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Permutation classes and polyomino classes with excluded submatrices

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This article introduces an analogue of permutation classes in the context of polyominoes. For both permutation classes and polyomino classes, we present an original way of characterizing them by avoidance constraints (namely, with excluded submatrices) and we discuss how canonical such a description by submatrix-avoidance can be. We provide numerous examples of permutation and polyomino classes which may be defined and studied from the submatrix-avoidance point of view, and conclude with various directions for future research on this topic.

1. Introduction

The concept of *substructure* (or *pattern*) within a *combinatorial structure* is an essential notion in combinatorics, whose study has had many developments in various branches of discrete mathematics. Among them, the research on permutation patterns and pattern-avoiding permutations – starting in the seventies with the work of Knuth (1975) – has become a very active field. Nowadays, the research on permutation patterns is being developed in several directions. One of them is to define and study analogues of the concept of pattern in permutations in other combinatorial objects such as set partitions (Goyt 2008; Klazar 1996; Sagan 2010), words (Björner 1990; Burstein 1998), trees (Dairyko 2012; Rowland 2010), and paths (Ferrari 2013). The work presented here goes into this direction, and is specifically interested in patterns in polyominoes.

Polyominoes[†] are discrete objects among the most studied in combinatorics, especially from an enumerative point of view. Indeed, their enumeration (w.r.t. the area or the semi-perimeter) is a difficult problem, and is still open. In order to probe further into their study, a large part of the research on polyominoes consists in studying restricted families of polyominoes – defined by imposing geometrical constraints, such convexity

[†] The term polyomino was introduced by Golomb in 1953 during a talk at the Harvard Mathematics Club, which was published later (Golomb 1954) and popularized by Gardner (1957).

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and directedness, see (Bousquet-Mélou 1996; Delest 1984). Our work provides a general framework to define such families of polyominoes, by the avoidance of patterns.

Both permutations and polyominoes may be represented in a natural way by (restricted) binary matrices, and it is on this simple fact that we will build our notion of pattern in a polyomino. Doing so, we have also been led to consider families of permutations and polyominoes that are defined by the avoidance of some patterns that are not themselves permutations or polyominoes, but rather less restricted binary matrices. In addition to this being an original way to look at restricted families of polyominoes, we believe it also brings new ideas to the research on permutation patterns.

The article is organized as follows. In Section 2, we recall the definitions of permutation patterns and permutation classes, define polyomino classes, and describe some of their basic properties. More details about permutations and patterns may be found in Bona (2004), and we refer the reader to Bousquet-Mélou (1996) for the basic definitions on polyominoes. Section 3 introduces submatrix-avoidance in permutations and polyominoes, and examines how a permutation or polyomino class may be characterized by the avoidance of submatrices, defining several notions of *matrix-bases* of classes. Section 4 investigates how these matrix-bases are related to the usual basis of a permutation class (and its analogue for a polyomino class). In Sections 5 and 6, we provide several examples of permutation and polyomino classes, both previously studied and new, that can be studied with our submatrix-avoidance approach. Finally, we open directions for future work in Section 7.

2. Permutation classes and polyomino classes

2.1. *Permutation patterns and permutation classes*

For any integer n, let us denote by $\mathfrak{S}_n$ the set of all permutations of the set $\{1, 2, \ldots, n\}$. The integer n is called the *size* of a permutation in $\mathfrak{S}_n$. We denote by $\mathfrak{S} = \cup_{n\in\mathbb{N}} \mathfrak{S}_n$ the set of all permutations. We write permutations in one-line notation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, meaning for instance that $\sigma = 53142$ is the permutation of $\mathfrak{S}_5$ such that $\sigma(1) = 5$, $\sigma(2) = 3, \ldots,$ $\sigma(5) = 2$.

Definition 1. Given two permutations $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ and $\pi = \pi_1\pi_2\ldots\pi_k$, π is a *pattern* of σ (denoted $\pi \leqslant_{\mathfrak{S}} \sigma$) when there exist indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that the sequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ is order-isomorphic to π. Such sequences $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ are called *occurrences* of π in σ.

A permutation π which is a pattern of a permutation σ is said to be *contained* or *involved* in σ. A permutation σ that does not contain π is said to *avoid* π.

The relation $\leqslant_{\mathfrak{S}}$ is a partial order on the set $\mathfrak{S}$ of all permutations. Moreover, properties of the poset $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$ have been described in the literature (Steingrìmsson 2010) and we recall some of the most well known here: $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$ is a well-founded poset (i.e. it does not contain infinite descending chains), but it is not well ordered, since it contains infinite *antichains* (i.e. infinite sets of pairwise incomparable elements); moreover, it is a graded poset (the rank function being the size of the permutations).

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Definition 2. A *permutation class* (sometimes called *pattern class* or *class* for short) is a set of permutations $\mathcal{C}$ that is downward closed for $\leqslant_{\mathfrak{S}}$: for all $\sigma \in \mathcal{C}$, if $\pi \leqslant_{\mathfrak{S}} \sigma$, then $\pi \in \mathcal{C}$.

Permutation classes are then just *order ideals* in the poset $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$. In this article, we will also consider order ideals in other posets, and we will call them *classes* as well. This choice in the terminology, which could be surprising to a reader more familiar with poset theory, is in accordance with the huge literature on permutation classes, where we took the motivations for our work.

For any set $\mathcal{B}$ of permutations, denoting $Av_{\mathfrak{S}}(\mathcal{B})$ the set of all permutations that avoid every pattern in $\mathcal{B}$, we clearly have that $Av_{\mathfrak{S}}(\mathcal{B})$ is a permutation class. The converse statement is also true. Namely:

Proposition 3. For every permutation class $\mathcal{C}$, there is a unique antichain $\mathcal{B}$ such that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{B})$. The set $\mathcal{B}$ consists of all minimal permutations (in the sense of $\leqslant_{\mathfrak{S}}$) that do not belong to $\mathcal{C}$.

In the usual terminology, $\mathcal{B}$ is called the *basis* of $\mathcal{C}$. Here, we shall rather call $\mathcal{B}$ the *permutation-basis* (or *p-basis* for short), to distinguish from other kinds of bases that we introduce in Section 3.

Proposition 3 is a classical fact in the permutation pattern field; for a proof, see for instance (Bona 2004). Notice that because $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$ contains infinite antichains, the *p*-basis of a permutation class may be infinite.

Actually, Proposition 3 does not hold only for permutation classes, but for downward closed sets in any *well-founded poset*, i.e. partially ordered set that do not contain infinite descending chains.

Proposition 4. Let $(\mathfrak{X}, \leqslant)$ be a well-founded poset. For any subset $\mathcal{C}$ of $\mathfrak{X}$ that is downward closed for $\leqslant$, there exists a unique antichain $\mathcal{B}$ of $\mathfrak{X}$ such that $\mathcal{C} = Av_{\mathfrak{X}}(\mathcal{B}) = \{x \in \mathfrak{X} : \text{for all } b \in \mathcal{B}, b \leqslant x \text{ does not hold}\}$. The set $\mathcal{B}$ consists of all minimal elements of $\mathfrak{X}$ (in the sense of $\leqslant$) that do not belong to $\mathcal{C}$.

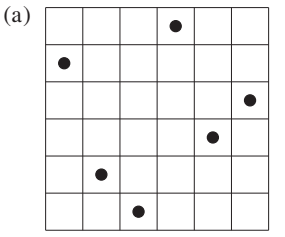
Proof. Let $\mathcal{C}$ be a subset of $\mathfrak{X}$ that is downward closed for $\leqslant$. The complement $\mathfrak{X} \setminus \mathcal{C}$ of $\mathcal{C}$ with respect to $\mathfrak{X}$ is upward closed for $\leqslant$. Let us define $\mathcal{B}$ to be the set of minimal elements of $\mathfrak{X} \setminus \mathcal{C}$: $\mathcal{B} = \{b \in \mathfrak{X} \setminus \mathcal{C} : \forall \mathfrak{x} \in \mathfrak{X} \setminus \mathcal{C}, \text{ if } \mathfrak{x} \leqslant \mathfrak{b} \text{ then } \mathfrak{x} = \mathfrak{b}\}$. This is equivalent to characterizing $\mathcal{B}$ as the set of minimal elements of $\mathfrak{X}$ (in the sense of $\leqslant$) that do not belong to $\mathcal{C}$. Because $\mathfrak{X}$ is well founded, we have that $x \in \mathfrak{X} \setminus \mathcal{C}$ if and only if $\exists b \in \mathcal{B}$ such that $b \leqslant x$. The contrapositive gives $\mathcal{C} = Av_{\mathfrak{X}}(\mathcal{B})$. In addition, by minimality, the elements of $\mathcal{B}$ are pairwise incomparable, so that $\mathcal{B}$ is indeed an antichain.

To ensure uniqueness, it is enough to notice that for two different antichains $\mathcal{B}$ and $\mathcal{B}'$ the sets $\mathcal{C} = Av_{\mathfrak{X}}(\mathcal{B})$ and $\mathcal{C}' = Av_{\mathfrak{X}}(\mathcal{B}')$ are also different. □

2.2. *Permutation matrices and the submatrix order*

Permutations are in (obvious) bijection with permutation matrices, i.e. (necessarily square) binary matrices with exactly one entry 1 in each row and in each column. To any

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(b)

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 1. (a) Graphical representation (or diagram) of the permutation $\sigma = 521,634$. (b) The permutation matrix corresponding to σ.

permutation σ of $\mathfrak{S}_n$, we may associate a permutation matrix M_σ of dimension n by setting $M_\sigma(i,j) = 1$ if $i = \sigma(j)$, and 0 otherwise. Throughout this work, we adopt the convention that rows of matrices are numbered from bottom to top, so that the 1 in M_σ are at the same positions than the dots in the diagram of σ – see an example on Figure 1.

Let $\mathfrak{M}$ be the class of binary matrices (i.e. with entries in $\{0,1\}$). We denote by $\preccurlyeq$ the usual submatrix order on $\mathfrak{M}$, i.e. $M' \preccurlyeq M$ if M' may be obtained from M by deleting any collection of rows and/or columns. Of course, whenever $\pi \preccurlyeq_{\mathfrak{S}} \sigma$, we have that M_π is a submatrix of M_σ.

It follows from identifying permutations with the corresponding permutation matrices that we may rephrase the definition of permutation classes as follows: A set $\mathcal{C}$ of permutations is a class if and only if, for every $\sigma \in \mathcal{C}$, every submatrix of M_σ which is a permutation is in $\mathcal{C}$. This does not say much by itself, but it allows to define analogues of permutation classes for other combinatorial objects that are naturally represented by matrices, like polyominoes.

2.3. *Polyominoes and polyomino classes*

Definition 5. A polyomino is a finite union of cells (i.e. unit squares in the plane $\mathbb{Z} \times \mathbb{Z}$) that is connected and has no cut point (i.e. the set of cells has to be connected according to the edge adjacency). Polyominoes are defined up to translation.

A polyomino P may be represented by a binary matrix M whose dimensions are those of the minimal bounding rectangle of P: drawing P in the positive quarter plane, in the unique way that P has contacts with both axes, an entry (i,j) of M is equal to 1 if the unit square $[j-1,j] \times [i-1,i]$ of $\mathbb{Z} \times \mathbb{Z}$ is a cell of P, 0 otherwise (see Figure 2). Notice that according to this definition, in a matrix representing a polyomino the first (resp. the last) row (resp. column) should contain at least a 1.

Let us denote by $\mathfrak{P}$ the set of polyominoes, viewed as binary matrices as explained above. We can consider the restriction of the submatrix order $\preccurlyeq$ on $\mathfrak{P}$. This defines the poset $(\mathfrak{P}, \preccurlyeq_{\mathfrak{P}})$ and the pattern order between polyominoes: a polyomino P is a *pattern* of a polyomino Q (which we denote $P \preccurlyeq_{\mathfrak{P}} Q$) when the binary matrix representing P is a submatrix of that representing Q.

We point out that the order $\preccurlyeq_{\mathfrak{P}}$ has already been studied in Castiglione (2005) under the name of *subpicture order*. The main point of focus of Castiglione (2005) is the family of

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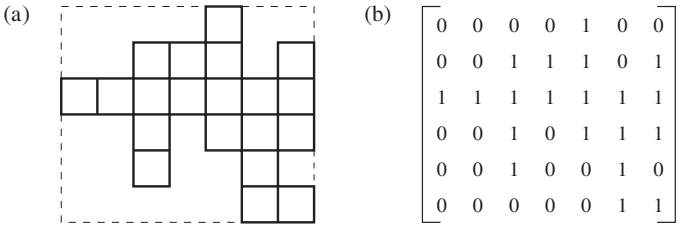


Fig. 2. A polyomino and its representation as a binary matrix.

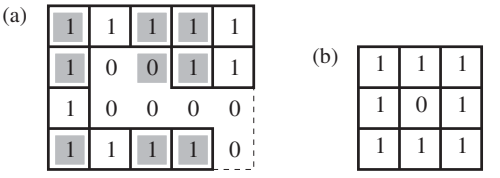


Fig. 3. (a) A polyomino P with no holes; (b) A polyomino $P' \leqslant_{\mathfrak{P}} P$ containing a hole.

L-convex polyominoes defined by the same authors in Castiglione (2003). But (Castiglione 2005) also proves that $\leqslant_{\mathfrak{P}}$ is not a partial well order, since $(\mathfrak{P}, \leqslant_{\mathfrak{P}})$ contains infinite antichains. Remark also that $(\mathfrak{P}, \leqslant_{\mathfrak{P}})$ is a graded poset (the rank function being the semi-perimeter of the bounding box of the polyominoes). This implies in particular that $(\mathfrak{P}, \leqslant_{\mathfrak{P}})$ is well founded. Notice that these properties are shared with the poset $(\mathfrak{S}, \leqslant_{\mathfrak{S}})$ of permutations.

A natural analogue of permutation classes for polyominoes is as follows:

Definition 6. A *polyomino class* is a set of polyominoes $\mathcal{C}$ that is downward closed for $\leqslant_{\mathfrak{P}}$: for all polyominoes P and Q, if $P \in \mathcal{C}$ and $Q \leqslant_{\mathfrak{P}} P$, then $Q \in \mathcal{C}$.

The reader can exercise in finding simple examples of polyomino classes, like the family of polyominoes having at most k columns, for any fixed k, or the family of polyominoes having a rectangular shape. Some of the most famous families of polyominoes are indeed polyomino classes, like the *convex polyominoes* and the L-convex polyominoes. This will be investigated in more details in Section 6. However, there are also well-known families of polyominoes which are not polyomino classes, like: the family of polyominoes having a square shape, or the family of polyominoes *with no holes* (i.e. polyominoes whose boundary is a simple path). Figure 3 shows that a polyomino in this class may contain a polyomino with a hole.

Similarly to the case of permutations, for any set $\mathcal{B}$ of polyominoes, let us denote by $Av_{\mathfrak{P}}(\mathcal{B})$ the set of all polyominoes that do not contain any element of $\mathcal{B}$ as a pattern. Every such set $Av_{\mathfrak{P}}(\mathcal{B})$ of polyominoes defined by pattern avoidance is a polyomino class. Conversely, like for permutation classes, every polyomino class may be characterized in this way.

Proposition 7. For every polyomino class $\mathcal{C}$, there is a unique antichain $\mathcal{B}$ of polyominoes such that $\mathcal{C} = Av_{\mathfrak{P}}(\mathcal{B})$. The set $\mathcal{B}$ consists of all minimal polyominoes (in the sense of $\leqslant_{\mathfrak{P}}$) that do not belong to $\mathcal{C}$.

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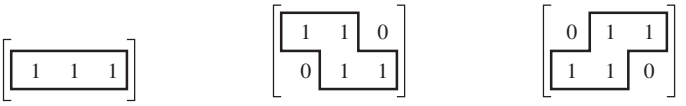


Fig. 4. The p-basis of the class $\mathcal{W}$.

Proof. It follows immediately from Proposition 4 and the fact that $(\mathfrak{P}, \preccurlyeq_{\mathfrak{P}})$ is a well-founded poset. □

As in the case of permutations we call $\mathcal{B}$ the *polyomino-basis* (also abbreviated as *p-basis*), to distinguish from other kinds of bases introduced in Section 3.

Example 8. The set $\mathcal{V}$ of vertical bars (i.e. polyominoes with only one column) is a polyomino class whose p-basis is $\{[\,1 \quad 1\,]\}$.

Let $\mathcal{W}$ be the set of polyominoes made of at most two columns. It is a polyomino class and its p-basis is shown in Figure 4.

Recall that $(\mathfrak{P}, \preccurlyeq_{\mathfrak{P}})$ contains infinite antichains (Castiglione 2005), so there exist polyomino classes with infinite p-basis. We will show an example of a polyomino class with an infinite p-basis in Proposition 30. However, we are not aware of *natural* polyomino classes whose p-basis is infinite.

3. Characterizing classes with excluded submatrices

3.1. *Submatrices as excluded patterns in permutations and polyominoes*

Obviously, not all submatrices of permutation matrices are themselves permutation matrices. More precisely:

Remark 9. The submatrices of permutation matrices are exactly those that contain at most one 1 in each row and each column. We will call such matrices *quasi-permutation matrices* in the rest of this paper. Denoting $\mathfrak{Q}$ the set of quasi-permutation matrices, $(\mathfrak{Q}, \preccurlyeq_{\mathfrak{M}})$ is a subposet of $(\mathfrak{M}, \preccurlyeq_{\mathfrak{M}})$, and is of course also well founded.

Quasi-permutation matrices are equivalently described as *rook placements* (of rectangular shape). Rook placements avoiding rook placements have recently been studied from an enumerative point of view, as well as rook placements avoiding permutations (Daly 2014). However, to our knowledge, permutations that avoid rook placements have never been investigated, and this is also one way of viewing our work on permutation classes defined by the avoidance of some quasi-permutation matrices (see in particular Section 5).

Similarly to Remark 9, not all submatrices of polyominoes are themselves polyominoes, but the situation is very different from that of permutations:

Remark 10. Every binary matrix is a submatrix of some polyomino.

Indeed, for every binary matrix M, it is always possible to add rows and columns of 1 to M in such a way that all 1 entries of the resulting matrix are connected.

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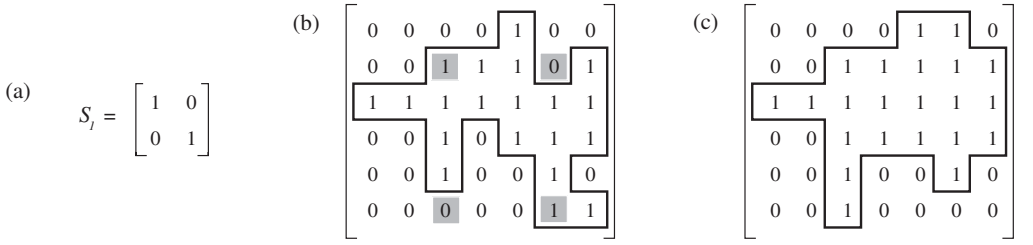


Fig. 5. (a) a matrix S_1; (b) a polyomino that contains S_1 as a submatrix, hence does not belong to $Av_{\mathfrak{P}}(S_1)$; (c) a polyomino that does not contain S_1, i.e. that belongs to $Av_{\mathfrak{P}}(S_1)$.

From Remarks 9 and 10, it makes sense to examine sets of permutations (resp. polyominoes) that avoid submatrices that are not themselves permutations (resp. polyominoes) but quasi-permutation matrices† (resp. binary matrices).

Definition 11. For any set $\mathcal{M}$ of quasi-permutation matrices (resp. of binary matrices), let us denote by $Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $Av_{\mathfrak{P}}(\mathcal{M})$) the set of all permutations (resp. polyominoes) that do not contain any submatrix in $\mathcal{M}$.

Figure 5 illustrates Definition 11 in the polyomino case.

The following facts, although immediate to prove, will be useful in our work:

Remark 12. When $\mathcal{M}$ contains only permutations (resp. polyominoes), these definitions of $Av_{\mathfrak{S}}(\mathcal{M})$ and $Av_{\mathfrak{P}}(\mathcal{M})$ coincide with the ones given in Section 2.

Remark 13. Denoting $Av_{\mathfrak{M}}(\mathcal{M})$ the set of binary matrices without any submatrix in $\mathcal{M}$, we have $Av_{\mathfrak{S}}(\mathcal{M}) = Av_{\mathfrak{M}}(\mathcal{M}) \cap \mathfrak{S}$ and $\mathfrak{Av}_{\mathfrak{P}}(\mathcal{M}) = \mathfrak{Av}_{\mathfrak{M}}(\mathcal{M}) \cap \mathfrak{P}$.

In addition†, denoting $Av_{\mathfrak{Q}}(\mathcal{M})$ the set of quasi-permutation matrices without any submatrix in $\mathcal{M}$, we also have $Av_{\mathfrak{S}}(\mathcal{M}) = Av_{\mathfrak{Q}}(\mathcal{M}) \cap \mathfrak{S}$.

Remark 14. Sets of the form $Av_{\mathfrak{S}}(\mathcal{M})$ are downward closed for $\preccurlyeq_{\mathfrak{S}}$, i.e. are permutation classes. Similarly, sets $Av_{\mathfrak{P}}(\mathcal{M})$ are polyomino classes.

We believe it is quite natural to characterize some permutation or polyomino classes by avoidance of submatrices, and will provide several examples in Sections 5 and 6. In the present section, we investigate further the description of permutation and polyomino classes by avoidance of matrices, and in particular how canonical and concise such a description can be.

3.2. *Matrix bases of permutation and polyomino classes*

From Propositions 3 and 7, every permutation (resp. polyomino) class $\mathcal{C}$ is characterized by the avoidance of a set of permutations (resp. polyominoes) which is uniquely determined.

† We may as well consider sets $\mathcal{M}$ containing arbitrary binary matrices. But excluding a matrix M which is not a quasi-permutation matrix is not actually introducing any restriction, since no permutation contains M as a submatrix.

† Like in footnote †, we may assume that $\mathcal{M}$ contains only quasi-permutation matrices here.

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With Proposition 4, we may similarly associate with every permutation (resp. polyomino) class $\mathcal{C}$ a canonical set $\mathcal{M}$ of matrices such that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $\mathcal{C} = Av_{\mathfrak{P}}(\mathcal{M})$) (see Definition 15). However, we shall see that for some permutation (resp. polyomino) classes $\mathcal{C}$, there exist several antichains $\mathcal{M}'$ such that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{M}')$ (resp. $\mathcal{C} = Av_{\mathfrak{P}}(\mathcal{M}')$).

Definition 15. Let $\mathcal{C}$ be a permutation (resp. polyomino) class. Denote by $\mathcal{C}^+$ be the set of matrices that appear as a submatrix of some element of $\mathcal{C}$, i.e.

$$\mathcal{C}^+ = \{M \in \mathfrak{M} : \exists \mathfrak{P} \in \mathcal{C}, \text{ such that } \mathfrak{M} \preccurlyeq \mathfrak{P}\}.$$

Notice that $\mathcal{C}^+$ contains only quasi-permutation (resp. binary) matrices.

Denote by $\mathcal{M}$ the set of all minimal quasi-permutation (resp. binary) matrices in the sense of $\preccurlyeq$ that do not belong to $\mathcal{C}^+$. $\mathcal{M}$ is called the *canonical matrix-basis* (or *canonical m-basis* for short) of $\mathcal{C}$.

Of course, the canonical m-basis of a class $\mathcal{C}$ is uniquely defined, and is always an antichain for $\preccurlyeq$. Moreover, Proposition 16 shows that it indeed provides a description of $\mathcal{C}$ by avoidance of submatrices.

Proposition 16. Let $\mathcal{C}$ be a permutation (resp. polyomino) class, and denote by $\mathcal{M}$ its canonical m-basis. We have $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $\mathcal{C} = Av_{\mathfrak{P}}(\mathcal{M})$).

Proof. In the case of a permutation class $\mathcal{C}$, we can work in the poset $(\mathfrak{Q}, \preccurlyeq)$ of quasi-permutation matrices. Proposition 4 ensures that $\mathcal{C}^+ = Av_{\mathfrak{Q}}(\mathcal{M})$. For a polyomino class, working in the poset of binary matrices $(\mathfrak{M}, \preccurlyeq)$, Proposition 4 ensures that $\mathcal{C}^+ = Av_{\mathfrak{M}}(\mathcal{M})$. And since $\mathcal{C} = \mathcal{C}^+ \cap \mathfrak{S}$ (resp. $\mathcal{C} = \mathcal{C}^+ \cap \mathfrak{P}$), Remark 13 yields the conclusion. □

Example 17 (Canonical *m*-basis).

i. For the (trivial) permutation class $\mathcal{T} = \{1, 12, 21\}$, we have

$$\mathcal{T}^+ = \left\{ [\,0\,], [\,1\,], [\,1 \quad 0\,], [\,0 \quad 1\,], \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

and the canonical m-basis of $\mathcal{T}$ is $\{[\,0 \quad 0\,], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$.

ii. Let $\mathcal{A}$ be the permutation class $Av_{\mathfrak{S}}(321, 231, 312)$. The canonical m-basis of $\mathcal{A}$ is $\{Q_1, Q_2\}$ with $Q_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $Q_2 = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$.

iii. The canonical m-basis of the class $\mathcal{V}$ of vertical bars, considered in Example 8, is $\{[\,0\,], [\,1 \quad 1\,]\}$.

iv. Let $\mathcal{R}$ be the polyomino class consisting of polyominoes of rectangular shape. The canonical m-basis of $\mathcal{R}$ consists only of the matrix $[\,0\,]$.

v. The canonical m-basis of the class $\mathcal{W}$ of polyominoes with at most two columns, considered in Example 8, is

$$\{[\,1 \quad 1 \quad 1\,], [\,0 \quad 0\,], [\,1 \quad 0 \quad 1\,], [\,1 \quad 1 \quad 0\,], [\,0 \quad 1 \quad 1\,]\}.$$

There is one important difference between p-basis and canonical m-basis. Every antichain of permutations (resp. polyominoes) is the p-basis of a class. On the contrary, every antichain $\mathcal{M}$ of quasi-permutation (resp. binary) matrices describes a permutation (resp.

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polyomino) class $Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $Av_{\mathfrak{P}}(\mathcal{M})$), but not every such antichain is the canonical m-basis of the corresponding permutation (resp. polyomino) class – see Example 19 below. Imposing the avoidance of matrices taken in an antichain being however a natural way of describing permutation and polyomino classes, let us define the following weaker notion of basis.

Definition 18. Let $\mathcal{C}$ be a permutation (resp. polyomino) class. Every antichain $\mathcal{M}$ of matrices such that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $Av_{\mathfrak{P}}(\mathcal{M})$) is called a *matrix-basis* (or *m-basis*) of $\mathcal{C}$.

Example 19 show several examples of m-bases of permutation and polyomino classes which are different from the canonical m-basis (see Example 17).

Example 19 (m-bases). Consider the set $\mathcal{M}$ consisting of the following four matrices: $M_1 = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$, $M_2 = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$, $M_3 = \left[\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right]$, $M_4 = \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]$. We may check that every permutation of size 3 contains a submatrix $M \in \mathcal{M}$, and that it actually contains each of these four M_i. Moreover, $\mathcal{M}$ is an antichain, and so is obviously each set $\{M_i\}$. Therefore, $\mathcal{T} = Av_{\mathfrak{S}}(\mathcal{M}) = Av_{\mathfrak{S}}(M_i)$, for each $1 \leqslant i \leqslant 4$, even though these antichains characterizing $\mathcal{T}$ are not the canonical m-basis of $\mathcal{T}$.

It is readily checked that $\mathcal{A} = Av_{\mathfrak{S}}(Q_1) = Av_{\mathfrak{S}}(Q_2)$ even though the canonical m-basis of $\mathcal{A}$ is $\{Q_1, Q_2\}$.

The canonical m-basis of the class $\mathcal{V}$ of vertical bars is $\{[0], [1\ 1]\}$. But we also have $Av_{\mathfrak{P}}([1\ 1]) = \mathcal{V}$.

Consider the sets $\mathcal{M}_1 = \{[1\ 0], [0\ 1], [0\ 0], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$ and $\mathcal{M}_2 = \{[1\ 0], [0\ 1], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$. We may easily check that $\mathcal{M}_1$ and $\mathcal{M}_2$ are antichains, and that their avoidance characterizes the rectangular polyominoes: $\mathcal{R} = Av_{\mathfrak{P}}(\mathcal{M}_1) = Av_{\mathfrak{P}}(\mathcal{M}_2)$.

Similarly, the sets $\mathcal{M}_3 = \{[1\ 1\ 1], [1\ 1\ 0]\}$ and $\mathcal{M}_4 = \{[1\ 1\ 1], [0\ 1\ 1]\}$ are m-bases of the class $\mathcal{W}$ of polyominoes with at most two columns.

Example 19 shows in addition that the canonical m-basis is not always the most concise way of describing a permutation (or polyomino) class by avoidance of submatrices. This motivates the following definition:

Definition 20. Let $\mathcal{C}$ be a permutation (resp. polyomino) class. A *minimal m-basis* of $\mathcal{C}$ is an m-basis of $\mathcal{C}$ satisfying the following additional conditions:

1. $\mathcal{M}$ is a minimal subset subject to $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $Av_{\mathfrak{P}}(\mathcal{M})$),
   i.e. for every strict subset $\mathcal{M}'$ of $\mathcal{M}$, $\mathcal{C} \neq Av_{\mathfrak{S}}(\mathcal{M}')$ (resp. $Av_{\mathfrak{P}}(\mathcal{M}')$);
2. for every submatrix M' of some matrix $M \in \mathcal{M}$, we have
   i. $M' = M$ or
   ii. with $\mathcal{M}' = \big(\mathcal{M} \setminus \{M\}\big) \cup \{M'\}$, $\mathcal{C} \neq Av_{\mathfrak{S}}(\mathcal{M}')$ (resp. $Av_{\mathfrak{P}}(\mathcal{M}')$).

Condition (1) ensures minimality in the sense of inclusion, while Condition (2) ensures that it is not possible to replace a matrix of the minimal m-basis by another one of smaller dimensions. For future reference, let us notice that with the notations of Definition 20, the

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statement $\mathcal{C} \neq Av_{\mathfrak{S}}(\mathcal{M}')$ (resp. $Av_{\mathfrak{P}}(\mathcal{M}')$) in Condition (2)ii is equivalent to $\mathcal{C} \nsubseteq Av_{\mathfrak{S}}(\mathcal{M}')$ (resp. $Av_{\mathfrak{P}}(\mathcal{M}')$), since the other inclusion always holds.

To illustrate the relevance of Condition (2), consider for instance the m-basis $\{M_1\}$ of $\mathcal{T}$, with $M_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (see Example 19). Of course it is minimal in the sense of inclusion, however noticing that $\mathcal{T} = Av_{\mathfrak{S}}([0\ 0]) = Av_{\mathfrak{S}}(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right])$, with these excluded submatrices being submatrices of M_1, it makes sense *not* to consider $\{M_1\}$ as a *minimal* m-basis. This is exactly the point of Condition (2). Actually, $\{[0\ 0]\}$ and $\{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\}$ both satisfy Conditions (1) and (2), i.e. are minimal m-basis of $\mathcal{T}$.

This also illustrates the somewhat undesirable property that a class may have several minimal m-bases. This is not only true for the trivial class $\mathcal{T}$.

Example 21. For the permutation class $\mathcal{A}$, the m-bases $\{Q_1\}$ and $\{Q_2\}$ of $\mathcal{A}$ (see Example 17) are minimal m-bases of $\mathcal{A}$.

For the polyomino class $\mathcal{W}$, the two m-bases $\mathcal{M}_3$ and $\mathcal{M}_4$ of Example 19 are minimal m-bases of $\mathcal{W}$. We can observe that in this case each polyomino containing a pattern of $\mathcal{M}_3$ must contain a pattern of $\mathcal{M}_4$ and conversely.

However, the minimal m-bases of a class are relatively constrained:

Proposition 22. Let $\mathcal{C}$ be a permutation (resp. polyomino) class and let $\mathcal{M}$ be its canonical m-basis. The minimal m-bases of $\mathcal{C}$ are the subsets $\mathcal{B}$ of $\mathcal{M}$ that are minimal (for inclusion) under the condition $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{B})$ (resp. $Av_{\mathfrak{P}}(\mathcal{B})$).

Proof. For simplicity of notations, let us forget the indices and write $Av(\mathcal{B})$ instead of $Av_{\mathfrak{S}}(\mathcal{B})$ (resp. $Av_{\mathfrak{P}}(\mathcal{B})$).

Consider a subset $\mathcal{B}$ of $\mathcal{M}$ that is minimal for inclusion under the condition $\mathcal{C} = Av(\mathcal{B})$, and let us prove that $\mathcal{B}$ is a minimal m-basis of $\mathcal{C}$. $\mathcal{B}$ is clearly an m-basis of $\mathcal{C}$ satisfying Condition (1). Assume that $\mathcal{B}$ does not satisfy Condition (2): there is some $M \in \mathcal{B}$ and some proper submatrix M' of M, such that $\mathcal{C} = Av(\mathcal{B}')$ for $\mathcal{B}' = (\mathcal{B} \setminus \{M\}) \cup \{M'\}$. By definition of the canonical m-basis, $M' \in \mathcal{C}^+$ (or M would not be minimal for $\preccurlyeq$), so there exists a permutation (resp. polyomino) $P \in \mathcal{C}$ such that $M' \preccurlyeq P$. But then $P \notin Av(\mathcal{B}') = \mathcal{C}$ bringing the contradiction which ensures that $\mathcal{B}$ satisfies Condition (2).

Conversely, consider a minimal m-basis $\mathcal{B}$ of $\mathcal{C}$ and a matrix $M \in \mathcal{B}$, and let us prove that M belongs to $\mathcal{M}$. Because of Condition (1), this is enough to conclude the proof. First, notice that $M \notin \mathcal{C}^+$. Indeed, otherwise there would exist a permutation (resp. polyomino) $P \in \mathcal{C}$ such that $M \preccurlyeq P$, and we would also have $P \notin Av(\mathcal{B}) = \mathcal{C}$, a contradiction. By definition, $\mathcal{C}^+ = Av_{\mathfrak{Q}}(\mathcal{M})$ (resp. $\mathcal{C}^+ = Av_{\mathfrak{M}}(\mathcal{M})$), so there exists $M' \in \mathcal{M}$ such that $M' \preccurlyeq M$. Since $\mathcal{B}$ is a minimal m-basis we either have $M = M'$, which proves that $M \in \mathcal{M}$, or we have $Av(\mathcal{B}') \subsetneq \mathcal{C}$ for $\mathcal{B}' = (\mathcal{B} \setminus \{M\}) \cup \{M'\}$, in which case we derive a contradiction as follows. If $Av(\mathcal{B}') \subsetneq \mathcal{C}$, then there is some permutation (resp. polyomino) $P \in \mathcal{C}$ which has a submatrix in $\mathcal{B}'$. It cannot be some submatrix in $\mathcal{B} \setminus \{M\}$, because $\mathcal{C} = Av(\mathcal{B})$. So $M' \preccurlyeq P$, which is a contradiction to $P \in \mathcal{C} = Av(\mathcal{M})$. □

Example 23. Proposition 22 (together with Examples 17 and 19) allows to compute all minimal m-bases for our running examples.

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Both permutation classes $\mathcal{T}$ and $\mathcal{A}$ each have exactly two minimal m-bases, namely $\{[\begin{smallmatrix}0 & 0\end{smallmatrix}]\}$ and $\{[\begin{smallmatrix}0\\0\end{smallmatrix}]\}$, and $\{Q_1\}$ and $\{Q_2\}$ respectively.

The polyomino classes $\mathcal{V}$ and $\mathcal{R}$ have unique minimal m-bases, respectively $\{[\begin{smallmatrix}1 & 1\end{smallmatrix}]\}$ and $\{[\begin{smallmatrix}0\end{smallmatrix}]\}$. But the polyomino class $\mathcal{W}$ has exactly two minimal m-bases, namely $\mathcal{M}_3$ and $\mathcal{M}_4$.

A natural problem is then to characterize the permutation (resp. polyomino) classes which have a unique minimal m-basis. Propositions 24 and 25 give partial answers to this problem.

Proposition 24. If the p-basis of a permutation (resp. polyomino) class $\mathcal{C}$ is also a minimal m-basis of $\mathcal{C}$, then it is the unique minimal m-basis of $\mathcal{C}$.

Proof. Consider a permutation (resp. polyomino) class $\mathcal{C}$, denote by $\mathcal{B}$ its p-basis, and assume that $\mathcal{B}$ is a minimal m-basis of $\mathcal{C}$. Let us also denote $\mathcal{M}$ the canonical m-basis of $\mathcal{C}$. From Proposition 22, $\mathcal{B}$ is a subset of $\mathcal{M}$, and we see that Proposition 24 will follow if we prove that: for all subsets $\mathcal{X}$ of $\mathcal{M}$,

i. either $\mathcal{X} = \mathcal{B}$,
ii. or $\mathcal{C} \neq Av_{\mathfrak{S}}(\mathcal{X})$ (resp. $Av_{\mathfrak{P}}(\mathcal{X})$),
iii. or $\mathcal{X}$ contains a strict subset $\mathcal{X}'$ such that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{X}')$ (resp. $Av_{\mathfrak{P}}(\mathcal{X}')$).

So, let $\mathcal{X}$ be a subset of $\mathcal{M}$, different from $\mathcal{B}$. Assume that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{X})$ (resp. $Av_{\mathfrak{P}}(\mathcal{X})$), and let us prove that (iii) holds.

First we claim that $\mathcal{B} \subseteq \mathcal{X}$. Indeed, assuming the contrary, there would exists $b \in \mathcal{B}$ such that $b \notin \mathcal{X}$. $\mathcal{B}$ being the p-basis of $\mathcal{C}$, we deduce that b is a permutation (resp. polyomino) such that $b \notin \mathcal{C}$. Because $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{X})$ (resp. $Av_{\mathfrak{P}}(\mathcal{X})$), we know that b contains a pattern in $\mathcal{X}$. And since $b \notin \mathcal{X}$, it has to be a strict pattern, that we denote b'. The claim is then proved, since $b \in \mathcal{B} \subseteq \mathcal{M}$ and $b' \in \mathcal{X} \subseteq \mathcal{M}$ are two distinct elements of $\mathcal{M}$, such that $b' \leqslant b$, contradicting that $\mathcal{M}$ is an antichain.

Knowing that $\mathcal{B} \subseteq \mathcal{X}$ holds, we conclude the proof as follows. Because $\mathcal{X} \neq \mathcal{B}$, $\mathcal{B}$ is a strict subset of $\mathcal{X}$. By hypothesis $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{B})$ (resp. $Av_{\mathfrak{P}}(\mathcal{B})$), so that $\mathcal{X}$ contains a strict subset $\mathcal{X}' = \mathcal{B}$ such that $\mathcal{C} = Av_{\mathfrak{S}}(\mathcal{X}')$ (resp. $Av_{\mathfrak{P}}(\mathcal{X}')$). □

Proposition 24 applies for instance to the class $\mathcal{V}$ of vertical bars (see Examples 8 and 23) or to the case of parallelogram polyominoes (in Section 6).

Proposition 25. Let $\mathcal{C}$ be a permutation (resp. polyomino) class and $\mathcal{M}$ be its canonical m-basis. Assuming that for all $M \in \mathcal{M}$, there exists a permutation (resp. polyomino) P such that $M \leqslant P$ and for all $M' \in \mathcal{M}$ (with $M' \neq M$) $M' \not\leqslant P$, then $\mathcal{M}$ is a minimal m-basis. Consequently, $\mathcal{C}$ has a unique minimal m-basis.

Proof. Considering a proper subset $\mathcal{M}'$ of $\mathcal{M}$, from Proposition 22 it is enough to prove that $Av_{\mathfrak{S}}(\mathcal{M}') \neq Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $Av_{\mathfrak{P}}(\mathcal{M}') \neq Av_{\mathfrak{P}}(\mathcal{M})$). Since $\mathcal{M}' \subsetneq \mathcal{M}$ there exists $M'' \in \mathcal{M}$ such that $M'' \notin \mathcal{M}'$. By assumption, there is a permutation (resp. polyomino) P such that $M'' \leqslant P$ but $M' \not\leqslant P$ for all $M' \in \mathcal{M}, M' \neq M''$. Hence $Av_{\mathfrak{S}}(\mathcal{M}') \neq Av_{\mathfrak{S}}(\mathcal{M})$ (resp. $Av_{\mathfrak{P}}(\mathcal{M}') \neq Av_{\mathfrak{P}}(\mathcal{M})$). □

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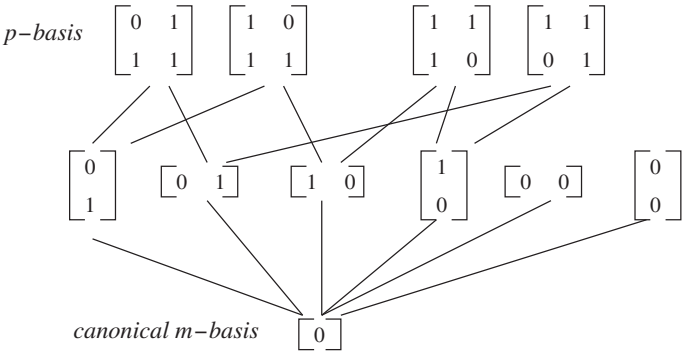


Fig. 6. The p-basis, an m-basis and the canonical m-basis of the class $\mathcal{R}$ of polyominoes having rectangular shape.

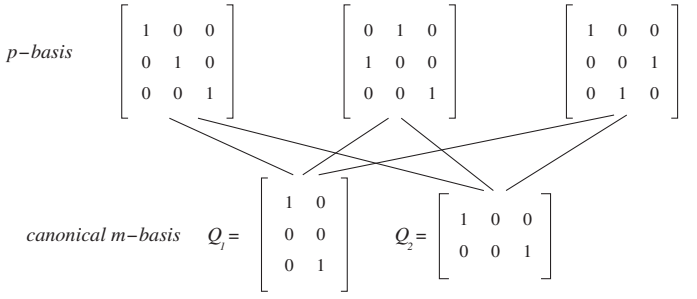


Fig. 7. The p-basis and the canonical m-basis of $\mathcal{A} = Av_{\mathfrak{S}}(321, 231, 312)$.

4. Relations between the p-basis and the m-bases

A permutation (resp. polyomino) class being now equipped with several notions of basis, we investigate how to describe one basis from another, and focus here on describing the p-basis from any m-basis.

Proposition 26. Let $\mathcal{C}$ be a permutation (resp. polyomino) class, and let $\mathcal{M}$ be an m-basis of $\mathcal{C}$. Then the p-basis of $\mathcal{C}$ consists of all permutations (resp. polyominoes) that contain a submatrix in $\mathcal{M}$, and that are minimal (w.r.t. $\preccurlyeq_{\mathfrak{S}}$ resp. $\preccurlyeq_{\mathfrak{P}}$) for this property.

Proof. This follows immediately from the description of the p-basis of $\mathcal{C}$ in Proposition 3 (resp. 7) and the definition of m-basis. □

Example 27. Figures 6 and 7 give the p-bases of the classes $\mathcal{A}$ and $\mathcal{R}$ of Example 17, and illustrate their relation to their canonical m-basis.

In the case of permutation classes, Proposition 26 allows to compute the p-basis of any class $\mathcal{C}$, given an m-basis of $\mathcal{C}$. Indeed, the minimal permutations (in the sense of $\preccurlyeq_{\mathfrak{S}}$) that contain a given submatrix M are straightforward to describe:

Proposition 28. Let M be a quasi-permutation matrix. The minimal permutations that contain M are exactly those that may be obtained from M by insertions of rows (resp.

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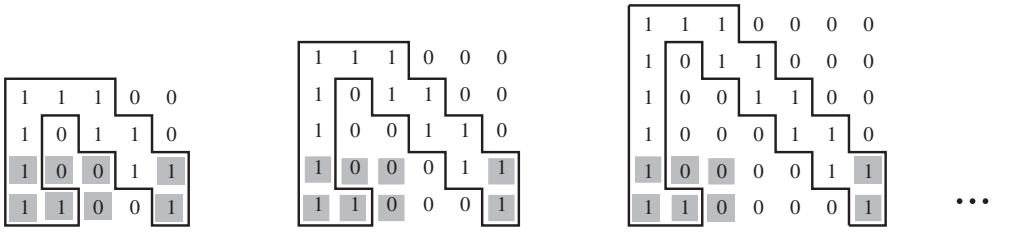


Fig. 8. An infinite antichain of polyominoes belonging to the p-basis of $Av_{\mathfrak{P}}(M_{\text{inf}})$.

columns) with exactly one entry 1, which should moreover fall into a column (resp. row) of 0 of M.

It follows from Proposition 28 that the p-basis of a permutation class $\mathcal{C}$ can be easily computed from an m-basis of $\mathcal{C}$. Also, Proposition 28 implies that:

Corollary 29. If a permutation class has a finite m-basis (i.e. is described by the avoidance of a finite number of submatrices) then it has a finite p-basis.

The situation is more complex if we consider polyomino classes. The description of the polyominoes containing a given submatrix is not as straightforward as in Proposition 28, and the analogue of Corollary 29 does not hold for polyomino classes, as illustrated in Proposition 30.

Proposition 30. The polyomino class $\mathcal{C} = Av_{\mathfrak{P}}(M_{\text{inf}})$ defined by the avoidance of $M_{\text{inf}} = \left[\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{smallmatrix}\right]$ has an infinite p-basis.

Proof. It is enough to exhibit an infinite sequence of polyominoes containing M_{inf}, and that are minimal (for $\preccurlyeq_{\mathfrak{P}}$) for this property. By minimality of its elements, such a sequence is necessarily an antichain, and it forms an infinite subset of the p-basis of $\mathcal{C}$. The first few terms of such a sequence are depicted in Figure 8, and the definition of the generic term of this sequence should be clear from the figure. We check by comprehensive verification that every polyomino P of this sequence contains M_{inf}, and additionally that occurrences of M_{inf} in P always involve the two bottommost rows of P, its two leftmost columns, and its rightmost column. Moreover, comprehensive verification also shows that every polyomino P of this sequence is minimal for the condition $M_{\text{inf}} \preccurlyeq_{\mathfrak{P}} P$, i.e. that every polyomino P' occurring in such a P as a proper submatrix avoids M_{inf}. Indeed, the removal of rows or columns from such a polyomino P either disconnects it or removes all the occurrences of M_{inf}. □

5. Some permutation classes defined by submatrix avoidance

Many notions of pattern avoidance in permutations have been considered in the literature. The one of submatrix avoidance that we have considered is yet another one. In this section, we start by explaining how it relates to other notions of pattern avoidance. Then, using this approach to pattern avoidance, we give simpler proofs of the enumeration of some

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permutation classes. Finally, we show how these proofs can be brought to a more general level, to prove Wilf-equivalences of many permutation classes.

5.1. *Connection of submatrix avoidance to the avoidance of other generalized permutation patterns*

A first generalization of pattern avoidance in permutations introduces *adjacency constraints* among the elements of a permutation that should form an occurrence of a (otherwise classical) pattern. Such patterns with adjacency constraints are known as *vincular* and *bivincular* patterns (Bousquet-Mélou 2010), and a generalization with additional border constraints has recently been introduced by Úlfarsson (2011).

The avoidance of submatrices in permutations can be seen as a dual notion to the avoidance of vincular and bivincular patterns, in the following sense. In an occurrence of some vincular (resp. bivincular) pattern in a permutation σ, we impose that some elements of the domain must be adjacent (resp. mapped by σ into consecutive values). Now, consider a quasi-permutation matrix M, and let π be the largest permutation contained in M. If there is a column (resp. row) of 0's in M, then an occurrence of M in a permutation σ is an occurrence of π in σ where some elements of the domain of σ are not allowed to be adjacent (resp. mapped by σ into consecutive values).

Example 31. $Av_{\mathfrak{S}}\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$ denotes the set of all permutations such that in any occurrence of 231 the elements mapped into '2' and '3' are at adjacent positions.

$Av_{\mathfrak{S}}\left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)$ denotes the set of all permutations such that in any occurrence of 231 the elements mapped into '2' and '3' are at adjacent positions, or the actual values of '1' and '2' are consecutive numbers.

$Av_{\mathfrak{S}}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$ denotes the set of all permutations such that in any occurrence of 231 the actual value of '3' is the maximum of the permutation.

As noticed in Remark 14, sets of permutations defined by avoidance of submatrices are permutation classes. On the contrary, avoidance of vincular or bivincular patterns (although very useful for characterizing important families of permutations, like Baxter permutations (Baxter 1964)) does not in general describe sets of permutations that are downward closed for $\preccurlyeq_{\mathfrak{S}}$. Hence, in the context of permutation *classes*, the above discussion suggests that it is more convenient to introduce *non-adjacency* (instead of adjacency) constraints in permutation patterns, which correspond to rows and columns of 0 in quasi-permutation matrices.

Mesh patterns are another generalization of permutation patterns that has been introduced more recently by Brändén and Claesson (2011). It has itself been generalized in several way, in particular by Úlfarsson who introduced in Úlfarsson (2011) the notion of marked mesh patterns. It is very easy to see that the avoidance of a quasi-permutation

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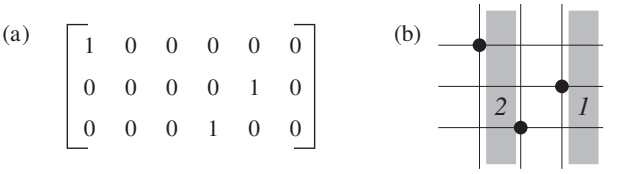


Fig. 9. A quasi-permutation matrix with no uncovered 0 entries and the corresponding marked-mesh pattern.

matrix with no uncovered 0 entries[†] can be expressed as the avoidance of a special form of marked-mesh pattern. Indeed, a row (resp. k consecutive rows) of 0 in a quasi-permutation matrix with no uncovered 0 entries corresponds to a mark, spanning the whole pattern horizontally, indicating the presence of at least one (resp. at least k) element(s). The same holds for columns and vertical marks. Figure 9 shows an example of a quasi-permutation matrix with no uncovered 0 entries with the corresponding marked-mesh pattern.

5.2. *A different look at some known permutation classes*

Several permutation classes avoiding three patterns of size 3 or four patterns of size 4 that have been studied in the literature (and are referenced in Guibert's catalog (Guibert 1995, Appendix A)) are easier to describe with the avoidance of just one submatrix, as we explain in the following. For some of these classes, the description by submatrix avoidance also allows to provide a simple proof of their enumeration.

In this paragraph, we do not consider classes that are equal up to symmetry (reverse, inverse, complement and their compositions). But the same results of course apply (up to symmetry) to every symmetry of each class considered.

5.2.1. *The class* $\mathcal{F} = Av_{\mathfrak{S}}(123, 132, 213)$. This class was studied by Simion and Schmidt (1985), in the context of the systematic enumeration of permutations avoiding patterns of size 3. An alternative description of $\mathcal{F}$ is $\mathcal{F} = Av_{\mathfrak{S}}(M_F)$ where $M_F = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right]$. It follows immediately since 123, 132 and 213 are exactly the permutations which cover M_F in the sense of Propositions 26 and 28.

Simion and Schmidt (1985) shows that $\mathcal{F}$ is enumerated by the Fibonacci numbers. Of course, it is possible to use the description of $\mathcal{F}$ by the avoidance of M_F to prove this enumeration result. However, in this case the proof would just rephrase the original one of Simion and Schmidt (1985).

5.2.2. *The class* $\mathcal{G} = Av_{\mathfrak{S}}(123, 132, 231)$. This class is also studied in Simion and Schmidt (1985), where it is shown that there are n permutations of size n in $\mathcal{G}$. The enumeration is obtained by a simple inductive argument, which relies on a recursive description of the permutations in $\mathcal{G}$.

[†] A 0 entry is *covered* when there is a 1 entry in its column or row (or both).

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For the same reasons as in the case of $\mathcal{F}$, $\mathcal{G}$ is alternatively described by $\mathcal{G} = Av_{\mathfrak{S}}(M_G)$ where $M_G = \left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$. Any occurrence of a pattern 12 in a permutation σ can be extended to an occurrence of M_G, as long as it does not involve the last element of σ. So from this characterization, it follows that the permutations of $\mathcal{G}$ are exactly the decreasing sequences followed by one element. This describes the permutations of $\mathcal{G}$ non-recursively, and gives immediate access to the enumeration of $\mathcal{G}$.

5.2.3. *The classes* $\mathcal{H} = Av_{\mathfrak{S}}(1234, 1243, 1423, 4123)$, $\mathcal{J} = Av_{\mathfrak{S}}(1324, 1342, 1432, 4132)$ *and* $\mathcal{K} = Av_{\mathfrak{S}}(2134, 2143, 2413, 4213)$. These three classes have been studied in Guibert (1995, Section 4.2), where it is proved that they are enumerated by the central binomial coefficients. The proof first gives a generating tree for these classes, and then the enumeration is derived analytically from the corresponding rewriting system. In particular, this proof does not provide a description of the permutations in $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{K}$ which could be used to give a *combinatorial* proof of their enumeration. Excluded submatrices can be used for that purpose.

As before, because the p-basis of $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{K}$ are exactly the permutations which cover the matrices M_H, M_J and M_K given below, introducing the matrices $M_H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $M_J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $M_K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we have $\mathcal{H} = Av_{\mathfrak{S}}(M_H)$, $\mathcal{J} = Av_{\mathfrak{S}}(M_J)$ and $\mathcal{K} = Av_{\mathfrak{S}}(M_K)$.

Similarly to the case of $\mathcal{G}$, any occurrence of a pattern 123 (resp. 132, resp. 213) in a permutation σ can be extended to an occurrence of M_H (resp. M_J, resp. M_K), as long as it does not involve the maximal element of σ. Conversely, if a permutation σ contains M_H (resp. M_J, resp. M_K), then there is an occurrence of 123 (resp. 132, resp. 213) in σ that does not involves its maximum. Consequently, the permutations of $\mathcal{H}$ (resp. $\mathcal{J}$, resp. $\mathcal{K}$) are exactly those of avoiding 123 (resp. 132, resp. 213) to which a maximal element has been added. This provides a very simple description of the permutations of $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{K}$. Moreover, recalling that for any permutation $\pi \in \mathfrak{S}_3$ $Av_{\mathfrak{S}}(\pi)$ is enumerated by the Catalan numbers, it implies that the number of permutations of size n in $\mathcal{H}$ (resp. $\mathcal{J}$, resp. $\mathcal{K}$) is $n \times Cat_{n-1} = \binom{2n-2}{n-1}$.

5.3. *Propagating enumeration results and Wilf-equivalences with submatrices*

The similarities that we observed between the cases of the classes $\mathcal{G}$, $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{K}$ are not a coincidence. Indeed, they can all be encapsulated in the following proposition, which simply pushes the same idea forward to a general setting.

Proposition 32. Let τ be a permutation. Let $M_{\tau,\text{top}}$ (resp. $M_{\tau,\text{bottom}}$) be the quasi-permutation matrix obtained by adding a row of 0 entries above (resp. below) the permutation matrix of τ. Similarly, let $M_{\tau,\text{right}}$ (resp. $M_{\tau,\text{left}}$) be the quasi-permutation matrix obtained by adding a column of 0 entries on the right (resp. left) of the permutation matrix of τ. The permutations of $Av_{\mathfrak{S}}(M_{\tau,\text{top}})$ (resp. $Av_{\mathfrak{S}}(M_{\tau,\text{bottom}})$, resp. $Av_{\mathfrak{S}}(M_{\tau,\text{right}})$, resp. $Av_{\mathfrak{S}}(M_{\tau,\text{left}})$)

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are exactly the permutations avoiding τ to which a maximal (resp. minimal, resp. last, resp. first) element has been added.

Proof. We prove the case of $M_{\tau,\text{top}}$ only, the other cases being identical up to symmetry.

Consider a permutation $\sigma \in Av_{\mathfrak{S}}(M_{\tau,\text{top}})$, and denote by σ' the permutation obtained deleting the maximum of σ. Assuming that σ' contains τ, then σ would contain $M_{\tau,\text{top}}$, which contradicts $\sigma \in Av_{\mathfrak{S}}(M_{\tau,\text{top}})$; hence $\sigma' \in Av_{\mathfrak{S}}(\tau)$.

Conversely, consider a permutation $\sigma' \in Av_{\mathfrak{S}}(\tau)$, and a permutation σ obtained by adding a maximal element to σ'. Assume that σ contains $M_{\tau,\text{top}}$, and consider an occurrence of $M_{\tau,\text{top}}$ in σ. This occurrence of $M_{\tau,\text{top}}$ cannot involve the maximum of σ, so it yields an occurrence of τ in σ', and hence a contradiction. Therefore, $\sigma \in Av_{\mathfrak{S}}(M_{\tau,\text{top}})$. □

Proposition 32 has two nice consequences in terms of enumeration. When the enumeration of the class is known, it allows to deduce the enumeration of four other permutation classes. Similarly, for any pair of Wilf-equivalent classes (i.e. of permutation classes having the same enumeration), it produces a set of eight classes which are all Wilf-equivalent.

Corollary 33. Let $\mathcal{C}$ be a permutation class whose p-basis is $\mathcal{B}$. Let $\mathcal{M} = \{M_{\tau,\text{top}} : \tau \in \mathcal{B}\}$. Denote by c_n the number of permutations of size n in $\mathcal{C}$. The permutation class $Av_{\mathfrak{S}}(\mathcal{M})$ is enumerated by the sequence $(n \cdot c_{n-1})_n$. The same holds replacing $M_{\tau,\text{top}}$ with $M_{\tau,\text{bottom}}$, $M_{\tau,\text{right}}$ or $M_{\tau,\text{left}}$.

Corollary 34. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two Wilf-equivalent permutation classes whose p-basis are respectively $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{M}_{1,\text{top}} = \{M_{\tau,\text{top}} : \tau \in \mathcal{B}_1\}$, $\mathcal{M}_{2,\text{top}} = \{M_{\tau,\text{top}} : \tau \in \mathcal{B}_2\}$, and define $\mathcal{M}_{1,\text{bottom}}$, $\mathcal{M}_{2,\text{bottom}}$, $\mathcal{M}_{1,\text{right}}$, $\mathcal{M}_{2,\text{right}}$, $\mathcal{M}_{1,\text{left}}$, $\mathcal{M}_{2,\text{left}}$ similarly. The permutation classes defined by the avoidance of one of these eight sets of matrices are all Wilf-equivalent.

6. Submatrix avoidance in polyominoes

In this section, we show that several families of polyominoes studied in the literature can be characterized in terms of submatrix avoidance. For more details on these families of polyominoes we address the reader to Bousquet-Mélou (1996) and Delest (1984). We also use submatrix avoidance to introduce new polyomino classes.

6.1. *Some known polyomino classes*

Quite a few known families of polyominoes defined by geometric constraints (see Figure 10) can be described by the avoidance of submatrices (which translate these geometric constraints in a natural way).

6.1.1. *Convex polyominoes.* These are defined by imposing one of the simplest geometrical constraints: the connectivity of rows/columns.

Definition 35. A polyomino is *horizontally convex* or *h-convex* (resp. *vertically convex* or *v-convex*), if each of its rows (resp. columns) is connected. A polyomino that is both *h*-convex and *v*-convex is said to be *convex*.

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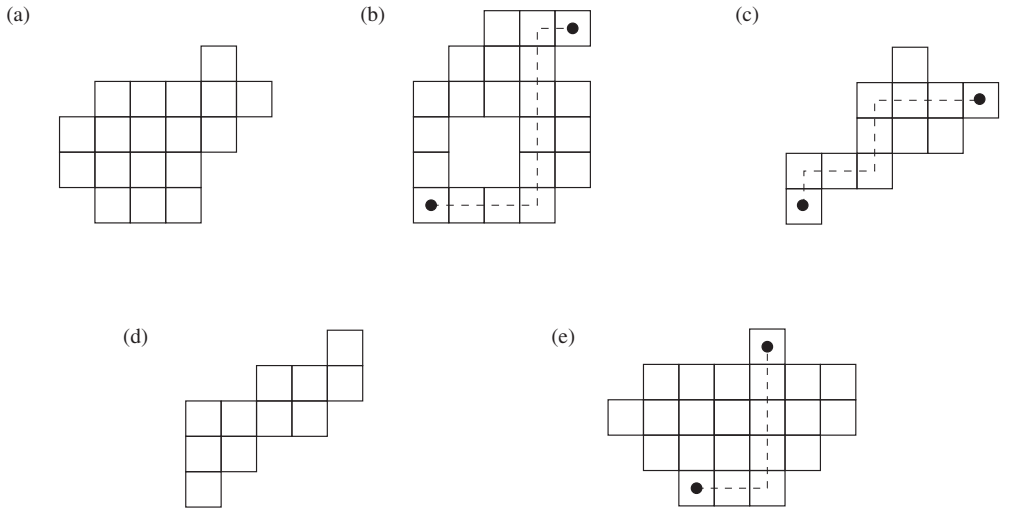


Fig. 10. (a) a convex polyomino; (b) a directed polyomino; (c) a directed-convex polyomino; (d) a parallelogram polyomino; (e) an L-convex polyomino.

Figure 10a shows a convex polyomino. The convexity constraint can be easily expressed in terms of excluded submatrices since its definition already relies on some specific configurations that the cells of each row and column of a polyomino have to avoid:

Proposition 36. Convex polyominoes can be represented by the avoidance of the two submatrices $H = [\,1\ \ 0\ \ 1\,]$ and $V = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

More precisely, the avoidance of the matrix H (resp. V) indicates the h-convexity (resp. v-convexity).

Remark 37. It is easily checked from Definition 20 that $\{H, V\}$ is a minimal m-basis of the class of convex polyominoes. By Proposition 22, this implies that $\{H, V\}$ is included in the canonical m-basis of this class. We would like to point out that this inclusion is strict, since the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ also belongs to this canonical m-basis. Indeed, (*i*) there is no convex polyomino that contains A, and (*ii*) every proper submatrix of A is included in some convex polyomino. The proof of (*ii*) is immediate by comprehensive verification, and (*i*) follows easily by noticing that every polyomino containing A must have some rows or columns which are not connected. Finding the canonical m-basis of the class of convex polyominoes remains an open problem.

From Proposition 26, it is also easy to determine the p-basis of the class of convex polyominoes: it is the set of four polyominoes $\{H_1, H_2, V_1, V_2\}$ depicted in Figure 11.

We advise the reader that, in the rest of the section, for each polyomino class, we will provide only a matrix description of the basis. Indeed, in all these examples the p-basis can easily be obtained from the given m-basis with Proposition 26, like in the case of convex polyominoes.

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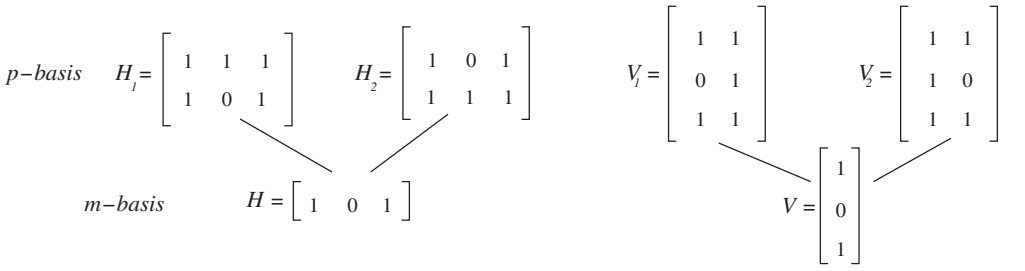


Fig. 11. The p-basis and a minimal m-basis of the class of convex polyominoes.

Directed-convex polyominoes. Directed-convex polyominoes are defined using the notion of internal path of a polyomino. An *(internal) path* of a polyomino is a sequence of distinct cells $(c_1, \ldots, c_n)$ of the polyomino such that every two consecutive cells in this sequence are edge-connected; according to the respective positions of the cells c_i and c_{i+1}, we say that the pair (c_i, c_{i+1}) forms a *north, south, east or west step* in the path.

Definition 38. A polyomino P is *directed* when every cell of P can be reached from a distinguished cell (called the source) by a path that uses only north and east steps.

Figure 10b depicts a directed polyomino. The reader can check (e.g. removing the fourth column on Figure 10b) that the set of directed polyominoes is not a polyomino class. However, the set of *directed-convex* polyominoes (i.e. of polyominoes that are both directed and convex – see Figure 10c) is a class:

Proposition 39. The class $\mathcal{D}$ of directed-convex polyominoes is characterized by the avoidance of the submatrices $H = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Proof. Let us prove that $\mathcal{D} = Av_{\mathfrak{P}}(H, V, D)$.

First, let $P \in Av_{\mathfrak{P}}(H, V, D)$. By Proposition 36, P is a convex polyomino which avoids the submatrix D. Let s be the bottom cell of the leftmost column of P. If P is directed, then necessarily s is its source. Let us proceed by contradiction assuming that P is not directed. Then there exists a cell c of P such that all the paths from s to c contain either a south step or a west step.

Consider a path, say p, which goes from s to c and is of minimal length (where the length of a path is just the number of its cells). Denote the length of p by ℓ. Suppose that, in p, the initial run of north and east steps is followed by a south step (if it is followed by a west step, a similar reasoning holds). Let us denote by i, with $1 \leqslant i < \ell$ the index such that the pair of cells (c_i, c_{i+1}) forms the first south step in p. Let us consider two cases.

i. If the first vertical step of p is a north step, then let (c_j, c_{j+1}) be the last north step before the first south step of p. By minimality of p, it holds that c_{j+1} and c_i are two distinct cells on the same row, that do not share an edge. Consequently, c_j and c_{i+1} are also distinct cells on the same row that do not share an edge. And by minimality of p, this row is not connected, contradicting the h-convexity of P.
ii. If the first vertical step of p is a south step, i.e. if (c_i, c_{i+1}) is the first vertical step of p, it means that s and c_i are on the same row of P. The cell q immediately below s is

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then on the row of c_{i+1}, and by definition of s, it does not belong to P. Therefore, the four cells s, q, c_i and c_{i+1} form the pattern D, a contradiction.

Conversely, let P be a directed convex polyomino. By Proposition 36, P avoids H and V. We will proceed by contradiction assuming that P contains the pattern D. Let q be the cell, not belonging to P, corresponding to the lower left element of D. Let c_1 and c_2 be the cells of P that correspond to the upper left and the lower right elements of D, respectively. Let K be the column of P containing the cell c_1. Since K is connected, all cells of K lie strictly above q. In other words, all cells of K lie strictly above c_2. The source cell s of P must lie weakly to the left of K; otherwise every path from s to c_1 would contain at least one west step, contradicting the fact that P is directed. Now consider any path leading from s to c_2. Such a path visits at least one cell of K. From a cell of K, this path cannot reach c_2 without making south steps. This again contradicts the fact that P is directed.

□

6.1.2. *Parallelogram polyominoes.* Another widely studied family of polyominoes – that can also be defined using a notion of path, this time of *boundary path* – is that of *parallelogram polyominoes* (see Figure 10d).

Definition 40. A parallelogram polyomino is a polyomino whose boundary can be decomposed in two paths, the upper and the lower paths, which are made of north and east unit steps and meet only at their starting and final points.

Proposition 41 below shows that parallelogram polyominoes can be described by submatrix avoidance, hence form a polyomino class. Since the proof of Proposition 41 resembles that of Proposition 39, it will be left to the reader.

Proposition 41. Parallelogram polyominoes are characterized by the avoidance of the submatrices $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$.

Remark 42. The set of two excluded submatrices of Proposition 41 is also the p-basis of the class of parallelogram polyominoes. And it can further be checked that it forms a minimal m-basis. Therefore, by Proposition 24, it is the unique minimal m-basis of the class of parallelogram polyominoes.

6.1.3. *L-convex polyominoes.* Parallelogram polyominoes and directed-convex polyominoes form subclasses of the class of convex polyominoes that are both defined in terms of paths. The relationship between paths and convexity is closer than it may appear. In Castiglione (2005), the authors observed that convex polyominoes have the property that every pair of cells is connected by a monotone path, and proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. More precisely, a convex polyomino is *k-convex* if every pair of its cells can be connected by a monotone path with at most k changes of direction, and k is called the *convexity degree* of the polyomino.

The 1-convex polyominoes are more commonly called L-convex polyominoes: any two cells can be connected by a path with at most one change of direction (see Figure 10e).

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In recent literature L-convex polyominoes have been considered from several points of view: under tomographical aspects (Castiglione 2005a) and from the enumeration perspective (Castiglione 2007, 2005b).

Here, we study how the constraint of being k-convex can be represented in terms of submatrix avoidance. In order to reach this goal, let us present some basic definitions and properties from the field of *discrete tomography* (Ryser 1963). Given a binary matrix, the vector of its *horizontal* (resp. *vertical*) *projections* is the vector of the row (resp. column) sums of its elements. Ryser (1963) established a fundamental result which, using our notation, can be reformulated as follows:

Theorem 43. A binary matrix is uniquely determined by its horizontal and vertical projections if and only if it does not contain $S_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $S_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ as submatrices.

Now, we show that the set $\mathcal{L}$ of L-convex polyominoes is a polyomino class.

Proposition 44. L-convex polyominoes are characterized by the avoidance of the submatrices H, V, S_1 and S_2. In other words, $\mathcal{L} = Av_{\mathfrak{P}}(H, V, S_1, S_2)$.

Proof. First, let $P \in Av_{\mathfrak{P}}(H, V, S_1, S_2)$. By Proposition 36, P is a convex polyomino that avoids the submatrices S_1 and S_2. Let us proceed by contradiction assuming that P is not L-convex. It means that there exists a pair of cells c_1 and c_2 of P, that are neither horizontally nor vertically aligned, and such that all the paths from c_1 to c_2 are not L-paths. In other words, all paths from c_1 to c_2 have at least two changes of directions. Up to symmetry, we may assume that c_2 lies below c_1 and on its right. Consider the path from c_1 that goes always right (resp. down) until it encounters a cell c' (resp. c'') which does not belong to P. This happens before reaching the column (resp. row) of c_2, otherwise by convexity there would be a path from c_1 to c_2 with only one change of direction. By convexity, all the cells on the right of c' in the same row (resp. below c'' in the same column) do not belong to P. Consequently, the 2×2-submatrix of P corresponding to the two rows and two columns of c_1 and c_2 is an occurrence of the pattern S_1, giving the desired contradiction.

Conversely, let P be an L-convex polyomino. By convexity, P does not contain the submatrices H and V. Moreover, in Castiglione (2005a) it is proved that an L-convex polyomino is uniquely determined by its horizontal and vertical projections, so by Theorem 43 it cannot contain S_1 nor S_2. □

6.1.4. *2-convex polyominoes.* Unlike L-convex polyominoes, 2-convex polyominoes do not form a polyomino class. Indeed, the 2-convex polyomino in Figure 12a contains the 3-but-not-2-convex polyomino (*b*) as a submatrix. Similarly, the set of k-convex polyominoes is not a polyomino class, for $k \geqslant 2$.

As already pointed out, there are several other families of polyominoes that are not polyomino classes, like directed polyominoes or polyominoes without holes. Consequently, these families cannot be expressed in terms of submatrix avoidance. This suggests to extend the notion of submatrix avoidance, in order to be able to represent more families of polyominoes. *Generalized submatrix avoidance* would impose adjacency constraints between rows or columns, in the same fashion as the avoidance of vincular and

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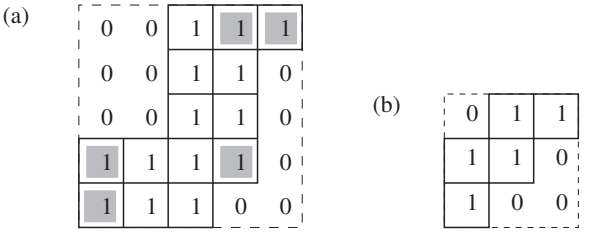


Fig. 12. (a) a 2-convex polyomino P; (b) a submatrix of P that is a polyomino of convexity degree 3.

bivincular patterns (Bona 2004) in permutations. For instance, 2-convex polyominoes can be characterized using such generalized excluded submatrices, although we omit the proof here, since we believe it goes beyond the aims of this paper. We refer the reader to (Battaglino 2014, Section 3.6.2) for more details about generalized excluded submatrices.

6.2. *Defining new polyomino classes by submatrix avoidance*

In addition to characterizing known classes, the approach of submatrix avoidance may be used to define new classes of polyominoes, the main question being then to give a combinatorial/geometrical characterization of these classes. We present some examples of such classes, with simple characterizations and interesting combinatorial properties. These examples illustrate that the submatrix avoidance approach in the study of families of polyominoes is promising.

6.2.1. *L-polyominoes.* Proposition 44 states that L-convex polyominoes can be characterized by the avoidance of four matrices: H and V, which impose the convexity constraint; and S_1 and S_2, which account for the L-property, or equivalently (by Theorem 43) indicate the uniqueness of the polyomino w.r.t its horizontal and vertical projections. So, it is quite natural to study the class $Av_{\mathfrak{P}}(S_1, S_2)$, which we call the class of *L-polyominoes*. From Theorem 43, it follows that:

Proposition 45. Every L-polyomino is uniquely determined by its horizontal and vertical projections.

From a geometrical point of view, the L-polyominoes can be characterized using the concept of (geometrical) inclusion between rows (resp. columns) of a polyomino. For any polyomino P with n columns, and any rows $r_1 = (r_{1;1} \dots r_{1;n})$, $r_2 = (r_{2;1} \dots r_{2;n})$ of the matrix representing P, we say that r_1 is *geometrically included* in r_2 (denoted $r_1 \leqslant r_2$) if, for all $1 \leqslant i \leqslant n$ we have that $r_{1;i} = 1$ implies $r_{2;i} = 1$. Geometric inclusion of columns is defined analogously. Two rows (resp. columns) r_1, r_2 (resp. c_1, c_2) of a polyomino P are said to be *comparable* if $r_1 \leqslant r_2$ or $r_2 \leqslant r_1$ (resp. $c_1 \leqslant c_2$ or $c_2 \leqslant c_1$). These definitions are illustrated in Figure 13.

The avoidance of S_1 and S_2 has an immediate interpretation in geometric terms, proving that:

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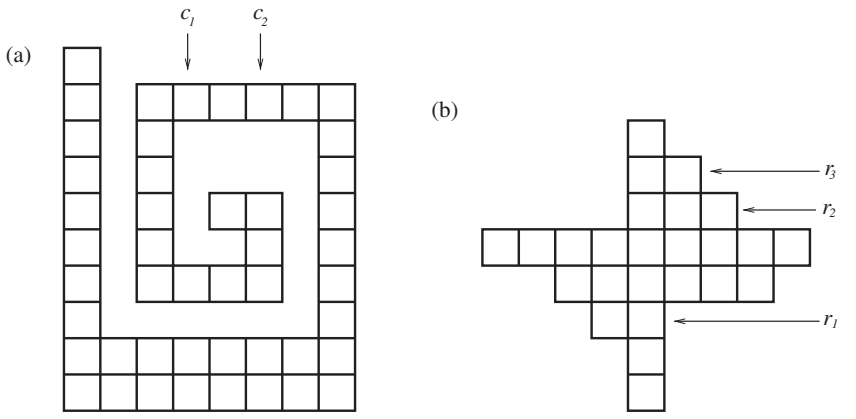


Fig. 13. (a) a L-polyomino, where the reader can check that every pair of rows and columns are comparable; for instance, $c_1 \leqslant c_2$; (b) an example of a polyomino which is not an L-polyomino, where row r_1 is comparable neither to row r_2 nor to row r_3.

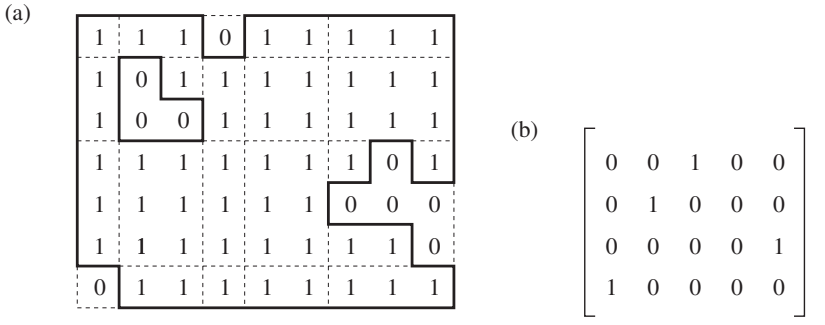


Fig. 14. (a) a polyomino in $\mathcal{C}'$ and its decomposition; (b) the corresponding quasi-permutation matrix.

Proposition 46. The class of L-polyominoes coincides with the set of the polyominoes where every pair of rows (resp. columns) are comparable.

We leave open the problem of studying further the class of L-polyominoes, in particular from an enumerative point of view (enumeration w.r.t. the area or the semi-perimeter).

The class $Av_{\mathfrak{P}}(H', V')$ with $H' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $V' = [\,0 \;\; 1 \;\; 0\,]$.

By analogy[†] with the class of convex polyominoes (characterized by the avoidance of H and V), we may consider the class $\mathcal{C}'$ of polyominoes avoiding the two submatrices H' and V' defined above. In the sense of the 0/1 duality, these objects can be viewed as a dual class to convex polyominoes. Figure 14a shows a polyomino in $\mathcal{C}'$.

[†] which essentially consists in exchanging 0 and 1 in the excluded submatrices

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(a)

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(b)

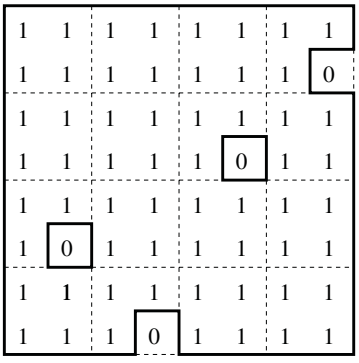


Fig. 15. (a) a permutation matrix of dimension 4; (b) the corresponding polyomino of dimension 8 in $\mathcal{C}'$.

The avoidance of H' and V' has a straightforward geometric interpretation, giving immediately that:

Proposition 47. A polyomino P belongs to $\mathcal{C}'$ if and only if every connected set of cells of maximal length in a row (resp. column) has a contact with the minimal bounding rectangle of P.

The avoidance of H' and V' also ensures that in a polyomino of $\mathcal{C}'$ every connected set of 0s has the shape of a convex polyomino, which we call – by abuse of notation – a *convex* 0*-polyomino* (contained) in P. Each of these convex 0-polyominoes has a minimal bounding rectangle, which individuates an horizontal (resp. vertical) strip of cells in P, where no other convex 0-polyomino of P can be found. Therefore, every polyomino P of $\mathcal{C}'$ can be uniquely decomposed in regions of two types: rectangles all made of 1 s (of type A) or rectangles bounding a convex 0-polyomino (of type B). Then, we can map P onto a quasi-permutation matrix as follows: each rectangle of type A is mapped onto a 0, and each rectangle of type B is mapped onto a 1. See an example in Figure 14b.

Although this representation is clearly nonunique, we believe it may be used for the enumeration of $\mathcal{C}'$. For a start, it provides a simple lower bound on the number of polyominoes in $\mathcal{C}'$ whose bounding rectangle is a square.

Proposition 48. Let c'_n be the number of polyominoes in $\mathcal{C}'$ whose bounding rectangle is an $n \times n$ square. For $n \geqslant 1$, $c'_n \geqslant \lfloor \frac{n}{2} \rfloor !$.

Proof. The statement directly follows from a mapping from permutations of size $m \geqslant 1$ to polyominoes in $\mathcal{C}'$ whose bounding rectangle is an $2m \times 2m$ square, and defined as follows. From a permutation π, we replace every entry of its permutation matrix by a 2×2 matrix according to the following rules: Every 0 entry is mapped onto a 2×2 matrix of type A, while every 1 entry is mapped onto $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. This mapping (illustrated in Figure 15) guarantees that the set of cells obtained is connected (hence is a polyomino), and avoids the submatrices H' and V', concluding the proof. □

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7. Some directions for future research

Because it introduces the new approach of submatrix avoidance in the study of permutation and polyomino classes, our work opens numerous and various directions for future research.

Both in the case of permutation and polyomino classes, we have described several notions of bases for these classes: p-basis, m-bases, canonical m-basis, minimal m-bases. Section 4 explains how to describe the p-basis from any m-basis. Conversely, we may ask how to transform the p-basis into an 'efficient' m-basis. Of course, the p-basis is itself an m-basis, but we may wish to describe the canonical one, or a minimal one.

Many questions may also be asked about the canonical and minimal m-bases themselves. For instance: When does a class have a unique minimal m-basis? Which elements of the canonical m-basis may belong to a minimal m-basis? May we describe (or compute) the minimal m-bases from the canonical m-basis?

Finally, we can study the classes for which the p-basis is itself a minimal m-basis of the class (see the examples of the polyomino classes of vertical bars, or of parallelogram polyominoes).

Submatrix avoidance in permutation classes has allowed us to derive a statement (Corollary 34) from which infinitely many Wilf-equivalences follow. Such general results on Wilf-equivalences are rare in the permutation patterns literature, and it would be interesting to explore how much further we can go in the study of Wilf-equivalences with the submatrix avoidance approach.

The most original concept of this article is certainly the introduction of the polyomino classes, which opens many directions for future research.

One is a systematic study of polyomino classes defined by pattern avoidance. Because enumeration is the biggest open question about polyominoes, we should study the enumeration of such classes, and see whether some interesting bounds can be provided. Notice that the Stanley–Wilf–Marcus–Tardos theorem (Marcus 2004) on permutation classes implies that the permutations in any given class represent a negligible proportion of all permutations. We don't know if a similar statement holds for polyomino classes.

As we have reported in Section 2.3, the poset $(\mathfrak{P}, \preccurlyeq_{\mathfrak{P}})$ of polyominoes was introduced in Castiglione (2005), where the authors proved that it is a ranked poset, and contains infinite antichains. There are however some combinatorial and algebraic properties of this poset which are still to explore, in particular w.r.t. characterizing some simple intervals in this poset.

Finally, we have used binary matrices to import some questions on permutation classes to the context of polyominoes. But a similar approach could be applied to any other family of combinatorial objects which are represented by binary matrices.

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