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Continuous singular measures equivalent to their convolution squares

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1. *Introduction.* Throughout this paper, G will denote a locally compact, non-discrete, Abelian group (subjected to various conditions) and $\mathbf{X}$ will denote the character group of G. All terminology and notation are as in (7). The measure algebra $\mathbf{M}(G)$, as is known, is a very complicated entity. We address ourselves here to some novel peculiarities of the subspace $\mathbf{M}_s(G)$ of continuous measures in $\mathbf{M}(G)$ that are singular with respect to Haar measure λ.

In 1968, B. M. Makarov (13) demonstrated the existence of a non-zero measure τ in $\mathbf{M}(\mathbf{R})$ such that $\tau \in \mathbf{M}_s^+(\mathbf{R})$, $\hat{\tau}(y) = o(1)$ as $y \to \pm\infty$, and the measures τ and $\tau * \tau$ are absolutely continuous with respect to each other. Hence in particular $\tau * \tau$ is singular. He did this in order to provide a certain example of a normal dynamical system (see §8 below). We are indebted to Professor Makarov for pointing out his paper to us and for conversations regarding it.

The phenomenon discovered by Makarov is antithetical to another phenomenon, which has its origins in a construction of Littlewood (12), and which was carried out for all G by Hewitt and Zuckerman (9). Namely, there are probability measures in $\mathbf{M}_s^+(G)$ for which $\mu * \mu$ is absolutely continuous and has Lebesgue–Radon–Nikodým derivative in the intersection $\bigcap_{1<p<\infty} L_p(G)$. (Extensions of this construction have been obtained by Stromberg (16).) Makarov's proof and the construction of Hewitt–Zuckerman led us to ask if an analogue of Makarov's result holds for all G. The answer is affirmative.

Here we replace Makarov's existence proof (which is based on a probabilistic existence theorem of Kahane and Salem (11), ch. VIII, Théorème 1) by an explicit construction using the generalized Riesz products introduced in (9). We are able also to establish a sharpened form of Makarov's result which remains valid for all G. At the same time we incorporate an analogue of Stromberg's construction (16). To state our main theorem, we need some notation.

(1·1) *Definition.* Write G as the product $\mathbf{R}^a \times G_0$, where a is a non-negative integer and the subgroup G_0 contains a compact open subgroup J_0. If G_0 is compact, let $J_0 = G_0$. For non-discrete G, either a is positive or J_0 is infinite. Let Υ_0 be the character group of J_0, and let r be the rank of Υ_0. Let G_1 be any subgroup of G_0 containing J_0 such that G_1/J_0 is countable. Let G_2 be $R^a \times G_1$.

(1·2) *Definition.* Let X be a set and $\mathscr{A}$ a σ-algebra of subsets of X. Given two measures μ and ν on $\mathscr{A}$, we write $\mu \ll \nu$ if μ is absolutely continuous with respect to ν,

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i.e. if $\nu(A) = 0$ implies $\mu(A) = 0$. If $\mu \ll \nu$ and $\nu \ll \mu$, we write $\mu \sim \nu$ and we say that μ and ν are *equivalent*. If μ and ν are mutually singular, we write $\mu \perp \nu$. (All terms are used as in (8), §19.) We state our final result.

(1·3) MAIN THEOREM. *There exists a set* $\mathbf{B}$ *of continuous probability measures in* $\mathbf{M}_s(G)$ *such that*:

(i) $\mu \sim \mu * \mu$ *for all* μ *in* $\mathbf{B}$;

(ii) $\mu\hat{}(\chi) \to 0$ *as* $\chi \to \infty$ *in* $\mathbf{X}$ *for all* μ *in* $\mathbf{B}$;

(iii) *the support of* μ *is* G_2 *for all* μ *in* $\mathbf{B}$;

(iv) *distinct elements of* $\mathbf{B}$ *are mutually singular.*

Further, $\mathbf{B}$ *can be expressed as a disjoint union* $\bigcup_{\iota \in I} \mathbf{B}_\iota$, *where* $\operatorname{card}(\mathbf{B}_\iota) = 2^{\aleph_0}$, $\operatorname{card}(I) = \max(r, 2^{\aleph_0})$, *and*:

(v) *for each* ι, $\mathbf{B}_\iota$ *is a convolution semigroup up to equivalence, in the sense that given* μ, ν *in* $\mathbf{B}_\iota$, *there exists* ω *in* $\mathbf{B}_\iota$ *such that* $\mu * \nu \sim \omega$;

(vi) *if* ι, ι' *are distinct indices in* I, $\mu \in \mathbf{B}_\iota$, *and* $\mu' \in \mathbf{B}_{\iota'}$, *then* $\mu * \mu'$ *and Haar measure on* G_2 *are equivalent.*

(1·4) VARIANT OF MAIN THEOREM. *Let* λ_2 *be a probability measure on* G_2 *which is equivalent to the Haar measure of* G_2. *Also valid is the statement obtained from* (1·3) *when* (i) *is replaced by*

(i)′ $\mu + \lambda_2 \sim (\mu + \lambda_2)$ *but* $\mu \neq \mu * \mu$ *for all* μ *in* $\mathbf{B}$;

and (v) *is replaced by*

(v)′ *for each* ι, $\{\mu + \lambda_2 : \mu \in \mathbf{B}_\iota\}$ *is a convolution semigroup up to equivalence.*

(1·5) *Comments on main theorem.* On many groups, it is very easy to find continuous singular measures that are equivalent to their own convolution squares. For example, on R^2, we may take the product measure $\epsilon_0 \times \lambda$, where λ is a Gaussian measure and ϵ_0 is the Dirac measure at 0, or on T^2 take the idempotent $\epsilon_0 \times \lambda$, where λ is Haar measure on the second factor. In general if τ is any non-discrete locally compact group topology on G strictly finer than the original topology, then $\mathbf{M}(G_\tau)$ can be regarded as the subalgebra of $\mathbf{M}_s(G)$ consisting of all measures in $\mathbf{M}(G)$ which are concentrated on a countable union of τ-compact sets; then we may find examples of the above type in $L_1(G_\tau)$. Note that condition (ii) rules out such a triviality, because $\mathbf{M}_0(G)$ is singular to $\mathbf{M}(G_\tau)$ for every LCA group topology τ strictly finer than the original one (see (4)). In particular, of course, this shows that every measure satisfying (ii) is automatically continuous.

We proceed to the proof.

2. *Facts about Riesz products.* A subset Δ of $\mathbf{X}$ is said to be *dissociate* if $1 \notin \Delta$ and the following weak form of independence holds. Given a finite subset

$$\{\chi_1, \chi_2, \ldots, \chi_m\} \subset \Delta \text{ and a sequence } (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$$

of numbers drawn from the set $\{-2, -1, 0, 1, 2\}$, the equality

$$\chi_1^{\epsilon_1} \chi_2^{\epsilon_2} \cdots \chi_m^{\epsilon_m} = 1$$

implies the equalities

$$\chi_1^{\epsilon_1} = \chi_2^{\epsilon_2} = \ldots = \chi_m^{\epsilon_m} = 1.$$

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Discussion and examples may be found in (2), (9), and in (7), §37. It is known that the values of Fourier–Stieltjes transform can be, roughly speaking, arbitrary on a dissociate set Δ. Specifically, the following is true. For $z \in \mathbf{C}$, let $z^{(1)} = z$ and $z^{(-1)} = \bar{z}$.

(2·1) THEOREM. *Let β be a complex-valued function on Δ such that $|\beta(\chi)| \leqslant \frac{1}{2}$ for all $\chi \in \Delta$ such that $o(\chi) > 2$ and such that $\beta(\chi)$ is real and $|\beta(\chi)| \leqslant 1$ for all $\chi \in \Delta$ such that $o(\chi) = 2$. (The symbol $o(\chi)$ denotes the order of the element χ.) There is one and only one measure μ in $\mathbf{M}(G)$ such that*

(i) $\mu\hat{}(\chi_1^{\delta_1}\chi_2^{\delta_2}\dots\chi_m^{\delta_m}) = \beta_1^{(\delta_1)}\beta_2^{(\delta_2)}\dots\beta_m^{(\delta_m)}$ *for every finite subset $\{\chi_1, \chi_2, \dots, \chi_m\}$ of Δ and sequence $(\delta_1, \delta_2, \dots, \delta_m)$ with values in $\{-1, 1\}$;*

(ii) $\mu\hat{}(\psi) = 0$ *for all $\psi \in \mathsf{X}$ not of the form*

$$\chi_1^{\delta_1}\chi_2^{\delta_2}\dots\chi_m^{\delta_m} \quad \textit{as in} \quad \text{(i)}.$$

The measure μ is necessarily in $\mathbf{M}^+(G)$ and $\mu(G) = 1$.

For a proof, see for example (7), (37.14). We denote the measure μ of (2·1) by $\mu_{\Delta,\beta}$ and call it the *Riesz product based on the set Δ and the function β*. A product $\chi_1^{\delta_1}\chi_2^{\delta_2}\dots\chi_m^{\delta_m}$, where the χ's are distinct elements of X and the δ_j's are in $\{-1, 1\}$, is called a *word from Δ*. Brown and Moran (2) have proved the following generalization of a theorem of Zygmund and Hewitt–Zuckerman.

(2·2) THEOREM. *Let $\mu_{\Delta,\beta}$ and $\mu_{\Delta',\beta'}$ be Riesz products in $\mathbf{M}(G)$. Consider the functions β and β' on $\Delta \cup \Delta'$, β being defined as zero on $\Delta'\backslash\Delta$, and β' as zero on $\Delta\backslash\Delta'$. Suppose that $\sup\{|\beta + \beta'|\} < 2$. The conditions*

(i) $\beta - \beta' \in l_2(\Delta \cup \Delta')$

and

(ii) $\mu_{\Delta,\beta} \sim \mu_{\Delta',\beta'}$

are equivalent. Also the conditions

(iii) $\beta - \beta' \notin l_2(\Delta \cup \Delta')$

and

(iv) $\mu_{\Delta,\beta} \perp \mu_{\Delta',\beta'}$

are equivalent.

Finally we note the following.

(2·3) THEOREM. *Let G be compact (and infinite). The discrete group X admits a family $\mathscr{D}$ of subsets such that*:

(i) *each Δ in $\mathscr{D}$ is infinite*;

(ii) *each Δ in $\mathscr{D}$ is dissociate*;

(iii) *$\Delta \cap \Delta'$ is finite for distinct Δ, Δ' in $\mathscr{D}$*;

(iv) $\operatorname{card}(\mathscr{D}) = 2^{\aleph_0}$.

Furthermore, let $\{\chi_1, \dots, \chi_m\}$ and $\{\psi_1, \dots, \psi_n\}$ be subsets of distinct sets Δ and Δ', respectively, in $\mathscr{D}$. Then equality can hold for words in the χ's and ψ's,

(v) $\chi_1^{\delta_1}\chi_2^{\delta_2}\dots\chi_m^{\delta_m} = \psi_1^{\epsilon_1}\psi_2^{\epsilon_2}\dots\psi_n^{\epsilon_n}$,

only if all of the χ's and ψ's belong to $\Delta \cap \Delta'$. Hence there are only finitely many words formed from elements of Δ and simultaneously from elements of Δ'.

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Proof. Let P be a countably infinite dissociate subset of X (see (7), Theorem (37·18)). Like every countably infinite set, P admits a continuum of infinite subsets Δ such that $\Delta \cap \Delta'$ is finite for any two distinct sets in the family. This is well known, but we note the proof, since it is very short. Identify P with the rational numbers. For every irrational number α, let Δ_α be a sequence of distinct rationals with limit α. Plainly $\Delta_\alpha \cap \Delta_{\alpha'}$ is finite if $\alpha \neq \alpha'$. Thus all of our sets Δ_α are subsets of a single dissociate set. The equality (v) can obtain only if the χ's and the ψ's are the same and appear to the same power. Since a subset of a dissociate set is dissociate, (ii) holds.

(2·4) *Note.* The last theorem shortens a construction given by Stromberg (16).

3. *The case of compact G.*

(3·1) *Definition.* Let Δ be a countably infinite dissociate subset of the discrete group X. Arrange Δ in any way in a sequence $(\chi_n)_{n=1}^\infty$, all of the χ_n's being of course distinct. Let $(r_n)_{n=1}^\infty$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$, i.e., $(r_n) \in c_0$, and $\sum_{n=1}^{\infty} r_n^\alpha = \infty$ for all positive real numbers α. (For example, we may take

$$r_n = (\log(n+1))^{-1}.)$$

For each positive real number α, let n_α be the smallest positive integer such that $r_n^\alpha \leqslant \frac{1}{2}$ for all $n \geqslant n_\alpha$. Let $\Delta_\alpha = (\chi_n)_{n=n_\alpha}^\infty$. Let σ_α be the Riesz product based on Δ_α and the function $\chi_n \to r_n^\alpha$.

(3·2) THEOREM. *If α and α' are distinct positive real numbers, then σ_α and $\sigma_{\alpha'}$ are mutually singular.*

Proof. Suppose that $\alpha' > \alpha$. We have

$$(r_n^\alpha - r_n^{\alpha'})^2 = r_n^{2\alpha}(1 - r_n^{-(\alpha-\alpha')})^2. \tag{3·2·1}$$

Write m for $\max\{n_\alpha, n_{\alpha'}\}$. The second factor on the right side of (3·2·1) has limit 1 as $n \to \infty$. Hence the series

$$\sum_{n=m}^{\infty} (r_n^\alpha - r_n^{\alpha'})^2$$

diverges. Hence condition (2·2·iii) holds, and so (2·2·iv) holds as well.

(3·3) THEOREM. *For arbitrary positive real numbers α and α', the measures $\sigma_\alpha * \sigma_{\alpha'}$ and $\sigma_{\alpha+\alpha'}$ are absolutely continuous with respect to each other.*

Proof. As above, write m for the integer $\max\{n_\alpha, n_{\alpha'}\}$. We first show that $\sigma_\alpha * \sigma_{\alpha'}$ is a Riesz product on the set $(\chi_n)_{n=m}^\infty$. For all $\psi \in X$, we of course have

$$(\sigma_\alpha * \sigma_{\alpha'})^\wedge(\psi) = \sigma_\alpha^\wedge(\psi)\, \sigma_{\alpha'}^\wedge(\psi). \tag{3·3·1}$$

Given a word $\chi_{s_1}^{\delta_1} \chi_{s_2}^{\delta_2} \dots \chi_{s_l}^{\delta_l}$ in the elements of $(\chi_n)_{n=1}^\infty$ with $s_1 < s_2 < \dots < s_l$, it is clear from (3·3·1) that

$$(\sigma_\alpha * \sigma_{\alpha'})^\wedge(\chi_{s_1}^{\delta_1} \chi_{s_2}^{\delta_2} \dots \chi_{s_l}^{\delta_l}) = \begin{cases} 0 & \text{if } s_1 < m, \\ r_{s_1}^{\alpha+\alpha'} r_{s_2}^{\alpha+\alpha'} \dots r_{s_l}^{\alpha+\alpha'} & \text{if } s_1 \geqslant m. \end{cases}$$

That is, $\sigma_\alpha * \sigma_{\alpha'}$ is the Riesz product on the set $(\chi_n)_{n=m}^\infty$ with the function $\beta(\chi_n) = r_n^{\alpha+\alpha'}$.

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The measure $\sigma_{\alpha+\alpha'}$ is the Riesz product on the set $(\chi_n)_{n=n_{\alpha+\alpha'}}^{\infty}$ with the function $\beta(\chi_n) = r_n^{\alpha+\alpha'}$. It follows from (2·2) that $\sigma_\alpha * \sigma_{\alpha'} \sim \sigma_{\alpha+\alpha'}$.

(3·4) THEOREM. *All of the measures σ_α are continuous, singular, and in $\mathbf{M}_0(G)$.*

Proof. Consider any word

$$\psi = \chi_{s_1}^{\delta_1}\chi_{s_2}^{\delta_2}\cdots\chi_{s_l}^{\delta_l}, \tag{3·4·1}$$

the notation being as in the proof of (3·3). only a finite There are number of these words for which $s_l < q$, q being a fixed integer. We have

$$\sigma_\alpha^\wedge(\psi) \leqslant r_l^\alpha. \tag{3·4·2}$$

Since $\lim_{n\to\infty} r_n = 0$, the right side of (3·4·2) is arbitrarily small when $s_l \geqslant q$ for sufficiently large q. Note that $\sigma_\alpha^\wedge(\omega)$ vanishes for characters ω that are not words (3·4·1). Hence σ_α is in $\mathbf{M}_0(G)$. As noted in (1·5), this guarantees that σ_α is continuous. All Riesz products $\mu_{\Delta,\beta}$ for which $\beta \notin l_2(\Delta)$ are singular with respect to Haar measure, as proved in (2), lemma 1. Thus, σ_α is in $\mathbf{M}_s^+(G)$.

We now make our final construction for the compact case.

(3·5) *Definition.* Let S be a countably infinite set of positive real numbers such that:

(i) if α and α' are in S, then $\alpha+\alpha'$ is in S;

(ii) if α is in S, then α has the form $\alpha'+\alpha''$ where α' and α'' are in S.

(3·6) THEOREM. *Let S be as in* (3·5) *and let $b = (b_\alpha)_{\alpha\in S}$ be a positive real-valued function on S such that $\sum_{\alpha\in S} b_\alpha = 1$. Let μ be the measure*

(i) $\mu = \sum_{\alpha\in S} b_\alpha\sigma_\alpha$.

Then μ is continuous and singular, and is in $\mathbf{M}_0(G)$. It has the further property that

(ii) $\mu \sim \mu * \mu$.

Proof. The first assertion follows from (3·4) and the facts that $\mathbf{M}_s(G)$ and $\mathbf{M}_0(G)$ are closed linear subspaces of $\mathbf{M}(G)$. To prove (ii), write

$$\begin{aligned}\mu * \mu &= \Big(\sum_{\alpha\in S} b_\alpha\sigma_\alpha\Big) * \Big(\sum_{\beta\in S} b_\beta\sigma_\beta\Big)\\ &= \sum_{\alpha\in S}\sum_{\beta\in S} b_\alpha b_\beta \sigma_\alpha * \sigma_\beta.\end{aligned} \tag{3·6·1}$$

By (3·5·1), $\alpha+\beta$ is in S for α, β in S. By (3·3), $\sigma_\alpha * \sigma_\beta$ is equivalent to $\sigma_{\alpha+\beta}$. If E is a set for which $\mu * \mu(E) = 0$, (3·6·1) shows that $\sigma_\alpha * \sigma_\beta(E) = 0$ for all α, β in S. It follows that $\sigma_{\alpha+\beta}(E) = 0$ for all α, β in S. From (3·5·ii), we infer that $\sigma_\gamma(E) = 0$ for all $\gamma \in S$, and so $\mu(E)$ vanishes as well. That is, we have $\mu \ll \mu * \mu$.

If $\mu(E) = 0$, then $\sigma_\alpha(E) = 0$ for all $\alpha \in S$ and so by (3·5·ii) we have $\sigma_{\alpha+\beta}(E) = 0$ for all $\alpha, \beta \in S$. By (3·3), $\sigma_\alpha * \sigma_\beta(E)$ vanishes and so (3·6·1) yields $\mu * \mu(E) = 0$. That is, we have $\mu * \mu \ll \mu$, and the proof is complete.

(3·7) THEOREM. *The support of μ (and hence the support of $\mu * \mu$) is the entire group G.*

Proof. It is proved in (1), Proposition 3, that $\operatorname{supp}(\mu_{\Delta,\beta}) = G$ provided that

$$\beta(\chi) \neq \pm\tfrac{1}{2} \quad \text{for all} \quad \chi \in \Delta.$$

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This is the case for all of the measures σ_α and so $\operatorname{supp}(\sigma_\alpha) = G$. Since

$$\operatorname{supp}(\mu) \supset \operatorname{supp}(\sigma_\alpha),$$

the theorem is proved.

(3·8) *Proof of* (1·3) *for compact* G. If rank $(X) \leqslant 2^{\aleph_0}$, we use (2·3) to find $2^{\aleph_0}$ infinite dissociate sets $\Delta \subset X$ every pair of which has finite intersection and for which there are only a finite number of common words. If $\mathfrak{r} = \operatorname{rank}(X) > 2^{\aleph_0}$, we follow Stromberg (16), p. 1021, in finding $\mathfrak{r}$ pairwise disjoint (except for $\{1\}$) countably infinite subgroups of X, in each of which we choose a countably infinite dissociate set. Any two of these dissociate sets, say Δ_1 and Δ_2, are disjoint. We thus have a family $\{\Delta_\iota : \iota \in I\}$ of cardinal number $\max(\mathfrak{r}, 2^{\aleph_0})$ of countably infinite dissociate subsets of X, every pair of which not only have finite intersection but have only a finite number of common words.

Next, given a Δ_ι, we consider a continuum of pairwise disjoint sets S as in (3·5). For example, let H be any set of positive real numbers that form a Hamel basis for $\mathbf{R}$ over $\mathbf{Q}$. For each $h \in H$, let $S_h = \{qh : q \in \mathbf{Q}_+\}$. Plainly $S_h \cap S_{h'}$ is void if $h \neq h'$. Thus all measures σ_α with $\alpha \in S_h$ are singular to all measures $\sigma_{\alpha'}$ with $\alpha' \in S_{h'}$: see (3·2). Write μ_h for the measure constructed as in (3·6) with the set S_h of positive real numbers and the dissociate set Δ_ι, and $\mathbf{B}_\iota$ for the set of measures μ_h as h runs through H.

For $\alpha \in S_h$ and $\alpha' \in S_{h'}$, the measures σ_α and $\sigma_{\alpha'}$ are singular. It follows that $\mu_h \perp \mu_{h'}$. If we form the convolution $\mu_h * \mu_{h'}$, we get a measure that is equivalent to the measure $\mu_{h+h'}$.

Finally, changing Δ_ι to $\Delta_{\iota'}$, we get measures μ and μ' for which $(\mu * \mu')\hat{}$ has finite support, since it is contained in the set of common words for Δ and Δ'. That is, $\mu * \mu'$ is a trigonometric polynomial times Haar measure on G. As this trigonometric polynomial is everywhere positive (a fact following from the construction of the original σ_α's), $\mu * \mu'$ is equivalent to Haar measure. This completes the construction for compact G.

4. *The group* $\mathbf{R}$. We extend the results of §3 from $\mathbf{T}$ to $\mathbf{R}$ by the 'lifting' process employed in (9).

(4·1) *Definition*. Regard $\mathbf{T}$ as the interval $[-\pi, \pi[$ with addition modulo 2π. Let ρ be any measure in $\mathbf{M}(\mathbf{T})$. For Borel sets E in $\mathbf{R}$, define $\rho_1(E) = \rho(E \cap [-\pi, \pi[)$. Next let b be an everywhere positive function in $l_1(\mathbf{Z})$. Define τ_b by

$$\tau_b = \sum_{s \in \mathbf{Z}} b_s \epsilon_{2\pi s},$$

and define ρ_2 by

$$\rho_2 = \rho_1 * \tau_b. \tag{4·1·1}$$

(4·2) *Remarks*. Formula (4·1·1) defines our lifting procedure. Plainly ρ_2 belongs to $\mathbf{M}(\mathbf{R})$, and to $\mathbf{M}^+(\mathbf{R})$ if ρ belongs to $\mathbf{M}^+(\mathbf{T})$. When ρ is positive, the different versions of ρ_2 obtained by varying b in (4·1·1) are equivalent, and accordingly we fix b. It is also true that ρ_2 is in $\mathbf{M}_0(\mathbf{R})$ if ρ is in $\mathbf{M}_0(\mathbf{T})$. We will prove this in (4·6). Throughout (4·3)–(4·7), convolutions of measures denoted by Greek letters without subscripts are in $\mathbf{M}(\mathbf{T})$; convolutions of measures denoted by Greek letters with subscripts are in $\mathbf{M}(\mathbf{R})$.

(4·3) THEOREM. *Let* ρ *and* σ *be measures in* $\mathbf{M}^+(\mathbf{T})$. *Then we have*

(i) $(\rho * \sigma)_2 \sim \rho_2 * \sigma_2$.

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Proof. For $t \in \mathbf{Z}$, let $c_t = \sum_{u \in \mathbf{Z}} b_{t-u} b_u$. Note that c_t is positive for all $t \in \mathbf{Z}$ and that $c \in l_1(\mathbf{Z})$. By (4·1·1), we have

$$\rho_2 * \rho_2 = \rho_1 * \sigma_1 * \tau_c. \tag{4·3·1}$$

Thus for all Borel sets in $\mathbf{R}$, we have

$$\rho_2 * \sigma_2(E) = \sum_{t \in \mathbf{Z}} c_t \rho_1 * \sigma_1(E - 2\pi t). \tag{4·3·2}$$

Similarly, we have

$$(\rho * \sigma)_2(E) = \sum_{t \in \mathbf{Z}} b_t (\rho * \sigma)_1 (E - 2\pi t).$$

We must prove that $\rho_2 * \sigma_2(E) = 0$ if and only if $(\rho * \sigma)_2(E) = 0$. Suppose that $\rho_2 * \sigma_2(E) = 0$, and suppose also that there is an integer k such that

$$E \subset [(2k-1)\pi, (2k+1)\pi[.$$

By (4·3·2) we have

$$\rho_1 * \sigma_1(E - 2\pi t) = 0$$

for all $t \in \mathbf{Z}$. That is, we have

$$\int_{[-\pi, \pi[} \rho((E - 2\pi t - x) \cap [-\pi, \pi[)\, d\sigma(x) = 0. \tag{4·3·3}$$

To compute $(\rho * \sigma)_1(E)$, write the following for $x \in [-\pi, \pi[$:

$$\text{for} \quad -\pi \leqslant x < \pi \quad \text{let} \quad E_1(x) = (E - 2k\pi - x) \cap [-\pi, \pi[;$$

$$\text{for} \quad 0 \leqslant x < \pi \quad \text{let} \quad E_2(x) = ((E - 2k\pi - x) \cap [-2\pi, -\pi[) + 2\pi;$$

$$\text{for} \quad -\pi \leqslant x < 0 \quad \text{let} \quad E_2(x) = ((E - 2k\pi - x) \cap [\pi, 2\pi[) - 2\pi.$$

From the definition of convolution in $\mathbf{M}(\mathbf{T})$ it is clear that

$$\begin{aligned}
(\rho * \sigma)_1 (E - 2k\pi) &= \rho * \sigma((E - 2k\pi) \cap [-\pi, \pi[) \\
&= \rho * \sigma(E - 2k\pi) \\
&= \int_{[-\pi, \pi[} \rho(E_1(x) \cup E_2(x))\, d\sigma(x) \\
&= \int_{[-\pi, \pi[} \rho(E_1(x))\, d\sigma(x) + \int_{[-\pi, \pi[} \rho(E_2(x))\, d\sigma(x).
\end{aligned} \tag{4·3·4}$$

Now in (4·3·3) take $t = k$. This gives

$$\begin{aligned}
\int_{[-\pi, \pi[} \rho((E - 2\pi k - x) \cap [-\pi, \pi[)\, d\sigma(x) &= \int_{[-\pi, \pi[} \rho(E_1(x))\, d\sigma(x) \\
&= 0.
\end{aligned} \tag{4·3·5}$$

Next take $t = k-1$ in (4·3·3). The set $(E - 2\pi(k-1))$ is contained in $[\pi, 3\pi[$. For $x \in [-\pi, \pi[$, the set $(E - 2\pi(k-1) - x)$ can intersect $[-\pi, \pi[$ only for $0 \leqslant x < \pi$ and then it can intersect only $[0, \pi[$. For $0 \leqslant x < \pi$ it is obvious that

$$\begin{aligned}
(E - 2(k-1)\pi - x) \cap [0, \pi[&= (E - 2k\pi - x) \cap [-2\pi, -\pi[+ 2\pi \\
&= E_2(x).
\end{aligned}$$

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By (4·3·3), therefore, with $t = k-1$, we have

$$\int_{[0,\,\pi[} \rho(E_2(x))\,d\sigma(x) = 0. \tag{4·3·6}$$

Similarly we take $t = k+1$ in (4·3·3) and obtain

$$\int_{[-\pi,\,0[} \rho(E_2(x))\,d\sigma(x) = 0. \tag{4·3·7}$$

We omit the details. Comparing (4·3·5)–(4·3·7) with (4·3·4), we find that

$$(\rho * \sigma)_1(E - 2\pi k) = 0. \tag{4·3·8}$$

Since $E - 2\pi t$ can have non-void intersection with $[-\pi, \pi[$ only for $t = k$, (4·3·8) shows that

$$(\rho * \sigma)_2(E) = 0.$$

Next, consider an arbitrary Borel set E for which $\rho_2 * \sigma_2(E) = 0$. Write E_k for

$$E \cap [\pi(2k-1),\ \pi(2k+1)[\quad \text{for all} \quad k \in \mathbf{Z}.$$

Then we have $\rho_2 * \sigma_2(E_k) = 0$, and by the above, $(\rho * \sigma)_2(E_k) = 0$. Adding over all k, we obtain $(\rho * \sigma)_2(E) = 0$ by countable additivity. That is, we have proved that $(\rho * \sigma)_2 \ll \rho_2 * \sigma_2$.

Next let E be a Borel set in $\mathbf{R}$ such that $(\rho * \sigma)_2(E) = 0$, and suppose first that $E \subset [(2k-1)\pi, (2k+1)\pi[$ for some $k \in \mathbf{Z}$. Referring again to (4·3·4), we see that

$$0 = \int_{[-\pi,\,\pi[} \rho(E_1(x))\,d\sigma(x) = \int_{[-\pi,\,0[} \rho(E_2(x))\,d\sigma(x) = \int_{[0,\,\pi[} \rho(E_2(x))\,d\sigma(x).$$

As above, we find that

$$(\rho_1 * \sigma_1)(E - 2\pi t) = 0 \tag{4·3·9}$$

for $t = k-1$, k, and $k+1$. Since the support of $\rho_1 * \sigma_1$ is contained in $[-2\pi, 2\pi[$ and $E - 2\pi t$ can intersect $[-2\pi, 2\pi[$ only for $t = k-1$, k, and $k+1$, (4·3·9) implies that $\rho_2 * \sigma_2(E) = 0$. For general Borel sets E, argue again as above. We have thus proved that $\rho_2 * \sigma_2 \ll (\rho * \sigma)_2$ and so (i) is proved.

(4·4) THEOREM. *Let ρ and σ be measures in $\mathbf{M}^+(\mathbf{T})$ such that $\rho \ll \sigma$. Then we also have $\rho_2 \ll \sigma_2$.*

Proof. Let E be any Borel set such that $\sigma_2(E) = 0$. We have

$$\begin{aligned} 0 &= \sum_{s \in \mathbf{Z}} b_s \sigma_1(E - 2\pi s) \\ &= \sum_{s \in \mathbf{Z}} b_s \sigma((E - 2\pi s) \cap [-\pi, \pi[) \\ &= \sum_{s \in \mathbf{Z}} b_s \rho((E - 2\pi s) \cap [-\pi, \pi[) \\ &= \rho_2(E). \end{aligned}$$

(4·5) THEOREM. *Let ρ and σ be measures in $\mathbf{M}^+(\mathbf{T})$ such that $\rho \perp \sigma$. Then we also have $\rho_2 \perp \sigma_2$.*

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Proof. Let E be a subset of $[-\pi, \pi[$ such that $\rho(E) = 0$ and $\sigma([-\pi, \pi[\backslash E) = 0$. Let $F = \bigcup_{s \in \mathbf{Z}} (E + 2s\pi)$. We have

$$\begin{aligned}\rho_2(F) &= \sum_{t \in \mathbf{Z}} b_t \rho_1(\bigcup_{s \in \mathbf{Z}} (E + 2s\pi) - 2\pi t) \\ &= \sum_{t \in \mathbf{Z}} b_t \rho(E) \\ &= 0;\end{aligned}$$

similarly we show that $\sigma_2(\mathbf{R}\backslash F) = 0$.

(4·6) THEOREM. *Let ρ be a measure in $\mathbf{M}_0(\mathbf{T})$. Then ρ_2 belongs to $\mathbf{M}_0(\mathbf{R})$.*

Proof. Because $\mathbf{M}_0(\mathbf{R})$ is a closed ideal in $\mathbf{M}(\mathbf{R})$, it suffices to show that ρ_1 is in $\mathbf{M}_0(\mathbf{R})$ if ρ is in $\mathbf{M}(\mathbf{T})$. This is a well-known fact. See, for example, (14) page 38, Theorem 1.

(4·7) THEOREM. *Let ρ be a measure in $\mathbf{M}^+(\mathbf{T})$. If* supp (ρ) *is* $\mathbf{T}$, *then* supp (ρ_2) *is* $\mathbf{R}$.

Proof. Let U be an open subset of $\mathbf{R}$ such that $\rho_2(U) = 0$. That is, suppose that

$$\sum_{s \in \mathbf{Z}} b_s \rho((U - 2\pi s) \cap [-\pi, \pi[) = 0.$$

Since all b_s are positive and supp $(\rho) = \mathbf{T}$, we infer that $U \cap](2k-1)\pi, (2k+1)\pi[= \varnothing$ for all $k \in \mathbf{Z}$. This implies that $U \subset 2\pi\mathbf{Z}$, so that U is $\varnothing$. ∎

We assemble the results of this section.

(4·8) *Proof of* (1·3) *for* $G = \mathbf{R}$. By (3·8), with $G = \mathbf{T}$, we have a set $\mathbf{B}$ of measures in $\mathbf{M}(\mathbf{T})$ that satisfy all of the conditions (1·3·i)–(1·3·vi). Let $\mathbf{B}_2 = \{\mu_2 : \mu \in \mathbf{B}\}$ and $\mathbf{B}_{2,\iota} = \{\mu_2 : \mu \in \mathbf{B}_\iota\}$. For each $\mu \in \mathbf{B}$, apply (1·3·i), (4·4), and (4·3) to write

$$\mu_2 \sim (\mu * \mu)_2 \sim \mu_2 * \mu_2.$$

Thus (1·3·i) holds for $\mathbf{B}_2$. For μ and ν distinct elements of $\mathbf{B}$, apply (1·3·iv) and (4·5), obtaining

$$\mu \perp \nu \quad \text{and} \quad \mu_2 \perp \nu_2.$$

Thus (1·3·iv) holds for $\mathbf{B}_2$. The validity of (1·3·ii) and (1·3·iii) for $\mathbf{B}_2$ follows from (4·6) and (4·7) respectively.

To verify (1·3·v) for each set $\mathbf{B}_{2,\iota}$ let μ, ν, and ω be as in (1·3·v) in $\mathbf{B}_\iota$. Then by (4·4) and (4·3), we have

$$\omega_2 \sim (\mu * \nu)_2 \sim \mu_2 * \nu_2.$$

Thus (1·3·v) holds for $\mathbf{B}_{2,\iota}$.

Finally, we verify (1·3·vi). Note that $G_2 = G = \mathbf{R}$ in this case. Let $\lambda_{\mathbf{T}}$ denote Haar (i.e. normalized Lebesgue) measure on $\mathbf{T}$. It is easy to verify that $(\lambda_{\mathbf{T}})_2$ is equivalent to Lebesgue measure $\lambda_{\mathbf{R}}$ on $\mathbf{R}$. For μ and μ' as in (1·3·vi) in $\mathbf{B}_\iota$ and $\mathbf{B}_{\iota'}$, we apply (4·4) and (4·3) again to write

$$\lambda_{\mathbf{R}} \sim (\lambda_{\mathbf{T}})_2 \sim (\mu * \mu')_2 \sim \mu_2 * \mu'_2.$$

This verifies (1·3·vi), and the proof of (1·3) for $G = \mathbf{R}$ is complete.

5. *Groups with compact open subgroups.* It is simple to extend our construction to groups with compact open subgroups. The idea is the same as in going from $\mathbf{T}$ to $\mathbf{R}$, but the details are somewhat different.

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(5·1) *Notation.* Throughout this section, our group G becomes the group G_0 of (1·1) with J_0 an infinite compact open subgroup, and G_1 is a subgroup of G_0 containing J_0 such that G_1/J_0 is countably infinite. (If G_1/J_0 is finite, then G_0 is compact and we are in the case analyzed in section 3.) Our constructions will be carried out in the group G_0, although we could just as well restrict ourselves to G_1. Write H for the countably infinite group G_1/J_0, realized as the family of cosets of J_0 in G_1 with the usual multiplication. Let ϕ be the canonical mapping of G_1 onto G_1/J_0: $\phi(x)$ is the coset xJ_0 for all $x \in G_1$.

(5·2) *Definition.* Let ρ be a measure in $\mathbf{M}(J_0)$. For Borel subsets E of G_0, let $\rho_1(E) = \rho(E \cap J_0)$. Plainly ρ_1 belongs to $\mathbf{M}(G_0)$, and to $\mathbf{M}^+(G_0)$ if ρ belongs to $\mathbf{M}^+(J_0)$. Now for each $h \in H$, choose an element $x_h \in \phi^{-1}(h)$. Let W be the subgroup of G_0 generated by $\{x_h : h \in H\}$. Plainly W is countable and is a subgroup of G_1. Also the equality $WJ_0 = G_1$ is obvious Let b be an everywhere positive function in $l_1(W)$. Let τ_b be the measure defined by

$$\tau_b = \sum_{w \in W} b_w \epsilon_w. \tag{5·2·1}$$

Define ρ_2 by

$$\rho_2 = \rho_1 * \tau_b. \tag{5·2·2}$$

(5·3) *Remarks.* Plainly ρ_2 belongs to $\mathbf{M}(G_0)$, and the support of ρ_2 is contained in G_1. If $\rho \in \mathbf{M}^+(J_0)$, then ρ_2 is in $\mathbf{M}^+(G_0)$. For positive ρ, the different versions of ρ_2 obtained by varying b in (5·2·1) are equivalent. We omit the proofs of these facts. If $\rho \in \mathbf{M}_0(J_0)$, then ρ_2 is in $\mathbf{M}_0(G_0)$. We will prove this in (5·7). Throughout (5·2)–(5·9), convolutions of measures denoted by Greek letters without subscripts are in $\mathbf{M}(J_0)$; convolutions of measures denoted by Greek letters with subscripts are in $\mathbf{M}(G_0)$.

(5·4) THEOREM. *Let ρ and σ be measures in $\mathbf{M}^+(J_0)$. Then we have*

(i) $(\rho * \sigma)_2 \sim \rho_2 * \sigma_2$.

Proof. First it is clear that

$$\rho_2 * \sigma_2 = \rho_1 * \sigma_1 * \tau_b * \tau_b. \tag{5·4·1}$$

Since τ_b can be regarded as an element of the group algebra $l_1(W)$, $\tau_b * \tau_b$ is the measure τ_c, where c is the function on W defined by

$$c_w = \sum_{u \in W} b_{wu^{-1}} b_u. \tag{5·4·2}$$

Plainly c is strictly positive on W and is in $l_1(W)$.

Suppose that E is a Borel subset of G_1 that is contained in a single coset $w_1 J_0$ of J_0 with $w_1 \in W$. (All of these cosets have the form $x_h J_0$, but this fact is irrelevant to our argument.) Suppose also that $\rho_2 * \sigma_2(E) = 0$. From (5·4·1) and (5·4·2), we see that

$$0 = \rho_2 * \sigma_2(E) = \sum_{w \in W} c_w \rho_1 * \sigma_1(Ew^{-1}),$$

so that

$$\rho_1 * \sigma_1(Ew^{-1}) = 0 \quad \text{for all} \quad w \in W. \tag{5·4·3}$$

The only w in W for which $(Ew^{-1}) \cap J_0$ can be non-void is w_1, and for $w = w_1$, we have $Ew^{-1} \subset J_0$. It follows that

$$0 = \rho_1 * \sigma_1(Ew_1^{-1}) = \rho * \sigma(Ew_1^{-1}) = (\rho * \sigma)_1(Ew_1^{-1}). \tag{5·4·4}$$

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Combine (5·4·4) and the definition of ρ_2 in general to find

$$\begin{aligned}(\rho * \sigma)_2(E) &= \sum_{w \in W} b_w (\rho * \sigma)_1 (Ew^{-1}) \\ &= b_{w_1} (\rho * \sigma)_1 (Ew_1^{-1}) \\ &= 0. \end{aligned} \tag{5·4·5}$$

Any Borel set in G_0 is the union of a Borel subset of $G_0 \backslash G_1$ and a countable family of Borel sets each contained in a single $w_1 J_0$. Both $(\rho * \sigma)_2$ and $\rho_2 * \sigma_2$ vanish on $G_0 \backslash G_1$, and for the others, we have (5·4·5). It follows that $(\rho * \sigma)_2 \ll \rho_2 * \sigma_2$.

Now let E be a Borel subset of some $w_1 J_0$ such that $(\rho * \sigma)_2(E) = 0$. The set $Ew^{-1} \cap J_0$ can be nonvoid only for $w = w_1$. Thus we have

$$\begin{aligned} 0 = (\rho * \sigma)_2(E) &= b_{w_1} (\rho * \sigma)_1 (Ew_1^{-1}) \\ &= b_{w_1} (\rho * \sigma)(Ew_1^{-1}) \\ &= b_{w_1} \rho_1 * \sigma_1 (Ew_1^{-1}). \end{aligned}$$

It follows that

$$\rho_2 * \sigma_2(E) = c_{w_1} \rho_1 * \sigma_1 (Ew_1^{-1}) = 0.$$

As above we infer that $\rho_2 * \sigma_2 \ll (\rho * \sigma)_2$, and so (i) is proved.

(5·5) THEOREM. *If ρ and σ are measures in $\mathbf{M}^+(J_0)$ such that $\rho \ll \sigma$, then $\rho_2 \ll \sigma_2$.*

Proof. We may limit ourselves to Borel sets E such that $E \subset w_1 J_0$ for some $w_1 \in W$. If $\sigma_2(E) = 0$, we have

$$0 = \sum_{w \in W} b_w \sigma_1 (Ew^{-1}) = b_{w_1} \sigma (Ew_1^{-1}),$$

and so $\rho(Ew_1^{-1}) = 0$. Therefore $\rho_2(E)$ is also zero.

(5·6) THEOREM. *If ρ and σ are measures in $\mathbf{M}^+(J_0)$ such that $\rho \perp \sigma$, then $\rho_2 \perp \sigma_2$.*

We omit the proof.

(5·7) THEOREM. *Let ρ be a measure in $\mathbf{M}_0(J_0)$. Then the measure ρ_2 is in $\mathbf{M}_0(G_0)$.*

Proof. Let X_0 denote the character group of G_0, and $\mathsf{A}(\mathsf{X}_0, J_0) = \mathsf{A}$ as usual the annihilator of J_0 in X_0. Let Y_0 be the character group of J_0. It is known that A is compact and open, since J_0 is closed and compact. (See, for example, (7), (23·24).) It is also known that Y_0 can be identified with the [discrete] quotient group X_0/A, the general character of J_0 being any restriction to J_0 of a character in $\chi\mathsf{A}$ for χ in X_0 (see (7), (24·11)). Let ϵ be a positive real number. For $\rho \in \mathbf{M}_0(J_0)$, we have $|\rho\hat{}(\chi\mathsf{A})| < \epsilon$ for all $\chi\mathsf{A}$ outside of a finite subset $\{\chi_1\mathsf{A}, \chi_2\mathsf{A}, \ldots, \chi_m\mathsf{A}\}$ of X_0/A. For the measure ρ_1 in $\mathbf{M}(G_0)$, we have

$$\rho_1\hat{}(\chi) = \rho\hat{}(\chi\mathsf{A}) \quad \text{for all} \quad \chi \in \mathsf{X}_0,$$

as one can easily see. Hence $|\rho_1\hat{}(\chi)| < \epsilon$ for all $\chi \notin \bigcup_{j=1}^{m} \chi_j\mathsf{A}$. Since $\bigcup_{j=1}^{m} \chi_j\mathsf{A}$ is compact, ρ_1 is in $\mathbf{M}_0(G_0)$. Finally

$$\rho_2\hat{}(\chi) = \sum_{w \in W} b_w \bar{\chi}(w) \rho_1\hat{}(\chi)$$

for all $\chi \in \mathsf{X}_0$, so that

$$|\rho_2\hat{}(\chi)| \leqslant |\rho_1\hat{}(\chi)| \sum_{w \in W} b_w.$$

That is, ρ_2 belongs to $\mathbf{M}_0(G_0)$.

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(5·8) THEOREM. *Let ρ be a measure in $\mathbf{M}^+(J_0)$ such that* $\operatorname{supp}(\rho) = J_0$. *Then we have* $\operatorname{supp}(\rho_2) = G_1$.

Proof. If U is an open subset of G_1 such that $\rho_2(U) = 0$, then we have

$$\sum_{w \in W} b_w \rho((Uw^{-1}) \cap J_0) = 0,$$

and so $(Uw^{-1}) \cap J_0 = \varnothing$ for all $w \in W$. This implies that $U \cap (wJ_0) = \varnothing$ for all w and so

$$\begin{aligned} \varnothing &= \bigcup_{w \in W} U \cap (wJ_0) = U \cap \Big(\bigcup_{w \in W} wJ_0\Big) \\ &= U \cap G_1 \\ &= U. \end{aligned}$$

(5·9) *Proof of* (1·3) *for* G_0. We take the set of measures $\mathbf{B} = \bigcup_{\iota \in I} \mathbf{B}_\iota$ on the compact group J_0, whose existence was proved in §3. The set $\mathbf{B}_{2,\iota}$ is defined as $\{\mu_2 : \mu \in \mathbf{B}_\iota\}$ and $\mathbf{B}_2$ as $\bigcup_{\iota \in I} \mathbf{B}_{2,\iota}$. Verification of (1·3·i)–(1·3·vi) proceeds as in (4·8). We omit the details. The only needed change is the observation that λ_{G_1} is equivalent to $\lambda_{J_0} * \tau_b$.

6. *The case of general* G. We follow (9) in producing our set of measures $\bigcup_{\iota \in I} \mathbf{B}_\iota$ for arbitrary G.

(6·1) *Proof of* (1·3). As noted in (1·1), since G is non-discrete, either a is positive or J_0 is infinite. If $a > 0$ and $\operatorname{rank}(\Upsilon_0) \leqslant 2^{\aleph_0}$, we carry out most of the construction on the first coordinate group of $\mathbf{R}^a$. Let $\mathbf{B}_1 = \bigcup_{\iota \in I} \mathbf{B}_{1,\iota}$ be a family of measures as in (1·3) for the group $\mathbf{R}$. Note that $\operatorname{card}(\mathbf{B}_\iota) = \operatorname{card}(I) = 2^{\aleph_0}$. Let γ be the measure on $\mathbf{R}$ such that $d\gamma(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)\,dx$. Let δ_0 be a measure on G_0 defined as follows. If G_0 is compact, δ_0 is normalized Haar measure λ_{G_0} on G_0. If G_0 is non-compact, δ_0 is $\lambda_{J_0} * \tau_b$ with τ_b defined as in (5·2). Now write G specifically as

$$\mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R} \times G_0 \tag{6·1·1}$$

with a copies of $\mathbf{R}$. For each $\mu_1 \in \mathbf{B}_1$, let μ be defined as a product measure:

$$\mu = \mu_1 \times \gamma \times \ldots \times \gamma \times \delta_0 \tag{6·1·2}$$

with $a - 1$ factors γ appearing. It is easy to check that

$$\mu * \mu = (\mu_1 * \mu_1) \times (\gamma * \gamma) \times \ldots \times (\gamma * \gamma) \times (\delta_0 * \delta_0)$$

and that

$$\mu^\wedge = \mu_1^\wedge \gamma^\wedge \ldots \gamma^\wedge \delta_0^\wedge$$

under the usual interpretation of X as $\mathbf{R}^a \times \mathsf{X}_0$. Now apply (8), p. 394, theorem (21·29). This theorem states that if one has two product measures $\rho = \rho_1 \times \ldots \times \rho_m$ and $\sigma = \sigma_1 \times \ldots \times \sigma_m$, and if all ρ_j are equivalent to σ_j, then $\rho \sim \sigma$. If any one of the ρ_j's is singular to the corresponding σ_j, then $\rho \perp \sigma$.

It follows that the set of all measures μ as μ_1 runs through $\mathbf{B}_{1,\iota}$ is a set of measures $\mathbf{B}_\iota$ with all of the properties attributed to $\mathbf{B}_\iota$ in (1·3). We also take $\mathbf{B} = \bigcup_{\iota \in I} \mathbf{B}_\iota$. It is easy to check all of (1·3·i)–(1·3·vi) for the measures in $\mathbf{B}_\iota$ and $\mathbf{B}$. We need only note that γ and $\gamma * \gamma$ are equivalent to $\lambda_{\mathbf{R}}$ and that δ_0 and $\delta_0 * \delta_0$ are equivalent to λ_{G_1}.

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Finally, suppose that either a is zero or that rank $(\mathsf{Y}_0) > 2^{\aleph_0}$. Here we carry out the construction on the G_0 factor, in which the group J_0 is necessarily infinite. Let

$$\mathbf{B}_0 = \bigcup_{\iota \in I} \mathbf{B}_{0,\iota}$$

be a set of measures on G_0 constructed as in §5. For each μ_0 in $\mathbf{B}_0$, let μ be a product measure:

$$\mu = \gamma \times \gamma \times \ldots \times \gamma \times \mu_0 \quad (a \text{ copies of } \gamma).$$

Let $\mathbf{B}_\iota = \{\mu : \mu_0 \in \mathbf{B}_{0,\iota}\}$ and $\mathbf{B} = \bigcup_{\iota \in I} \mathbf{B}_\iota$. All of (1·3) is true for $\mathbf{B}$ and the $\mathbf{B}_\iota$; the details are as in the preceding paragraph.

(6·2) *Remarks.* The main theorem is proved but it remains to check the variant (1·4). This will be done in section 7; as a preliminary we abstract the common element of the procedures of sections 4–6. In the following statement we use the notation of (1·1) and may choose K to be either J_0 or $\mathbf{T}$ as appropriate.

(6·3) THEOREM. *There exist an infinite compact group K and an affine-linear norm-preserving norm-continuous map $\theta : \mathbf{M}^+(K) \to \mathbf{M}^+(G)$ which satisfy the following conditions*:

(i) $\theta(\sigma) * \theta(\tau) \sim \theta(\sigma * \tau)$;
(ii) $\sigma \sim \tau$ *implies* $\theta(\sigma) \sim \theta(\tau)$;
(iii) $\sigma \perp \tau$ *implies* $\theta(\sigma) \perp \theta(\tau)$;
(iv) $|\theta(\sigma) - \theta(\tau)| = \theta(|\sigma - \tau|)$;
(v) $\sigma \in \mathbf{M}_0(K)$ *implies* $\theta(\sigma) \in \mathbf{M}_0(G)$;
(vi) *if* σ *has support* K, *then* $\theta(\sigma)$ *has support* G_2;
(vii) *the image under* θ *of Haar measure on* K *is equivalent to Haar measure on* G_2.

Proof. We construct θ in a special way with an eye towards use in section 7. Suppose first that a is positive and that rank $(\mathsf{Y}_0) \leqslant 2^{\aleph_0}$. In this case, we take K to be $\mathbf{T}$. For $\sigma \in \mathbf{M}^+(\mathbf{T})$, lift σ to $\sigma_2 \in \mathbf{M}^+(\mathbf{R})$ by the procedure of section 4. Let $\theta(\sigma)$ be the product measure

$$\sigma_2 \times \gamma \times \ldots \times \gamma \times \delta_0$$

as in (6·1·2).

Suppose next that $a = 0$ or that rank $(\mathsf{Y}_0) > 2^{\aleph_0}$, so that J_0 is an infinite compact group. Let K be J_0. If $J_0 = G_0$, let $\sigma_0 = \sigma$. If $J_0 \neq G_0$, let σ_0 be obtained by lifting from J_0 to G_0 as in §5. Let $\theta(\sigma)$ be the product measure with last factor σ_0 and all others γ.

Thus the mapping θ is defined in all cases. All assertions about θ except for (iv) and the claim that θ is continuous are either obvious or have previously been checked. To establish (iv), let the Hahn decomposition of $\sigma - \tau$ be given by

$$\sigma - \tau = \rho_+ - \rho_-, \quad |\sigma - \tau| = \rho_+ + \rho_-. \tag{6·3·1}$$

The equality $\sigma + \rho_- = \tau + \rho_+$, together with the additivity of θ, gives

$$\theta(\sigma) - \theta(\tau) = \theta(\rho_+) - \theta(\rho_-). \tag{6·3·2}$$

Because $\rho_+ \perp \rho_-$ we see by means of (iii) that $\theta(\rho_+) \perp \theta(\rho_-)$ and that (6·3·2) corresponds to the Hahn decomposition of $\theta(\sigma) - \theta(\tau)$. Accordingly we may write

$$|\theta(\sigma) - \theta(\tau)| = \theta(\rho_+) + \theta(\rho_-) = \theta(\rho_+ + \rho_-). \tag{6·3·3}$$

The combination of (6·3·3) and (6·3·1) proves (iv).

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Taking the norm of each side of (iv), we find that

$$\|\theta(\sigma)-\theta(\tau)\| = \|\sigma-\tau\|,$$

and continuity of θ is an immediate consequence.

7. *The variant of the main theorem.* We verify (1·4) by changing the construction in a simple way. In fact let $(r_n)_{n=1}$ be a sequence of positive real numbers which belongs to some but *not all* l_p for $p > 0$. (One suitable choice is $r_n = n^{-1}$.) Since $\lim_{n\to\infty} r_n = 0$, the preceding constructions make sense although the resulting measures are not purely singular. It turns out that their singular parts (after renormalization) have the required properties.

(7·1) *Proof of* (1·4). We fix a sequence $(r_n)_{n=1}^{\infty}$ of the type just described and note that $\alpha_0 = \inf\{\alpha > 0: (r_n^{\alpha}) \in l_p\}$ must be finite and positive. It follows that the interval $]0, \infty[$ may be expressed as the union $I_l \cup I_r$ of disjoint non-void intervals such that $(r_n^{\alpha}) \in l_2$ if and only if $\alpha \in I_r$. Now consider an arbitrary set S as in (3·5). It is easy to see that $\inf S = 0$ and hence that S is dense in $]0, \infty[$. Now write $S_l = S \cap I_l$ and $S_r = S \cap I_r$, and choose a strictly positive function (b_α) on S_l such that $\sum_{\alpha \in S_l} b_\alpha = 1$.

Now choose K and θ as in (6·3). For an infinite dissociate subset Δ of the dual of K, define σ_α as in (3·1). The set $\mathbf{B}$ consists of the measures

$$\mu = \sum_{\alpha \in S_l} b_\alpha \theta(\sigma_\alpha) = \theta\Big(\sum_{\alpha \in S_l} b_\alpha \sigma_\alpha\Big) \tag{7·1·1}$$

as Δ, S vary as in the proof of (1·3). (See (3·8)).

An application of (2·2) shows that $\mu \in \mathbf{M}_s(G)$ and the fact that (ii), (iii), (iv), (vi) of (1·4) are satisfied is clear in the light of earlier arguments. It is also clear from (2·2) and (vii) of (6·3) that

$$\theta(\sigma_\alpha) \sim \lambda_2 \quad \text{for all} \quad \alpha \quad \text{in} \quad S_r. \tag{7·1·2}$$

Because $(S_l + S_l) \cap S_r \neq \varnothing$ it follows from (7·1·2) that $\mu * \mu$ has a non-zero absolutely continuous component, and hence that

$$\mu * \mu \nmid \mu. \tag{7·1·3}$$

Using (7·1·1) and (7·1·2) again we see that

$$\mu + \lambda_2 \sim \sum_{\alpha \in S_l} b_\alpha \theta(\sigma_\alpha) + \sum_{\alpha \in S_l} c_\alpha \theta(\sigma_\alpha) \sim \sum_{\alpha \in S} d_\alpha \theta(\sigma_\alpha), \tag{7·1·4}$$

whenever $\sum_{\alpha \in S_l} c_\alpha < \infty$, $\sum_{\alpha \in S} d_\alpha < \infty$, and the c_α's, d_α's are strictly positive. The remaining assertions of (i)′, (v)′ follow as in (3·6), (3·8).

8. *Application to dynamical systems*

(8·1) *Terminology and background.* We confine our attention to the standard example of a normal dynamical system as defined for example in (15), pp. 37–38. Let X be the product space $\underset{n \in \mathbf{Z}}{P} \mathbf{R}_{(n)}$, each factor $\mathbf{R}_{(n)}$ being the real line. Let γ be any nonzero positive–definite function on $\mathbf{Z}$. For $\{n_1, \ldots, n_s\} \subset \mathbf{Z}$, let Γ be the matrix $(\gamma(n_j - n_k))_{j,k=1}^{s}$ regarded also as a linear operator on $\mathbf{R}^s$. For Borel subsets Y of $\mathbf{R}^s$, define

$$\sigma(\{x \in X: (x_{n_1}, \ldots, x_{n_s}) \in Y\})$$

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as

$$\det(\Gamma)^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}s}\int_Y \exp(-\tfrac{1}{2}(\Gamma x, x))\,d\lambda(x),$$

λ being here Lebesgue measure on $\mathbf{R}^s$. There is a unique extension of the set function σ to a countably additive non-negative measure on the Borel sets $\mathscr{B}$ of X. We again denote this measure by σ. The shift operator T on X is defined by $T(x)_n = x_{n+1}$ for all $x \in X$ and $n \in Z$. The quadruple $(X, \mathscr{B}, T, \sigma)$ is called a *normal dynamical system*.

By Herglotz's theorem, there is a unique measure $\mu \in \mathbf{M}^+(\mathbf{T})$ such that $\mu\hat{}(n) = \gamma(n)$ for all $n \in Z$. Conversely, every non-zero measure $\mu \in \mathbf{M}^+(\mathbf{T})$ defines a positive-definite function $\mu\hat{}$ and thereby a normal dynamical system. The measure μ is called the *spectral measure* of the system $\mathfrak{S} = (X, \mathscr{B}, T, \sigma)$. Systems $\mathfrak{S}, \mathfrak{S}'$ with continuous spectral measures μ, μ' are isomorphic if and only if $\mu \sim \mu'$. See (15), p. 43.

For a dynamical system $\mathfrak{S} = (X, \mathscr{B}, T, \sigma)$, let $L_2'(X, \sigma)$ be the orthogonal complement in $L_2(X, \sigma)$ of the function identically 1. Let U be the unitary operator on $L_2'(X, \sigma)$ defined by $Uf(x) = f(Tx)$. Since $L_2'(X, \sigma)$ is separable, it admits an ordered representation relative to the unitary operator U. See (3), definition X.5.9 (p. 916). There are a measure $\nu \in \mathbf{M}^+(\mathbf{T})$ and a descending sequence $(B_n)_{n=1}$ of Borel subsets of $\mathbf{T}$ that define this representation. Let ν_n be the restriction of ν to B_n. Consider the Hilbert space $L_2(B_n, \nu_n)$ and on it the unitary operator that is multiplication by the function $t \to \exp(it)$: $V_n f(t) = \exp(it) f(t)$. The ordered representation theorem (also called the spectral multiplicity theorem) asserts that U is unitarily equivalent to the direct sum $\bigoplus_{n=1}^{\infty} V_n$ on the direct sum of Hilbert spaces $\bigoplus_{n=1}^{\infty} L_2(B_n, \nu_n)$. The class of all measures equivalent to ν_1 is called the *maximal spectral type of* U. The sequence of measures $(\nu_n)_{n=1}^{\infty}$ (each up to equivalence) is called *the spectral type of* U. It is customary to speak also of the maximal spectral type of $\mathfrak{S}$ and the spectral type of $\mathfrak{S}$.

The constructions of sections 1–7 yield dynamical systems with curious spectral types. Fomin ((5), pp. 44–45) stated that if μ is equivalent to $\mu * \mu$, then the normal dynamical system with spectral measure μ has maximal spectral type represented by μ. He also showed that if $\mu \in \mathbf{M}_0(\mathbf{T})$, then the corresponding dynamical system enjoys the strong mixing property. Makarov (13) showed by his existence theorem that singular measures μ of this kind exist. Girsanov (6) announced that the maximal spectral type of the system with spectral measure μ is $\exp(\mu) - \epsilon_0$ for arbitrary μ. A proof of this fact, using a decomposition of $L_2(X, \sigma)$ due to K. Itô (10), appears in Sinaĭ (15), pp. 37–43. For $\mu \in \mathbf{M}^+(\mathbf{T})$, we have $\exp(\mu) - \epsilon_0 \sim \mu$ if and only if $\mu * \mu \sim \mu$. It follows that $\mu \sim \nu_1$ if and only if $\mu \sim \mu * \mu$. Accordingly we say that the normal dynamical system $\mathfrak{S}$ is *determined by its maximal spectral type* if the spectral measure of $\mathfrak{S}$ is equivalent to its own convolution square, in which case the spectral measure represents the maximal spectral type of $\mathfrak{S}$.

Any dynamical system $\mathfrak{S}$, whose maximal spectral type is represented by ν_1, is said to have *pure Lebesgue spectrum* if $\nu_1 \ll \lambda$ or *purely singular spectrum* if $\nu_1 \perp \lambda$ (λ being Haar measure on $\mathbf{T}$). If neither of these properties holds then $\mathfrak{S}$ has *impure spectrum*. In the particular case where $\mathfrak{S}$ is determined by its maximal spectral type there is (up to isomorphism) only one example of a normal dynamical system with pure

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Lebesgue spectrum and this system has strong mixing in consequence of the Riemann–Lebesgue lemma. (In fact we must have $\mu \sim \nu_1 \sim \lambda$. To see this note that the assumptions $\mu \in \mathbf{M}^+(\mathbf{T})$, $\mu \ll \lambda$ imply that the Radon–Nikodým derivative of $\mu * \mu$ can be chosen continuous and non-negative. Because μ is non-zero this demonstrates the existence of some non-void open subset U of $\mathbf{T}$ such that $(\mu * \mu)|_U \sim \lambda|_U$. The remaining hypothesis that $\mu \sim \mu * \mu$ now leads quickly to the equivalence of μ and λ.) The existence of a strong mixing $\mathfrak{S}$ with $\mu \sim \nu_1$ and purely singular spectrum is Makarov's result. If μ is the spectral measure of such a system then the (strong mixing) system $\mathfrak{S}'$ whose spectral measure is $\mu + \lambda$ is determined by its maximal spectral type and has impure spectrum. However, spectral impurity need not arise from such simple conjunction of pure types. Suppose, in fact, that $\mathfrak{S}$ is a dynamical system determined by its maximal spectral type which has the further property that the singular component of the spectral measure of $\mathfrak{S}$ is *not* equivalent to its own convolution square. Then $\mathfrak{S}$ has impure spectrum but cannot be obtained in the same manner as $\mathfrak{S}'$. We say that such a system has *essentially impure spectrum.*

We interpret a special case of the results of sections 3, 7 in terms of dynamical systems by defining an explicit map from the set of all positive sequences in $\mathbf{c}_0$ to normal dynamical systems. All systems in the image set have strong mixing and are determined by their maximal spectral types. Moreover this set contains uncountably many non-isomorphic systems with purely singular spectrum and uncountably many non-isomorphic systems with essentially impure spectrum. So far as we are aware essential impurity has not previously been exhibited.

(8·2) *Definition.* We define a map $r \to \mathfrak{S}(r)$ from the set of all positive sequences in $\mathbf{c}_0$ to normal dynamical systems by specifying a positive-definite function R on $\mathbf{Z}$ corresponding to r and demanding that the Fourier–Stieltjes transform of the spectral measure of $\mathfrak{S}(r)$ should be R.

In fact fix an integer $a \geqslant 3$. Let $\mathbf{Q}_+$ denote the set of positive rational numbers. Consider a strictly positive function $(b_q)_{q \in \mathbf{Q}_+}$ such that $\sum_{b \in \mathbf{Q}_+} b_q = 1$. Given r and $q \in \mathbf{Q}_+$, let $k(r, q)$ be the least positive integer such that $r_k^q < \frac{1}{2}$ for all $k \geqslant k(r, q)$. Now define, for m in $\mathbf{N}$, q in $\mathbf{Q}_+$,

$$r(m, q) = \begin{cases} 0 & \text{if} \quad m \leqslant k(r, q), \\ r_m^q & \text{if} \quad m > k(r, q). \end{cases}$$

We now define R as the following function on $\mathbf{Z}$. For every sequence

$$m_1 < m_2 < \ldots < m_s$$

of positive integers, let

$$R(\pm a^{m_1} \pm a^{m_2} \pm \ldots \pm a^{m_s}) = \sum_{q \in \mathbf{Q}_+} b_q \left(\sum_{j=1}^{s} r(m_j, q) \right).$$

Let $R(0) = 1$ and $R(n) = 0$ for all non-zero integers not of the form

$$\pm a^{m_1} \pm a^{m_2} \pm \ldots \pm a^{m_s}.$$

An integer has at most one representation of the form

$$\pm a^{m_1} \pm a^{m_2} \pm \ldots \pm a^{m_s},$$

and so R is well defined.

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(8·3) THEOREM. *For each r, $\mathfrak{S}(r)$ is a normal dynamical system which is determined by its maximal spectral type and which satisfies strong mixing. There are three spectral classes*:

(i) *$\mathfrak{S}(r)$ has pure Lebesgue spectrum if and only if $r \in \bigcap_{p>0} l_p$;*

(ii) *$\mathfrak{S}(r)$ has essentially impure spectrum if and only if $r \in (\bigcup_{p>0} l_p) \setminus (\bigcap_{p>0} l_p)$;*

(iii) *$\mathfrak{S}(r)$ has purely singular spectrum if and only if $r \in \mathbf{c}_0 \setminus (\bigcup_{p>0} l_p)$.*

Moreover in cases (ii) *and* (iii), *the systems $\mathfrak{S}(tr)$, $\mathfrak{S}(t'r)$ are non-isomorphic whenever t, t' are distinct positive real numbers.*

Proof. Let r be a positive sequence in $\mathbf{c}_0$ and q a positive rational number. In the notation of (2·1), let σ_q denote the Riesz product based on the dissociate set

$$\{a^m : m = 1, 2, \ldots\}$$

(regarded as a subset of the dual group of $\mathbf{T}$) and on the function $\beta^{(q)}$ which takes the value $r(m, q)$ at a^m. Comparison of (2·1) and (8·2) shows that the spectral measure μ of $\mathfrak{S}(r)$ can be written in the form

$$\mu = \sum_{q \in \mathbf{Q}_+} b_q \sigma_q. \tag{8·3·1}$$

Because $r \in \mathbf{c}_0$ it follows, as in (3·4), that every σ_q is in $\mathbf{M}_0(\mathbf{T})$ and hence that $\mathfrak{S}(r)$ enjoys strong mixing. For $q, q' \in \mathbf{Q}_+$, we have $\sigma_q * \sigma_{q'} \sim \sigma_{q+q'}$ as in (3·3), and this leads to the conclusion that $\mu * \mu \sim \mu$.

The remaining assertions can be obtained by successive applications of (2·2). In fact the discussion of the classification into cases (i), (ii), (iii) is facilitated by consideration of the following simple corollaries of (2·2):

$$\sigma_q \perp \lambda \quad \text{if and only if} \quad \beta^{(q)} \notin l_2; \tag{8·3·2}$$

$$\sigma_q \ll \lambda \quad \text{if and only if} \quad \beta^{(q)} \in l_2. \tag{8·3·3}$$

(Here λ is Haar measure on $\mathbf{T}$.) Statements (i) and (iii) now follow from (8·2·1), (8·2·2), (8·2·3) and the equivalences $\beta^{(q)} \in l_2$ for all q in $\mathbf{Q}_+$ if and only if $r \in \bigcap_{p>0} l_p$, $\beta^q \notin l_2$ for all q in $\mathbf{Q}_+$ if and only if $r \notin l_p$, for all $p > 0$. Statement (ii) is a little more delicate, but is obtained as in (7·1). The relevant observation is that if r belongs to some but not all l_p, there exist non-void intervals I_l, I_r whose union is $]0, \infty[$ and such that $\sigma_q \perp \lambda$ for $q \in I_l$, $\sigma_q \ll \lambda$ for q in I_r. The essential nature of the spectral impurity corresponds to the fact that there exist q, q' in I_l such that $q + q' \in I_r$.

Suppose finally that $0 < t' < t$. To show that $\mathfrak{S}(tr)$ is not isomorphic with $\mathfrak{S}(t'r)$, it suffices to find some σ_q in the expansion (8·3·1) corresponding to the spectral measure of $\mathfrak{S}(tr)$ which is singular to every σ_q appearing in the expansion of the spectral measure of $\mathfrak{S}(t'r)$. By means of (2·2), we translate this as the following requirement: there exists q in $\mathbf{Q}_+$, such that for all q' in $\mathbf{Q}_+$,

$$\sum_{n=1}^{\infty} ((tr_n)^q - (t'r_n)^{q'})^2 = \infty. \tag{8·3·4}$$

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The hypothesis that $r \notin \bigcap_{p>0} l_p$ ensures that we may choose q in $\mathbf{Q}_+$ such that $\sum_{n=1}^{\infty} r_n^{2q} = \infty$. We complete the proof by showing that (8·3·4) holds for this choice of q and all q' in $\mathbf{Q}_+$. If $q' > q$, then $(t'r_n)^{q'}(tr_n)^{-q}$ tends to zero as $n \to \infty$ and the tail of the sum in (8·3·4) may be compared with a multiple of the tail of Σr_n^{2q}. If $q' < q$, then $\Sigma r_n^{2q'}$ diverges and the tail of the series in (8·3·4) may be compared with this. If $q' = q$ then the nth term of the series in (8·3·4) is $(t^q - (t')^q)^2 r_n^{2q}$ and the required result follows.

(8·4) *Definition.* The operator U has *homogeneous spectrum* if the spectral type of U can be represented by $(\nu_n)_{n=1}^{\infty}$ with $\nu_n = \nu_1$ for all n. The dynamical system $\mathfrak{S}$ has *homogeneous spectrum* if the associated unitary operator U does.

(8·5) *Remark.* We need the result of the next theorem to be able to say that all the dynamical systems considered in (8·3) have homogeneous spectrum. It is presumably a well-known fact of the spectral multiplicity theory of normal operators but we have been unable to find a proof in the literature. The application to normal dynamical systems is stated without proof in (5), p. 45.

(8·6) THEOREM. *Let H be a separable Hilbert space and U a unitary operator on H. Suppose that U is unitarily equivalent to $\bigoplus_{k=1}^{\infty} U_k$ on a direct sum decomposition $\bigoplus_{k=1}^{\infty} H_k$ of H and that each U_k has the same maximal spectral type. Then U has homogeneous spectrum. In particular, every normal dynamical system which is determined by its maximal spectral type has homogeneous spectrum.*

Proof. It is convenient to use the symbol $\sim$ to denote both unitary equivalence of operators and equivalence of measures. Also for each μ in $\mathbf{M}^+(\mathbf{T})$, we denote the operator on $L_2(\mathbf{T}, \mu)$ determined by multiplication by $\exp(it)$, by $U[\mu]$.

Let ν represent the maximal spectral type of each of the operators U_k. Let the spectral type of U_k be represented by $(\nu_{k,n})_{n=1}^{\infty}$ for each k. As noted in (3), corollary X.5.5, p. 912, the measure ν may be taken finite. It is necessarily non-zero, as U is not the zero operator.

By our hypothesis, we may take $\nu = \nu_{k,1}$ for $k = 1, 2, 3, \ldots$. Each sequence $(\nu_{k,n})_{n=1}^{\infty}$ is defined by a descending sequence $(B_{k,n})_{n=1}^{\infty}$ of Borel subsets of $\mathbf{T}$ such that all $B_{k,1}$ are the same set, say B, and $\nu_{k,n}$ is ν restricted to $B_{k,n}$ for all k and n. Write

$$D_{k,n} = B_{k,n} \backslash B_{k,n+1}$$

for all positive integers k and n. Plainly the family $\{D_{k,n}\}_{n=1}^{\infty}$ is pairwise disjoint for each k, and $\bigcup_{n=1}^{\infty} D_{k,n} = B$. Some sets $D_{k,1}, D_{k,2}, \ldots$ may be void. In any case the non-void sets in $\{D_{k,n}\}_{n=1}^{\infty}$ form a countable (perhaps finite) dissection of B into Borel sets. It is easy to see that there is a unique coarsest dissection of B that is a refinement of each $\{D_{k,n}\}_{n=1}^{\infty}$. This dissection is countable. Removing all sets of ν-measure zero from this dissection, we write $B_0 = \bigcup_{j \in J} C_j$, where J is either $\{1, 2, \ldots, m\}$ for a positive integer m or is $\{1, 2, 3, \ldots\}$, each C_j has positive ν-measure, the C_j's are pairwise disjoint, and B_0 differs from B by a set of ν-measure zero.

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We write $\nu^{(j)}$ for the measure ν restricted to C_j. We may now write

$$U \sim \bigoplus_{k=1}^{\infty} U_k \sim \bigoplus_{k=1}^{\infty} \left(\bigoplus_{n=1}^{\infty} U[\nu_{k,n}] \right) \sim \bigoplus_{k=1}^{\infty} \left(\bigoplus_{n=1}^{\infty} \left(\bigoplus_{j \in L(k,n)} U[\nu^{(j)}] \right) \right), \tag{8·6·1}$$

where $L(k,n)$ is the set of all $j \in J$ such that $C_j \subset B_{k,n}$. Every j in J appears in $\bigcup_{n=1}^{\infty} L(k,n)$ for every k, since $\{D_{k,n}\}_{n=1}^{\infty}$ is a dissection of B_0 (up to a set of ν-measure zero). The last direct sum in (8·6·1) can be accordingly rearranged as

$$\bigoplus_{j \in J} \left(\bigoplus U[\nu^{(j)}] \right), \tag{8·6·2}$$

the inner direct sum being countably infinite. Reversing the order of the two direct sums in (8·6·2), we get

$$U \sim \bigoplus \left(\bigoplus_{j \in J} U[\nu^{(j)}] \right), \tag{8·6·3}$$

the outer direct sum being countably infinite. Since

$$\bigoplus_{j \in J} U[\nu^{(j)}] \sim U[\nu],$$

(8·6·3) shows that U is unitarily equivalent to the direct sum of a countably infinite number of operators $U[\nu]$. It follows that H has an ordered representation relative to U, $H \sim \bigoplus_{n=1}^{\infty} L_2(B_{(n)}, \nu)$, each $B_{(n)}$ being equal to B. The uniqueness theorem for such representations ((3), theorem X.5.10, pp. 916–917) shows that U has homogeneous spectrum. (Note too that the condition that U be unitary can be replaced by the condition that U be normal.)

For the application to dynamical systems we make use of the Itô decomposition described in (15), pp. 41–42. Suppose that $\mathfrak{S}$ is a normal dynamical system with spectral measure μ and let U be the associated unitary operator on $L_2(X, \sigma)$. According to Itô's decomposition we may write $U \sim \bigoplus_{k=1}^{\infty} W_k$, in such a way that the maximal spectral type of W_k is the measure $\mu^k (= \mu * \mu * \ldots * \mu$ k times). Under the added hypothesis that $\mathfrak{S}$ is determined by its spectral measure, we know that $\mu^k \sim \mu$ for $k = 1, 2, \ldots$. The first part of the proof implies that U and therefore $\mathfrak{S}$ has homogeneous spectrum.

(8·7) *Problem.* The last theorem can scarcely be regarded as adding to our knowledge of normal dynamical systems beyond what was obtained in (8·3), although it is of some interest from the point of view of general dynamical systems to know that spectral homogeneity can occur side by side with singularity or essential impurity. It is, however, possible to ask whether a more stringent homogeneity condition is satisfied for normal dynamical systems $\mathfrak{S}$ of the kind discussed in (8·3). The question is whether each operator W_k appearing in the Itô decomposition has homogeneous spectrum. (Since W_1 has simple spectrum, the problem arises only for $k \geqslant 2$.) In the case where $\mathfrak{S}$ has pure Lebesgue spectrum, we are able to answer in the affirmative. In no other case can we make any significant progress.

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