# 2
# Asymptotic freedom and confinement

This chapter introduces the two concepts of asymptotic freedom and confinement. Confinement explains why the quark and gluon degrees of freedom are not observed as states which propagate over macroscopic distances. Asymptotic freedom explains why the methods of perturbation theory are useful at high energy; the coupling constant is said to run, being large at low energy and becoming smaller at high energy.

## 2.1 The running coupling constant

In order to introduce the concept of a running coupling, consider as an example a dimensionless physical observable $R$ which depends on a single energy scale $Q$. By assumption the scale $Q$ is much bigger than all other dimensionful parameters such as quark masses and we shall therefore set the masses to zero. This step requires the additional assumption that $R$ has a sensible zero-mass limit – we discuss this point in more detail in Section 2.4.

Naive scaling would suggest that because there is a single large scale, $R$ should have a constant value independent of $Q$. This result is not however true in a renormalizable quantum field theory. When we calculate $R$ as a perturbation series in the coupling $\alpha_S = g^2/4\pi$, (defined in analogy with the fine structure constant of QED), the perturbation series requires renormalization to remove ultraviolet divergences. Because this renormalization procedure introduces a second mass scale $\mu$ – the point at which the subtractions which remove the ultraviolet divergences are performed – $R$ depends in general on the ratio $Q^2/\mu^2$ and is therefore not constant. It follows also that the renormalized coupling $\alpha_S$ depends on the choice made for the subtraction point $\mu$.



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However $\mu$ is an arbitrary parameter. The Lagrangian of QCD makes no mention of the scale $\mu$, even though a choice of $\mu$ is required to define the theory at the quantum level. Therefore, if we hold the bare coupling fixed, physical quantities such as $R$ cannot depend on the choice made for $\mu$. Since $R$ is dimensionless, it can only depend on the ratio $Q^2/\mu^2$ and the renormalized coupling $\alpha_S$. Mathematically, the $\mu$ independence of $R$ may be expressed by

$$\mu^2 \frac{d}{d\mu^2} R(Q^2/\mu^2, \alpha_S) \equiv \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S} \right] R = 0 \,. \tag{2.1}$$

To rewrite this equation in a more compact form we introduce the notations

$$t = \ln \left( \frac{Q^2}{\mu^2} \right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \,. \tag{2.2}$$

The derivative of the coupling in the definition of the $\beta$ function is performed at fixed bare coupling. We rewrite Eq. (2.1) as

$$\left[ -\frac{\partial}{\partial t} + \beta(\alpha_S) \frac{\partial}{\partial \alpha_S} \right] R(e^t, \alpha_S) = 0 \,. \tag{2.3}$$

This first order partial differential equation is solved by implicitly defining a new function – the *running coupling* $\alpha_S(Q^2)$ – as follows:

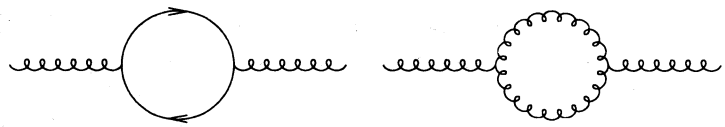
$$t = \int_{\alpha_S}^{\alpha_S(Q^2)} \frac{dx}{\beta(x)}, \quad \alpha_S(\mu^2) \equiv \alpha_S. \tag{2.4}$$

By differentiating Eq. (2.4) we see that

$$\frac{\partial \alpha_S(Q^2)}{\partial t} = \beta(\alpha_S(Q^2)), \quad \frac{\partial \alpha_S(Q^2)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q^2))}{\beta(\alpha_S)} \tag{2.5}$$

and hence that $R(1, \alpha_S(Q^2))$ is a solution of Eq. (2.3). The above analysis shows that all of the scale dependence in $R$ enters through the running of the coupling constant $\alpha_S(Q^2)$. It follows that knowledge of the quantity $R(1, \alpha_S)$, calculated in fixed-order perturbation theory, allows us to predict the variation of $R$ with $Q$ if we can solve Eq. (2.4). In the next section, we shall show that QCD is an *asymptotically free* theory. This means that $\alpha_S(Q^2)$ becomes smaller as the scale $Q$ increases. For sufficiently large $Q$, therefore, we can always solve Eq. (2.4) using perturbation theory.

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a) b)

Fig. 2.1. Graphs which contribute to the $\beta$ function in the one loop approximation.

## 2.2 The $\beta$ function

The running of the coupling constant $\alpha_S$ is determined by the renormalization group equation,

$$Q^2 \frac{\partial \alpha_S}{\partial Q^2} = \beta(\alpha_S). \tag{2.6}$$

In QCD, the $\beta$ function has the perturbative expansion [1,2,3]

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S + b''\alpha_S^2 + O(\alpha_S^3)) \,, \tag{2.7}$$

where

$$\begin{aligned}
b &= \frac{(11C_A - 2n_f)}{12\pi} = \frac{(33 - 2n_f)}{12\pi} \\
b' &= \frac{(17C_A^2 - 5C_A n_f - 3C_F n_f)}{2\pi(11C_A - 2n_f)} = \frac{(153 - 19n_f)}{2\pi(33 - 2n_f)} \,, \\
b'' &= \frac{(2857C_A^3 + (54C_F^2 - 615C_F C_A - 1415C_A^2)n_f + (66C_F + 79C_A)n_f^2)}{288\pi^2(11C_A - 2n_f)} \\
&= \frac{(77139 - 15099n_f + 325n_f^2)}{288\pi^2(33 - 2n_f)}
\end{aligned} \tag{2.8}$$

and $n_f$ is the number of active light flavours.[1] We shall define this concept more precisely in Section 2.4. The $\beta$ function coefficients are extracted from the higher-order (loop) corrections to the bare vertices of the theory, as in QED. Fig. 2.1 shows some of the diagrams which contribute to the $\beta$ function of QCD in the one-loop approximation. Here we see for the first time the effect of the non-Abelian interactions in QCD. In QED (with one fermion flavour) the $\beta$ function is

$$\beta_{\rm QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots \tag{2.9}$$

[1] The three-loop coefficient $b''$ depends on the scheme used for the renormalization of the theory. The result given here is for the $\overline{\rm MS}$ scheme. See Section 2.5 for a discussion of scheme dependence.

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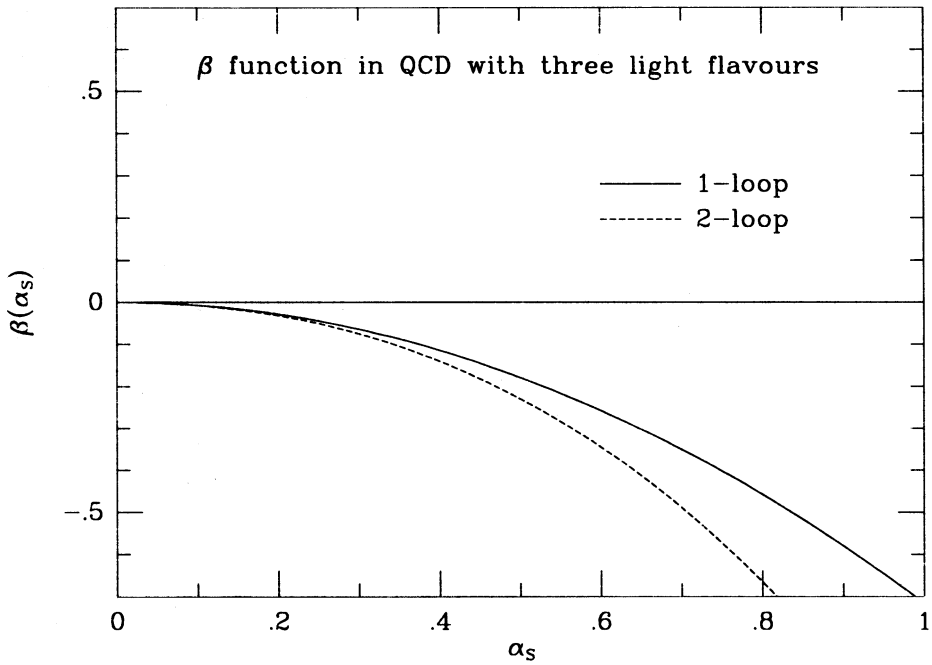



Fig. 2.2. The $\beta$ function in the one- and two-loop approximations.

and thus the $b$ coefficients in QED and QCD have the opposite sign for $n_f \leq 16$.

Fig. 2.2 shows the beta function with three light flavours, when only the first (one-loop) term, or the first and second (two-loop) terms in the perturbative expansion in Eq. (2.7) are retained. So far, the three-loop coefficient $b''$ has not been generally used for phenomenology, because most other quantities are not known to this order, and so we shall neglect it. The curve including the three-loop coefficient has not been shown in Fig. 2.2 because it would be indistinguishable from the two-loop curve. An alternative notation which is sometimes used is

$$\beta(\alpha_S) = -\alpha_S \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_S}{4\pi}\right)^{(n+1)}$$

$$\beta_0 = 4\pi b = 11 - \frac{2}{3}n_f, \quad \beta_1 = 16\pi^2 bb' = 102 - \frac{38}{3}n_f,$$

$$\beta_2 = 64\pi^3 bb'' = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2, \quad \ldots \tag{2.10}$$

From Eqs. (2.6,2.7) we may write

$$Q^2 \frac{\partial \alpha_S(Q^2)}{\partial Q^2} = -b\alpha_S^2(Q^2)\left[1 + b'\alpha_S(Q^2) + O(\alpha_S^2(Q^2))\right]. \tag{2.11}$$

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If both $\alpha_S(\mu^2)$ and $\alpha_S(Q^2)$ are in the perturbative region it makes sense to truncate the series on the right-hand-side and solve the resulting differential equation for $\alpha_S(Q^2)$. For example, neglecting the $b'$ and higher coefficients in Eq. (2.11) gives the solution

$$\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + \alpha_S(\mu^2)bt}, \quad t = \ln\frac{Q^2}{\mu^2}. \tag{2.12}$$

This gives the relation between $\alpha_S(Q^2)$ and $\alpha_S(\mu^2)$, if both are in the perturbative region. Evidently as $t$ becomes very large, the running coupling $\alpha_S(Q^2)$ decreases to zero. This is the property of *asymptotic freedom*. The approach to zero is rather slow since $\alpha_S$ only decreases like an inverse power of $\ln Q^2$. Notice that the sign of $b$ is crucial. With the opposite sign of $b$ the coupling would *increase* at large $Q^2$, as it does in QED.

It is relatively straightforward to show that including the next-to-leading order coefficient $b'$ yields the solution

$$\frac{1}{\alpha_S(Q^2)} - \frac{1}{\alpha_S(\mu^2)} + b'\ln\Big(\frac{\alpha_S(Q^2)}{1 + b'\alpha_S(Q^2)}\Big) - b'\ln\Big(\frac{\alpha_S(\mu^2)}{1 + b'\alpha_S(\mu^2)}\Big) = bt. \tag{2.13}$$

Note that this is now an implicit equation for $\alpha_S(Q^2)$ as a function of $t$ and $\alpha_S(\mu^2)$. In practice, given values for these parameters, $\alpha_S(Q^2)$ can easily be obtained numerically to any desired accuracy. We discuss in Section 2.5 the expressions for $\alpha_S$ that are normally used in practice.

Returning to the physical quantity $R$, we can now demonstrate the type of terms which the renormalization group resums. Assume that in perturbation theory $R$ has the expansion

$$R = R_1\alpha_S + \ldots \tag{2.14}$$

where $\ldots$ represents terms of order $\alpha_S^2$ and higher. The solution of Eq. (2.3), $R(1, \alpha_S(Q^2))$, can be re-expressed in terms of $\alpha_S(\mu^2)$ using Eq. (2.12):

$$\begin{aligned} R(1, \alpha_S(Q^2)) &= R_1\alpha_S(\mu^2)\sum_{j=0}^{\infty}\left[-\alpha_S(\mu^2)bt\right]^j \\ &= R_1\alpha_S(\mu^2)\Big[1 - \alpha_S(\mu^2)bt + \alpha_S^2(\mu^2)(bt)^2 + \ldots\Big] \end{aligned} \tag{2.15}$$

Thus order by order in perturbation theory there are logarithms of $Q^2/\mu^2$ which are automatically resummed by using the running coupling. Higher-order terms in $R$ – represented by the dots in Eq. (2.14) – when expanded give terms with fewer logarithms per power of $\alpha_S$. For example, a term $R_2\alpha_S^2$ would give $R_2\alpha_S(\mu^2)^2[1 - 2\alpha_S(\mu^2)bt + \ldots]$, with one less logarithm in each term. An example of how this works in practice will be discussed in the next chapter.

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## 2.3 Asymptotic freedom

There is no simple intuitive explanation of the property of asymptotic freedom in QCD. This is in contrast to QED, where the idea that the observed charge of the electron is smaller at large distances because of screening of its electric charge by vacuum polarization is quite intuitive. It would be nice to be able to extend this argument to QCD. The new feature in QCD is the presence of the gluons, which are spin one particles that carry colour charge.

There are two types of arguments which aim to give a simple explanation of asymptotic freedom [4,5]. They describe the phenomenon either as a dielectric or a paramagnetic effect. The first line of argument [4] calculates the dielectric properties of the vacuum and ascribes the asymptotic freedom of the theory to the self-interaction of the gluon field. The basic idea is that the emission of virtual gluons by static colour sources causes their colour charges to 'leak out' into the surrounding vacuum. The interaction between distributed charges is weaker than that between point charges when the charge distributions overlap, leading to a reduction of the colour force at short distances, which is the essence of asymptotic freedom.

Here we sketch in more detail an argument of the second type [5] which describes asymptotic freedom as a paramagnetic effect due to the spin of the gluons. The appealing feature of the magnetic explanation is that polarizable materials that are attracted towards a magnetic field (paramagnetism) and repelled from one (diamagnetism) are both familiar. This is in contrast to electric polarization effects in matter which are always attractive (dielectric).

We start by reviewing the situation in QED. Here the effective charge grows at small distances (large momenta) as one probes more of the unshielded central charge. This can be represented by a scale-dependent dielectric constant. One can define a running charge at scale $q^2$ in terms of the charge at the ultraviolet cut-off scale $\Lambda_{\rm UV}^2 \gg q^2$ as

$$\alpha(q^2) = \frac{\alpha(\Lambda_{\rm UV}^2)}{\varepsilon(q^2)} \tag{2.16}$$

where

$$\frac{1}{\varepsilon(q^2)} = 1 - \frac{\beta(\alpha)}{\alpha} \ln \frac{\Lambda_{\rm UV}^2}{q^2} \tag{2.17}$$

and the running charge satisfies the equation

$$\frac{d\alpha(q^2)}{d \ln q^2} = \beta(\alpha) \, . \tag{2.18}$$

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| Electric | | Magnetic | | |
|---|---|---|---|---|
| Screening | $\varepsilon > 1$ | $\mu < 1$ | $\chi < 0$ | diamagnetism |
| Anti-screening | $\varepsilon < 1$ | $\mu > 1$ | $\chi > 0$ | paramagnetism |

Table 2.1. Relationship between the dielectric and paramagnetic explanation of asymptotic freedom.

If the $\beta$ function is positive, as in QED, then $\varepsilon(q^2) > 1$, which corresponds to a screening of the central charge.

We now assume that the vacuum of a relativistic quantum field theory can be treated as a polarizable medium. For this special medium the dielectric constant $\varepsilon(q^2)$ and the magnetic permeability $\mu(q^2)$ must satisfy the relation

$$\mu(q^2)\varepsilon(q^2) = 1 \tag{2.19}$$

(in units in which $c = 1$) for all $q$. This is a consequence of the Lorentz invariance of the vacuum. We are free to exploit this relationship by considering the properties of the QCD vacuum in a background colour magnetic field, rather than a colour electric field, if it is simpler to do so. We choose to consider the behaviour of the magnetic susceptibility, $\chi$ related to the permeability by the relation $\mu = 1 + \chi$. From Eq. (2.17) we have

$$\chi(q^2) = -\frac{\beta(\alpha)}{\alpha} \ln \frac{\Lambda_{\mathrm{UV}}^2}{q^2} \,. \tag{2.20}$$

The relationship between electric and magnetic properties is summarized in Table 2.1.

Before we go on to describe the paramagnetic properties of the vacuum, a brief reminder of the magnetic properties of a free electron quantum gas may be useful [6]. The magnetic susceptibility of a free electron gas may be divided into two parts. The first term, known as Pauli paramagnetism, represents the effect of electrons coupling to the magnetic field through their spin magnetic moments. It therefore describes a gas of particles with magnetic moments but no charges. The second term, known as Landau diamagnetism, describes the coupling of the field to the orbital motion of the electrons. It can be shown that the two contributions are related by

$$\chi_{\mathrm{Landau}} = -\tfrac{1}{3}\, \chi_{\mathrm{Pauli}} \,. \tag{2.21}$$

In this case the system is paramagnetic because the Pauli paramagnetism is larger in magnitude than the Landau diamagnetism.

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Returning to the case of the vacuum of a relativistic field theory, we find that both paramagnetic and diamagnetic contributions are present. The Pauli term has to be generalized slightly to take into account particles of spin $S$ different from one half: it becomes proportional to $S^2$. The final result is given by Eq. (2.20) where

$$\beta(\alpha) = -b\alpha^2 \tag{2.22}$$

and the contribution to $b$ from particles of spin $S$ is

$$b = \frac{(-1)^{2S}}{2\pi}\left[(2S)^2 - \tfrac{1}{3}\right] . \tag{2.23}$$

For spins $0, \frac{1}{2}$ and 1 the result from Eq. (2.23) is

$$b = -\frac{1}{6\pi} , \; -\frac{1}{3\pi} , \; \frac{11}{6\pi} \quad \text{for } S = 0 , \; \frac{1}{2} , \; 1 . \tag{2.24}$$

We see that Eq. (2.23) can be considered as a generalization of Eq (2.21) to arbitrary spin. As expected, the Pauli paramagnetic term grows with the spin of the particle. The overall sign $(-1)^{2S}$ is due to the fact that the fermionic vacuum energy, including the energy due to an external magnetic field, is negative, whereas that for bosons is positive. This is a consequence of the anticommutation and commutation relations, respectively, of the field operators. Because of this overall sign, spin-$\frac{1}{2}$ particles give a diamagnetic contribution to the $\beta$ function (unlike the case of the free electron gas), while spin-1 particles make a larger paramagnetic contribution.

Inserting the appropriate colour factors of $C_A/2$ for gluons with $S = 1$ and $n_f/2$ for quarks with $S = \frac{1}{2}$, we obtain

$$b = \frac{1}{12\pi}(33 - 2n_f) , \tag{2.25}$$

which exhibits asymptotic freedom as long as $n_f < 17$, because the colour paramagnetic contribution of the gluons overcomes the colour diamagnetic quark contribution.

## 2.4 Quark masses

In the discussion up to now we have ignored the masses of the quarks. We shall now investigate the theoretical basis for this assumption. Note that from the point of view of the Lagrangian the quark mass is just another parameter like the coupling. Let us consider for simplicity the case where we have one flavour of quark with renormalized mass $m$. As before the $\mu$ independence of the physical result implies a renormalization group equation for $R$ which is now a function of the renormalized mass

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$m$. The form of the equation depends on the choice of renormalization procedure. In the following we shall use the scheme of Weinberg [7] which ensures that the $\beta$ function and the mass anomalous dimension $\gamma_m$ are only functions of the renormalized coupling $\alpha_S$.

In this scheme the renormalization group equation including the mass parameter is of the form

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha_S)\frac{\partial}{\partial\alpha_S}-\gamma_m(\alpha_S)m\frac{\partial}{\partial m}\right]R(Q^2/\mu^2,\alpha_S,m/Q)=0\,. \quad (2.26)$$

The partial derivatives in Eq. (2.26) are taken at fixed values of the other two parameters. The function $\gamma_m$ is the mass anomalous dimension, a perturbatively calculable quantity:

$$\gamma_m(\alpha_S)=c\alpha_S(1+c'\alpha_S+\ldots)\,. \quad (2.27)$$

The coefficients of the mass anomalous dimension are also dependent on the renormalization scheme: in the scheme that is most often used (the $\overline{\rm MS}$ scheme – see Section 2.5), they are [8]

$$c=\frac{1}{\pi}\,,\quad c'=\frac{(303-10n_f)}{72\pi}\,. \quad (2.28)$$

Since $R$ is a dimensionless function we have the result

$$\left[Q^2\frac{\partial}{\partial Q^2}+\mu^2\frac{\partial}{\partial\mu^2}+m^2\frac{\partial}{\partial m^2}\right]R(Q^2/\mu^2,\alpha_S,m/Q)=0\,. \quad (2.29)$$

Subtracting Eq.(2.26) from Eq.(2.29) we obtain an equation which fixes the dependence of $R$ on $Q$:

$$\left[Q^2\frac{\partial}{\partial Q^2}-\beta(\alpha_S)\frac{\partial}{\partial\alpha_S}+\left(\tfrac{1}{2}+\gamma_m(\alpha_S)\right)m\frac{\partial}{\partial m}\right]R(Q^2/\mu^2,\alpha_S,m/Q)=0\,. \quad (2.30)$$

This equation is solved by introducing a running mass $m(Q^2)$ in analogy with the running coupling, such that

$$Q^2\frac{\partial m}{\partial Q^2}=-\gamma_m(\alpha_S)m(Q^2)\,. \quad (2.31)$$

The solution of this equation is

$$\begin{aligned} m(Q^2) &= m(\mu^2)\exp\left[-\int_{\mu^2}^{Q^2}\frac{dQ^2}{Q^2}\gamma_m(\alpha_S(Q^2))\right] \\ &= m(\mu^2)\exp\left[-\int_{\alpha_S(\mu^2)}^{\alpha_S(Q^2)}d\alpha_S\frac{\gamma_m(\alpha_S)}{\beta(\alpha_S)}\right]\,. \end{aligned} \quad (2.32)$$

In terms of the running mass and running coupling, the solution of the renormalization group equation is the same as before. All of the non-

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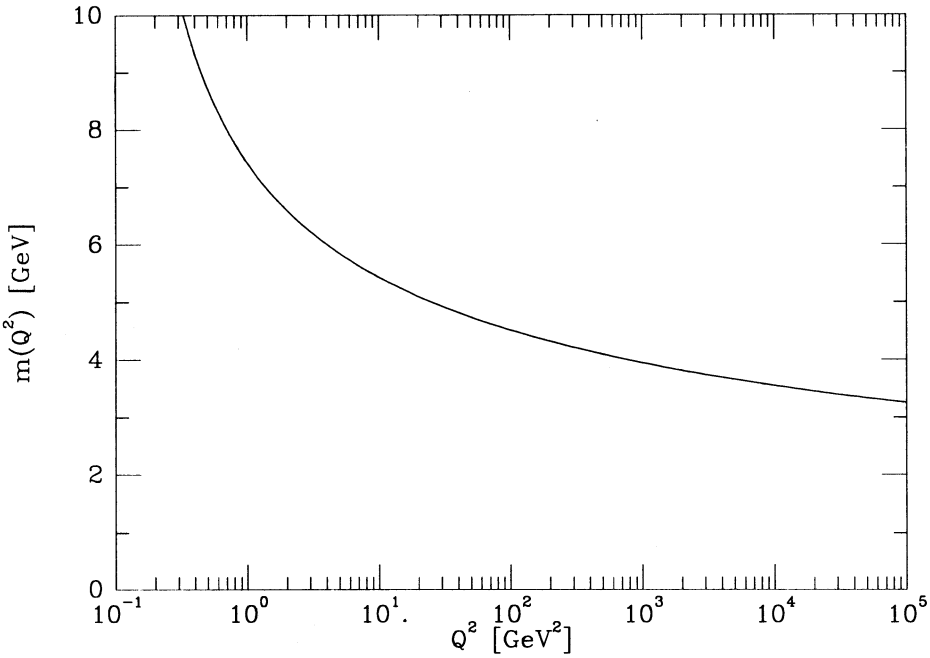



Fig. 2.3. Running of the bottom quark mass.

trivial scale dependence of $R$ can be included in the running of the mass and the coupling:

$$R(Q^2/\mu^2, \alpha_S, m/Q) \to R(1, \alpha_S(Q^2), m(Q^2)/Q) \; . \tag{2.33}$$

Using the asymptotic freedom of the theory at large $Q$, we can calculate the running mass in perturbation theory. From Eq. (2.31) we obtain

$$m(Q^2) = \hat{m} \left[\alpha_S(Q^2)\right]^{c/b} \left[1 + \frac{c(c'-b')}{b} \alpha_S(Q^2) + O(\alpha_S^2)\right] . \tag{2.34}$$

The mass $\hat{m}$ is a renormalization-group invariant constant of integration (similar to $\Lambda$). The running of the $b$ quark mass is illustrated in Fig. 2.3. We see from Eqs. (2.33) and (2.34) that the effect of the mass is suppressed at high $Q^2$, both by its 'engineering dimension' which leads to a suppression of $1/Q$ and by its anomalous dimension $\gamma_m$ $(c/b > 0)$, which leads to a further logarithmic suppression.

To investigate mass effects more quantitatively, consider the case in which $R$ and its first $N$ derivatives are finite at $m = 0$. We can write

$$\begin{aligned} R(1, \alpha_S(Q^2), m(Q^2)/Q) \; &\sim \; R(1, \alpha_S(Q^2), 0) \\ &+ \sum_{n=1}^{N} \frac{1}{n!} \left[\frac{m(Q^2)}{Q}\right]^n R^{(n)}(1, \alpha_S(Q^2), 0) \end{aligned} \tag{2.35}$$

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where $R^{(n)}$ denotes the $n$th derivative of $R$ with respect to its third argument. The derivative terms are suppressed by at least an inverse power of $Q$. At large $Q$ we therefore have a theoretical justification to drop the mass. The important point is that in an asymptotically free theory the anomalous dimension leads to a change of the mass with $Q$ which varies like a power of $\ln Q$. This can never overcome the inverse power of $Q$ from the engineering dimension, and in fact leads to a further suppression when $Q \gg m(Q^2)$.

Consider now the opposite case, of a heavy quark with mass much greater than the relevant scale $Q$. An example would be the influence of the top quark on physics at scales well below $m_t$. In this case it can be shown that the effects of the the heavy quark on cross sections are suppressed by inverse powers of $m$ and can therefore be ignored for $Q \ll m$ [9]. That is why we specified that $n_f$ in Eq. (2.7) represents the number of 'active' light flavours, *i.e.* the number with masses much smaller than the characteristic scale $Q$ at which $\alpha_S(Q^2)$ is required.

When $Q$ is neither much smaller nor much larger than a particular quark mass, but still large enough to be in the perturbative domain (for example, $Q \sim m_b$), then the mass of that quark has to be taken into account explicitly in calculations. We shall see examples of this in Chapter 10. There remains, however, the problem of relating the values of $\alpha_S$ measured in the two domains $Q \ll m$ and $Q \gg m$. Technically this is achieved by matching the coupling of the full theory which includes the heavy quark with an effective theory which involves only the light degrees of freedom. In particular, the value of $n_f$ increases by one unit as we pass from the effective theory to the full theory and so the $\beta$ function changes, altering the running of $\alpha_S$.

The simplest prescription is to require that $\alpha_S(Q^2)$ should be continuous at $Q = m$, the heavy quark mass scale. In this approach, $\alpha_S$ is simply regarded as the solution of the first-order differential equation (2.11), with a step-function discontinuity in the right-hand side at $Q = m$. Writing the solutions above and below the discontinuity as $\alpha_+$ and $\alpha_-$, respectively, and expanding the $\beta$ function to leading order, we have

$$\frac{1}{\alpha_+(Q^2)} - b_+ \ln \frac{Q^2}{m^2} = \frac{1}{\alpha_-(Q^2)} - b_- \ln \frac{Q^2}{m^2} = \frac{1}{\alpha_S(m^2)} , \tag{2.36}$$

where $b_- - b_+ = 1/6\pi$ from Eq. (2.7), and hence

$$\alpha_+(Q^2) = \alpha_-(Q^2) \left[ 1 - \frac{1}{6\pi} \alpha_-(Q^2) \ln \frac{Q^2}{m^2} \right]^{-1} . \tag{2.37}$$

The same approach can be carried through to any order; for example, Eq. (2.13) yields an implicit next-to-leading order relation between $\alpha_+$ and $\alpha_-$.

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At this point one might wonder why we should match the two expressions for $\alpha_S$ at $Q^2 = m^2$ rather than, say, $4m^2$, which might even seem more natural in terms of the energy required to create a heavy quark-antiquark pair. It should be recalled, however, that our expressions for $\alpha_+(Q^2)$ and $\alpha_-(Q^2)$ are only valid for $Q^2 \gg m^2$ and $Q^2 \ll m^2$ respectively, and so a physical argument about what should happen at $Q \sim m$, or even whether $\alpha_S$ should be continuous in that region, is not strictly relevant. What matters is the consistency of the calculation in the full theory including the heavy quark, (appropriate when $Q \gg m$) and the calculation performed in an effective theory with only light degrees of freedom, (appropriate when $Q \ll m$). This is enforced by computing the coefficient functions $C_k$ in the generalized version of Eq. (2.37):

$$\alpha_+(Q^2) = \alpha_-(Q^2) + \sum_{k=1}^{\infty} C_k \left(\ln \frac{Q^2}{m^2}\right) [\alpha_-(Q^2)]^{k+1} \ . \tag{2.38}$$

One finds [10] that in the usual renormalization scheme (the $\overline{\rm MS}$ scheme – see the discussion in Section 2.5)

$$C_1(x) = \frac{1}{6\pi} x \tag{2.39}$$

$$C_2(x) = \frac{1}{72\pi^2}(2x^2 + 33x - 7) \ . \tag{2.40}$$

Taking into account that $m = m(Q^2)$ is a running mass, this is almost equivalent to the prescription of matching at the quark mass, defined implicitly by $m_q = m(m_q^2)$ [11]. At this point Eq. (2.38) becomes

$$\alpha_+(m_q^2) = \alpha_-(m_q^2) - \frac{7}{72\pi^2}\left[\alpha_-(m_q^2)\right]^3 \ , \tag{2.41}$$

a difference that is negligible for all practical purposes. Eq. (2.39) explains the choice of performing the matching of the $\overline{\rm MS}$ coupling at $Q = m$. The absence of a constant term in Eq. (2.39) indicates that at one loop order the coupling $\alpha_+$ is equal to $\alpha_-$ if the matching is performed at $Q = m$. If the matching were performed at another scale, such as $Q = 2m$, the $\overline{\rm MS}$ coupling of the full theory including the heavy quark would differ at the matching point from the coupling of the effective theory with only the light degrees of freedom. As emphasized above, the continuity of the coupling at the matching point is not in itself a requirement of the theory. The choice of matching at $Q = m$ leads to approximate continuity of the running coupling and hence an appealing physical picture for $\alpha_S$.

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## 2.5 The Λ parameter

Perturbative QCD tells us how the coupling constant varies with the scale, not the absolute value itself. The latter has to be obtained from experiment. Thus we can choose as 'the' fundamental parameter of the theory the value of the coupling constant at a convenient reference scale which is large enough to be in the perturbative domain, $M_Z^2$ for example. Setting $\mu = M_Z$ in Eq. (2.13), we can then deduce the value at any other large scale $Q^2$. This has become the standard approach: when we review the measurements of $\alpha_S$ in Chapter 12, we shall compare them mainly in terms of $\alpha_S(M_Z^2)$.

An alternative approach, which was adopted historically and is still convenient for many purposes, is to introduce a dimensionful parameter directly into the definition of $\alpha_S(Q^2)$. By convention this parameter is called $\Lambda$ and is the constant of integration defined by

$$\ln \frac{Q^2}{\Lambda^2} = -\int_{\alpha_S(Q^2)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_S(Q^2)}^{\infty} \frac{dx}{bx^2(1 + b'x + \ldots)}. \tag{2.42}$$

$\Lambda$ represents the scale at which the coupling would diverge, if extrapolated outside the perturbative domain. More qualitatively, it indicates the order of magnitude of the scale at which $\alpha_S(Q^2)$ becomes strong. Depending on the precise definition, to be discussed below, its value is in the neighbourhood of 200 MeV. Thus $\alpha_S(Q^2)$ becomes large, and perturbation theory breaks down, for scales comparable with the masses of the light hadrons, *i.e.* $Q \simeq 1$ GeV. This could be an indication that the *confinement* of quarks and gluons inside hadrons is actually a consequence of the growth of the coupling at low scales, which is a corollary of the decrease at high scales that leads to asymptotic freedom.

The introduction of $\Lambda$ allows us to write the asymptotic solution for $\alpha_S$ in terms of this parameter. In leading order (LO), *i.e.* retaining only the $b$ coefficient in the $\beta$ function, we can perform the integration in Eq. (2.42) to obtain

$$\alpha_S(Q^2) = \frac{1}{b \ln(Q^2/\Lambda^2)}. \tag{2.43}$$

Note that for large $Q^2$ this solution agrees with Eq. (2.12) as it must. The definition of $\Lambda$ is extended to next-to-leading order (NLO) by including also the $b'$ coefficient in the integral:

$$\frac{1}{\alpha_S(Q^2)} + b' \ln\Big(\frac{b'\alpha_S(Q^2)}{1 + b'\alpha_S(Q^2)}\Big) = b \ln\Big(\frac{Q^2}{\Lambda^2}\Big). \tag{2.44}$$

Again, this allows a numerical determination of $\alpha_S(Q^2)$ for a given value of $\Lambda$. Alternatively, we can obtain an approximate solution of Eq. (2.44)

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| | $\alpha_S(n_f = 4)$ | $\alpha_S(n_f = 5)$ |
|---|---|---|
| LO | 0.234 | 0.255 |
| NLO | 0.184 | 0.206 |

Table 2.2. $\alpha_S(Q^2)$ for $Q = 5$ GeV and $\Lambda = 200$ MeV.

in terms of inverse powers of $\ln(Q^2/\Lambda^2)$:

$$\alpha_S(Q^2) = \frac{1}{b\ln(Q^2/\Lambda^2)}\left[1 - \frac{b'}{b}\frac{\ln\ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)} + \ldots\right]. \tag{2.45}$$

Note, however, that this expression corresponds to a slightly different definition of Λ to Eq. (2.44). The *true* expansion of $\alpha_S(Q^2)$ in inverse powers of $\ln(Q^2/\Lambda^2)$ would contain a term of order $k/\ln^2(Q^2/\Lambda^2)$ where $k$ is a constant. To this order, the two definitions differ by a constant factor. Specifically, if we call Eqs. (2.44) and (2.45) definitions 1 and 2 respectively, then for the *same* value of $\alpha_S(Q^2)$ the two Λ's are related by

$$\Lambda_1 \simeq \left(\frac{b}{b'}\right)^{\frac{b'}{2b}}\Lambda_2 \simeq 1.1\Lambda_2 \qquad (n_f = 5). \tag{2.46}$$

In this book we shall follow the convention of using Eq. (2.45) as described in ref. [12].

It will be clear from the above discussion that the use of Λ as the fundamental parameter of QCD presents a number of traps which can ensnare the unwary. First, Λ can be defined to leading or next-to-leading order and in each case multiplying Λ by a constant gives an equally acceptable definition. The differences induced in $\alpha_S(Q^2)$ are one order higher in perturbation theory. Nowadays, all precision QCD phenomenology is performed at next-to-leading order. Either Eq. (2.44) or Eq. (2.45) can be used to define Λ in this case, and both definitions are used in the literature. Since in practice it is usually $\alpha_S$ which is measured experimentally, it is important when comparing Λ values to check that the same equation has been used to determine Λ from the coupling constant. Differences between the results obtained using different conventions – although small – can be comparable to present-day measurement errors.

A second difficulty with the above definitions is that Λ depends on the number of active flavours. At two loop order values of Λ for different numbers of flavours are defined by imposing the continuity of $\alpha_S$ at the scale $\mu = m$, where $m$ is the mass of the heavy quark, as described in the previous section. This is illustrated in Table 2.2 where the LO and NLO couplings are calculated using Eqs.(2.43) and (2.45) respectively. The matching prescription is determined by the conditions that for all

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values of the momenta the coupling constant must be both a solution of the renormalization group equation and also a continuous function. From Eq. (2.45) for $\mu > m_b$ we have

$$\alpha_S(Q^2, 5) = \frac{1}{b(5)\ln(Q^2/\Lambda(5)^2)}\Big[1 - \ldots\Big]. \tag{2.47}$$

For $m_c < Q < m_b$, the coupling evolves with four active flavours, and the form to use is

$$\frac{1}{\alpha_S(Q^2, 4)} = \frac{b(4)\ln(Q^2/\Lambda(5)^2)}{\Big[1 - \ldots\Big]} + \text{constant} \tag{2.48}$$

where the square bracket is the same as in Eq. (2.47). The constant is fixed by the continuity condition,

$$\alpha_S(m_b^2, 4) = \alpha_S(m_b^2, 5). \tag{2.49}$$

Using the next-to-leading order form for $\alpha_S(Q^2)$ one can show then that

$$\Lambda(4) \approx \Lambda(5)\Big(\frac{m_b}{\Lambda(5)}\Big)^{\frac{2}{25}}\Big[\ln\Big(\frac{m_b^2}{\Lambda(5)^2}\Big)\Big]^{\frac{963}{14375}}. \tag{2.50}$$

Fig. 2.4 shows the relation between $\Lambda(4)$ and $\Lambda(5)$. In summary, it is important when comparing different $\Lambda$ values to establish the number of light quark flavours assumed and also whether the LO or NLO expressions have been used. The change in the value of $\alpha_S$ at $Q = 5$ GeV with the different definitions of $\Lambda$ is illustrated in Table 2.2. An error can lead to a substantial change in $\alpha_S$.

The third troubling property of $\Lambda$ is that it depends on the renormalization scheme. Consider two calculations of the renormalized coupling constant which start from the same bare parameters,

$$\begin{aligned} \alpha_S^A &= Z^A \alpha_S^0 \,, \\ \alpha_S^B &= Z^B \alpha_S^0 \,. \end{aligned} \tag{2.51}$$

The two schemes start from the same bare coupling $\alpha_S^0$. The infinite parts of the renormalization constants $Z^A$ and $Z^B$ must be the same in all orders of perturbation theory. Therefore the two renormalized coupling constants must be related by a finite renormalization:

$$\alpha_S^B = \alpha_S^A(1 + c_1\alpha_S^A + \ldots). \tag{2.52}$$

Note that the first two coefficients of the $\beta$ function, $b$ and $b'$, are unchanged by such a transformation. They are therefore independent of the renormalization scheme. From Eq. (2.42) we see that the two values of $\Lambda$ are related by,

$$\ln\frac{\Lambda^B}{\Lambda^A} = \frac{1}{2}\int_{\alpha_S^A(Q^2)}^{\alpha_S^B(Q^2)} \frac{dx}{bx^2(1 + \ldots)} = \frac{c_1}{2b}. \tag{2.53}$$

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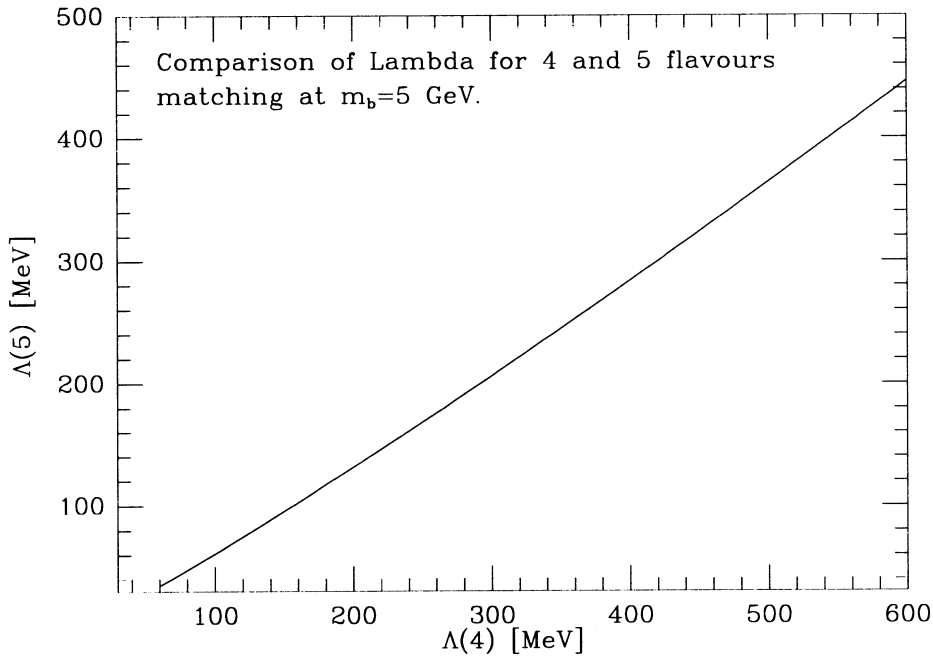



Fig. 2.4. Comparison of Λ for 4 and 5 light quark flavours, with matching at $m_b = 5$ GeV.

The last equality follows from taking the limit $Q \to \infty$, because the relation must be true for all values of $Q^2$. Therefore relations between different definitions of Λ are always determined by the one loop calculation which fixes $c_1$:

$$\Lambda^B = \Lambda^A \exp \frac{c_1}{2b} . \tag{2.54}$$

Nowadays, most calculations in fixed-order QCD perturbation theory are performed in the *modified minimal subtraction* renormalization scheme. In this approach, ultraviolet loop divergences are regulated by reducing the number of space-time dimensions to $n < 4$:

$$\frac{d^4k}{(2\pi)^4} \longrightarrow (\mu)^{2\epsilon} \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}} \tag{2.55}$$

where $\epsilon = 2 - \frac{n}{2}$. For a detailed discussion of renormalization and dimensional regularization we refer the reader to ref. [13]. Note that the renormalization scale $\mu$ preserves the dimensions of the couplings and the fields. Loop integrals of the form $d^nk/[k^2 + m^2]^2$ then lead to poles at $\epsilon = 0$. The *minimal subtraction* renormalization prescription is to subtract off these poles and to replace the bare coupling by the renormalized

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coupling $\alpha_S(\mu^2)$. In practice the poles always appear in the combination

$$\frac{\Gamma(1+\epsilon)}{\epsilon}(4\pi)^{\epsilon} = \frac{1}{\epsilon} + \ln(4\pi) - \gamma_E + O(\epsilon) \,. \tag{2.56}$$

The $\Gamma$ function is defined as

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1}, \text{ Re } z > 0 \,, \tag{2.57}$$

and has the properties

$$\Gamma(1+z) = z\Gamma(z) \tag{2.58}$$

$$\Gamma(1+\epsilon) = 1 - \gamma_E\, \epsilon + (\frac{\pi^2}{12} + \frac{1}{2}\gamma_E^2)\, \epsilon^2 + O(\epsilon^3) \tag{2.59}$$

where $\gamma_E = 0.5772\ldots$ is the Euler constant. In the *modified minimal subtraction* scheme the additional constants in Eq. (2.56) which accompany the pole are subtracted off as well. These two schemes are therefore examples of schemes A and B introduced above, and it is straightforward to show using Eqs.(2.52) and (2.54) that

$$\Lambda^2_{\overline{\text{MS}}} = \Lambda^2_{\text{MS}} e^{(\ln(4\pi) - \gamma_E)}. \tag{2.60}$$

Lastly, the expression of the experimentally measured coupling $\alpha_S$ in terms of $\Lambda$ leads to an error which is both exponentially magnified and asymmetric. This is mathematically correct but depressing for an experimenter since most experiments actually measure $\alpha_S$. A compilation of measurements is shown in Fig. 2.5. The data points will be described in detail in Chapter 12. The errors in Fig. 2.5 are too large to measure the slope of the logarithmic dependence of $1/\alpha_S$ on $Q$ precisely, but the coupling constant clearly decreases with increasing scale. An analysis of jet data in $e^+e^-$ annihilation demonstrates that $\alpha_S$ decreases with scale using a single reaction type (see Chapter 3).

Guided by Fig. 2.5, for the purpose of making phenomenological predictions one may assume that

$$100 \text{ MeV} < \Lambda_{\overline{\text{MS}}}(5) < 400 \text{ MeV}. \tag{2.61}$$

Using Eq. (2.45), this corresponds to

$$0.105 < \alpha_S(M_Z^2) < 0.129 \,. \tag{2.62}$$

*i.e.* $\alpha_S \simeq 0.117$, with about a $\pm 10\%$ uncertainty, at the mass of the $Z$. As we shall see in Chapter 12, the current world average value of $\alpha_S(M_Z^2)$ is $0.116 \pm 0.005$. The wider range of uncertainty in Eq. (2.62) is chosen to cover all the highest-precision measurements. The uncertainty in $\alpha_S$ translates directly into an uncertainty in the prediction of the size of QCD cross sections. Thus we should expect errors of the order of $\pm 10\%$ in the prediction of cross sections that begin in order $\alpha_S$.

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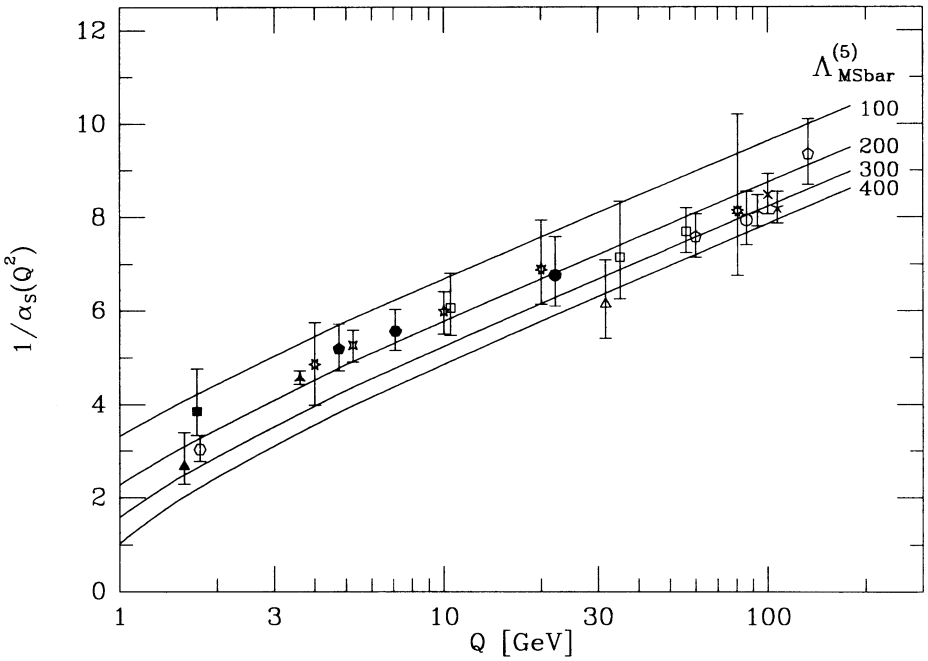



Fig. 2.5. Measurements of $\alpha_S$ compared with predictions for various values of $\Lambda(5)$.

## 2.6 Lattice QCD

We have seen in the preceding section that at high momenta (or equivalently small distances) the coupling constant of QCD becomes small. On the other hand at large distances (of about 1 fermi), it becomes large. In this latter regime perturbation theory is not useful. Furthermore, when we actually calculate amplitudes in perturbation theory using quarks and gluons we find collinear and infra-red singularities which indicate the inapplicability of the perturbative method. Another signal of the inadequacy of perturbation theory is the failure to observe the free quarks and gluons which are used as asymptotic states. The physical asymptotic states are the hadrons observed in detectors.

Although the principal tool which we shall use in this book is the method of perturbation theory, there are clearly situations in which this method is inapplicable. It is therefore important to have a non-perturbative formulation of the theory determined by the QCD Lagrangian. One of the most developed and systematic methods is the method of lattice QCD, which has been extensively used to study hadronic masses and matrix elements.

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The Lagrangian of QCD does not make sense without an ultraviolet cut-off, because of divergences which occur at high momenta or equivalently at short distances. In order to perform calculations with the theory one has to introduce a regulator. A description of the regulation procedure used in perturbation theory was given briefly in Section 2.5 and will be amplified in the next chapter. In this section we will describe a regulation of the Euclidean version of QCD implemented by defining the theory on a four-dimensional hypercubic lattice [14,15,16,17,18]. The continuation to the Euclidean region gives a real, positive action for the gauge fields. This allows one easily to identify the most important field configurations, which give the smallest action.

The basic idea is to separate the field operators by discretizing Euclidean space-time into hypercubes with sides of length $a$. Full Lorentz invariance is reduced to a hypercubic symmetry by this procedure, but the lattice version of the theory preserves local gauge invariance. We are ultimately interested in the limit of small lattice spacing when the full symmetry should be restored. The lattice spacing provides a cut-off of order $\pi/a$ on all momenta so that ultraviolet divergences are no longer present. As long as the lattice spacing $a$ is small compared to the physical extent of the hadrons one is studying, the discrete version of the theory should be an adequate approximation at that scale.

The continuum Euclidean action can be calculated from the corresponding expression in Minkowski space, Eq. (1.9). Performing the Wick rotation $x^4 = ix^0, \partial^4 = -i\partial^0$ we find that

$$i \int d^4x \, \mathcal{L}(x) \to -S_E \tag{2.63}$$

where

$$S_E = \int d^4x_E \left[ \sum_{\mu,\nu} \tfrac{1}{2} \mathrm{Tr}\{t \cdot F^{\mu\nu}(x) \, t \cdot F^{\mu\nu}(x)\} + \bar{q}(x)(\sum_{\mu} \hat{\gamma}^\mu \partial^\mu + m) q(x) \right]. \tag{2.64}$$

The indices $\mu$ and $\nu$ run from 1 to 4 and the four gamma matrices used in the Euclidean version of the theory satisfy the anticommutation relation

$$\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = 2\delta^{\mu\nu}. \tag{2.65}$$

They are related to the gamma matrices used in Minkowski space by the transformation,

$$\hat{\gamma}^4 = \gamma^0, \; \hat{\gamma}^k = i\gamma^k, \; k = 1, 2, 3. \tag{2.66}$$

To motivate the structure of the discrete version of this theory we will first describe the construction of a gauge-invariant operator made from two quark fields at different space-time points. Consider a quark field which belongs to the fundamental representation of SU(3). If we form a

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composite operator for two separated points $x$ and $y$, gauge invariance may be preserved by linking the two fields by a *parallel transporter* which we denote by $\mathcal{U}$,

$$O = \bar{q}_j(y)\,\mathcal{U}(y,x)_{ji}\,q_i(x) \tag{2.67}$$

where the parallel transporter is expressed in terms of the gauge field $\mathcal{A}$ as

$$\mathcal{U}(y,x)_{ji} = \mathcal{P}\left\{\exp\left[-ig\int_x^y dz_\mu\, t\cdot\mathcal{A}^\mu(z)\right]\right\}_{ji}. \tag{2.68}$$

The line integral is along a path connecting $x$ with $y$ and $\mathcal{P}$ defines a path ordering. Each choice of path defines a different gauge-invariant operator. The path ordering is defined by the power series expansion of the exponential.

$$\begin{aligned}\mathcal{U}(y,x) &= 1 - ig\int_x^y dz_\mu\, t\cdot\mathcal{A}^\mu(z)\\ &\quad -g^2\int_z^y dz_\mu\, t\cdot\mathcal{A}^\mu(z)\int_x^z dw_\mu\, t\cdot\mathcal{A}^\mu(w) + \ldots .\end{aligned} \tag{2.69}$$

If $s$ denotes the position along the path, fields with larger $s$ stand to the left of those with smaller $s$.

We may investigate the gauge transformation properties of the parallel transporter by considering the infinitesimal version for $y = x + dx$. Using Eq. (1.24) we find that under a gauge transformation the infinitesimal parallel transport transforms as

$$\begin{aligned}\mathcal{U}(x+dx,x) &= 1 - igdx_\mu\, t\cdot\mathcal{A}^\mu(x)\\ &\to 1 - ig\,dx_\mu\,\Omega(x)t\cdot\mathcal{A}^\mu(x)\Omega^{-1}(x)\\ &\quad +dx_\mu\,(\partial^\mu\Omega(x))\Omega^{-1}(x)\end{aligned} \tag{2.70}$$

and hence for the infinitesimal transporter we derive

$$\mathcal{U}(x+dx,x) \to \Omega(x+dx)\mathcal{U}(x+dx,x)\Omega^{-1}(x). \tag{2.71}$$

Under a gauge transformation $\Omega$ the parallel transporter over a finite path transforms as

$$\mathcal{U}(y,x) \to \Omega(y)\mathcal{U}(y,x)\Omega^{-1}(x). \tag{2.72}$$

When combined with the gauge transformation property of the quark fields, Eq. (1.21), this is sufficient to demonstrate the gauge invariance of Eq. (2.67).

The formulation of QCD on the lattice uses a link variable $U$ which is a discretized version of the parallel transporter as the fundamental quantity to represent the gauge fields. Fig. 2.6 shows a two-dimensional lattice with spacing $a$ and linear size $L = na$, where for the case illustrated $n = 5$.

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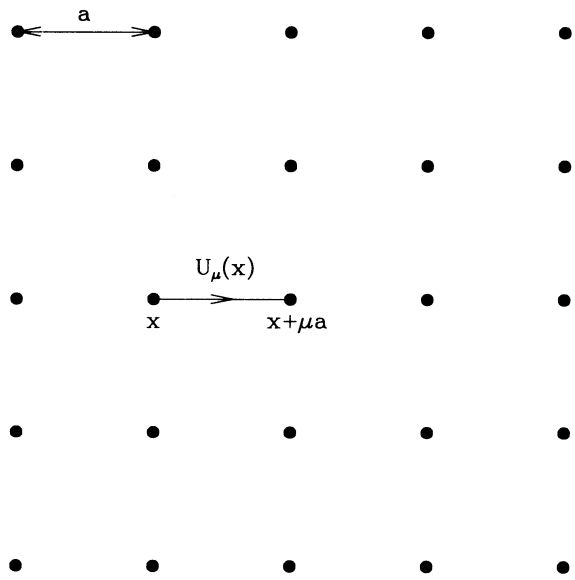



Fig. 2.6. Two-dimensional lattice showing a link variable.

The link variables are the parallel transporters between adjacent lattice sites as shown,

$$U_\mu(x) \equiv \mathcal{U}(x, x + \mu a) \tag{2.73}$$

where $\mu$ is a unit vector directed along one of the links of the lattice. In the lattice implementation of QCD the quark fields reside on the lattice sites and the gauge fields are associated with the links between lattice sites. Because of the gauge transformation property, Eq. (2.72), the product of matrices along a closed path or along a path terminated at the ends by quark and antiquark fields is gauge invariant.

We begin by considering the lattice action for the gauge field alone. The simplest action for the gauge field is determined by the product of link variables around a square path of side length $a$, called a plaquette.

$$\begin{aligned} U_\square &= U_\mu(x)U_\nu(x + a\mu)U_{-\mu}(x + a\mu + a\nu)U_{-\nu}(x + a\nu) \\ &\equiv U_\mu(x)U_\nu(x + a\mu)U_\mu^\dagger(x + a\nu)U_\nu^\dagger(x). \end{aligned} \tag{2.74}$$

The total action for SU(3) pure gauge theory is given by a sum over plaquettes,

$$S_G = \beta \sum_\square \left[1 - \frac{1}{3}\mathrm{ReTr}U_\square(x)\right] \tag{2.75}$$

where $\beta$ is a constant related to the gauge coupling. The sum over pla-

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quettes includes every plaquette with only one orientation:

$$\sum_{\square} \equiv \sum_{x} \sum_{1 \le \mu < \nu \le 4} \equiv \tfrac{1}{2} \sum_{x} \sum_{\mu,\nu=1}^{4} . \tag{2.76}$$

To show the correspondence of Eq. (2.75) with the continuum Euclidean action we take the limit $a \to 0$ and replace the link variable by

$$U_\mu(x) = 1 - i g_{\rm L} a \; t.\mathcal{A}_\mu(x) - \tfrac{1}{2} g_{\rm L}^2 a^2 \; t.\mathcal{A}_\mu(x) \; t.\mathcal{A}_\mu(x) + O(a^3). \tag{2.77}$$

The displaced gauge field may also be expanded about $x$:

$$\mathcal{A}_\nu(x + a\mu) = 1 + a\partial_\mu \mathcal{A}_\nu(x) + O(a^2). \tag{2.78}$$

We find that the plaquette variable may be written as

$$\begin{aligned} U_\square(x) &\approx 1 - i a^2 g_{\rm L} \; t \cdot F_{\mu\nu}(x) - \tfrac{1}{2} a^4 g^2 \; t \cdot F_{\mu\nu}(x) \; t \cdot F_{\mu\nu}(x) \\ \mathrm{Tr} U_\square(x) &\approx 3 - \tfrac{1}{2} a^4 g^2 \; \mathrm{Tr}\{t \cdot F_{\mu\nu}(x) \; t \cdot F_{\mu\nu}(x)\}, \end{aligned} \tag{2.79}$$

where the field strength tensor is

$$t \cdot F_{\mu\nu}(x) = t \cdot \partial_\mu \mathcal{A}_\nu(x) - t \cdot \partial_\nu \mathcal{A}_\mu(x) + i g_{\rm L} [t \cdot \mathcal{A}_\mu(x), t \cdot \mathcal{A}_\nu(x)]. \tag{2.80}$$

The action for the pure gauge theory in the continuum limit is thus

$$\begin{aligned} S_G &\to a^4 \frac{\beta g_{\rm L}^2}{6} \sum_{\square} \mathrm{Tr}\{t \cdot F^{\mu\nu} \; t \cdot F^{\mu\nu}\} \\ &= a^4 \sum_x \left\{ \frac{\beta g_{\rm L}^2}{12} \sum_{\mu,\nu} \mathrm{Tr}\{t \cdot F^{\mu\nu} \; t \cdot F^{\mu\nu}\} \right\}, \end{aligned} \tag{2.81}$$

in agreement with Eq. (2.64) provided we interpret $\beta = 6/g_{\rm L}^2$ and replace $a^4 \sum_x$ by an integral over $x$. The $a \to 0$ limit of the quantum theory is more subtle than the formal limit given in Eq. (2.81) because of the phenomenon of renormalization and will be considered below.

We have therefore constructed a regulated version of Euclidean QCD without quarks on a space-time lattice. Gauge invariance has been preserved, although the full Lorentz invariance has been reduced to a hypercubic symmetry. Note also that there is considerable ambiguity in the formulation of the action of QCD on the lattice. For example, any action which differs from Eq. (2.75) only by terms which vanish as $a \to 0$ would also be acceptable.

The construction of an action for quark fields follows the pattern of the above. The quark fields are placed on the sites of the lattice. A Euclidean action for the free quark fields on the lattice can be obtained by replacing the derivative in the action by a finite difference, and the integral over $x$ by a sum over sites. The simplest prescription, which we shall call the

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'naive fermion' action, is

$$\begin{aligned} S_{\rm NF} &= \int d^4x_E\ \bar{q}(x)(\hat{\gamma}_\mu\partial_\mu + m)q(x) \\ &\to a^4\sum_x\left\{\frac{1}{2a}\sum_{\mu=1}^{\mu=4}\bar{q}_x(t_\mu - t_{-\mu})\hat{\gamma}_\mu q_x + m\bar{q}_x q_x\right\} \end{aligned} \tag{2.82}$$

where $t$ is a displacement operator, $t_\mu q_x = q_{x+a\mu}$.

If we take the Fourier transform of the inverse propagator derived from Eq. (2.82) we obtain

$$S_{\rm NF}(p) = \sum_{\mu=1}^{\mu=4}\frac{i}{a}\sin(ap_\mu)\hat{\gamma}_\mu + m. \tag{2.83}$$

We must examine all values of this inverse propagator which are of order $a^0$ in the limit $a \to 0$. The range of momenta is restricted to an interval of length $2\pi/a$ which we can take to be $-\pi/(2a) < p_\mu < 3\pi/(2a)$. Terms which are more singular than $a^0$ lead to a vanishing propagator. There are 16 contributions which lead to terms of order $a^0$, corresponding to choosing each of the four components of $p_\mu$ to be either in the vicinity of zero or in the vicinity of $\pi/a$. Therefore rather than representing one fermion, Eq. (2.82) actually represents 16 fermions. This is referred to as the fermion doubling problem; it is a consequence of the fact that the Dirac equation is first order. The solution to the fermion doubling problem advocated by Wilson [14] is to add another term to the fermionic action to remove the degeneracy of the unwanted states,

$$S_{\rm WF} = S_{\rm NF} - a^4\sum_x\left\{\frac{a}{2}\sum_{\mu=1}^{\mu=4}\bar{q}_x\Delta q_x\right\} \tag{2.84}$$

where $\Delta = (t_\mu + t_{-\mu} - 2)/a^2$. The inverse propagator is now given by

$$S_{\rm WF}(p) = \sum_{\mu=1}^{\mu=4}\frac{i}{a}\sin(ap_\mu)\hat{\gamma}_\mu + m + \frac{1}{a}\sum_{\mu=1}^{\mu=4}(1-\cos(ap_\mu)). \tag{2.85}$$

The propagator is of order $O(a^0)$ only for $p_\mu \approx 0$; in the vicinity of $p_\mu = 0$ the Wilson term is of order $a$ and therefore innocuous in the naive continuum limit. It is only important for $p_\mu \approx 1/a$. As a consequence the Wilson prescription gives all but one of the fermions a mass $m + 2j/a$ where $j = 1, 2, 3$ or 4 and the degeneracy is lifted. Combining terms in the action we find that

$$S_{\rm WF} = a^4\sum_x\left\{\left(m+\frac{4}{a}\right)\bar{q}_x q_x - \frac{1}{2a}\sum_{\mu=1}^{\mu=4}\bar{q}_x\left[(1-\hat{\gamma}_\mu)t_\mu + (1+\hat{\gamma}_\mu)t_{-\mu}\right]q_x\right\}. \tag{2.86}$$

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This Lagrangian is conventionally re-written by changing the normalization of the quark fields to simplify the coefficient of the mass term. Including the link variables we obtain the full gauge-invariant expression for the quark terms in the action,

$$S_F = a^4 \sum_x \left\{ \bar{q}_x q_x - K \sum_{\mu=1}^{\mu=4} \left[ \bar{q}_x (1-\hat{\gamma}_\mu) U_\mu(x) q_{x+a\mu} + \bar{q}_x (1+\hat{\gamma}_\mu) U_\mu^\dagger(x) q_{x-a\mu} \right] \right\}. \tag{2.87}$$

The coefficient of the term connecting adjacent lattice sites, $K$, is called the hopping parameter and is given by

$$K = \frac{1}{2(ma+4)}. \tag{2.88}$$

Note that chiral symmetry has been destroyed by the introduction of the $\Delta$ term in the action, Eq. (2.84), even in the absence of an explicit mass term $m$.

In the above discussion of the fermion lattice action we have followed the treatment by Wilson. An alternative solution to the fermion doubling problem is provided by Kogut-Susskind fermions [19].

Physical predictions are made by evaluating the integral over all configurations of the fields, weighted by the complete action. The fermion fields normally appear as a bilinear term in the action and can be integrated out

$$\begin{aligned} &\int \{dU_\mu\} dq d\bar{q} \, \exp[-S_G(U) + \bar{q} M(U) q] \\ &= \int \{dU_\mu\} \, \det M(U) \, \exp[-S_G(U)]. \end{aligned} \tag{2.89}$$

The above integral is a functional integral which is approximated by integrating over all the fields and links at all points of the lattice. Although several techniques have been used to obtain numerical evaluation of the lattice integrals, the most frequently used is direct Monte Carlo evaluation of the integral with importance sampling. The computation of the fermion determinant is very costly in computer time. As a result most calculations are currently performed in the *quenched approximation* [20] which replaces the determinant in Eq. (2.89) by a constant independent of the link variables $U$. Replacing $M$ by a constant is equivalent to the neglect of closed fermion loops and leads to an enormous saving in computer time since the probability of any configuration of the gauge field is calculated with the pure gauge theory action alone. Note that the quenched approximation is made only to save computing time. Its validity will not be the same for all physical quantities and can only be established *a posteriori.*

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In the quenched approximation the expectation value of any operator $O(U)$ is

$$\langle O \rangle = \frac{\int \{dU_\mu\}\, O(U)\, \exp[-S_G(U)]}{\int \{dU_\mu\}\, \exp[-S_G(U)]}. \tag{2.90}$$

In order for the lattice theory to be an adequate representation of the continuum theory we must satisfy two conditions. The lattice must be large enough that the object which we wish to describe fits on it. If this not the case the result obtained will be subject to effects from the boundaries called finite size effects. In addition the lattice spacing must be small enough that the physical quantity to be calculated is insensitive to the granularity of the lattice.

Calculations proceed by evaluating the integral in Eq. (2.90) by Monte Carlo methods. The two variables that one has at one's disposal are the bare lattice coupling $g_{\rm L}$ and the number of lattice sites. As we shall see, the value of the coupling controls the physical size of the lattice spacing. As an ideal one would like to have a large number of sites so that the system being modelled fits comfortably on to the lattice. On the other hand one would like to have a small value of the coupling so that the granularity of the lattice is small compared to the object under study. Therefore for a hadron of radius $R$ we require that

$$a \ll R \ll L = na. \tag{2.91}$$

The relationship between the value of the coupling and the size of lattice spacing cannot be determined *a priori*.

Consider any physical quantity such as the mass of a hadron, $m$, which is measured on the lattice. If we assume that quarks are massless, the mass of the hadron is fixed by the inverse lattice spacing which is the only parameter with dimension

$$m = \frac{1}{a} f(g_{\rm L}). \tag{2.92}$$

In order to yield a sensible answer the $a \to 0$ limit must clearly be taken with some care. The proper continuum limit is achieved when we take $a$ to zero holding the physical quantities fixed. In practice this means that as $a \to 0$

$$\lim \frac{1}{a} f(g_{\rm L}) \to \text{constant}. \tag{2.93}$$

This means that $g_{\rm L}$ must also be changed as $a$ goes to zero so that a sensible continuum theory is reached. The continuum limit of the theory cannot be approached at fixed $g_{\rm L}$. Fortunately the behaviour of the coupling of the theory at small distances is predicted in QCD as we discussed

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earlier in this chapter. That this should happen is a consequence of the renormalizability of the theory.

We shall now use our knowledge of the short-distance behaviour of the coupling to determine the relationship between the coupling and the lattice spacing which leads to a sensible continuum limit. In the limit $a \to 0$ we must have that

$$\frac{d}{da} m = 0 \tag{2.94}$$

since a physical quantity cannot depend on the cut-off. Differentiating Eq. (2.92) using the chain rule,

$$\left\{ a \frac{\partial}{\partial a}\bigg|_{g_L} + \left( a \frac{\partial g_L}{\partial a} \right) \frac{\partial}{\partial g_L}\bigg|_a \right\} m = 0 \,, \tag{2.95}$$

we get

$$f(g_L) = \left( a \frac{\partial g_L}{\partial a} \right) \frac{\partial}{\partial g_L} f(g_L) \,. \tag{2.96}$$

The scale dependence of the bare coupling is determined by the QCD $\beta$ function (not to be confused with the $\beta$ of Eq. (2.75)),

$$a \frac{\partial g_L}{\partial a} = \frac{b}{4\pi} g_L^3 \left( 1 + b' \frac{g_L^2}{4\pi} + \ldots \right) \tag{2.97}$$

where the coefficients $b$ and $b'$ are given in Eq. (2.7). We therefore find that all dimensionful quantities are proportional to a fixed scale $\Lambda_L$:

$$m = c\Lambda_L \tag{2.98}$$

where

$$\Lambda_L = \frac{1}{a} \exp\left( -\frac{2\pi}{b g_L^2} \right) \left( \frac{4\pi + b' g_L^2}{4\pi g_L^2} \right)^{b'/(2b)} \,. \tag{2.99}$$

Despite the fact that we start with a theory which is scale free (if we ignore quark masses) we end up with a theory which has one scale, to which all other dimensionful quantities are related. This is the phenomenon of *dimensional transmutation*. The lattice spacing goes to zero exponentially in $1/g_L^2$ as $g_L$ tends to zero. This relationship, which is known as asymptotic scaling, serves two functions. As the output of a Monte Carlo simulation one obtains the mass in lattice units as a function of the coupling. One can then check that the Monte Carlo data satisfy Eq. (2.99). It can also be used to determine $\Lambda_L$ which is related to the $\Lambda$ parameter measured in the continuum.

How does a practical Monte Carlo simulation proceed? We shall use as an illustration the calculation of hadronic masses. The first step is to

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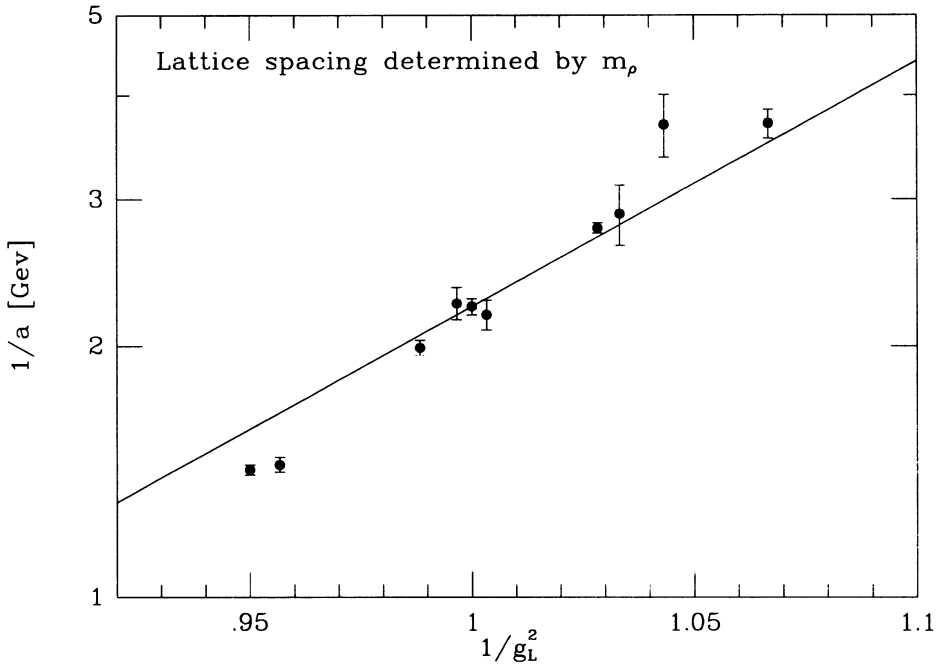



Fig. 2.7. Lattice spacing as a function of coupling. Adapted from ref. [21].

calculate the masses in lattice units at several values of the bare coupling and to verify that the relationship between the masses and the couplings is as predicted by asymptotic scaling, Eq. (2.99). Fig. 2.7 shows a compilation of data points which are the result of Monte Carlo measurements of the lattice spacing $a$ as a function of the lattice coupling. The quantity measured on the lattice is the mass of the $\rho$ meson. The value of the $\rho$ mass measured by experiment and given in the particle data tables is used to transform this measurement into GeV. The curve in Fig. 2.7 shows the prediction of Eq. (2.99) with the value of the intercept adjusted to the data (corresponding to $\Lambda_L = 1.74$ MeV). The data are in fair agreement with the slope predicted by asymptotic scaling, which gives assurance that the results are indeed close to the continuum limit.

Having chosen the mass of the $\rho$ to fix the scale, one can then go on to make predictions for other physical quantities. In the quenched approximation it is found that $g_L^2 = 1$ corresponds to $a^{-1} \approx 2$ GeV, or equivalently $a \approx 1/10$ fermi. This lattice spacing should be much less than the size of the hadron whose mass one is computing. If this condition is satisfied it makes sense to predict the other hadron masses in terms of the first.

Fig. 2.8 shows the results of a lattice calculation of the hadron masses

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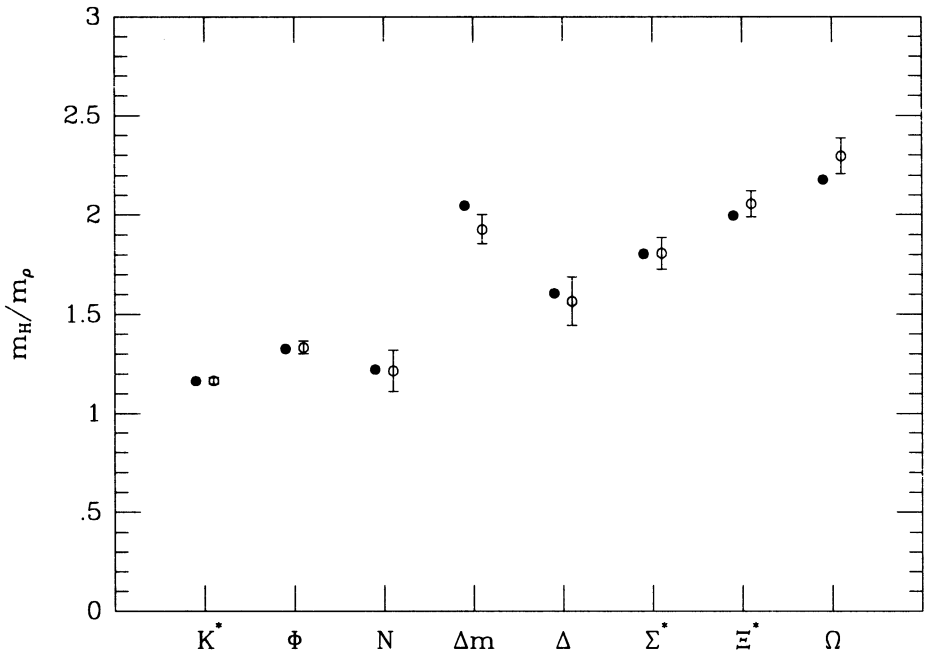



Fig. 2.8. Hadron masses in units of the $\rho$ mass from the lattice (open points) compared with experiment (solid points).

[22] performed with Wilson fermions in the quenched approximation. The open points, which are the lattice predictions, have been obtained from calculations performed at values of $\beta = 6/g_L^2$ between 5.70 and 6.17. The quantity $\Delta m$ refers to the combination of masses $m_\Xi + m_\Sigma - m_N$. The lattice results have been extrapolated to zero lattice spacing at finite volume and subsequently extrapolated to infinite volume. The details of the procedure can be found in ref. [22]. For comparison Fig. 2.8 also shows the measured values for hadron masses (solid points). The agreement between theory and experiment is extremely good. This indicates that the quenched approximation is reliable for the calculation of hadron masses.

As a second example of the precision now available from lattice gauge theory, we present in Fig. 2.9 a lattice calculation of the static $q\bar{q}$ potential [23], performed again in the quenched approximation. At small distances the potential is expected to be of the form

$$V(R) = V_0 - \alpha_S(aR)/R. \tag{2.100}$$

At large distances we have a linearly rising potential of the form,

$$V(R) = V_0 + KR. \tag{2.101}$$

The constant $K$ is known as the string tension. Fig. 2.9 shows the poten-

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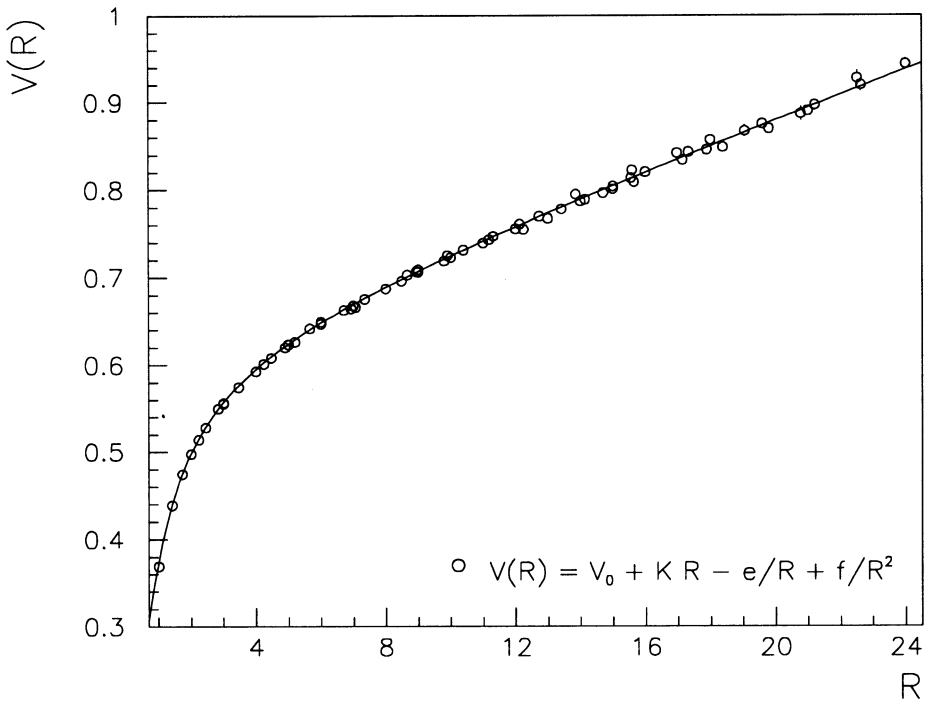



Fig. 2.9. QCD potential *vs.* $R$ (in lattice units) from lattice QCD. Figure from ref. [23].

tial as calculated in lattice QCD with $\beta \equiv 6/g_L^2 = 6.4$ as a function of the distance $R$ measured in lattice units. The curve is a fit to the Monte Carlo data and the fitted value of $Ka^2 \approx 0.0147$. Using the value of the string tension determined from experiment [24], which is $(440\ \text{MeV})^2$, we can determine the lattice spacing in physical units to be $a = 0.0544$ fermi. The existence of the linearly rising term with $K > 0$ is a signal for confinement, since it costs infinite energy to separate quarks an infinite distance. The linearly rising term alone is not sufficient to demonstrate confinement, since it refers to a calculation performed at finite lattice spacing in the quenched approximation. However the observation that the string tension varies as we change the coupling in accordance with the renormalization group scaling relation, Eq. (2.99), makes it plausible that confinement survives in the continuum limit.

For reasons of simplicity we have described only the calculation of hadron masses and the string tension in the above discussion. If this were all that one could accomplish with lattice gauge theory it would be of limited utility, since both of these quantities are well determined by experiment. In fact much of the modern research on lattice QCD is devoted

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to the calculation of quantities of great interest which are not so directly measured by experiment. Examples of this latter type of quantities are the parameters of the Lagrangian (such as quark masses and the running coupling) and the hadronic matrix elements of quark fields, which are important to calculate weak decays. The hadronic matrix elements of quark operators are poorly known and therefore even the relatively crude information which is presently available from lattice gauge theory is of great importance.

## References


1) H.D. Politzer, *Phys. Rev. Lett.* **30** (1973) 1346;
D.J Gross and F. Wilczek, *Phys. Rev. Lett.* **30** (1973) 1343.

2) W.A. Caswell, *Phys. Rev. Lett.* **33** (1974) 244.

3) O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *Phys. Lett.* **B93** (1980) 429;
S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **B303** (1993) 334.

4) V.N. Gribov, Lectures at the 12th Winter School of the Leningrad Nuclear Physics Institute, SLAC-TRANS-176 (1977);
S. D. Drell, in a *Festschrift for Maurice Goldhaber*, Trans. NY Acad. Sci., Series II, **40** (1980) 76.

5) N.K. Nielsen, *Am. J. Phys.* **49** (1981) 1171;
R.J. Hughes, *Phys. Lett.* **B97** (1980) 246; *Nucl. Phys.* **B186** (1981) 376;
K. Johnson, in *Asymptotic Realms of Physics*, edited by A. Guth, K. Huang and R.L. Jaffe, MIT press, Cambridge, Mass. (1983).

6) R. E. Peierls, *Quantum Theory of Solids*, Oxford (1955).

7) S. Weinberg, *Phys. Rev.* **D8** (1973) 3497.

8) R. Tarrach, *Nucl. Phys.* **B183** (1981) 384;
O. Nachtmann and W. Wetzel, *Nucl. Phys.* **B187** (1981) 333.

9) T. Appelquist and J. Carazzone, *Phys. Rev.* **D11** (1975) 2856.

10) W. Bernreuther and W. Wetzel, *Nucl. Phys.* **B197** (1982) 228;
W. Bernreuther, *Ann. Phys. (NY)* **151** (1983) 127.

11) W. Marciano, *Phys. Rev.* **D29** (1984) 580.

12) Particle Data Group, L. Montanet *et al.*, *Review of Particle Properties*, *Phys. Rev.* **D50** (1994) 1173.

13) J.C. Collins *Renormalization*, Cambridge University Press (1984).

14) K. Wilson, *Phys. Rev.* **D10** (1974) 2445;
K. Wilson in *New Phenomena in Subnuclear Physics*, edited by A. Zichichi, Plenum, New York (1977).

15) M. Creutz, *Quarks, gluons and lattices*, Cambridge University Press (1983).

16) C. Rebbi, *Lattice gauge theories and Monte Carlo simulations* , World Scientific (1983).

17) I. Montvay and G. Munster, *Quantum Fields on a Lattice*, Cambridge University Press (1994).

18) A.S. Kronfeld and P.B. Mackenzie, *Ann. Rev. Nucl. Part. Sci.* **43** (1993) 793.

19) L. Susskind, *Phys. Rev.* **D16** (1977) 3031;
T. Banks, J.B. Kogut and L. Susskind, *Phys. Rev.* **D13** (1976) 1043.

20) D. Weingarten, *Phys. Lett.* **B109** (1982) 57.

21) C.R. Allton, *Nucl. Phys.* **B437** (1995) 641.

22) F. Butler *et al.*, *Nucl. Phys.* **B430** (1994) 179.


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23) G.S. Bali and K. Schilling, *Phys. Rev.* **D47** (1993) 661, *Intl. J. Mod. Phys.* **C4** (1993) 1167.

24) E.J. Eichten *et al.*, *Phys. Rev.* **D21** (1980) 203;
P. Goddard *et al.*, *Nucl. Phys.* **B56** (1973) 109.

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