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IMPROVED AVAILABILITY BOUNDS FOR BINARY AND MULTISTATE MONOTONE SYSTEMS WITH INDEPENDENT COMPONENT PROCESSES

JØRUND GÅSEMYR * ** AND
BENT NATVIG,* *** *University of Oslo*

Abstract

Multistate monotone systems are used to describe technological or biological systems when the system itself and its components can perform at different operationally meaningful levels. This generalizes the binary monotone systems used in standard reliability theory. In this paper we consider the availabilities of the system in an interval, i.e. the probabilities that the system performs above the different levels throughout the whole interval. In complex systems it is often impossible to calculate these availabilities exactly, but if the component performance processes are independent, it is possible to construct lower bounds based on the component availabilities to the different levels over the interval. In this paper we show that by treating the component availabilities over the interval as if they were availabilities at a single time point, we obtain an improved lower bound. Unlike previously given bounds, the new bound does not require the identification of all minimal path or cut vectors.

Keywords: Availability bound; binary monotone system; flow network system; minimal cut vector; minimal path vector; multistate monotone system

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1. Introduction

In multistate reliability theory we consider multistate monotone systems (MMSs), as defined in Block and Savits (1982). An MMS (C, ϕ) consists of a set $C = \{1, 2, \ldots, n\}$ of components and a structure function ϕ, defining the state of the system as an element in the set $S = \{0, 1, 2, \ldots, M\}$. Here n and M are arbitrary natural numbers. The state of component i at time t is denoted by $X_i(t)$, and belongs to a subset S_i of S, assumed in Natvig (2011) to contain 0 and M. We will, however, allow $M_i = \max(S_i)$ to be smaller than M in general. By setting $M = M_i = 1$, all the results of this paper cover the binary case.

The system state is supposed to be a nondecreasing function of the component states, and is given by $\phi(\boldsymbol{X}(t))$, where, by definition, $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$. We assume that $\phi(0, \ldots, 0) = 0$ and $\phi(M_1, \ldots, M_n) = M$. In accordance with tradition in the field, we consider time points t in some subset $\tau(I)$ of an interval I of interest, with $\tau(I)$ finite and $\tau(I) = I$ being typical special cases. The interval I is typically chosen for operational considerations.

The concept of an MMS generalizes the concept of a binary monotone system (BMS), studied in the binary reliability theory. It allows a more refined description of a system than the concept

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* Postal address: Department of Mathematics, University of Oslo, PO Box 1053 Blindern, 0316 Oslo, Norway.
** Email address: gaasemyr@math.uio.no
*** Email address: bent@math.uio.no

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of a BMS, which is often necessary in order to handle complex systems that can perform at different levels. The system could, for example, be an oil or gas transportation system, or an electrical power grid, where the probabilities that the system can deliver above certain given levels at any given time point within a certain time period are of interest. In specific applications it may be natural to let S and S_i consist of arbitrary real numbers that are directly interpretable as some kind of measurable quantities. However, by the use of discrete approximations restricting attention to state spaces consisting of natural numbers hardly represents a limitation. In general, the elements of S and S_i are thought of as representing an ordering of meaningful performance levels.

We will illustrate the various concepts through the following simple example.

Example 1. Let $C = \{1, 2, 3, 4\}$, $S_i = S = \{0, 1, 2\}$, and $\phi(\boldsymbol{x}) = \max(\min(x_1, x_2), \min(x_3, x_4))$. This system consists of two modules $\{1, 2\}$ and $\{3, 4\}$, both being series systems, and an organizing structure being a parallel system.

The component performance processes $\{X_i(t),\ t \in \tau(I)\}$, are random, possibly stochastically dependent processes involving repair at fixed or random points of time. The processes are assumed to be continuous from the right. A full probabilistic analysis of a multistate monotone system over an interval I requires the specification of a full dynamic model of the joint component process $\{\boldsymbol{X}(t),\ t \in \tau(I)\}$. A framework for the specification of such a parametric model was given by Gåsemyr and Natvig (2005). In all but very simple cases analytic calculations are intractable. Gåsemyr and Natvig (2005) outlined a procedure for simulating the process $\{\boldsymbol{X}(t),\ t \geq 0\}$, and also a data augmentation procedure for using such simulations in Bayesian estimation of the parameters of the model. A program for simulation of a binary system with independent component processes was presented by Huseby *et al.* (2010), while a similar program for the simulation of a multistate system with independent components was given by Huseby and Natvig (2013).

In complex systems, the above mentioned simulation based probabilistic analysis of the system may be computationally prohibitively costly. In many cases there is also insufficient information to model the dynamic behaviour of the marginal component processes, and even more so the joint process of dependent components. The analysis then has to be based on less accurate information about the system. In this paper we will consider methods based on the component availabilities

$$p_i^j = \mathbb{P}(X_i(t) \geq j \text{ for all } t \in \tau(I)) = \mathbb{P}\Big(\min_{t\in\tau(I)} X_i(t) \geq j\Big), \qquad i = 1, \ldots, n,\ j = 0, \ldots, M.$$

Here, $1 = p_i^0 \geq p_i^1 \geq \cdots \geq p_i^{M_i} \geq 0$, and $p_i^j = p_i^{j-1}$ if $j - 1 \notin S_i$, $p_i^j = 0$ if $M_i < j$. We denote by $\boldsymbol{p}$ the vector consisting of all the availabilities for all the components. The determination of p_i^j may be based on expert opinion, test data or operational data, or on a combination of these information sources. In the case of a fully specified dynamic model, the component availabilities are, in principle, given by the model, but the determination of p_i^j can be a difficult task in practice. Section 3.6 of Natvig (2011) provides an example where such a calculation is performed in a very simple system. The component availabilities can also be estimated by Monte Carlo simulation.

The component availabilities over an interval I do not determine the corresponding system availabilities

$$p_\phi^j = \mathbb{P}(\phi(\boldsymbol{X}(t)) \geq j \text{ for all } t \in \tau(I)), \qquad j = 1, \ldots, M.$$

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These system availabilities can then not be calculated, even in the case of independent components,and we have to resort to bounds. For the binary case, such bounds were studied by Bodin (1970), Esary and Proschan (1970), Barlow and Proschan (1975) and Natvig (1980). The multistate case was considered by Block and Savits (1982), Natvig (1982), Butler (1982), Funnemark and Natvig (1985), Natvig (1986), Natvig (1993) and Gåsemyr (2012). A comprehensive treatment was given by Natvig (2011). The basic bounds given in these publications are based on the sets of minimal path vectors and minimal cut vectors to level j, i.e. vectors $\boldsymbol{y}$, respectively $\boldsymbol{z}$, that are minimal, respectively maximal, in the natural ordering on $S_1 \times \cdots \times S_n$ with respect to the properties that $\phi(\boldsymbol{y}) \geq j$, respectively $\phi(\boldsymbol{z}) < j$.

Both upper and lower bounds were provided in these papers, the upper bounds generally being of poorer quality than the lower ones. From the point of view of a cautionary principle, lower bounds are much more important, and these are often quite good. However, in some cases the lower bounds are also poor, and this may have the unfortunate consequence that a disproportionate amount of resources are invested in improving the availability of the system.

This paper, concentrating on independent component processes, adds a new tool to the toolkit for this case. We define a lower bound for the system availabilities that, unlike the existing ones, is not based on the minimal path or cut vectors. Equation (3.17) of Natvig (2011), appearing in the proof of Theorem 3.7, can be used to define a corresponding lower bound for systems with dependent components.

The new lower bound is introduced in Section 2 of this paper. Despite the apparent simplicity of the idea behind this suggestion, it turns out that we obtain a lower bound that is an improvement on the existing ones generally, and a strict improvement in many cases. This is proved in Section 3. In Section 4 we introduce flow networks and shed light on the theoretical results by comparing the different lower bounds in a simple example. Section 5 contains some concluding remarks. The applicability of the new lower bound depends on computational issues, which will be discussed in a separate paper along with a case study.

2. The new lower bound

Consider first the special case when I collapses to a point, i.e. $I = [t, t]$. In this special case, assuming, in addition, that the component states at t are independent, the system availabilities are deterministic functions $h_\phi^j(\boldsymbol{p})$ of the component availabilities. To see this, define, for $i = 1, \ldots, n, k \in S_i$ (with $p_i^{M_i+1} = 0$),

$$r_i^k = \mathbb{P}(X_i(t) = k) = p_i^k - p_i^{k+1},$$

collected in the vector $\boldsymbol{r}$. Then by the independence of $X_i(t)$, $i = 1, \ldots, n$,

$$p(\boldsymbol{X}(t) = \boldsymbol{x}) = \prod_{i=1}^{n} r_i^{x_i}.$$

For each $j = 1, \ldots, M$, we may then write

$$\begin{aligned} p_\phi^j &= \mathbb{E}(\mathbf{1}(\phi(\boldsymbol{X}(t)) \geq j) \mid \boldsymbol{r}) \\ &= \sum_{\boldsymbol{x} \in S_1 \times \cdots \times S_n} \mathbf{1}(\phi(\boldsymbol{x}) \geq j) \prod_{i=1}^{n} r_i^{x_i} \end{aligned}$$

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$$= \sum_{\boldsymbol{x} \in S_1 \times \cdots \times S_n} \mathbf{1}(\phi(\boldsymbol{x}) \geq j) \prod_{i=1}^{n} (p_i^{x_i} - p_i^{x_i+1})$$
$$\coloneqq h_\phi^j(\boldsymbol{p}), \tag{1}$$

where $\mathbf{1}(\cdot)$ is the indicator function. If we are able to calculate (1) numerically, the need for a lower bound is eliminated in this special case.

Returning to a general interval I, we may still evaluate the function h_ϕ^j at $\boldsymbol{p}$, even though $\boldsymbol{p}$ now represents the component availabilities in I, rather than the availabilities at a specific point of time. In the case of independent component processes, our idea is to use the number $h_\phi^j(\boldsymbol{p})$ as a lower bound. We show in the following theorem that a lower bound is in fact obtained in this way.

Theorem 1. *Assume that the component processes are independent. Define*

$$\check{X}_i = \min_{t \in \tau(I)} X_i(t), \qquad i = 1, \ldots, n,$$

and let $\check{\boldsymbol{X}} = (\check{X}_1, \ldots, \check{X}_n)$. *Let* $\boldsymbol{p}$ *be the vector with components*

$$p_i^k = \mathbb{P}(\check{X}_i \geq k), \qquad i = 1, \ldots, n, k \in S_i.$$

Define

$$\tilde{l}_\phi^j(\boldsymbol{p}) = h_\phi^j(\boldsymbol{p}), \qquad j = 1, \ldots, M. \tag{2}$$

Then $p_\phi^j \geq \tilde{l}_\phi^j(\boldsymbol{p})$.

Proof. Note that $\mathbf{1}(\phi(\boldsymbol{X}(t)) \geq j \text{ for all } t \in \tau(I)) \geq \mathbf{1}(\phi(\check{\boldsymbol{X}}) \geq j)$.

By taking expectations, the inequality

$$p_\phi^j \geq \mathbb{P}(\phi(\check{\boldsymbol{X}}) \geq j) \tag{3}$$

follows, even in the case of dependent components. Now, if the component processes are independent in I then it is fairly obvious that $\check{X}_1, \ldots, \check{X}_n$ are independent, with $\mathbb{P}(\check{X}_i = k) = r_i^k$, where now

$$r_i^k = (p_i^k - p_i^{k+1}) = \mathbb{P}(\check{X}_i = k), \qquad i = 1, \ldots, n,\ k \in S_i. \tag{4}$$

Comparing with (1), it then follows that the right-hand side of (3) is equal to $h_\phi^j(\boldsymbol{p})$. □

In nonrepairable systems, $\check{X}_i = X_i(t_B)$ for every i, where t_B is the right end point of I. It follows that $\tilde{l}_\phi^j(\boldsymbol{p})$ is exact. If there is a high probability that the processes $X_i(t), i = 1, \ldots, n$, attain their minima simultaneously, the bound $\tilde{l}_\phi^j(\boldsymbol{p})$ is close to p_ϕ^j. In fact, defining $\tau_i = \{t\colon X_i(t) = \check{X}_i\} \cap \tau(I)$, we find, by conditioning on the events that the set $\bigcap_{i=1}^n \tau_i$ is respectively empty or nonempty, that

$$p_\phi^j - \tilde{l}_\phi^j(\boldsymbol{p}) = \mathbb{E}(\mathbf{1}([\min_{t \in \tau(I)} \phi(\boldsymbol{X}(t))] \geq j) - \mathbf{1}(\phi(\check{\boldsymbol{X}}) \geq j))$$
$$= \mathbb{E}\left(\mathbf{1}\left(\left[\min_{t \in \tau(I)} \phi(\boldsymbol{X}(t))\right] \geq j\right) - \mathbf{1}(\phi(\check{\boldsymbol{X}}) \geq j) \;\middle|\; \bigcap_{i=1}^n \tau_i = \varnothing\right) \mathbb{P}\left(\bigcap_{i=1}^n \tau_i = \varnothing\right)$$

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$$+\mathbb{E}\left(\mathbf{1}\left(\left[\min_{t\in\tau(I)}\phi(\boldsymbol{X}(t))\right]\geq j\right)-\mathbf{1}(\phi(\check{\boldsymbol{X}})\geq j)\;\Big|\;\bigcap_{i=1}^{n}\tau_i\neq\varnothing\right)\mathbb{P}\left(\bigcap_{i=1}^{n}\tau_i\neq\varnothing\right)$$

$$\leq\mathbb{P}\left(\bigcap_{i=1}^{n}\tau_i=\varnothing\right).$$

This latter probability is, for instance, small if we consider an interval that is so short that repairs are unlikely to take place, implying that $\check{\boldsymbol{X}}=\boldsymbol{X}(t_B)$.

Example 2. (*Example 1 continued.*) We assume that $\boldsymbol{p}_1=\ldots=\boldsymbol{p}_4=\boldsymbol{p}=(p^{(1)},p^{(2)})$. The deviating notation $p^{(1)}$, $p^{(2)}$ instead of p^1, p^2 is chosen here to avoid confusion in formulas involving squared availabilities.

Denoting the organizing parallel structure function by ψ and the modular series structure functions by χ_1, χ_2, we find, for $j=1,2$,

$$\begin{aligned}\tilde{l}_\phi^j(\boldsymbol{p})&=h_\phi^j(\boldsymbol{p})\\&=h_\psi^j(h_{\chi_1}^j(\boldsymbol{p}),h_{\chi_2}^j(\boldsymbol{p}))\\&=h_{\chi_1}^j(\boldsymbol{p})+h_{\chi_2}^j(\boldsymbol{p})-h_{\chi_1}^j(\boldsymbol{p})h_{\chi_2}^j(\boldsymbol{p})\\&=2(p^{(j)})^2-(p^{(j)})^4,\qquad j=1,2.\end{aligned}$$

The computation of $\tilde{l}_\phi^j(\boldsymbol{p})$ can be quite difficult in more complex systems than that of Example 1. For moderately sized systems, we can use the conceptually simple but computationally usually inefficient expression

$$h_\phi^j(\boldsymbol{p})=\sum_{\boldsymbol{x}\in S_1\times\cdots\times S_n\,|\,\phi(\boldsymbol{x})\geq j}\;\prod_{i=1}^{n}r_i^{x_i},$$

building directly on (1), with $r_i^{x_i}$ defined through (4). Still not feasible in very complex systems, with a suitable choice of i, the expression

$$h_\phi^j(\boldsymbol{p})=\sum_{k\in S_i}r_i^k\,\mathbb{P}(\phi(\check{\boldsymbol{X}})\geq j\mid\check{X}_i=k)\tag{5}$$

can sometimes simplify the computation. Using a modular decomposition, as we did in Example 1, can be a very efficient way to simplify calculations when it is available.

In more complex systems, more realistically we may estimate $\tilde{l}_\phi^j(\boldsymbol{p})$ with arbitrary accuracy by means of a simple Monte Carlo simulation technique, provided we are able to calculate $\phi(\boldsymbol{x})$ for any $\boldsymbol{x}\in S_1\times\cdots\times S_n$ reasonably efficiently. Indeed, knowing the probabilities $r_i^k=\mathbb{P}(\check{X}_i=k)$, $i=1,\ldots,n$, $k\in S_i$,, we may easily simulate $\check{\boldsymbol{X}}$. We estimate $\tilde{l}_\phi^j(\boldsymbol{p})$ by the mean of $\mathbf{1}(\phi(\check{\boldsymbol{X}}_r^*)\geq j)$ taken over a large sample of independent $\check{\boldsymbol{X}}_r^*$, $r=1,2,\ldots$. The computational issue will be further discussed in a separate paper.

3. Comparison with the established lower bounds

We start by reviewing the relevant lower bounds given in the literature. A generally valid lower bound using the minimal path vectors $\boldsymbol{y}^m$, $m=1,\ldots,M_p$, to level j was given by

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Funnemark and Natvig (1985) as

$$l_\phi^{\prime\prime j} = \max_{1\le m\le M_p} \mathbb{P}\Big(\bigcap_{i=1}^{n}(X_i(t) \ge y_i^m \text{ for all } t \in \tau(I))\Big) = \max_{1\le m\le M_p}\Big(\mathbb{P}\Big(\bigcap_{i=1}^{n}(\check{X}_i \ge y_i^m)\Big)\Big).$$

In the case of independent component processes, this takes the form

$$l_\phi^{\prime j}(\boldsymbol{p}) = \max_{1\le m\le M_p}\Big(\prod_{i=1}^{n}\mathbb{P}(\check{X}_i \ge y_i^m)\Big) = \max_{1\le m\le M_p}\Big(\prod_{i=1}^{n} p_i^{y_i^m}\Big). \tag{6}$$

Based on the minimal cut vectors $\boldsymbol{z}^m$, $m = 1, \ldots, M_c$, still under the assumption of independent component processes, we have the lower bound (see Butler (1982) and Funnemark and Natvig (1985))

$$l_\phi^{**j}(\boldsymbol{p}) = \prod_{m=1}^{M_c}\coprod_{i=1}^{n}\mathbb{P}(\check{X}_i > z_i^m) = \prod_{m=1}^{M_c}\coprod_{i=1}^{n} p_i^{z_i^m+1}, \tag{7}$$

where, for real numbers $p_i \in [0, 1]$, we define $\coprod_{i=1}^{n} p_i = 1 - \prod_{i=1}^{n}(1 - p_i)$.

Example 3. Let $C = \{1, 2\}$, $S_1 = S_2 = \{0, 1, 2\}$, $S = \{0, 1, 2, 3, 4\}$, and $\phi(\boldsymbol{x}) = x_1 + x_2$. This is an example of a flow network system, which is discussed more generally in Section 4. Assume that the component states are independent. Let $\boldsymbol{p}_1 = \boldsymbol{p}_2 = (p^{(1)}, p^{(2)}) = (0.9, 0.8)$. As in Example 1, the deviating notation $p^{(1)}$ and $p^{(2)}$ is chosen here to avoid confusion in formulas involving squared availabilities.

Below we state the minimal path and cut vectors and lower bounds to all the levels. In order to familiarize the reader with the theory, we state both analytic and numerical formulas for the calculations, in addition to the numerical value of the lower bounds. We use (5) to calculate $\tilde{l}_\phi^j(\boldsymbol{p})$.

To level 4, (2, 2) is the minimal path vector. We use the notation

$$l_\phi^{\prime 4}(\boldsymbol{p}) = (p^{(2)})^2 = 0.8^2 = 0.64.$$

Similarly, (2, 1) and (1, 2) are the minimal cut vectors with notation

$$l_\phi^{**4}(\boldsymbol{p}) = [1 - (1 - p^{(2)})]^2 = (p^{(2)})^2 = 0.8^2 = 0.64,$$
$$\tilde{l}_\phi^4(\boldsymbol{p}) = r^2 p^{(2)} = (p^{(2)})^2 = 0.8^2 = 0.64.$$

To level 3, (2, 1) and (1, 2) are the minimal path vectors with notation

$$l_\phi^{\prime 3}(\boldsymbol{p}) = p^{(2)}p^{(1)} = 0.9 \cdot 0.8 = 0.72.$$

Similarly, (2, 0), (1, 1), and (0, 2) are the minimal cut vectors with notation

$$l_\phi^{**3}(\boldsymbol{p}) = (p^{(1)})^2[1 - (1 - (p^{(2)}))^2] = 0.81 \cdot 0.96 = 0.78,$$
$$\tilde{l}_\phi^3(\boldsymbol{p}) = r^1 p^{(2)} + r^2 p^{(1)} = 0.1 \cdot 0.8 + 0.8 \cdot 0.9 = 0.80.$$

To level 2, (2, 0), (1, 1), and (0, 2) are the minimal path vectors with notation

$$l_\phi^{\prime 2}(\boldsymbol{p}) = (p^{(1)})^2 = 0.81.$$

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Similarly, $(1,0)$ and $(0,1)$ are the minimal cut vectors with notation

$$l_\phi^{**2}(\boldsymbol{p}) = [1-(1-p^{(2)})(1-p^{(1)})]^2 = 0.96, \qquad \tilde{l}_\phi^2(\boldsymbol{p}) = r^0 p^{(2)} + r^1 p^{(1)} + r^2 = 0.97.$$

To level 1, $(1,0)$ and $(0,1)$ are the minimal path vectors with notation

$$l_\phi'^1 = p^{(1)} = 0.9.$$

Similarly, $(0,0)$ is the minimal cut vector with notation

$$l_\phi^{**1}(\boldsymbol{p}) = 1-(1-p^{(1)})^2 = 0.99, \qquad \tilde{l}_\phi^1(\boldsymbol{p}) = r^0 p^{(1)} + r^1 + r^2 = 0.99.$$

In a complex system it may be quite difficult to identify all minimal path and cut vectors. It is worth noting that (6) is a valid, but possibly suboptimal, bound if the maximization is taken over a subset of the minimal path vectors or over a set including nonminimal path vectors. The bound (7) is, however, invalid unless all minimal cut vectors are identified and included in the product.

It depends on the structure function ϕ and on $\boldsymbol{p}$ whether $l_\phi^{**j}(\boldsymbol{p})$ or $l_\phi'^j(\boldsymbol{p})$ is the better bound. Moreover, contrary to intuition, the l^{**j}-bound is not necessarily nonincreasing in j. Hence, the best possible bounds based on (6) and (7) are obtained by maximization, and is defined as follows (see Funnemark and Natvig (1985)):

$$B_\phi^{*j}(\boldsymbol{p}) = \max_{j' \geq j} \max(l_\phi^{**j'}(\boldsymbol{p}), l_\phi'^{j'}(\boldsymbol{p})). \tag{8}$$

In order to compare $\tilde{l}_\phi^j(\boldsymbol{p})$ to the bounds (6)–(8) given above, we need some properties of associated random variables, given in Theorem 3.1 of Barlow and Proschan (1975). We have the following properties.

Property 1. Nondecreasing functions of independent variables are associated.

Property 2. If $X_1, \ldots, X_n$ are associated random variables taking values in $[0,1]$ then $\mathbb{E}(X_1 \cdots X_n) \geq \mathbb{E}(X_1) \cdots \mathbb{E}(X_n)$.

Theorem 2. *Assume that the component processes are independent. We then have*

$$\tilde{l}_\phi^j(\boldsymbol{p}) \geq B_\phi^{*j}(\boldsymbol{p}). \tag{9}$$

Proof. Choose a minimal path vector $\boldsymbol{y}$ for ϕ to level j. Clearly,

$$\mathbf{1}(\phi(\check{\boldsymbol{X}}) \geq j) \geq \mathbf{1}(\check{X}_i \geq y_i \text{ for all } i = 1, \ldots, n) = \mathbf{1}\Big(\bigcap_{i=1}^n (\check{X}_i \geq y_i)\Big).$$

Taking expectations, we obtain $\mathbb{P}(\phi(\check{\boldsymbol{X}}) \geq j) \geq \mathbb{P}(\bigcap_{i=1}^n (\check{X}_i \geq y_i))$. Maximizing over minimal path vectors $\boldsymbol{y} = \boldsymbol{y}^m$, $m = 1, \ldots, M_p$, we obtain

$$\mathbb{P}(\phi(\check{\boldsymbol{X}}) \geq j) \geq l_\phi''^j, \tag{10}$$

valid for any joint distribution of the component processes. With independent component processes, it follows that $\tilde{l}_\phi^j(\boldsymbol{p}) \geq l_\phi'^j(\boldsymbol{p})$.

Expressing the system state by means of the minimal cut vectors, we furthermore have

$$\mathbf{1}(\phi(\check{\boldsymbol{X}}) \geq j) = \prod_{m=1}^{M_c} \mathbf{1}\Big(\bigcup_{i=1}^n (\check{X}_i > z_i^m)\Big). \tag{11}$$

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Note that the indicator functions in this product are nondecreasing functions of the independent variables $\check{X}_1, \ldots, \check{X}_n$. Taking expectations, and using properties 1 and 2 of associated random variables, we obtain

$$\tilde{l}_\phi^j(\boldsymbol{p}) \geq \prod_{m=1}^{M_c} \mathbb{P}\left(\bigcup_{i=1}^{n} (\check{X}_i > z_i^m)\right) = \prod_{m=1}^{M_c} \coprod_{i=1}^{n} p_i^{z_i^m+1} = l_\phi^{**j}(\boldsymbol{p}).$$

Furthermore, $\tilde{l}_\phi^j(\boldsymbol{p})$ is clearly nonincreasing in j. Combining these facts, (9) follows. □

Remark 1. The probability that the right-hand side of (11) is equal to 1 appears, in a somewhat different notation, in Equation (3.17) of Natvig (2011) as part of the proof of Theorem 3.7. It is shown that this probability, which is equal to $\mathbb{P}(\phi(\check{\boldsymbol{X}}) \geq j)$, is smaller than or equal to p_ϕ^j and greater than or equal to $l_\phi^{**j}(\boldsymbol{p})$. It was, however, not realised that it could, in fact, be expressed as a function of $\boldsymbol{p}$, and could, hence, be used as a lower bound for the availability of systems with independent components.

The following theorem, giving necessary and sufficient conditions for $\tilde{l}_\phi^j(\boldsymbol{p})$ to be strictly better than $B_\phi^{*j}(\boldsymbol{p})$, is a counterpart to Theorem 3.24 of Natvig (2011). The proof uses arguments similar to arguments used in the proofs of Theorem 3.22 and 3.23 of Natvig (2011).

For a minimal cut vector $\boldsymbol{z}$ to level j, we define the corresponding minimal cut set to be the set $\{i \in C \mid z_i < M_i\}$. Similarly, the minimal path set corresponding to a minimal path vector $\boldsymbol{y}$ to level j is the set $\{i \in C \mid y_i > 0\}$.

Theorem 3. *Assume that the component processes are independent in I. Suppose that, for each $i \in S_i$, p_i^k is a strictly decreasing function of $k \in S_i$, with $p_i^{M_i} > 0$. Then, for a given level j, $\tilde{l}_\phi^j(\boldsymbol{p}) = B_\phi^{*j}(\boldsymbol{p})$ if and only if, for all minimal cut vectors to level j, all the corresponding minimal cut sets are disjoint. Otherwise, i.e. if at least two minimal cut sets to level j overlap, inequality (9) is strict.*

Proof. Let D_m, $m = 1, \ldots, M_c$, be the minimal cut sets corresponding to the minimal cut vectors to level j. For the corresponding minimal cut vectors $\boldsymbol{z}^m$, we have $z_i^m < M_i$ for $i \in D_m$, $z_i^m = M_i$ for $i \in D_m^c$, $m = 1, \ldots, M_c$, and $x_i \leq z_i^m$ for all $i \in D_m$ implies that $\phi(\boldsymbol{x}) < j$. The indicator functions $\mathbf{1}(\bigcup_{i \in D_m} (\check{X}_i \geq z_i^m + 1))$ are associated by property 1, being nondecreasing functions of the independent random variables $\check{X}_i$, $i = 1, \ldots, n$. If the sets D_m are disjoint, these indicator functions are also independent, by the independence of the vectors $\check{\boldsymbol{X}}_{D_m}$ whose components are the variables $\check{X}_i$, $i \in D_m$. In this case, it follows that

$$\begin{aligned}
\tilde{l}_\phi^j(\boldsymbol{p}) &= \mathbb{E}\left(\prod_{m=1}^{M_c} \mathbf{1}\left(\bigcup_{i \in D_m} (\check{X}_i \geq z_i^m + 1)\right)\right) \\
&= \prod_{m=1}^{M_c} \mathbb{E}\left(\mathbf{1}\left(\bigcup_{i \in D_m} (\check{X}_i \geq z_i^m + 1)\right)\right) \\
&= \prod_{m=1}^{M_c} \coprod_{i \in D_m} p_i^{z_i^m+1} \\
&= l_\phi^{**j}(\boldsymbol{p}).
\end{aligned}$$

Since $l_\phi^{**j}(\boldsymbol{p}) \leq B_\phi^{*j}(\boldsymbol{p})$, it follows that $\tilde{l}_\phi^j(\boldsymbol{p}) \leq B_\phi^{*j}(\boldsymbol{p})$. Since we also have the opposite inequality by (9), this proves the equality part.

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Now assume that the sets D_m are not disjoint. We may assume that $D_1 \cap (D_2 \cup \ldots \cup D_{M_c}) \neq \varnothing$. Then, as in the proof of Theorem 3.23 of Natvig (2011), the variables $\mathbf{1}(\bigcup_{i\in D_1}(\check{X}_i \geq z_i^1+1))$ and $\prod_{m=2}^{M_c}\mathbf{1}(\bigcup_{i\in D_m}(\check{X}_i \geq z_i^m+1))$ are dependent and associated, and, hence, they have a strictly positive covariance by Exercise 6 of Barlow and Proschan (1975, p. 31). It follows that

$$\begin{aligned}
\tilde{l}_\phi^j(\boldsymbol{p}) &= \mathbb{E}\Big(\mathbf{1}\Big(\bigcup_{i\in D_1}(\check{X}_i \geq z_i^1+1)\Big)\Big(\prod_{m=2}^{M_c}\Big(\mathbf{1}\Big(\bigcup_{i\in D_m}(\check{X}_i \geq z_i^m+1)\Big)\Big)\Big)\\
&> \mathbb{E}\Big(\mathbf{1}\Big(\bigcup_{i\in D_1}(\check{X}_i \geq z_i^1+1)\Big)\Big)\mathbb{E}\Big(\prod_{m=2}^{M_c}\Big(\mathbf{1}\Big(\bigcup_{i\in D_m}(\check{X}_i \geq z_i^m+1)\Big)\Big)\Big)\\
&\geq \mathbb{E}\Big(\mathbf{1}\Big(\bigcup_{i\in D_1}(\check{X}_i \geq z_i^1+1)\Big)\Big)\prod_{m=2}^{M_c}\mathbb{E}\Big(\mathbf{1}\Big(\bigcup_{i\in D_m}(\check{X}_i \geq z_i^m+1)\Big)\Big)\\
&= \prod_{m=1}^{M_c}\coprod_{i\in D_m} p_i^{z_i^m+1}\\
&= l_\phi^{**j}(\boldsymbol{p}).
\end{aligned}$$

Note that if there is only one minimal path vector $\boldsymbol{y}$ to level j, with corresponding minimal path set of the form $P = \{i_1, \ldots, i_k\}$, the minimal cut sets are necessarily $\{i_1\}, \ldots, \{i_k\}$, and are, hence, disjoint. Hence, by assumption there are at least two minimal path vectors for ϕ to level j. Furthermore, we may assume that $l_\phi'^j(\boldsymbol{p}) = \prod_{i=1}^n p_i^{y_i^1}$. Then, similar to the proof of Theorem 3.22 of Natvig (2011),

$$\begin{aligned}
\tilde{l}_\phi^j(\boldsymbol{p}) &= \mathbb{P}\Big(\bigcup_{m=1}^{M_p}\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^m)\Big)\Big)\\
&\geq \mathbb{P}\Big(\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^1)\Big)\cup\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^2)\Big)\Big)\\
&= \mathbb{P}\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^1)\Big)+\mathbb{P}\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^2)\Big)-\mathbb{P}\Big(\bigcap_{i=1}^n(\check{X}_i \geq \max(y_i^1, y_i^2))\Big)\\
&> \mathbb{P}\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^1)\Big)\\
&= \prod_{i=1}^n p_i^{y_i^1}\\
&= l_\phi'^j(\boldsymbol{p}).
\end{aligned}$$

Here we have used that $p_i^{\max(y_i^1,y_i^2)} < p_i^{y_i^2}$ for at least one i and $p_i^k > 0$ for all i, k, and, hence,

$$\mathbb{P}\Big(\bigcap_{i=1}^n(\check{X}_i \geq y_i^2)\Big)-\mathbb{P}\Big(\bigcap_{i=1}^n(\check{X}_i \geq \max(y_i^1, y_i^2))\Big) > 0.$$

Also, since $\tilde{l}_\phi^j(\boldsymbol{p})$ is nonincreasing in j, the strict inequality in (9) follows. □

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Corollary 1. *Assume that the component processes are independent. Suppose there is only one minimal path vector to level j. Then all the bounds (2), (6), and (7) for system availability to level j are exact, and we have*

$$p_\phi^j = \prod_{i=1}^n p_i^{y_i} = l_\phi'^j(\boldsymbol{p}) = \tilde{l}_\phi^j(\boldsymbol{p}) = l_\phi^{**j}(\boldsymbol{p}).$$

Proof. If there is only one minimal path vector $\boldsymbol{y}$ to level j, we have

$$p_\phi^j = \mathbb{P}\left(\bigcap_{i=1}^n (\check{X}_i \geq y_i)\right) = \prod_{i=1}^n p_i^{y_i} = l_\phi'^j(\boldsymbol{p}).$$

Obviously, we also have

$$\tilde{l}_\phi^j(\boldsymbol{p}) = h_\phi^j(\boldsymbol{p}) = \prod_{i=1}^n p_i^{y_i}.$$

Hence, the first three equalities follow. If P is the minimal path set corresponding to the minimal path vector $\boldsymbol{y}$, then there is one minimal cut vector for each $i \in P$, being of the form $(M_1, \ldots, M_{i-1}, y_i - 1, M_{i+1}, \ldots, M_n)$. Hence, (7) takes the form

$$l_\phi^{**j}(\boldsymbol{p}) = \prod_{i \in P}(1 - (1 - p_i^{(y_i-1)+1})) = \prod_{i=1}^n p_i^{y_i} = \tilde{l}_\phi^j(\boldsymbol{p}),$$

proving the last equality. □

An example of a system with one minimal path vector to each level is given by the structure function of a series system $\phi(\boldsymbol{x}) = \min_{1 \leq i \leq n} x_i$, for which $(j, \ldots, j)$ is the only minimal path vector to level j, $j = 1, \ldots, M$. For the system of Example 3, $(2, 2)$ is the only minimal path vector to level 4, which explains the equality of all the lower bounds in this case.

Under the conditions of Corollary 1, we also obviously have $p_\phi^j = B_\phi^{*j}(\boldsymbol{p})$. This identity does not hold under the weaker condition of Theorem 3, that the minimal cut sets are disjoint. A simple counterexample is a binary two-component parallel system, which has a single minimal cut set $\{1, 2\}$. By the equality part of Theorem 3, $p_\phi^1 - B_\phi^{*1}(\boldsymbol{p}) = p_\phi^1 - \tilde{l}_\phi^1(\boldsymbol{p})$ is equal to the probability that the two components are in state 0 on nonempty disjoint subsets of $\tau(I)$, which is positive due to independent component processes.

Remark 2. The identity $p_\phi^j = l_\phi'^j(\boldsymbol{p})$ and the corresponding identity $p_\phi^j = B_\phi^{*j}(\boldsymbol{p})$ are also stated in respectively Theorems 3.22 and 3.24 of Natvig (2011), but are erroneously based on the assumption that there is only one minimal path set to level j rather than the stronger assumption that there is only one minimal path vector to level j. For a counterexample, see the $j = 3$ case of Example 3. The two minimal path vectors $(2, 1)$ and $(1, 2)$ have a single common minimal path set $\{1, 2\}$, but overlapping minimal cut sets $\{2\}$, $\{1, 2\}$, $\{1\}$ corresponding to the minimal cut vectors $(2, 0)$, $(1, 1)$, $(0, 2)$, and, hence, the strict inequality of Theorem 3 holds.

In Example 1, there are four minimal cut sets $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, and $\{2, 4\}$, each intersecting with two others. Although the conditions for the strict inequality in Theorem 3 are, hence, abundantly satisfied, we find that, in this example, for reasonably large values of p^j, $l_\phi^{**j}(\boldsymbol{p}) = B_\phi^{*j}(\boldsymbol{p})$ is very close to $\tilde{l}_\phi^j(\boldsymbol{p})$. If relevant components of $\boldsymbol{p}$ are large, this is the case also in the two-component flow network system studied in the next section. However, for

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the larger k and correspondingly lower values of the relevant p_i^k involved when j approaches M, $l_\phi^{**j}(\boldsymbol{p})$ becomes increasingly poor compared to $\tilde{l}_\phi^j(\boldsymbol{p})$ when the number of states in S_i and S increases.

4. Comparison of the lower bounds in simple flow network systems

In this section we consider the special case of a flow network system. Such systems in general consist of a source node s, a terminal node t, and a varying number of other nodes between s and t. The nodes are connected to each other by directed edges. These edges $\{1, 2, \ldots, n\}$ represent the components of the system. Each edge i has a flow capacity that changes over time, representing the random state $X_i(t)$ of the component, while the state of the system is the flow from s to t. This flow is determined through the min-cut-max-flow theorem, see Ford and Fulkerson (1956), by what we will call minimal flow cut sets, to distinguish them from the minimal cut sets defined just before Theorem 3. A subset of components is called a flow cut set if there is no directed path from s to t if the set is removed from the graph. Such a set is called minimal if no proper subset is a flow cut set. The capacity of a flow cut set is the sum of the capacities of the components in the set. The flow through the system from s to t is then equal to the capacity of the minimal flow cut set with the smallest capacity. Denoting by $K_1, \ldots, K_k$ the set of minimal flow cut sets, the flow from s to t is, hence, given by

$$\phi(\boldsymbol{x}) = \min_{1 \le r \le k} \Big(\sum_{i \in K_r} x_i \Big). \tag{12}$$

Comparison of the different lower bounds in an extended version of Example 3 provides a good illustration of their properties. More complex systems will be analysed in a separate paper, focusing on computational aspects. Based on Example 3, one might think that although the minimal path lower bound is relatively poor in this type of system, the minimal cut lower bound compares quite favorably with $\tilde{l}_\phi^j(\boldsymbol{p})$. In Example 4 we show that this is not the case in general.

Example 4. Let $C = \{1, 2\}$, $S_1 = S_2 = \{0, 1, 2, \ldots, 5\}$, $S = \{0, 1, 2, \ldots, 10\}$, and $\phi(\boldsymbol{x}) = x_1 + x_2$. Assume that the component states are independent. Let $\boldsymbol{p}_1 = \boldsymbol{p}_2 = (0.95, 0.90, 0.85, 0.80, 0.75)$. We will limit attention to lower bounds for the system availability to levels 9, 6, and 3, and skip the analytic formulas for this example. We still use (5) to calculate $\tilde{l}_\phi^j(\boldsymbol{p})$.

To level 9, (4, 5) and (5, 4) are the minimal path vectors with notation

$$l_\phi'^9(\boldsymbol{p}) = 0.75 \cdot 0.80 = 0.60.$$

Similarly, (3, 5), (4, 4), and (5, 3) are the minimal cut vectors with notation

$$l_\phi^{**9}(\boldsymbol{p}) = 0.8^2 \cdot (1 - 0.0625) = 0.60, \qquad \tilde{l}_\phi^9(\boldsymbol{p}) = 0.05 \cdot 0.75 + 0.75 \cdot 0.80 = 0.64.$$

To level 6, (1, 5), (2, 4), (3, 3), (4, 2), and (5, 1) are the minimal path vectors with notation

$$l_\phi'^6(\boldsymbol{p}) = 0.85^2 = 0.72.$$

Similarly, (0, 5), (1, 4), (2, 3), (3, 2), (4, 1), and (5, 0) are the minimal cut vectors with notation

$$l_\phi^{**6}(\boldsymbol{p}) = 0.95^2 \cdot 0.975^2 \cdot 0.97^2 = 0.81,$$
$$\tilde{l}_\phi^6(\boldsymbol{p}) = 0.05 \cdot (0.75 + 0.80 + 0.85 + 0.90) + 0.75 \cdot 0.95 = 0.88.$$

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To level 3, (1, 2) and (2, 1) are the minimal path vectors with notation

$$l_\phi'^3(\boldsymbol{p}) = 0.95 \cdot 0.90 = 0.86.$$

Similarly, (0, 2), (1, 1), and (2, 0) are the minimal cut vectors with notation

$$l_\phi^{**3}(\boldsymbol{p}) = (1 - 0.0075)^2 \cdot 0.99 = 0.975,$$

$$\tilde{l}_\phi^3(\boldsymbol{p}) = 0.05 \cdot (0.85 + 0.90 + 0.95) + 0.85 = 0.985.$$

We see that $l_\phi'^j(\boldsymbol{p})$ is consistently the poorest lower bound. For the smallest value $j = 3$, $l_\phi^{**j}(\boldsymbol{p})$ is close to $\tilde{l}_\phi^j(\boldsymbol{p})$. For this value of j, the factors $\coprod_{i=1}^n p_i^{z_i^m+1} = 1 - \prod_{i \in D_m}(1 - p_i^{z_i^m+1})$ of (7) depend on values of $p_i^{z_i^m+1}$, $i \in D_m$, $1 \le m \le M_c$, that are close to 1. When j increases, smaller values of $p_i^{z_i^m+1}$ are involved in these factors, and this apparently makes $l_\phi^{**j}(\boldsymbol{p})$ more sensitive to this increase than $\tilde{l}_\phi^j(\boldsymbol{p})$. This partly explains a generally poorer performance of $l_\phi^{**j}(\boldsymbol{p})$ at the higher system levels compared to $\tilde{l}_\phi^j(\boldsymbol{p})$. However, the number M_c of factors in (7) is also important, and M_c attains its highest values at the intermediate values of j. This explains why $l_\phi^{**j}(\boldsymbol{p})$ compares least favourably with $\tilde{l}_\phi^j(\boldsymbol{p})$ for $j = 6$ in Example 4, where M_c attains its maximal value 6. A further increase in M_i and M would aggravate this problem.

The negative effect a large value of M_c can have on $l_\phi^{**j}(\boldsymbol{p})$ can also be expected to be quite significant in more complex systems. Consider, in general, a flow network system where the component state spaces are of the form $S_i = \{0, 1, 2, \ldots, M_i\}$, whereas the system state space is $S = \{0, 1, 2, \ldots, M\}$. We may assume that M is equal to the maximal flow through the system, which by (12) is $\min_{1 \le r \le k}(\sum_{i \in K_r} M_i)$.

It is easy to see that $\boldsymbol{z}$ is a minimal cut vector to level j if and only if there exists r such that $\sum_{i \in K_r} z_i = j - 1$, and $z_i = M_i$ for every $i \notin K_r$. Hence, an increased number of minimal flow cut sets due to an increased complexity of the system can be expected to lead to $l_\phi^{**j}(\boldsymbol{p})$ being relatively poor.

5. Concluding remarks

In this paper we have introduced a new lower bound $\tilde{l}_\phi^j(\boldsymbol{p})$ for system availability to level j over an interval I, valid in the case of independent component processes. We show both theoretically and in some simple examples that the bound is an improvement over existing ones. In addition, unlike these bounds, it does not require the identification of all minimal path or cut vectors. However, beyond simple cases like those analysed in this paper, the computation of $\tilde{l}_\phi^j(\boldsymbol{p})$ may be difficult. The most promising solution to this problem is the use of simulation based methods; a theme we pursue in another paper. Simulation techniques could also be used to extend the methodology to dependent component processes. If a reasonable full dynamic model for the joint component process $\boldsymbol{X}(t)$ can be specified, and if we could simulate sample paths $\boldsymbol{x}(t)$, $t \in \tau(I)$, from this model, extract the corresponding $\check{\boldsymbol{X}}$, and calculate each $\phi(\check{\boldsymbol{x}})$, we could, possibly at a very high computational cost, estimate a lower bound $\tilde{l}_\phi^j$ defined as the right-hand side of (3). Inequality (10) shows that this would be an improvement over $l_\phi''^j$. However, one could then also, in principle, estimate p_ϕ^j itself, rather than a lower bound, but at an even higher and possibly prohibitive computational cost.

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