}) \mid \beta < \gamma \text{ nonzero even ordinal} \rangle$. Then let $\eta_{\gamma} := \eta_{T^{\gamma}}$.

In all cases, $(T^{\gamma}, \vec{f}^{\gamma}, \eta_{\gamma})$ belongs to D and requirements (i)–(iii) are satisfied for γ by the induction and the construction of $(T^{\gamma}, \vec{f}^{\gamma}, \eta_{\gamma})$. This completes the description of our winning strategy for **II**. ⊣

LEMMA 3.11. *For every $\xi < \zeta$, $\mathbb{S}^{\kappa}_{\mathbf{X}} \Vdash \dot{\mathbb{A}}^{\kappa}_{\mathbf{X},\xi}$ has the κ-cc.*

In particular, $\mathbb{S}^{\kappa}_{\mathbf{X}} \Vdash \dot{\mathbb{A}}^{\kappa}_{\mathbf{X}}$ has the κ-cc.

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PROOF. The "in particular" part follows from the fact that $\zeta < \theta^+ < \kappa$. Now, fix a $\xi < \zeta$, and let $\dot{A}$ be an $\mathbb{S}^\kappa_{\mathbf{X}}$-name for a maximal antichain in $\dot{\mathbb{A}}^\kappa_{\mathbf{X},\xi}$. We shall need the following key claim concerning the poset $\mathbb{S}^\kappa_{\mathbf{X}}$.

CLAIM 3.11.1. *Let $\langle T, \vec{f}\rangle \in \mathbb{S}^\kappa_{\mathbf{X}}$. Then there exists $\langle T', \vec{f}'\rangle \leq \langle T, \vec{f}\rangle$ with $\circledcirc(\vec{f}(\eta_T), \vec{f}'(\eta_{T'})) = \theta$ such that for every triple $(\vec{x}, Y, \pi)$ satisfying the following two conditions*:

(A) *$\vec{x} = \langle x_\tau \mid \tau < \theta\rangle$ consists of mutually exclusive nodes of T_{η_T};*
(B) *$Y \in \mathcal{I}$ and $\pi : Y \to Y$ is an injection such that*:
- *$x_\tau = \vec{f}(\eta_T)(\tau)$ for every $\tau \in \theta \setminus Y$;*
- *$x_\tau =^* \vec{f}(\eta_T)(\pi(\tau))$ for every $\tau \in Y$.*

there exists an $\alpha < \eta_{T'}$ such that:

- $\langle T', \vec{f}'\rangle \Vdash_{\mathbb{S}^\kappa_{\mathbf{X}}} \alpha \in \dot{A}$;
- $X_\xi \setminus Y \subseteq \circledcirc(\vec{f}'(\alpha), \vec{f}'(\eta_{T'}))$;
- *$\vec{f}'(\alpha)(\tau) \subseteq x_\tau * \vec{f}'(\eta_{T'})(\pi(\tau))$ for every $\tau \in X_\xi \cap Y$.*

PROOF. By our assumptions, $\mu := \max\{|T_{\eta_T}|, \theta\}^\theta$ is smaller than κ. Fix a bijection h from the set of all even ordinals in μ to the set of all triples $(\vec{x}, Y, \pi)$ satisfying (A) and (B). We shall construct a sequence $\langle (T^j, \vec{f}^j, \vec{z}^j) \mid j < \mu\rangle$ in such a way that all of the following hold:

(i) $\langle T^0, \vec{f}^0\rangle = \langle T, \vec{f}\rangle$;
(ii) for all $i < j < \mu$, $\langle T^j, \vec{f}^j\rangle \leq \langle T^i, \vec{f}^i\rangle$, and $X_\xi \subseteq \circledcirc(\vec{f}^i(\eta_{T^i}), \vec{f}^j(\eta_{T^j}))$;
(iii) for every pair $i < j$ of even ordinals below μ, $\circledcirc(\vec{f}^i(\eta_{T^i}), \vec{f}^j(\eta_{T^j})) = \theta$;
(iv) for every nonzero even $j < \mu$, $\vec{z}^j : (j, \mu] \to T^j$, where $j < \iota < \mu$ implies:
- $\vec{z}^j(\iota)$ belongs to the top level of T^j;
- $\vec{z}^j(\iota) \neq^* \vec{z}^j(\iota')$ for every $\iota' \in (j, \mu] \setminus \{\iota\}$;
- $\vec{z}^j(\iota) \neq^* \vec{f}^j(\eta_{T^j})(0)$;
- $\vec{z}^j(\iota)$ and $\vec{f}^j(\eta_{T^j})(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$;
- $\vec{z}^i(\iota) \subseteq \vec{z}^j(\iota)$ for every nonzero even $i < j$ and every $\iota \in (j, \mu]$.

(v) for every even $i < \mu$, letting $(\langle x_\tau \mid \tau < \theta\rangle, Y, \pi) := h(i)$, there exists an $\alpha \leq \eta_{T^{i+1}}$ such that:
(a) $\langle T^{i+1}, \vec{f}^{i+1}\rangle \Vdash_{\mathbb{S}^\kappa_{\mathbf{X}}} \alpha \in \dot{A}$;
(b) $X_\xi \setminus Y \subseteq \circledcirc(\vec{f}^{i+1}(\alpha), \vec{f}^{i+1}(\eta_{T^{i+1}}))$;
(c) $\vec{f}^{i+1}(\alpha)(\tau) \subseteq x_\tau * \vec{f}^{i+1}(\eta_{T^{i+1}})(\pi(\tau))$ for every $\tau \in X_\xi \cap Y$.

We start by setting $(T^0, \vec{f}^0, \vec{z}^0) := (T, \vec{f}, \emptyset)$. Next, suppose that $j \in (0, \mu]$ is such that $\langle (T^i, \vec{f}^i, \vec{z}^i) \mid i < j\rangle$ has already been defined.

Case 1: $j = i + 1$ for an even ordinal i: Let $(\langle x_\tau \mid \tau < \theta\rangle, Y, \pi) := h(i)$. By (B), $\langle \vec{f}(\eta_T)(\tau) \mid \tau \in \theta \setminus Y\rangle^\frown \langle x_\tau \mid \tau \in Y\rangle$ coincides with $\langle x_\tau \mid \tau < \theta\rangle$. So, by (A), it consists of mutually exclusive nodes. By Clauses (i) and (ii), $\circledcirc(\vec{f}(\eta_T), \vec{f}^i(\eta_{T^i})) = \theta$. So, since $Y \in \mathcal{I}$, we can take a one-step extension $\langle S, \vec{g}\rangle$ of $\langle T^i, \vec{f}^i\rangle$ such that:

- $\theta \setminus Y \subseteq \circledcirc(\vec{f}^i(\eta_{T^i}), \vec{g}(\eta_S))$, and
- for every $\tau \in Y$, $x_\tau * \vec{f}^i(\eta_{T^i})(\pi(\tau)) \subseteq \vec{g}(\eta_S)(\tau)$.

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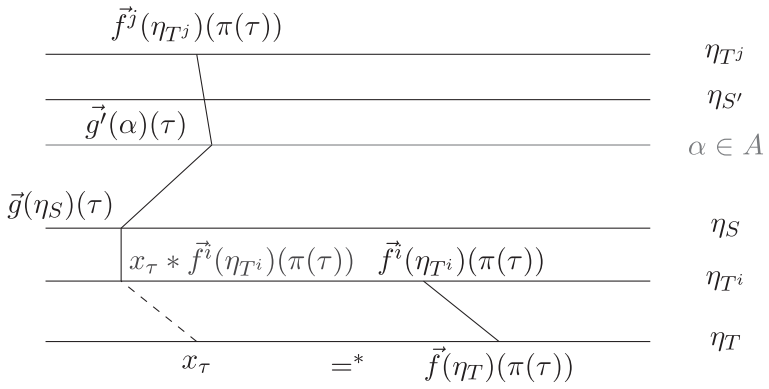


FIGURE 2. Case: $\tau \in X_\xi \cap Y$.

Recalling Definition 3.9, and as A is a maximal antichain, we may find a condition $\langle S', \vec{g}' \rangle \leq \langle S, \vec{g} \rangle$ such that, for some $\alpha \leq \eta_{S'}$,

- $\langle S', \vec{g}' \rangle \Vdash_{\mathbb{S}^\kappa_{\mathbf{x}}} \alpha \in \dot{A}$, and
- $X_\xi \subseteq \circledcirc(\vec{g}(\eta_S), \vec{g}'(\alpha))$.

Fix an injection $\psi : \theta \setminus ((X_\xi \setminus Y) \cup (X_\xi \cap \mathrm{Im}(\pi))) \to \theta \setminus X_\xi$.[8] Denote $\alpha' := \max\{\eta_S, \alpha\}$. Pick a one-step extension $\langle T^j, \vec{f}^{\,j} \rangle$ of $\langle S', \vec{g}' \rangle$ such that, for every $\tau < \theta$,

$$\vec{f}^{\,j}(\eta_{T^j})(\tau) = \begin{cases} (\vec{g}'(\alpha')(\tau) * \vec{g}'(\eta_{S'})(\tau))^\frown\langle\tau\rangle, & \text{if } \tau \in X_\xi \setminus Y; \\ (\vec{g}'(\alpha')(\pi^{-1}(\tau)) * \vec{g}'(\eta_{S'})(\tau))^\frown\langle\pi^{-1}(\tau)\rangle, & \text{if } \tau \in X_\xi \cap \mathrm{Im}(\pi); \\ (\vec{g}'(\alpha')(\psi(\tau)) * \vec{g}'(\eta_{S'})(\tau))^\frown\langle\psi(\tau)\rangle, & \text{otherwise.} \end{cases}$$

This ensures that the following two hold (Figure 2):

- $X_\xi \setminus Y \subseteq \circledcirc(\vec{g}'(\alpha), \vec{f}^{\,j}(\eta_{T^j}))$;
- $\vec{g}'(\alpha)(\tau) \subseteq x_\tau * \vec{f}^{\,j}(\eta_{T^j})(\pi(\tau)) = \vec{f}^{\,j}(\eta_{T^j})(\pi(\tau))$ for every $\tau \in X_\xi \cap Y$.

Case 2: $j = 2$: Using Lemma 3.5, fix a one-step extension $\langle T^2, \vec{f}^{\,2} \rangle$ of $\langle T^1, \vec{f}^{\,1} \rangle$ such that $\nu_{T^2} \geq \mu$. In particular, we may fix a map $\vec{z}^{\,2} : (2, \mu] \to T^2$, where $2 < \iota < \mu$ implies:

- $\vec{z}^{\,2}(\iota)$ belongs to the top level of T^2;
- $\vec{z}^{\,2}(\iota) \neq^* \vec{z}^{\,2}(\iota')$ for every $\iota' \in (2, \mu] \setminus \{\iota\}$;
- $\vec{z}^{\,2}(\iota) \neq^* \vec{f}^{\,2}(\eta_{T^2})(0)$;
- $\vec{z}^{\,2}(\iota)$ and $\vec{f}^{\,2}(\eta_{T^2})(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$.

Case 3: $j = i + 2$ *for an even ordinal* i: Using Lemma 3.5, let $\langle T^j, \vec{f}^{\,j} \rangle$ be a one-step η_{T^i}-extension of $\langle T^{i+1}, \vec{f}^{\,i+1} \rangle$ with $\nu_{T^j} \geq \mu$. As $\circledcirc(\vec{f}^{\,i}(\eta_{T^i}), \vec{f}^{\,j}(\eta_{T^j})) = \theta$, we may then fix a map $\vec{z}^{\,j} : (j, \mu] \to T^j$, where $j < \iota < \mu$ implies:

[8] This is precisely where the hypothesis $|\theta \setminus X_0| = \theta$ comes into play.

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- $\vec{z}^j(\iota)$ belongs to the top level of T^j;
- $\vec{z}^j(\iota) \neq^* \vec{z}^j(\iota')$ for every $\iota' \in (j,\mu] \setminus \{\iota\}$;
- $\vec{z}^i(\iota) \subseteq \vec{z}^j(\iota)$ and $\vec{z}^j(\iota) \neq^* \vec{f}^j(\eta_{T^j})(\tau)$;
- $\vec{z}^j(\iota)$ and $\vec{f}^j(\eta_{T^j})(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$.

Case 4: $j \in \mathrm{acc}(\mu+1)$: Obtain $(T^j, \vec{f}^j, \vec{z}^j)$ by appealing to Lemma 3.6 with the sequence $\langle (T^i, \vec{f}^i, \vec{z}^i) \mid i < j \text{ nonzero even ordinal}\rangle$, using $\delta := \mu+1$.

Once we are done with the recursion, it is clear that $\langle T', \vec{f}'\rangle := \langle T^\mu, \vec{f}^\mu\rangle$ is as sought. ⊣

Let $\langle T, \vec{f}\rangle$ be an arbitrary condition in $\mathbb{S}^\kappa_{\mathbf{X}}$, and we shall find an extension of it forcing that A is bounded in κ. To this end, we shall construct a sequence $\langle (T^j, \vec{f}^j, \vec{z}^j) \mid j < \theta^+\rangle$ in such a way that all of the following hold:

(i) $\langle T^0, \vec{f}^0\rangle \leq \langle T, \vec{f}\rangle$;
(ii) for all $i < j < \theta^+$, $\langle T^j, \vec{f}^j\rangle \leq (T^i, \vec{f}^i\rangle$ and $\circledcirc(\vec{f}^i(\eta_{T^i}), \vec{f}^j(\eta_{T^j})) = \theta$;
(iii) for every $j < \theta^+$, $\vec{z}^j : (j,\theta^+] \to T^j$, where $j < \iota < \theta^+$ implies:
- $\vec{z}^j(\iota)$ belongs to the top level of T^j;
- $\vec{z}^j(\iota) \neq^* \vec{z}^j(\iota')$ for every $\iota' \in (j,\theta^+] \setminus \{\iota\}$;
- $\vec{z}^j(\iota) \neq^* \vec{f}^j(\eta_{T^j})(0)$;
- $\vec{z}^j(\iota)$ and $\vec{f}^j(\eta_{T^j})(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$;
- $\vec{z}^i(\iota) \subseteq \vec{z}^j(\iota)$ for every $i < j$ and every $\iota \in (j,\theta^+]$.

We start by appealing to Lemma 3.5 to find a condition $\langle T^0, \vec{f}^0\rangle \leq \langle T, \vec{f}\rangle$ with $v_{T^0} \geq \theta^+$. Then, fix a map $\vec{z}^0 : (0,\theta^+] \to T^0$, where $0 < \iota < \theta^+$ implies:

- $\vec{z}^0(\iota)$ belongs to the top level of T^0;
- $\vec{z}^0(\iota) \neq^* \vec{z}^0(\iota')$ for every $\iota' \in (0,\theta^+] \setminus \{\iota\}$;
- $\vec{z}^0(\iota) \neq^* \vec{f}^0(\eta_0)(0)$;
- $\vec{z}^0(\iota)$ and $\vec{f}^0(\eta_0)(\tau)$ are mutually exclusive for every nonzero $\tau < \theta$.

Next, for every $i < \theta^+$ such that $(T^i, \vec{f}^i, \vec{z}^i)$ has already been determined, obtain $\langle T^{i+1}, \vec{f}^{i+1}\rangle$ by appealing to Claim 3.11.1 with $\langle T^i, \vec{f}^i\rangle$. By possibly replacing $\langle T^{i+1}, \vec{f}^{i+1}\rangle$ with a one-step $\eta_{T^{i+1}}$-extension of it (using Lemma 3.5), we may also assume that $\eta_{T^{i+1}}$ is a successor ordinal and $v_{T^{i+1}} \geq \theta^+$. Consequently, we may construct $\vec{z}^{i+1}$ satisfying requirement (iii) for $j := i+1$. At a limit stage i, we obviously use Lemma 3.6 with $\gamma := i$ and $\delta := \theta^+ + 1$. Hereon, for notational simplicity, denote η_{T^i} by η_i.

We claim that $\langle T^{\theta^+}, \vec{f}^{\theta^+}\rangle$ forces that A is bounded in κ, and in fact $A \subseteq \eta_{\theta^+}$. Suppose not, and pick a condition $\langle S, \vec{g}\rangle \leq \langle T^{\theta^+}, \vec{f}^{\theta^+}\rangle$ that forces some $\beta \geq \eta_{\theta^+}$ to be in A. Without loss of generality, $\beta \leq \eta_S$.

Consider $Y := \theta \setminus \circledcirc(\vec{g}(\beta), \vec{f}^{\theta^+}(\eta_{\theta^+}))$ which is an element of $\mathcal{I}$. Recalling that we have invoked Lemma 3.6 with $\delta := \theta^+ + 1$, every element of $T_{\eta_{\theta^+}}$ is equal modulo bounded to $\vec{f}^{\theta^+}(\eta_{\theta^+})(\tau)$ for some $\tau < \theta$. Consequently, we may fix a map $\pi : Y \to \theta$ such that $\vec{g}(\beta)(\tau) \restriction \eta_{\theta^+} =^* \vec{f}^{\theta^+}(\eta_{\theta^+})(\pi(\tau))$ for every $\tau \in Y$. As the elements of $\vec{g}(\beta)$ are mutually exclusive, it follows that π is an injection from Y to Y. To summarize, we have found a $Y \in \mathcal{I}$ and an injection $\pi : Y \to Y$ such that the following two hold:

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(1) $\vec{g}(\beta)(\tau) \restriction \eta_{\theta^+} = \vec{f}^{\theta^+}(\eta_{\theta^+})(\tau)$ for every $\tau \in \theta \setminus Y$;
(2) $\vec{g}(\beta)(\tau) \restriction \eta_{\theta^+} =^* \vec{f}^{\theta^+}(\eta_{\theta^+})(\pi(\tau))$ for every $\tau \in Y$.

Next, as $\mathrm{dom}(\vec{g}(\beta)) = \theta$, we may find a large enough $i < \theta^+$ such that:

(3) $\vec{g}(\beta)(\tau) \restriction \eta_{\theta^+} = (\vec{g}(\beta)(\tau) \restriction \eta_i) * \vec{f}^{\theta^+}(\eta_{\theta^+})(\pi(\tau))$ for every $\tau \in Y$.

In particular:

(4) $\vec{g}(\beta)(\tau) \restriction \eta_{i+1} = \vec{f}^{i+1}(\eta_{i+1})(\tau)$ for every $\tau \in \theta \setminus Y$, and
(5) $\vec{g}(\beta)(\tau) \restriction \eta_{i+1} =^* \vec{f}^{i+1}(\eta_{i+1})(\pi(\tau))$ for every $\tau \in Y$.

Consider $\vec{x} := \langle \vec{g}(\beta)(\tau) \restriction \eta_{i+1} \mid \tau < \theta \rangle$. Then the triple $(\vec{x}, Y, \pi)$ satisfies requirements (A) and (B) with respect to $\langle T^{i+1}, \vec{f}^{i+1} \rangle$. Therefore, there exists an $\alpha < \eta_{i+2}$ such that:

(6) $\langle T^{i+2}, \vec{f}^{i+2} \rangle \Vdash_{\mathbb{S}^\kappa_{\mathbf{X}}} \alpha \in \dot{A}$;
(7) $X_\xi \setminus Y \subseteq \circledcirc(\vec{f}^{i+2}(\alpha), \vec{f}^{i+2}(\eta_{i+2}))$;
(8) $\vec{f}^{i+2}(\alpha)(\tau) \subseteq x_\tau * \vec{f}^{i+2}(\eta_{i+2})(\pi(\tau))$ for every $\tau \in X_\xi \cap Y$.

By Clauses (1) and (7), for every $\tau \in X_\xi \setminus Y$,

$$\vec{f}^{i+2}(\alpha)(\tau) \subseteq \vec{f}^{i+2}(\eta_{i+2})(\tau) \subseteq \vec{f}^{\theta^+}(\eta_{\theta^+})(\tau) \subseteq \vec{g}(\beta)(\tau).$$

By Clauses (3) and (8), for every $\tau \in X_\xi \cap Y$, we have

$$\begin{aligned}
\vec{f}^{i+2}(\alpha)(\tau) &\subseteq x_\tau * \vec{f}^{i+2}(\eta_{i+2})(\pi(\tau)) \\
&= (\vec{g}(\beta)(\tau) \restriction \eta_{i+1}) * \vec{f}^{i+2}(\eta_{i+2})(\pi(\tau)) \\
&\subseteq (\vec{g}(\beta)(\tau) \restriction \eta_{i+1}) * \vec{f}^{\theta^+}(\eta_{\theta^+})(\pi(\tau)) \\
&= \vec{g}(\beta)(\tau) \restriction \eta_{\theta^+} \subseteq \vec{g}(\beta)(\tau).
\end{aligned}$$

As $\vec{f}^{i+2} \subseteq \vec{f}^{\theta^+} \subseteq \vec{g}$, we infer that $\langle S, \vec{g} \rangle$ forces that $\beta \in A$ is a proper extension of $\alpha \in A$, contradicting the fact that A is an antichain. ⊣

COROLLARY 3.12. *For every $\mathbb{S}^\kappa_{\mathbf{X}}$-generic filter G:*

(1) *$T(G)$ is a uniformly homogeneous κ-Souslin tree*;
(2) *$V(T(G))$ is a club in κ*;
(3) *$\vec{f}^G$ is a mutually exclusive $\mathcal{F}$-ascent path through $T(G)$*;
(4) *$T(G)$ coincides with the tree generated by its ascent path. Furthermore, for every $t \in T(G)$ and every $\xi < \zeta$, there are $\alpha < \kappa$ and $\tau \in X_\xi$ such that $t \subseteq \vec{f}^G(\alpha)(\tau)$.*

PROOF. (1) $T(G)$ is a uniformly homogeneous κ-tree thanks to Lemmas 3.5 and 3.7. Towards a contradiction, suppose that $T(G)$ admits a κ-sized antichain. It then follows from the upcoming Clause (4) that there exists an injective sequence $\langle (\alpha_\gamma, \tau_\gamma) \mid \gamma < \kappa \rangle$ of elements of $\kappa \times X_0$ such that $\{\vec{f}^G(\alpha_\gamma)(\tau_\gamma) \mid \gamma < \kappa\}$ is an antichain in $T(G)$. Find a $\tau \in X_0$ for which $\Gamma := \{\gamma < \kappa \mid \tau_\gamma = \tau\}$ has size κ. Then $\{\alpha_\gamma \mid \gamma \in \Gamma\}$ is a κ-sized antichain in $\mathbb{A}^\kappa_{\mathbf{X},0}$, contradicting Lemma 3.11.

(2) By Definition 3.2(3), it suffices to prove that for every $\varepsilon < \kappa$, there exists a $\langle T, \vec{f} \rangle \in G$ such that $\eta_T \in \mathrm{acc}(\kappa \setminus \varepsilon)$. Now, this follows from Lemma 3.6, as demonstrated by the proof of Lemma 3.10.

(3) By Definition 3.2(2).

(4) By a density argument, using Definition 3.2(4). ⊣

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§4. On the verge of weak compactness. For the purpose of this section, we fix an $\mathbf{X} = \langle X_n \mid n < \omega \rangle$ satisfying the following requirements:

- $\mathbf{X}$ is a $\subseteq$-decreasing sequence of nonempty subsets of ω;
- $|\omega \setminus X_0| = |X_0 \setminus X_1| = \omega$;
- $\bigcap_{n<\omega} X_n = \emptyset$.

As in the previous section, we denote by $\mathcal{F}$ the filter generated by $\mathbf{X}$, and we derive the corresponding notions of forcing of Definitions 3.2 and 3.9. The next result is the main theorem of the article.

THEOREM 4.1. *Suppose that κ is a weakly compact cardinal. In some forcing extension, κ is a strongly inaccessible cardinal satisfying $\chi(\kappa) = 1$, and there exists a κ-Souslin tree T such that $V(T) = \mathrm{acc}(\kappa)$.*

PROOF. For an ordinal τ, let $\mathbb{P}_\tau$ be the Easton-support iteration of length τ which forces with $\mathbb{S}^\eta_{\mathbf{X}} * \mathbb{A}^\eta_{\mathbf{X}}$ at every strongly inaccessible cardinal $\eta < \tau$. As usual, at $\eta < \tau$ that is not strongly inaccessible, we use trivial forcing.

Let G_κ be $\mathbb{P}_\kappa$-generic over V. Let g be $\mathbb{S}^\kappa_{\mathbf{X}}$-generic over $V[G_\kappa]$ and let h be $\mathbb{A}^\kappa_{\mathbf{X},0}$-generic over $V[G_\kappa][g]$. Note that $G_{\kappa+1} := G_\kappa * g * h$ is $\mathbb{P}_{\kappa+1}$-generic over V.

By Corollary 3.12, in $V[G_\kappa][g]$, $T(g)$ is a uniformly homogeneous κ-Souslin tree admitting an $\mathcal{F}$-ascent path f^g, and $V(T(g))$ covers a club in κ. By [13, Lemma 2.15], then,

$$V[G_\kappa][g] \models \text{“there exists a } \kappa\text{-Souslin tree } T \text{ such that } V(T) = \mathrm{acc}(\kappa)\text{.”}$$

It thus remains to prove that $\chi(\kappa) \leq 1$ holds in $V[G_\kappa][g]$. To this end, let $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ be an arbitrary C-sequence in $V[G_\kappa][g]$.

Work in V. Let $\dot{C}$ be a $\mathbb{P}_\kappa * \dot{\mathbb{S}}^\kappa_{\mathbf{X}}$-name for $\vec{C}$. Take a κ-model M containing $\mathbb{P}_\kappa * \dot{\mathbb{S}}^\kappa_{\mathbf{X}} * \dot{\mathbb{A}}^\kappa_{\mathbf{X}}$ and $\dot{C}$. As κ is weakly compact, we may now pick a κ-model N and a nontrivial elementary embedding $j_0 : M \to N$ with critical point κ. Hereafter, work in $V[G_{\kappa+1}]$.

CLAIM 4.1.1. *j_0 may be lifted to an elementary embedding*

$$j_1 : M[G_\kappa] \to N[G_{\kappa+1} * G_{tail}].$$

PROOF. By [3, Proposition 7.13 and Remark 7.14], $\mathbb{P}_\kappa$ has the κ-cc. In addition, by Lemma 3.10, in $V[G_\kappa]$, $\mathbb{S}^\kappa_{\mathbf{X}} * \mathbb{A}^\kappa_{\mathbf{X}}$ has a dense set that is $<\kappa$-strategically closed. So, from ${}^{<\kappa}N \subseteq N$, we infer that ${}^{<\kappa}N[G_{\kappa+1}] \subseteq N[G_{\kappa+1}]$. Since we use nontrivial forcing only at strongly inaccessibles, Lemma 3.10 implies that the quotient forcing $j_0(\mathbb{P}_\kappa)/G_{\kappa+1}$ is κ-strategically closed in $N[G_{\kappa+1}]$. But ${}^{<\kappa}N[G_{\kappa+1}] \subseteq N[G_{\kappa+1}]$, and hence $j_0(\mathbb{P}_\kappa)/G_{\kappa+1}$ is κ-strategically closed. As N has no more than κ many $\mathbb{P}_{\kappa+1}$-names, the model $N[G_{\kappa+1}]$ has no more than κ many dense sets of $j_0(\mathbb{P}_\kappa)/G_{\kappa+1}$. It thus follows that a $j_0(\mathbb{P}_\kappa)/G_{\kappa+1}$-generic over $N[G_{\kappa+1}]$ may be recursively constructed, say G_{tail}, so that $N[G_{\kappa+1}][G_{tail}]$ is a $j_0(\mathbb{P}_\kappa)$-generic extension of N. As j_0 is the identity map over $\mathbb{P}_\kappa$, Silver's criterion holds vacuously, so we may lift j_0 to a $j_1 : M[G_\kappa] \to N[G_{\kappa+1} * G_{tail}]$, as sought. ⊣

CLAIM 4.1.2. *j_1 may be lifted to an elementary embedding*

$$j_2 : M[G_\kappa * g] \to N[G_{\kappa+1} * G_{tail} * g_{tail}].$$

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PROOF. For every $n \in X_0$, define $b_n : \kappa \to \kappa$ via:

$$b_n := \bigcup\{\vec{f}^{\,g}(\alpha)(n) \mid \alpha \in h\}.$$

As h is $\mathbb{A}^{\kappa}_{\mathbf{X},0}$-generic over $V[G_\kappa][g]$, each such b_n is a cofinal branch through $T(g)$. Furthermore, the elements of $\langle b_n \mid n \in X_0\rangle$ are mutually exclusive. Take $n_0 \in X_0 \setminus X_1$ and then let

$$T := T(g) \cup \{x * b_n \mid x \in T(g), n \in X_0 \,\&\, n \neq n_0\}.$$

Then T is a uniformly homogeneous tree for which b_{n_0} is a vanishing κ-branch. Thus, $V(T) = V(T(g)) \cup \{\kappa\}$. Next, we extend the ascent path $\vec{f}^{\,g}$ by setting $\vec{f} := \vec{f}^{\,g} \cup \{(\kappa, \langle b_{\pi(n)} \mid n < \omega\rangle)\}$, where $\pi : \omega \to X_0 \setminus \{n_0\}$ is some bijection such that $\pi \restriction X_1$ is the identity function.

Altogether, $\langle T, \vec{f}\rangle$ is a legitimate condition of $\mathbb{S}^{j_1(\kappa)}_{\mathbf{X}}$ as computed in $N[G_{\kappa+1}]$, in particular, it is an element of the last iterand of $j_1(\mathbb{P}_\kappa * \dot{\mathbb{S}}^{\kappa}_{\mathbf{X}})/G_{\kappa+1} * G_{tail}$. In addition, $\langle T, \vec{f}\rangle$ extends every condition lying in $j_1[g]$.

Since ${}^{<\kappa}N[G_{\kappa+1}] \subseteq N[G_{\kappa+1}]$ and G_{tail} is a generic for a κ-strategically closed forcing, ${}^{<\kappa}N[G_{\kappa+1} * G_{tail}] \subseteq N[G_{\kappa+1} * G_{tail}]$. By Lemma 3.10, $\mathbb{S}^{j_1(\kappa)}_{\mathbf{X}}$ is $<j_1(\kappa)$-strategically closed in $N[G_{\kappa+1} * G_{tail}]$. But ${}^{<\kappa}N[G_{\kappa+1} * G_{tail}] \subseteq N[G_{\kappa+1} * G_{tail}]$, and hence $\mathbb{S}^{j_1(\kappa)}_{\mathbf{X}}$ is κ-strategically closed. As N has no more than κ many $j_1(\mathbb{P}_\kappa)$-names, the model $N[G_{\kappa+1} * G_{tail}]$ has no more than κ many dense sets of $j_1(\mathbb{P}_\kappa * \dot{\mathbb{S}}^{\kappa}_{\mathbf{X}})/G_{\kappa+1} * G_{tail}$. It thus follows that a $j_1(\mathbb{P}_\kappa * \dot{\mathbb{S}}^{\kappa}_{\mathbf{X}})/G_{\kappa+1} * G_{tail}$-generic over $N[G_{\kappa+1} * G_{tail}]$ may be recursively constructed, say g_{tail}, so that $N[G_{\kappa+1} * G_{tail}][g_{tail}]$ is a $j_1(\mathbb{P}_\kappa * \dot{\mathbb{S}}^{\kappa}_{\mathbf{X}})$-generic extension of N and $\langle T, f\rangle \in g_{tail}$. By Silver's criterion, we may now lift j_1 to a $j_2 : M[G_\kappa][g] \to N[G_{\kappa+1} * G_{tail}][g_{tail}]$, as sought. ⊣

Let $j_2 : M[G_\kappa][g] \to N[G_{\kappa+1} * G_{tail} * g_{tail}]$ be given by the preceding claim. Clearly, $\vec{C}$ is in $M[G_\kappa][g]$, so that $j_2(\vec{C})$ is a C-sequence over $j_2(\kappa)$. Let C_κ denote $j_2(\vec{C})(\kappa)$.

CLAIM 4.1.3. *For every $\alpha < \kappa$, there exists $\beta \geq \alpha$ such that $C_\kappa \cap \alpha = C_\beta \cap \alpha$.*

PROOF. Suppose not, and let $\alpha < \kappa$ be a counterexample. Denote $A := C_\kappa \cap \alpha$, and define a map $\varphi : \kappa \setminus \alpha \to \alpha$ via:

$$\varphi(\beta) := \min\{\varepsilon < \alpha \mid A \cap \varepsilon \neq C_\beta \cap \varepsilon\}.$$

By elementarity, $j_2(A) \cap j_2(\varphi)(\kappa) \neq C_\kappa \cap j_2(\varphi)(\kappa)$. However, $j_2(A) = A = C_\kappa \cap \alpha \subseteq \alpha > j_2(\varphi)(\kappa)$. This is a contradiction. ⊣

Recall that C_κ is a club in κ lying in $V[G_\kappa][g][h]$. As h is $\mathbb{A}^{\kappa}_{\mathbf{X}}$-generic over $V[G_\kappa][g]$, and as $\mathbb{A}^{\kappa}_{\mathbf{X}}$ satisfies the κ-cc, it follows that there exists a club $D \subseteq C_\kappa$ with $D \in V[G][g]$. Evidently, for every $\alpha < \kappa$, there exists a $\beta \geq \alpha$ such that $D \cap \alpha \subseteq C_\beta \cap \alpha$. ⊣

REMARK 4.2. Since $\mathbb{A}^{\kappa}_{\mathbf{X},\xi}$ only adds κ-many cofinal branches to $T(g)$, it follows from Proposition 2.7 that it does not resurrect the weak compactness of κ in $V[G_\kappa][g]$.

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COROLLARY 4.3. *Assuming the consistency of a weakly compact cardinal, it is consistent that for some inaccessible cardinal* κ, (II) *holds, but* (I) *fails*:

(I) *For every C-sequence* $\langle C_\beta \mid \beta < \kappa\rangle$, *there exists a cofinal* $\Delta \subseteq \kappa$ *such that, for every* $\alpha < \kappa$, *there exists a* $\beta < \kappa$ *with* $\Delta \cap \alpha \subseteq \mathrm{nacc}(C_\beta)$.
(II) *For every C-sequence* $\langle C_\beta \mid \beta < \kappa\rangle$, *there exists a cofinal* $\Delta \subseteq \kappa$ *such that, for every* $\alpha < \kappa$, *there exists a* $\beta < \kappa$ *with* $\Delta \cap \alpha \subseteq C_\beta$.

PROOF. Work in the model of Theorem 4.1. We have $\chi(\kappa) = 1$ which implies that $\chi(\kappa) \leq 1$, i.e., Clause (II) holds. In addition, there exists a κ-Souslin tree T such that $V(T) = \mathrm{acc}(\kappa)$. An inspection of the proof of [12, Theorem 2.23] makes it clear that there exists a club $D \subseteq \kappa$ such that for every partition $\mathcal{S}$ of D into stationary sets, $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \omega, {<}\omega)$ holds. In particular, $\clubsuit_{\mathrm{AD}}(\{D\}, 1, 1)$ holds. An inspection of the proof of [12, Theorem 2.30] then yields that Clause (I) must fail. ⊣

QUESTION 4.4. *Is Postulate* (I) *strictly weaker than weak compactness*?

4.1. Concluding remarks. It is not hard to check that the final model $V[G_\kappa][g]$ in the proof of Theorem 4.1 is an intersection model in the sense of [14]. In particular, an adaptation of the proof of [14, Corollary 4.7] to generic elementary embeddings yields that in this model, every κ-Aronszajn tree contains an ω-ascent path. In addition, this is the first example of a model obtained as the decreasing intersection of countably many models in which $\chi(\kappa) < \omega$. Furthermore, by strengthening the initial large cardinal hypotheses, this yields the following two corollaries.

COROLLARY 4.5. *Suppose that* κ *is a measurable cardinal. Then there is a forcing extension in which* κ *carries a* κ*-saturated ideal and an indecomposable ultrafilter, yet* κ *is not subtle.* ⊣

The reader is referred to [7, Section 3] for a construction of the κ-saturated ideal over κ, to [14, Corollary 3.10] for a construction of the indecomposable ultrafilter over κ, and to [13, Corollary 2.22] for the verification that κ is not subtle.

COROLLARY 4.6. *Suppose that* $\theta < \kappa$ *is a pair of cardinals such that* θ *is measurable and* κ *is supercompact. Then there is a cardinal-preserving forcing extension in which* κ *is the least* θ*-strongly compact cardinal and* $\chi(\kappa) = 1$. ⊣

The verification that κ is the least θ-strongly compact cardinal is essentially the same as that of [14, Corollary 3.8].

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DEPARTMENT OF MATHEMATICS
BAR-ILAN UNIVERSITY
ISRAEL
E-mail: rinotas@math.biu.ac.il
E-mail: zhixingy121@gmail.com

INDEPENDENT RESEARCHER
E-mail: yuan.jachen@gmail.com

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