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A Ratio Inequality for Binary Trees and the Best Secretary

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Let $\mathbf{T}^n$ be the complete binary tree of height n considered as the Hasse diagram of a poset with its root $\mathbf{1_n}$ as the maximum element. Define $A(n;T) = |\{S \subseteq \mathbf{T^n} : \mathbf{1_n} \in S,\ S \cong T\}|$, and $B(n;T) = |\{S \subseteq \mathbf{T^n} : \mathbf{1_n} \notin S,\ S \cong T\}|$. In this note we prove that $\frac{A(n;T_1)}{B(n;T_1)} \leqslant \frac{A(n;T_2)}{B(n;T_2)}$ for any fixed n and rooted binary trees T_1, T_2 such that T_2 contains a subposet isomorphic to T_1. We conjecture that the ratio A/B also increases with T for arbitrary trees. These inequalities imply natural behaviour of the optimal stopping time in a poset extension of the secretary problem.

1. Introduction

The question we propose here is motivated by an on-line decision problem, a partial order analogue of the celebrated secretary problem. The classical secretary problem (see [1, 2, 3]) goes as follows. An administrator is interviewing a totally ordered set of N candidates for a secretary position. Candidates are showing up in a random order for the interview. The administrator may stop the search procedure at any time by offering the

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job to the latest candidate. The decision should be based on the relative ranking of the already interviewed candidates, and once the next candidate is invited for the interview no previous ones may be offered the job. The administrator's aim is to choose the top candidate with highest probability. When should the search procedure be stopped?

A partial order analogue of this widely known optimal stopping problem has been discussed recently in [7]. There are $N = 2^n - 1$ candidates ranked according to a complete binary tree $\mathbf{T}^n$ of height n rooted at $\mathbf{1_n}$; the 'top candidate' is represented by $\mathbf{1_n}$, which is the maximum element of the poset whose Hasse diagram is $\mathbf{T^n}$. The $N = 2^n - 1$ points of $\mathbf{T}^n$ appear in a random sequence before an observer. The permutations $\omega = (\omega_1, \ldots, \omega_N)$ are considered as equiprobable elementary events of the procedure. After time k the observer may determine nothing but the subposet of $\mathbf{T^n}$ determined by $\{\omega_1, \ldots, \omega_k\}$ (*i.e.*, just the relative ranking of the first k candidates). The process of investigating the next point can be stopped at any time $\tau \leqslant N$ by selecting the last point ω_τ as a final choice. The goal is to find $\mathbf{1_n}$ in this way with the greatest success probability. Let $\tau_n(\omega)$ be the optimal stopping time that maximizes the probability $P[\omega_\tau = \mathbf{1_n}]$ over all stopping times τ. In [7] it was proved that

$$\tau_n(\omega) = \min\{m \ : \ \omega_m \text{ is maximal in the subposet determined by } \omega_1, \ldots, \omega_m \text{ that is either not totally ordered, or totally ordered and } m > n/2\}$$

(with $\tau_n(\omega) = 2^n - 1$ if the minimum on the right is undefined). It was also observed in [7] that $P[\omega_{\tau_n} = \mathbf{1_n}] \to 1$ when $n \to \infty$.

During this on-line decision procedure the observer sees the progress of the observed subposet until time τ_n. For fixed k, $1 \leqslant k \leqslant 2^n - 1$, and for a fixed tree T, a subposet of $\mathbf{T^n}$, let H be the set of all permutations ω such that $\tau_n(\omega) = k$, for each i, $1 \leqslant i \leqslant k$, the subposets determined by $\{\omega_1, \ldots, \omega_i\}$ for all ω are isomorphic, and the subposet determined by $\{\omega_1, \ldots, \omega_k\}$ is isomorphic to T.

The results in [7] show that the conditional probability $P[\omega_{\tau_n} = \mathbf{1_n} \mid H]$ is always greater than $1/2$ provided $\tau_n(\omega) < 2^n - 1$. The results obtained here allow us to address the question of more precise estimates of $P[\omega_{\tau_n} = \mathbf{1_n} \mid H]$. In [7] it was proved that

$$P[\omega_{\tau_n} = \mathbf{1_n} \mid H] = \frac{A(n;T)}{A(n;T) + B(n;T)},$$

where

$$A(n;T) = |\{S \subseteq \mathbf{T^n} : \mathbf{1_n} \in S,\ S \cong T\}|, \quad \text{and}$$
$$B(n;T) = |\{S \subseteq \mathbf{T^n} : \mathbf{1_n} \notin S,\ S \cong T\}|.$$

Note that in the formula for $P[\omega_{\tau_n} = \mathbf{1_n} \mid H]$ we do not need the complete history of the progressing posets. We investigate here the ratio $p = \frac{A(n;T)}{B(n;T)}$ that is crucial to evaluating $P[\omega_{\tau_n} = \mathbf{1_n} \mid H] = \frac{1}{1+1/p}$. Let T_1, T_2 be rooted trees considered as the Hasse diagrams of posets with their roots as maximum elements. In this note we prove that

$$\frac{A(n;T_1)}{B(n;T_1)} \leqslant \frac{A(n;T_2)}{B(n;T_2)},$$

for any fixed n and for rooted binary trees T_1, T_2 such that T_2 has a subposet isomorphic

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to T_1 (see Theorem 3.5). We conjecture that the ratio $A(n;T)/B(n;T)$ increases with T also for arbitrary rooted trees (and n fixed). This monotonicity was proved for chains in [7] and for stars rooted at the centre in [6].

Tools for proving the main inequality in Section 3 are introduced in Section 2. The paper concludes with Section 4, discussing some examples of the monotonicity of conditional probabilities in the poset extension of the secretary problem. For further extensions of the best choice problem see [4].

2. Tools

Let T be a tree considered as the Hasse diagram of a tree poset with its root as the maximum element. The minimum elements of a tree poset will be called leaves. Tree T is binary if it is rooted and every vertex has at most two immediate predecessors. For any point $x \in T$, the Hasse diagram of the subposet $T \setminus \{x\}$ is called a *point removal* of T. Note that the point removal is a graph determined by the subposet $T \setminus \{x\}$. In general, for any rooted trees T_1, T_2 we write $T_1 \subseteq T_2$ provided that T_1 is a subposet of T_2.

Proposition 2.1. *If T_1, T_2 are binary trees and T_1 is a subposet of T_2, then there is a sequence of point removals leading from T_2 to T_1 through binary trees.*

Proof. A point of a binary tree with two immediate predecessors is called a branching. Let uv be an edge of T_2 . If one of u and v is not a branching, and one of u and v is not in T_1, then we say that uv is a contractible edge of T_2. If uv is contractible, then $T_2 \setminus \{v\}$ (or $T_2 \setminus \{u\}$) is clearly a binary tree containing T_1. Assume that $T_1 \subseteq T_2'$, and T_2' has no contractible edge. We show that $T_2' = T_1$.

Suppose to the contrary that $T_2' \setminus T_1 \neq \emptyset$. Let x be a leaf in $T_2' \setminus T_1$. By the assumption, x is not a leaf of T_2'. Moreover, x has two immediate predecessors, x_1 and x_2, in T_2', both belonging to T_1 as well, because xx_i is not contractible ($i = 1, 2$). Observe that the T_1-join $z = x_1 \vee x_2$ is greater than x in T_2'. Because T_1 is a binary tree, any immediate predecessor of z in T_2' is also in $T_2' \setminus T_1$. If z was not a branching in T_2', then it would have a unique immediate predecessor w, and then wz would be a contractible edge of T_2'. Thus we may assume that z is a branching in T_2'. Let y be an immediate predecessor of z in T_2' not comparable with x. Then any smallest point less than or equal to y would be a leaf of T_2' belonging to $T_2' \setminus T_1$, contradiction. □

A real sequence $a = \{a_n\}_0^\infty$ is *logarithmically concave*, or log-concave for short, if $a_i^2 \geqslant a_{i-1}a_{i+1}$, for all $i > 1$. Sequences we examine here enumerate combinatorial objects; thus their terms are nonnegative. Furthermore, these *combinatorial sequences* have no internal zeros, that is, each term is positive between any two positive terms. We shall frequently refer to the following more general inequality, $a_i a_j \geqslant a_{i-k}a_{j+k}$, for every $k \geqslant 0$ and $i \leqslant j$, which is actually equivalent to the definition of log-concavity in the case of the combinatorial sequences mentioned above. For the use of log-concavity in different branches of mathematics including combinatorics, see [9].

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Given nonnegative sequences $a = \{a_n\}_0^\infty$ and $b = \{b_n\}_0^\infty$, we will say that a *ratio inequality* $\frac{a_i}{b_i} \leqslant \frac{a_j}{b_j}$ holds true if and only if $a_i b_j \leqslant a_j b_i$, for all $i, j \geqslant 0$. For instance, using ratio inequality notation, a possible definition of a log-concave sequence is $\frac{a_i}{a_{i-1}} \geqslant \frac{a_{i+1}}{a_i}$.

A formal series generating function (or a polynomial) is log-concave if the sequence of its coefficients has this property. If the definition of a sequence is not otherwise restricted we can include leading or trailing zero terms. In particular, a linear polynomial is always log-concave. A fundamental operation on sequences maintaining log-concavity is convolution. We state this well-known result in terms of generating functions (as the convolution of two sequences corresponds to the multiplication of their ordinary power series generating functions). For a standard proof using the Binet–Cauchy formula for convolution matrices, see [5].

Convolution Lemma. *If $F(z)$ and $G(z)$ are log-concave formal power series with nonnegative coefficients, then $F(z) \cdot G(z)$ is log-concave as well.*

We say that a/b is nonincreasing if the ratio inequality $\frac{a_i}{b_i} \geqslant \frac{a_{i+1}}{b_{i+1}}$ holds true for each i. In induction proofs concerning ratio inequalities, the following technical lemma will be useful.

Lemma 2.2. *If p, q, r, s are sequences of length k, and both p/q and r/s are nonincreasing, then*

$$\frac{\sum_{i=1}^{k} p_i r_i}{\sum_{i=1}^{k} q_i r_i} \geqslant \frac{\sum_{i=1}^{k} p_i s_i}{\sum_{i=1}^{k} q_i s_i}.$$

Proof. By the conditions, $p_i q_j - p_j q_i \geqslant 0$ and $r_i s_j - r_j s_i \geqslant 0$ hold for every $1 \leqslant i \leqslant j \leqslant k$. The product of these two inequalities yields

$$p_i r_i q_j s_j + p_j r_j q_i s_i \geqslant p_i s_i q_j r_j + p_j s_j q_i r_i,$$

and the claim follows by summing up these inequalities, for all $1 \leqslant i \leqslant j \leqslant k$. □

3. Proof of the ratio inequality

The purpose of this section is to prove the ratio inequality $\frac{A(n;T_1)}{B(n;T_1)} \leqslant \frac{A(n;T_2)}{B(n;T_2)}$ for binary trees $T_1 \subseteq T_2$. The proof of Proposition 2.1 suggests that it is sufficient to consider the particular case when $T_1 = T_2 \setminus \{x\}$, and either x is a leaf or x has exactly one immediate predecessor. We shall verify these particular cases in Propositions 3.2 and 3.4 below by rewriting the ratio inequality in terms of the number of isomorphic *labelled* subtrees of $\mathbf{T^n}$.

Assume that T is a rooted tree of order k, and label its points with $\{1, \ldots, k\}$. Let S be a k-element subposet of $\mathbf{T^n}$, and assume a labelling of its points with $\{1, \ldots, k\}$. Extending isomorphism to labelled structures, we write $S \cong T$, provided S and T are isomorphic as

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labelled rooted trees. Define

$$A_T^{(n)} = |\{S \subseteq \mathbf{T^n} : S \cong T,\ \mathbf{1_n} \in S\}|, \quad \text{and}$$
$$B_T^{(n)} = |\{S \subseteq \mathbf{T^n} : S \cong T,\ \mathbf{1_n} \notin S\}|.$$

Clearly, $A(n;T) = |G|^{-1}A_T^{(n)}$, and $B(n;T) = |G|^{-1}B_T^{(n)}$, where G is the group of symmetries of the rooted (unlabelled) tree T.

A labelled subposet $S \subseteq \mathbf{T^n}$ with $S \cong T$ will be called an *embedding* of T into $\mathbf{T^n}$. For example, if the unlabelled 3-star T in Figure 3.1 is embedded into $\mathbf{T^3}$, then $A(3;T) = 6$ and $B(3;T) = 0$. However, for any fixed labelling of T, we have $A_T^{(3)} = 36$.

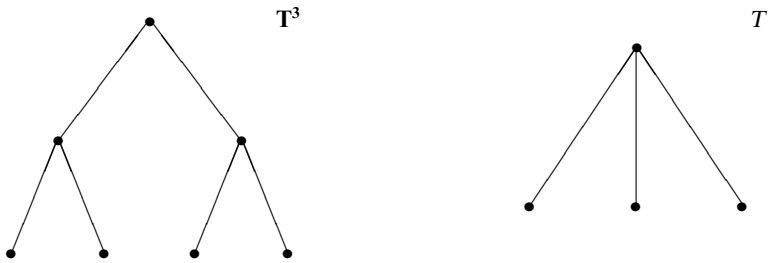


Figure 3.1

Let x be a vertex of the labelled tree T, and let p be an arbitrary point of $\mathbf{T^n}$ from level i ($1 \leqslant i \leqslant n$). The functions defined below count all embeddings of T with a fixed vertex x placed at a fixed position i in $\mathbf{T^n}$:

$$A_T^{(n)}(i;x) = |\{S \subseteq \mathbf{T^n} : p \text{ is labelled with } x,\ S \cong T,\ \mathbf{1_n} \in S\}|,$$
$$B_T^{(n)}(i;x) = |\{S \subseteq \mathbf{T^n} : p \text{ is labelled with } x,\ S \cong T,\ \mathbf{1_n} \notin S\}|.$$

Because of the symmetry of $\mathbf{T^n}$, these functions do not depend on the choice of p. Furthermore, we can count the embeddings of T into $\mathbf{T^n}$ by fixing a vertex of T labelled with x at one of the 2^{i-1} points p of level i, for $i = 1,\ldots,n$. Thus we have

$$A_T^{(n)} = \sum_{i=1}^{n} 2^{i-1} A_T^{(n)}(i;x),$$
$$B_T^{(n)} = \sum_{i=1}^{n} 2^{i-1} B_T^{(n)}(i;x).$$

For the tree T in Figure 3.1, where x is a leaf of T, we have $A_T^{(3)}(1;x) = 0$, $A_T^{(3)}(2;x) = 2$, and $A_T^{(3)}(3;x) = 8$.

Lemma 3.1. *If x is any vertex of a labelled binary tree T, then*

$$\frac{A_T^{(n)}(k;x)}{B_T^{(n)}(k;x)} \geqslant \frac{A_T^{(n)}(k+1;x)}{B_T^{(n)}(k+1;x)}$$

for every $k \geqslant 1$.

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Proof. If x is the root of T, then, for every $k \geqslant 1$, $A_T^{(n)}(k+1;x) = 0$. Thus the ratio inequality is trivial. So we may assume that there is an immediate T-successor of x we denote by z. For $k = 1$, $B_T^{(n)}(k;x) = 0$, and the ratio inequality is trivial. So we may also assume that $k \geqslant 2$. Let X be the labelled subtree of T determined by all points of T less than or equal to x.

Case 1: z is the root of T. If z has just one immediate predecessor, namely x, then $A_T^{(n)}(k;x) = A_X^{(n-k+1)}$. Consider an embedding of T into $\mathbf{T^n}$ such that x is a fixed point of level k and $z \neq \mathbf{1_n}$. Then z is one of the $k-2$ points of $\mathbf{T^n}$ larger than x, thus $B_T^{(n)}(k;x) = (k-2)A_X^{(n-k+1)}$. Therefore, the ratio inequality is equivalent to the obvious inequality $1/(k-2) > 1/(k-1)$. So we may assume that z has another immediate T-predecessor y different from x.

Let Y be the subtree of T determined by all points of T less than or equal to y. Consider an embedding of T into $\mathbf{T^n}$ such that x is a fixed point of level k, $z = \mathbf{1_n}$, and the $\mathbf{T^n}$-join, $t = x \vee y$, is a point of level i, for some $1 \leqslant i \leqslant k-1$. This embedding of T is uniquely determined by one of the $A_X^{(n-k+1)}$ embeddings of X into a copy of the subtree $\mathbf{T^{n-k+1}} \leqslant \mathbf{T^n}$, and an embedding of Y into the corresponding copy of $\mathbf{T^{n-i}} \leqslant \mathbf{T^n}$ below t. Because $A_Y^{(n-i)} + B_Y^{(n-i)} = \frac{1}{2}B_Y^{(n-i+1)}$, we obtain

$$A_T^{(n)}(k;x) = \frac{1}{2}A_X^{(n-k+1)} \sum_{i=1}^{k-1} B_Y^{(n-i+1)}.$$

Next we show the identity

$$B_T^{(n)}(k;x) = \frac{1}{2}A_X^{(n-k+1)} \sum_{i=1}^{k-1} (i-1)B_Y^{(n-i+1)}.$$

Consider all embeddings of T into $\mathbf{T^n}$ such that x is a fixed point of level k, and $z \neq \mathbf{1_n}$. As before, we count these embeddings by choosing the $\mathbf{T^n}$-join, $t = x \vee y$, at a point of level i of $\mathbf{T^n}$, for each $2 \leqslant i \leqslant k-1$. Because $t \leqslant z < \mathbf{1_n}$, there are $i-1$ choices for z. Using the argument above, for each of the $A_X^{(n-k+1)}$ embeddings of X into $\mathbf{T^{n-k+1}}$, the subtree Y has $\frac{1}{2}B_Y^{(n-i+1)}$ embeddings below t, and the formula for $B_T^{(n)}(k;x)$ follows. From the identities above we have

$$\begin{aligned} A_T^{(n)}(k+1;x) &= \left(A_T^{(n)}(k;x) + \frac{1}{2}A_X^{(n-k+1)}B_Y^{(n-k+1)}\right)\frac{A_X^{(n-k)}}{A_X^{(n-k+1)}}, \\ B_T^{(n)}(k+1;x) &= \left(B_T^{(n)}(k;x) + (k-1)\frac{1}{2}A_X^{(n-k+1)}B_Y^{(n-k+1)}\right)\frac{A_X^{(n-k)}}{A_X^{(n-k+1)}}. \end{aligned}$$

Thus the statement of the lemma is equivalent to

$$\frac{A_T^{(n)}(k;x)}{B_T^{(n)}(k;x)} \geqslant \frac{1}{k-1}.$$

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This last inequality follows from

$$\frac{A_T^{(n)}(k;x)}{B_T^{(n)}(k;x)} = \frac{\sum_{i=1}^{k-1} B_Y^{(n-i+1)}}{\sum_{i=1}^{k-1}(i-1)B_Y^{(n-i+1)}} \geqslant \frac{1}{k-1}.$$

In the proof of the second case we use induction on the order of T.

Case 2: z is not the root of T. Let Z be the subtree of T determined by all points of T less than or equal to z, and let W be the subtree determined by $(T \setminus Z) \cup \{z\}$. Assuming that z is at level i of $\mathbf{T^n}$, for some $i < k$, there are $A_Z^{(n-i+1)}(k-i+1;x)$ embeddings of Z such that x is at some point of level k of $\mathbf{T^n}$. Hence we have

$$\frac{A_T^{(n)}(k;x)}{B_T^{(n)}(k;x)} = \frac{\sum_{i=1}^{k-1} A_W^{(n)}(i;z)A_Z^{(n-i+1)}(k-i+1;x)}{\sum_{i=1}^{k-1} B_W^{(n)}(i;z)A_Z^{(n-i+1)}(k-i+1;x)}.$$

The ratio inequality required can be written in the form

$$\frac{\sum p_i r_i}{\sum q_i r_i} \geqslant \frac{\sum p_i s_i}{\sum q_i s_i}$$

by including zero summands as follows:

$$\frac{\sum_{i=1}^{k} A_W^{(n)}(i;z)A_Z^{(n-i+1)}(k-i+1;x)}{\sum_{i=1}^{k} B_W^{(n)}(i;z)A_Z^{(n-i+1)}(k-i+1;x)} \geqslant \frac{\sum_{i=1}^{k} A_W^{(n)}(i;z)A_Z^{(n-i+1)}(k-i+2;x)}{\sum_{i=1}^{k} B_W^{(n)}(i;z)A_Z^{(n-i+1)}(k-i+2;x)}.$$

To prove this inequality we verify the conditions in Lemma 2.2, that is, p/q and r/s are nonincreasing. Because W is a proper subtree of T, the first condition

$$\frac{A_W^{(n)}(i;z)}{B_W^{(n)}(i;z)} \geqslant \frac{A_W^{(n)}(i+1;z)}{B_W^{(n)}(i+1;z)}$$

follows from the induction hypothesis. The second condition amounts to the ratio inequality

$$\frac{A_Z^{(m)}(j;x)}{A_Z^{(m)}(j+1;x)} \geqslant \frac{A_Z^{(m-1)}(j-1;x)}{A_Z^{(m-1)}(j;x)}.$$

Let Y be the subtree determined by $Z \setminus (X \cup \{z\})$. If Y is empty, then $A_Z^{(m)}(j;x) = A_X^{(m-j+1)}$ and the inequality is trivial. Otherwise, as in Case 1, we have

$$A_Z^{(m)}(j;x) = \frac{1}{2}A_X^{(m-j+1)} \sum_{i=1}^{j-1} B_Y^{(m-i+1)}.$$

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Thus the second condition has the following equivalent form:

$$\frac{\sum_{i=1}^{j-1} B_Y^{(m-i+1)}}{\sum_{i=1}^{j} B_Y^{(m-i+1)}} \geqslant \frac{\sum_{i=1}^{j-2} B_Y^{(m-i)}}{\sum_{i=1}^{j-1} B_Y^{(m-i)}}.$$

This inequality verifies easily, concluding the proof of the lemma. □

Proposition 3.2. *If x is a leaf of a binary tree T, and $T' = T \setminus \{x\}$, then*

$$\frac{A_T^{(n)}}{B_T^{(n)}} \geqslant \frac{A_{T'}^{(n)}}{B_{T'}^{(n)}}.$$

Proof. Let z be the immediate successor of x. Let D be the subtree of T determined by all points of T less than or equal to z, set $D' = D \setminus \{x\}$, and let U be the subtree determined by $(T \setminus D) \cup \{z\}$. We count the embeddings of T or T' into $\mathbf{T^n}$ by fixing z at one of the 2^{i-1} points of level i, for $i = 1, \ldots, n$. Thus we obtain

$$A_T^{(n)} = \sum_{i=1}^{n} 2^{i-1} A_U^{(n)}(i;z) A_D^{(n-i+1)}, \qquad A_{T'}^{(n)} = \sum_{i=1}^{n} 2^{i-1} A_U^{(n)}(i;z) A_{D'}^{(n-i+1)},$$

$$B_T^{(n)} = \sum_{i=1}^{n} 2^{i-1} B_U^{(n)}(i;z) A_D^{(n-i+1)}, \qquad B_{T'}^{(n)} = \sum_{i=1}^{n} 2^{i-1} B_U^{(n)}(i;z) A_{D'}^{(n-i+1)}.$$

By Lemma 3.1, we have

$$\frac{A_U^{(n)}(i;z)}{B_U^{(n)}(i;z)} \geqslant \frac{A_U^{(n)}(i+1;z)}{B_U^{(n)}(i+1;z)}.$$

Next we verify the inequality

$$\frac{A_D^{(n-i+1)}}{A_{D'}^{(n-i+1)}} \geqslant \frac{A_D^{(n-i)}}{A_{D'}^{(n-i)}} \qquad \text{in the form} \qquad \frac{A_D^{(m+1)}}{A_{D'}^{(m+1)}} \geqslant \frac{A_D^{(m)}}{A_{D'}^{(m)}}.$$

Embeddings of D into $\mathbf{T^{m+1}}$ with $z = \mathbf{1_{m+1}}$ include any embedding of $D' \setminus \{z\}$ into one copy $\mathbf{T^m}$ in $\mathbf{T^{m+1}}$ and $2^m - 1$ choices for x in the symmetric copy $\mathbf{T^m}$. This justifies the bound $A_D^{(m+1)} \geqslant A_{D'}^{(m+1)}(2^m - 1)$. The bound $A_{D'}^{(m)}(2^m - 1) \geqslant A_D^{(m)}$ is also obvious, and the claim follows by combining these two inequalities.

The conditions of Lemma 2.2 are satisfied; thus we obtain the ratio inequality

$$\frac{\sum_{i=1}^{n} 2^{i-1} A_U^{(n)}(i;z) A_D^{(n-i+1)}}{\sum_{i=1}^{n} 2^{i-1} B_U^{(n)}(i;z) A_D^{(n-i+1)}} \geqslant \frac{\sum_{i=1}^{n} 2^{i-1} A_U^{(n)}(i;z) A_{D'}^{(n-i+1)}}{\sum_{i=1}^{n} 2^{i-1} B_U^{(n)}(i;z) A_{D'}^{(n-i+1)}}.$$

This concludes the proof of the proposition. □

Lemma 3.3. *If T is a binary tree, then the sequence $\{B_T^{(n)}\}_{n=1}^{\infty}$ is log-concave.*

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Proof. The proof is by induction on the order of T. If T is a singleton, then $B_T^{(n)} = 2^n - 2$ is clearly log-concave. Suppose that the claim is true for any subtree of T. Let z be the root of T and let X and Y denote the subtree components of $T \setminus \{z\}$.

First assume that one of these subtrees, say Y, is empty. We count the embeddings of T into $\mathbf{T^n}$ by fixing z at one of the 2^{i-1} points of level i of $\mathbf{T^n}$, for $i = 2, \ldots, n-1$. Thus we obtain

$$B_T^{(n)} = \sum_{i=2}^{n-1} 2^{i-1} B_X^{(n-i+1)}.$$

The sequence $\{2^n\}$ is log-concave, and by the induction hypothesis, $\{B_X^{(n)}\}$ is log-concave. Thus the logarithmic concavity of $\{B_T^{(n)}\}$ follows from the Convolution Lemma.

Now assume that X and Y are both nonempty, and let x and y be their roots, respectively. We count the embeddings of T into $\mathbf{T^n}$ by fixing the $\mathbf{T^n}$-join $t = x \vee y$ at one of the 2^{i-1} points of level i. There are $i-1$ choices for z ($t \leqslant z < \mathbf{1_n}$), and, for each embedding of X into one copy of $\mathbf{T^{n-i}}$ below t, there remain $\frac{1}{2} B_Y^{(n-i+1)}$ ways of embedding Y into the symmetric copy $\mathbf{T^{n-i}}$. Thus we obtain

$$B_T^{(n)} = \frac{1}{2} \sum_{i=2}^{n-1} (i-1) 2^{i-1} B_X^{(n-i+1)} B_Y^{(n-i+1)}.$$

The sequence $\{n2^n\}$ is clearly log-concave; furthermore, the sequence $\{B_X^{(n)} B_Y^{(n)}\}$ is log-concave, because both $\{B_X^{(n)}\}$ and $\{B_Y^{(n)}\}$ are log-concave, by the induction hypothesis. Hence the logarithmic concavity of $\{B_T^{(n)}\}$ follows from the Convolution Lemma. □

Proposition 3.4. *If z is a point of the binary tree T with one immediate predecessor x, and $T' = T \setminus \{x\}$, then*

$$\frac{A_T^{(n)}}{B_T^{(n)}} \geqslant \frac{A_{T'}^{(n)}}{B_{T'}^{(n)}}.$$

Proof. Let D be the tree determined by all points of T less than or equal to z, and let $D' = D \setminus \{x\}$. Let U be the tree determined by the set $(T \setminus D) \cup \{z\}$. We count the embeddings of T or T' into $\mathbf{T^n}$ by fixing z at one of the 2^{i-1} points of level i, for $i = 1, \ldots, n$. Thus we obtain

$$A_T^{(n)} = \sum_{i=1}^{n} 2^{i-1} A_U^{(n)}(i; z) A_D^{(n-i+1)}, \qquad A_{T'}^{(n)} = \sum_{i=1}^{n} 2^{i-1} A_U^{(n)}(i; z) A_{D'}^{(n-i+1)},$$

$$B_T^{(n)} = \sum_{i=1}^{n} 2^{i-1} B_U^{(n)}(i; z) A_D^{(n-i+1)}, \qquad B_{T'}^{(n)} = \sum_{i=1}^{n} 2^{i-1} B_U^{(n)}(i; z) A_{D'}^{(n-i+1)}.$$

By Lemma 3.1, we have

$$\frac{A_U^{(n)}(i; z)}{B_U^{(n)}(i; z)} \geqslant \frac{A_U^{(n)}(i+1; z)}{B_U^{(n)}(i+1; z)}.$$

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To apply Lemma 2.2 we also verify the second condition

$$\frac{A_D^{(n-i+1)}}{A_{D'}^{(n-i+1)}} \geqslant \frac{A_D^{(n-i)}}{A_{D'}^{(n-i)}} \qquad \text{in the form} \qquad \frac{A_{D'}^{(m-1)}}{A_D^{(m-1)}} \geqslant \frac{A_{D'}^{(m)}}{A_D^{(m)}}.$$

Using the obvious identities

$$A_D^{(m)} = B_{D'}^{(m)} \qquad \text{and} \qquad B_{D'}^{(m+1)} = 2(A_{D'}^{(m)} + B_{D'}^{(m)}),$$

we can rewrite the inequality in the following equivalent form:

$$\frac{B_{D'}^{(m)}}{B_{D'}^{(m-1)}} \geqslant \frac{B_{D'}^{(m+1)}}{B_{D'}^{(m)}}.$$

This holds true, by Lemma 3.3, concluding the proof of the proposition. □

Theorem 3.5. *If T_1 and T_2 are binary trees and T_1 is a subposet of T_2, then*

$$\frac{A(n;T_1)}{B(n;T_1)} \leqslant \frac{A(n;T_2)}{B(n;T_2)}.$$

Proof. We may assume that T_1 is a proper subposet of T_2. Because

$$\frac{A(n;T)}{B(n;T)} = \frac{A_T^{(n)}}{B_T^{(n)}},$$

the ratio inequality

$$\frac{A(n;T_1)}{B(n;T_1)} \leqslant \frac{A(n;T_2)}{B(n;T_2)}$$

is equivalent to

$$\frac{A_{T_1}^{(n)}}{B_{T_1}^{(n)}} \leqslant \frac{A_{T_2}^{(n)}}{B_{T_2}^{(n)}}.$$

Observe that the ratio inequality is trivial provided that $B_{T_2}^{(n)} = 0$. Assuming that $B_{T_2}^{(n)} \neq 0$, we use induction on the order of T_2.

By Proposition 2.1, T_2 has a point x such that $T' = T_2 \setminus \{x\}$ is a binary tree containing T_1. By Proposition 3.2 or 3.4, we have

$$\frac{A_{T'}^{(n)}}{B_{T'}^{(n)}} \leqslant \frac{A_{T_2}^{(n)}}{B_{T_2}^{(n)}}.$$

If $T' = T_1$, then the ratio inequality of the theorem follows. Otherwise, by the induction hypothesis, we also have

$$\frac{A_{T_1}^{(n)}}{B_{T_1}^{(n)}} \leqslant \frac{A_{T'}^{(n)}}{B_{T'}^{(n)}}.$$

Observe that the assumption $B_{T_2}^{(n)} \neq 0$ implies that both $B_{T'}^{(n)}$ and $B_{T_1}^{(n)}$ are positive. Therefore, the ratio inequalities

$$\frac{A_{T_1}^{(n)}}{B_{T_1}^{(n)}} \leqslant \frac{A_{T'}^{(n)}}{B_{T'}^{(n)}} \leqslant \frac{A_{T_2}^{(n)}}{B_{T_2}^{(n)}}$$

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imply

$$\frac{A_{T_1}^{(n)}}{B_{T_1}^{(n)}} \leqslant \frac{A_{T_2}^{(n)}}{B_{T_2}^{(n)}}. \qquad \square$$

4. Monotonicity of probabilities

Let us now state the ratio inequality of Theorem 3.5 in terms of conditional probabilities for the partial order secretary problem described in the introduction. Assume that the search process that used the optimal stopping time τ_n was repeated twice in two independent situations. Let the candidates in the first and the second process form posets T_1 and T_2, respectively, after stopping the search. As an example, let T_1 and T_2 be the trees depicted in Figure 4.1, and assume that the search is on $\mathbf{T^4}$.

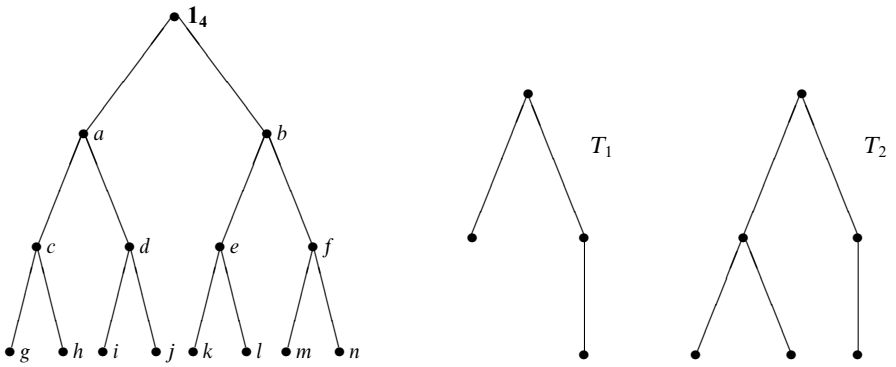


Figure 4.1

Assuming that the candidates appear in the order of the permutations $(c, d, b, a, k, \mathbf{1_4}, \ldots)$ or $(d, i, j, h, c, a, \ldots)$ the candidate chosen according to the optimal stopping time τ_4 is $\mathbf{1_4}$ in the first case and a in the second. These are examples of searches stopping at tree T_2. Examples of searches stopping at T_1 are permutations $(k, e, f, b, \ldots)$ or $(c, d, j, \mathbf{1_4}, \ldots)$, b is chosen in the first case and $\mathbf{1_4}$ in the second.

The inequality proved in Theorem 3.5 makes it possible to compare conditional probabilities of the choice of $\mathbf{1_n}$ when we know only the tree formed by all the candidates that preceded the moment of choice according to τ_n. In other words, we can state the following proposition.

Proposition 4.1. *If T_1, T_2 are proper subposets of $\mathbf{T^n}$, T_1 and T_2 are binary trees, and T_1 is a subposet of T_2, then*

$$P[\omega_{\tau_n(\omega)} = \mathbf{1_n} \mid \{\omega_1, \ldots, \omega_{\tau_n(\omega)}\} \cong T_1] \leqslant P[\omega_{\tau_n(\omega)} = \mathbf{1_n} \mid \{\omega_1, \ldots, \omega_{\tau_n(\omega)}\} \cong T_2].$$

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Proof. Note that

$$P[\omega_{\tau_n(\omega)} = \mathbf{1_n} \mid \{\omega_1, \ldots, \omega_{\tau_n(\omega)}\} \cong T_1] = \frac{A(n, T_1)}{A(n, T_1) + B(n, T_1)}$$

and

$$P[\omega_{\tau_n(\omega)} = \mathbf{1_n} \mid \{\omega_1, \ldots, \omega_{\tau_n(\omega)}\} \cong T_2] = \frac{A(n, T_2)}{A(n, T_2) + B(n, T_2)},$$

then apply Theorem 3.5. □

We conjecture that Theorem 3.5 remains true for arbitrary trees T_1 and T_2. If this is the case, then Proposition 4.1 extends to arbitrary trees as well.

The case of the complete binary tree is, though natural, only one instance of many similar decision problems. The subject of partial order optimal searches has also been considered in recent literature in a more general context: see [8]. It therefore seems important to notice that, if the candidates do not form a complete binary tree, then the monotonicity results of Theorem 3.5 and Proposition 4.1 do not necessarily hold true. As an example, consider the tree poset $\mathbf{S}$ with root $\mathbf{1_S}$ and its subposets T_1 and T_2, given in Figure 4.2.

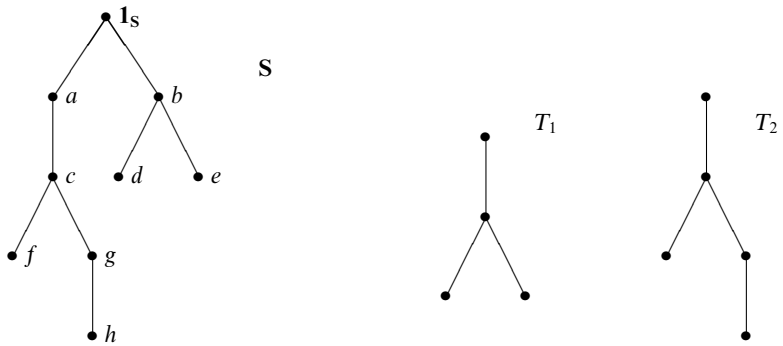


Figure 4.2

For any subposet D of $\mathbf{S}$ let $A(\mathbf{S}, D) = |\{U \subseteq \mathbf{S} : \mathbf{1_S} \in U, U \cong D\}|$ and $B(\mathbf{S}, D) = |\{U \subseteq \mathbf{S} : \mathbf{1_S} \notin U, U \cong D\}|$.

If, in a search process, the candidates form the permutation $(c, g, f, h, a, \ldots)$ or $(a, f, h, g, \mathbf{1_S}, \ldots)$, then it is obvious that an optimal strategy τ (whatever it is) will not stop the process until the fifth element appears. At that time the candidates that have already been examined form the tree T_2.

The administrator taking the decision at the stage when he has not chosen the candidate so far, and the candidates who have been examined form the tree T_2, knows that the probability of the fifth candidate being $\mathbf{1_S}$ is equal to

$$P[\omega_5 = \mathbf{1_S} \mid \{\omega_1, \ldots, \omega_5\} \cong T_2 \text{ and } \omega_5 \text{ is maximal}] = \frac{A(\mathbf{S}, T_2)}{A(\mathbf{S}, T_2) + B(\mathbf{S}, T_2)} = \frac{2}{3} > 1/2.$$

This means that, according to an optimal strategy, we must stop the search process here (note that if the search continues, the probability of success can never get larger than

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1/3). So the expression $P[\omega_{\tau(\omega)} = \mathbf{1}_{\mathbf{S}} \mid \{\omega_1, \ldots, \omega_{\tau(\omega)}\} \cong T_2]$ is meaningful, because the event which is the condition is not of probability zero. Because all permutations are equiprobable, it is equal to the quantity above, namely

$$P[\omega_{\tau(\omega)} = \mathbf{1}_{\mathbf{S}} \mid \{\omega_1, \ldots, \omega_{\tau_n(\omega)}\} \cong T_2] = \frac{A(\mathbf{S}, T_2)}{A(\mathbf{S}, T_2) + B(\mathbf{S}, T_2)} = \frac{2}{3}.$$

Assume now that the candidates already examined form tree T_1, where the last seen is the best so far. For instance, consider permutations $(b, d, e, \mathbf{1}_{\mathbf{S}}, \ldots)$ or $(c, h, f, a, \ldots)$. A similar argument as above yields

$$P[\omega_{\tau_n(\omega)} = \mathbf{1}_{\mathbf{S}} \mid \{\omega_1, \ldots, \omega_{\tau(\omega)}\} \cong T_1] = \frac{A(\mathbf{S}, T_1)}{A(\mathbf{S}, T_1) + B(\mathbf{S}, T_1)} = \frac{5}{7}.$$

Thus the conditional probability we obtain for the smaller tree T_1 is greater than the probability we have for T_2.

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