Math. Proc. Camb. Phil. Soc. (1991), **110**, 353
Printed in Great Britain

On homogeneous chaos

BY NIGEL CUTLAND AND SIU-AH NG
Department of Pure Mathematics, University of Hull, Hull HU6 7RX

(*Received* 4 *December* 1990)

Abstract

This paper discusses the Wiener–Itô chaos decomposition of an L^2 function ϕ over Wiener space, and is concerned in particular with the identification of the integrands f_n in the chaos decomposition

$$\phi(b) = \phi_0 + \sum_{n=1}^{\infty} \int f_n \, db^{(n)}.$$

First these are identified as Radon–Nikodým derivatives. Two elementary non-standard proofs of the Wiener–Itô chaos decomposition are given, based on Anderson's construction of Brownian motion and Itô integration. The non-standard framework then gives rigorous sense to Wiener's intuitive recipe

$$f_n(t_1, \ldots, t_n) = E(\phi(b)\, \dot{b}_{t_1} \dot{b}_{t_2} \ldots \dot{b}_{t_n})$$

where

$$\dot{b}_t = \frac{db_t}{dt},$$

which is meaningless in standard terms, but conveys the right idea. Finally, a Loeb-space chaos decomposition result is used to give a generalization of a result of Nualart, Ustunel and Zakai.

1. *Introduction*

The Wiener–Itô theory of homogeneous (or polynomial) chaos [**6**], [**12**] gives the following structure theorem for $L^2(W)$, the space of square-integrable functionals over the Wiener space:

$$L^2(W) = \bigoplus_{n=0}^{\infty} Z_n \tag{1·1}$$

where $Z_0 =$ constants and Z_n denotes the subspace of n-fold multiple Wiener integrals for $n \geqslant 1$, i.e. functionals of the form

$$\int_0^{\infty} \int_0^{t_n} \ldots \int_0^{t_2} f(t_1, \ldots, t_n)\, db_{t_1}\, db_{t_2} \ldots db_{t_n} = \int f \, db^{(n)}$$

for short where $f \in L^2$ (this is explained in more detail below). Wiener's intuitive idea (see [**8**] for example) was that for $n \geqslant 1$ the coefficient f_n in an expansion

$$\phi(b) = \phi_0 + \sum_{n=1}^{\infty} \int f_n \, db^{(n)}$$

for a given $\phi \in L^2(W)$ is computed by the formal 'recipe'

$$f_n(t_1, \ldots, t_n) = E(\phi(b)\, \dot{b}_{t_1} \dot{b}_{t_2} \ldots \dot{b}_{t_n}) \tag{1·2}$$

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

where $$\dot{b}_t = \frac{db_t}{dt},$$

which is of course meaningless in standard terms, but conveys the right idea. (Of course $f_0 = E(\phi)$.) This decomposition plays an important role in one of the approaches to Malliavin calculus (see [3] or [10]), and has been established in a number of ways. In the paper [11] Stroock proposed a rigorous meaning for Wiener's recipe by using Sobolev theory to show that

$$f_m = (D^m \phi(1))'$$

where D^m is a kind of differential operator on Wiener space.

In this note we begin by providing an alternative way to make Wiener's formula rigorous, showing that f_n is a certain Radon–Nikodým derivative. This is followed by two elementary non-standard proofs of the decomposition (1·1), together with a natural non-standard identification of the integrands f_n, that gives rigorous sense to Wiener's recipe. In fact, by taking a lifting Φ of ϕ and Anderson's hyperfinite random walk B (which gives $b = {}^\circ B$ = Brownian motion) we have

$$f_n = {}^\circ E(\Phi \dot{B}_{t_1} \dot{B}_{t_2} \dots \dot{B}_{t_n}).$$

It is well known that the Wiener–Itô decomposition has as a simple corollary the representation theorem for Brownian martingales, so this latter result thus has an elementary non-standard proof (which generalizes easily to the multidimensional situation). Finally we use the non-standard framework to give an elementary proof of a Loeb-space generalization of Nualart, Ustunel & Zakai's result [9] that the conditional expectation of even chaos Z_{2m} given an odd chaos Z_{2n+1} is zero.

2. *Identification of the integrands in the Wiener–Itô decomposition*

First let us clarify some terminology and set up our notation.

Let $b = (b_t)_{t \geqslant 0}$ be a 1-dimensional Brownian motion and let $\mathbf{W} = (\mathscr{C}, \mathscr{B}, W)$ be Wiener space, i.e. $\mathscr{C}$ is the space of continuous functions $x : [0, \infty[\to \mathbb{R}$ with $x(0) = 0$, and $W(A) = P(b \in A)$ for $A \in \mathscr{B}$, the Borel algebra on $\mathscr{C}$ (generated by cylinder sets). The space of square integrable functions ϕ defined on $\mathbf{W}$ is denoted by $L^2(W)$.

The multiple Wiener integral $\int f db^{(n)}$ is defined as follows. For $n \geqslant 1$ and $t \in [0, \infty]$ let

$$\Delta_n(t) = \{(t_1, \dots, t_n) : 0 \leqslant t_1 < t_2 < \dots < t_n < t\}$$

and let $\Delta_n = \Delta_n(\infty)$. For $f \in L^2(\Delta_n)$ we define

$$\int_{\Delta_n} f db^{(n)} = \int_{t_n=0}^{\infty} \left(\int_{t_{n-1}=0}^{t_n} \dots \left(\int_{t_2=0}^{t_3} \left(\int_{t_1=0}^{t_2} f(t_1, \dots, t_n)\, db_{t_1} \right) db_{t_2} \right) \dots \right) db_{t_n}$$

which we may also write as $\int_{\Delta_n} f(t_1, \dots, t_n)\, db_{t_1} \dots db_{t_n}$ as an intermediate abbreviation. We will often omit Δ_n on the integral where it is clear. Note that the first integral (i.e. with respect to db_{t_1}) is a Wiener integral but all others are Itô integrals.

The following facts are well known.

THEOREM 2·1. *Let $f \in L^2(\Delta_n)$ and $g \in L^2(\Delta_m)$. Then*
(*a*) $E(\int f db^{(n)}) = 0$;

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

(*b*) $E\left(\int f\,db^{(n)} \int g\,db^{(m)}\right) = \begin{cases} 0 & \text{if } n \neq m \\ (f,g) & \text{if } n = m \end{cases}$

where $(f,g) = \int_{\Delta_n} fg$ *is the inner product in* $L^2(\Delta_n)$.

Definition 2·2. (*a*) $Z_0 = \mathbb{R}$ (= constant functions in $L^2(W)$).

(*b*) For $n \geqslant 1$ the nth *Wiener chaos* Z_n is the closed linear subspace of $L^2(W)$ given by

$$Z_n = \left\{ \int_{\Delta_n} f\,db^{(n)} : f \in L^2(\Delta_n) \right\}.$$

From (2·1) it is clear that $Z_n \perp Z_m$ if $n \neq m$. The Wiener–Itô decomposition theorem ([**6**], [**12**]) is now:

THEOREM 2·3.

$$L^2(W) = \bigoplus_{n=0}^{\infty} Z_n,$$

i.e. every $\phi \in L^2(W)$ *has a unique representation as*

$$\begin{aligned} \phi &= \phi_0 + \sum_{n=1}^{\infty} \phi_n \qquad \text{with} \quad \phi_n \in Z_n \\ &= \phi_0 + \sum_{n=1}^{\infty} \int_{\Delta_n} f_n\,db^{(n)} \quad \text{with} \quad f_n \in L^2(\Delta_n). \end{aligned} \tag{2·4}$$

We will give an elementary non-standard proof of this result in Section 3.

It is clear that in the above expansion $\phi_0 = E(\phi)$. For $n \geqslant 1$, suppose we take Wiener's formal recipe (1·2) for f_n and formally integrate over a set $A \subseteq \Delta_n$. Then we would have

$$\begin{aligned} \int_A f_n(t_1, \ldots, t_n)\,dt_1 \ldots dt_n &= E\left(\int_A \phi \dot{b}_{t_1} \ldots \dot{b}_{t_n}\,dt_1 \ldots dt_n \right) \\ &= E\left(\phi \int_A db_{t_1} \ldots db_{t_n} \right) \\ &= E\left(\phi \int_{\Delta_n} \chi_A(t_1, \ldots, t_n)\,db^{(n)} \right) \end{aligned}$$

where χ_A is the indicator function of A. This suggests the following result.

THEOREM 2·5. *Let* $\phi \in L^2(W)$ *with decomposition*

$$\phi = \phi_0 + \sum_{n=1}^{\infty} \int_{\Delta_n} f_n\,db^{(n)}.$$

For $n \geqslant 1$ *and measurable* $A \subseteq \Delta_n$ *with Lebesgue measure* $m(A) < \infty$ *define*

$$\mu_n(A) = E\left(\phi \int_{\Delta_n} \chi_A(t_1, \ldots, t_n)\,db^{(n)} \right).$$

Then μ_n *is a signed measure on* Δ_n *with* $\mu_n \ll m$ *and*

$$f_n(t_1, \ldots, t_n) = \frac{d\mu_n}{dm}$$

(*where* m = *Lebesgue measure*). *Moreover,* $|\mu_n(A)| \leqslant \|\phi\|\,(m(A))^{\frac{1}{2}}$.

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

Proof. It is immediate from Theorem 2·1 (*b*) that for any A with $m(A) < \infty$, since $\chi_A \in L^2(\Delta_n)$, we have

$$\begin{aligned}\int_A f_n &= (f_n, \chi_A)\\ &= E\left(\int f_n\, db^{(n)} \int \chi_A\, db^{(n)}\right)\\ &= E\left(\phi \int \chi_A\, db^{(n)}\right) \quad \text{(by orthogonality)}\\ &= \mu_n(A).\end{aligned}$$

From this it follows that μ_n is a signed measure with $\mu_n \ll m$ and Radon–Nikodým derivative f_n. For the last part we have

$$\begin{aligned}\mu_n(A)^2 &= E\left(\phi \int \chi_A\, db^{(n)}\right)^2\\ &\leqslant \|\phi\|^2 E\left(\left(\int \chi_A\, db^{(n)}\right)^2\right)\\ &= \|\phi\|^2 m(A) \quad \text{by Theorem 2·1 (}b\text{).}\end{aligned}$$

∎

3. *Proof of Wiener–Itô decomposition*

We give two elementary proofs of the Wiener–Itô chaos decomposition, using Anderson's non-standard construction of Brownian motion as an infinitesimal random walk [**2**]. We do not claim that these proofs are necessarily any easier than any of the classical proofs, but they do help to see what is going on.

The basics of non-standard analysis and Loeb theory are assumed – see [**1**], [**4**] or [**7**] for instance. We briefly review Anderson's construction (see [**1**], [**4**] or [**7**] for a fuller exposition). Working in an ω_1-saturated non-standard universe, let $\Delta t = N^{-1}$ with N infinite and let

$$T = \{k\Delta t : 0 \leqslant k < N^2\}.$$

Anderson took an internal independent family $(\Delta B_t)_{t \in T}$ with $\Delta B_t = \pm\sqrt{\Delta t}$, each with probability $\frac{1}{2}$ under the internal counting measure ν on the internal hyperfinite space $\Omega = \{\pm 1\}^T$. Then, defining

$$B_t = \sum_{s<t} \Delta B_s \quad (B_0 = 0)$$

Anderson showed that B_t is S-continuous (for finite t) almost surely with respect to the Loeb measure $\nu_L = P$, say, and the process $b = {}^\circ B$ is Brownian motion on the Loeb space $(\Omega, \mathscr{F}, P)$.

Anderson [**2**] also developed a natural non-standard construction of the Itô integral which means that we can represent a multiple Wiener integral $\int f\, db^{(n)}$ as

$$\int f\, db^{(n)} = {}^\circ\left(\sum_{0 \leqslant t_1 < t_2 < \ldots < t_n} F(t_1, \ldots, t_n)\, \Delta B_{t_1}\, \Delta B_{t_2} \ldots \Delta B_{t_n}\right)$$

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

where each $t_i \in T$ and $F : T^n \to {}^*\mathbb{R}$ is an SL^2-lifting of $f \in \Delta_n$. In fact, we need only define F on the set $\Delta_n^T = \{(t_1, \ldots, t_n) \in T^n : t_1 < t_2 < \ldots < t_n\}$.

From this representation it is easy to see why Theorem 2·1 holds: the independence of the increments ΔB_t and the fact that $\Delta B_t^2 = \Delta t$ means that if $t_1 < t_2 < \ldots < t_n$ and $s_1 < s_2 < \ldots < s_m$, then

$$E(\Delta B_{t_1} \ldots \Delta B_{t_n} \Delta B_{s_1} \ldots \Delta B_{s_m}) = \begin{cases} \Delta t^n & \text{if} \quad n = m \text{ and } t_i = s_i \text{ for all } i \\ 0 & \text{otherwise.} \end{cases} \tag{3·1}$$

Turning now to our first elementary proof of Theorem 2·3, let

$$0 < u_1 < u_2 < \ldots < u_k < \infty$$

in $\mathbb{R}$ and let

$$\phi(b) = b_{u_1}^{\alpha_1}(b_{u_2} - b_{u_1})^{\alpha_2} \ldots (b_{u_k} - b_{u_{k-1}})^{\alpha_k} \tag{3·2}$$

where $\alpha_1, \ldots, \alpha_k$ are positive integers. Brownian functionals of this form span a dense subset of $L^2(W)$, so the Wiener–Itô decomposition (1·1) is proved once we have the following lemma.

LEMMA 3·3. *For ϕ defined as in* (3·2) *above*

$$\phi \in \bigoplus_{n=0}^{m} Z_n$$

where $m = \alpha_1 + \ldots + \alpha_k$.

Proof. Pick $s_1, \ldots, s_k \in T$ with $s_i \approx u_i$. Taking Anderson's Brownian motion $b = {}^\circ B$ we have

$$\begin{aligned} \phi(b) \approx \Phi(B) &= B_{s_1}^{\alpha_1}(B_{s_2} - B_{s_1})^{\alpha_2} \ldots (B_{s_k} - B_{s_{k-1}})^{\alpha_k} \quad \text{a.s.} \\ &= \Big(\sum_{t < s_1} \Delta B_t\Big)^{\alpha_1} \Big(\sum_{s_1 \leqslant t < s_2} \Delta B_t\Big)^{\alpha_2} \ldots \Big(\sum_{s_{k-1} \leqslant t < s_k} \Delta B_t\Big)^{\alpha_k} \\ &= \sum_{n=0}^{m} \sum_{t_1 < t_2 < \ldots < t_n < s_k} F_n(t_1, \ldots, t_n)\, \Delta B_{t_1} \ldots \Delta B_{t_n} \end{aligned}$$

for certain coefficients $F_n(t_1, \ldots, t_n)$ obtained by expanding the brackets and using $\Delta B_t^2 = \Delta t$. (The t's all range over T.) We will see that each F_n is an SL^2 lifting of a function $f_n \in L^2(\Delta_n)$; then by Anderson's Itô theory we have

$$\Phi(B) \approx {}^\circ F_0 + \sum_{n=1}^{m} \int f_n \, db^{(n)}$$

and the lemma is proved. In fact, each F_n is constant and finite within each of a finite number of rectangles. To see this, first note that

$$\infty > E(\Phi(B)^2) = F_0^2 + \sum_{n=1}^{m} \sum_{\Delta_n^T(s_k)} F_n^2(t_1, \ldots, t_n)\, \Delta t^n$$

so F_0 *is* finite and so is each $\sum_{\Delta_n^T(s_k)} F_n^2(t_1, \ldots, t_n)\, \Delta t^n$, where

$$\Delta_n^T(s_k) = \{(t_1, \ldots, t_n) : t_1 < \ldots < t_n < s_k, \text{ all in } T\}.$$

Now the intervals $[s_{i-1}, s_i[$ partition T, giving a partition of T^n into $(k+1)^n$ rectangles R_j. From the definition of Φ, $\Phi(B)$ is invariant under any permutation of t's within

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

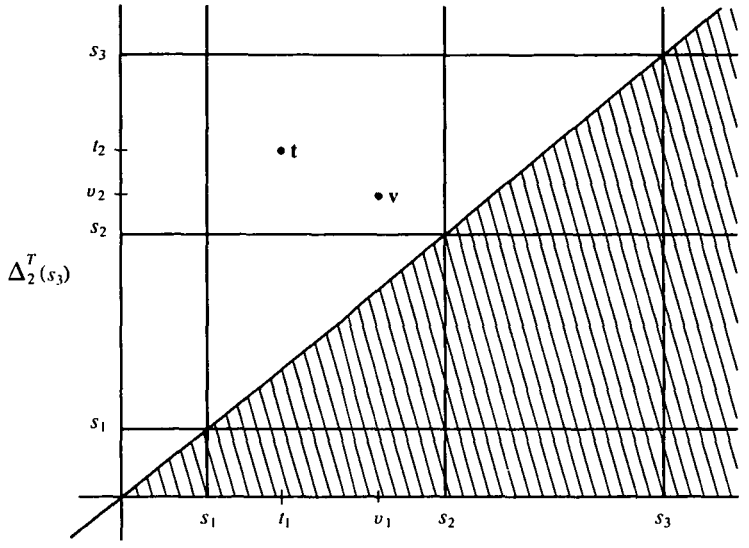


Fig. 1.

each interval $[s_{i-1}, s_i[$ and so F_n is constant within each $R_j \cap \Delta_n^T(s_k)$. This is best illustrated by considering the case $n = 2$ and $k = 3$ (see Fig. 1).

If $\mathbf{t} = (t_1, t_2)$ and $\mathbf{v} = (v_1, v_2)$ are in the same rectangle, say $s_1 \leqslant t_1$, $v_1 < s_2$ and $s_2 \leqslant t_2$, $v_2 < s_3$, then

$$F_2(t_1, t_2) = \text{coefficient of } \Delta B_{t_1} \Delta B_{t_2} \text{ in the expansion of}$$
$$\left(\sum_{t<s_1} \Delta B_t\right)^{\alpha_1}\left(\sum_{s_1 \leqslant t<s_2} \Delta B_t\right)^{\alpha_2}\left(\sum_{s_2 \leqslant t<s_3} \Delta B_t\right)^{\alpha_3}$$
$$= F_2(v_1, v_2).$$

Thus F_n is constant on each of the finite number of rectangles given by the intervals $[s_{i-1}, s_i[$; and the fact that $\sum_{\Delta_n^T(s_k)} F_n^2(t_1, \ldots, t_n)\Delta t^n$ is finite means that the values of F_n are finite.

Hence F_n is an SL^2 lifting of a function f_n defined on $\Delta_n(u_k) \subseteq \Delta_n$, with f_n constant on the members of the partition of Δ_n given by the intervals $[u_{i-1}, u_i[$. Taking $f_n = 0$ outside $\Delta_n(u_k)$ we have

$$\sum_{\Delta_n^T(s_k)} F_n(t_1, \ldots, t_n)\Delta B_{t_1} \ldots \Delta B_{t_n} \approx \int_{\Delta_n} f_n \, db^{(n)} \quad \text{a.s.}$$

and the result is proved. ∎

Our second elementary derivation of the Wiener–Itô decomposition is not so far from one of the classical standard proofs. It follows from two lemmas.

LEMMA 3·4. *Functionals of the form*

$$\exp\left(\int_0^\infty f(t)\, db_t - \frac{1}{2}\int_0^\infty f^2(t)\, dt\right)$$

where $f \in L^2[0, \infty[$, *span a dense subset of* $L^2(W)$.

Proof. This is standard, and is omitted. ∎

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

LEMMA 3·5. *If* $f \in L^2[0, \infty[$ *then*

$$\exp\left(\int_0^\infty f(t)\,db_t - \frac{1}{2}\int_0^\infty f^2(t)\,dt\right) = 1 + \sum_{n=1}^{\infty}\int_{\Delta_n} f^{(n)}\,db^{(n)}$$

where $f^{(n)}(t_1, t_2, \ldots, t_n) = f(t_1)f(t_2)\ldots f(t_n)$.

Proof. Let $F(t)$ be an SL^2 lifting of f. Then $\sum F^2(t)\,\Delta t \approx \|f\|^2 < \infty$. We may assume that $|F(t)|^4 < N$. Define an internal functional $\Phi(B)$ by

$$\Phi(B) = \prod_{t \in T}(1 + F(t)\,\Delta B_t).$$

Then we claim that

$$\Phi(B) \approx \exp\left(\int_0^\infty f(t)\,db_t - \frac{1}{2}\int_0^\infty f^2(t)\,dt\right) \quad \text{a.s.} \tag{3·6}$$

$$\Phi(B) \approx 1 + \sum_{n=1}^{\infty}\int_{\Delta_n} f^{(n)}\,db^{(n)} \quad \text{a.s.} \tag{3·7}$$

which establishes the lemma.

Proof of (3·6). We know (from Anderson) that

$$\sum F(t)\,\Delta B_t \approx \int f(t)\,db_t < \infty \quad \text{for a.a. } B.$$

Take any such B; then

$$\begin{aligned}\log \Phi(B) &= \sum_T \log(1 + F(t)\,\Delta B_t)\\ &= \sum_T (F(t)\,\Delta B_t - \tfrac{1}{2}F^2(t)\,\Delta t + \epsilon_t)\end{aligned}$$

where $|\epsilon_t| \leqslant |F(t)\,\Delta B_t|^3 \leqslant \Delta t^{\frac{1}{2}} F^2(t)\,\Delta t$ (using $|\log(1+x) - (x - \frac{1}{2}x^2)| \leqslant |x^3|$). Hence

$$\begin{aligned}\log \Phi(B) &\approx \sum_T (F(t)\,\Delta B_t - \tfrac{1}{2}F^2(t)\,\Delta t)\\ &\approx \int f(t)\,db_t - \frac{1}{2}\int f^2(t)\,dt\end{aligned}$$

from which (3·6) follows, since both sides are finite.

Proof of (3·7). Expanding we have

$$\begin{aligned}\Phi(B) &= 1 + \sum_{n=1}^{N}\sum_{\Delta_n^T} F(t_1)\ldots F(t_n)\,\Delta B_{t_1}\ldots\Delta B_{t_n}\\ &= \sum_{n=0}^{N}\Phi_n(B), \quad \text{say.}\end{aligned}$$

By Anderson's theory, for finite n

$$\begin{aligned}\Phi_n(B) &\approx \int_{\Delta_n} f^{(n)}\,db^{(n)}\\ &= \phi_n(b), \quad \text{say.}\end{aligned}$$

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

Moreover
$$E\left(\left({}^\circ\Phi(B)-\sum_{k=0}^{n}\phi_k\right)^2\right)=E\left(\left({}^\circ\sum_{k=n+1}^{N}\Phi_k\right)^2\right)$$
$$\leqslant {}^\circ E\left(\left(\sum_{k=n+1}^{N}\Phi_k\right)^2\right)$$
$$={}^\circ\sum_{k=n+1}^{N}\sum_{\Delta_k^T}F^2(t_1)\dots F^2(t_k)\,\Delta t^k$$
$$={}^\circ\sum_{k=n+1}^{N}(k!)^{-1}\|f\|^{2k}$$
$$\to 0 \quad \text{as} \quad n\to\infty,$$

which completes the proof. ∎

4. *A non-standard chaos decomposition and Wiener's recipe*

A non-standard justification of Wiener's recipe will follow from the following non-standard chaos decomposition.

THEOREM 4·1. *Let $\Phi(B)$ be an internal function defined on the paths B of Anderson's random walk. Then there are unique internal functions $F_n:\Delta_n^T\to{}^*\mathbb{R}$ for $n\leqslant N$ with*

$$\Phi(B)=\sum_{n\leqslant N}\sum_{t_1<\dots<t_n}F_n(t_1,\dots,t_n)\,\Delta B_{t_1}\dots\Delta B_{t_n}$$

where $t_i\in T$ for all i, and $F_0\in{}^*\mathbb{R}$.

Moreover, F_n is given by

$$F_n(t_1,\dots,t_n)=E(\Phi(B)\dot{B}_{t_1}\dots\dot{B}_{t_n}) \tag{4·2}$$

where $\dot{B}_t=\Delta B_t/\Delta t$.

Proof. Regarding $\Phi(B)$ as a function $\Phi((\Delta B_t)_{t\in T})$ of the increments $\Delta B_t\,(=\pm\sqrt{\Delta t})$, *either* proceed by induction on $|T|$ noting that any function $\theta(\Delta B_t)$ can be written

$$\theta(\Delta B_t)=a+b\Delta B_t$$

for suitable $a,b\in{}^*\mathbb{R}$; *or* note that the internal vector space of internal functions of the form $\Phi((\Delta B_t)_{t\in T})$ has dimension $2^{|T|}$. The functions **1** together with $\Delta B_{t_1}\dots\Delta B_{t_n}(t_1<\dots<t_n$ and $n\leqslant N)$ are independent and $2^{|T|}$ in number, hence form an internal basis for this space.

The remark (3·1) shows that

$$E(\Phi(B)\,\Delta B_{t_1}\dots\Delta B_{t_n})=F_n(t_1,\dots,t_n)\,\Delta t^n$$

from which the non-standard version of Wiener's recipe (4·2) follows. ∎

We now have

THEOREM 4·3. *Let $\phi\in L^2(W)$ and let $\Phi(B)$ be an SL^2-lifting of ϕ. Then for $n\in\mathbb{N}$ the integrands f_n in the Wiener–Itô decomposition of ϕ are given by*

$$f_n({}^\circ t_1,\dots,{}^\circ t_n)={}^\circ F_n(t_1,\dots,t_n)\quad a.s.\ for\ finite\quad (t_1,\dots,t_n)\in\Delta_n^T$$

where F_n are obtained from Φ by the recipe (4·2).

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

Proof. Let
$$\phi = \phi_0 + \sum_{n=1}^{\infty} \int f_n \, db^{(n)}$$
be the Wiener–Itô decomposition of ϕ, and take SL^2 liftings G_n of f_n. Then for sufficiently small infinite M
$$\bar{\Phi}(B) = \phi_0 + \sum_{n=1}^{M} \sum_{\Delta_n^T} G_n(t_1, \ldots, t_n) \Delta B_{t_1} \ldots \Delta B_{t_n}$$
is an SL^2 lifting of ϕ. Hence
$$E((\bar{\Phi} - \Phi)^2) \approx 0,$$
i.e. putting $G_n = 0$ for $n > M$,
$$(\phi_0 - F_0)^2 + \sum_{n=1}^{N^2} \sum_{\Delta_n^T} (G_n(t_1, \ldots, t_n) - F_n(t_1, \ldots, t_n))^2 \Delta t^n \approx 0.$$
In particular, for each finite n
$$\sum_{\Delta_n^T} (G_n - F_n)^2 \Delta t^n \approx 0$$
so F_n is also an SL^2 lifting of f_n as required. ∎

5. *Even and odd Loeb functionals*

In the paper [9] it was shown that if $\phi \in Z_{2k+1}$ and $\mathscr{H}_e$ denotes $\sigma\{Z_{2m} : m \geqslant 0\}$ then
$$E(\phi \mid \mathscr{H}_e) = 0 \quad \text{a.s.} \tag{5·1}$$

We shall give an elementary proof of a Loeb space generalization of this result (Theorem 5·6). First, some definitions.

Let $\mathbf{\Omega} = (\Omega, \mathscr{F}, P)$ be the Anderson–Loeb space described at the beginning of Section 3, carrying Anderson's Brownian motion $b = {}^\circ B$, i.e. $\Omega = \{\pm 1\}^T$, with P the Loeb-counting measure on Ω and $\mathscr{F}$ the P-completion of $\sigma({}^*\mathscr{P}(\Omega))$.

Definitions 5·2. (*a*) Let $\mathscr{E} \subseteq {}^*\mathscr{P}(\Omega)$ denote the algebra of *even* internal subsets of Ω, i.e.
$$A \in \mathscr{E} \quad \text{if and only if} \quad \omega \in A \Leftrightarrow -\omega \in A, \quad \text{for all } \omega \in A;$$

(*b*) Let $\mathscr{F}_e$ denote the P-completion of $\sigma(\mathscr{E})$.

(*c*) Let $\psi : \Omega \to \bar{\mathbb{R}} (= \mathbb{R} \cup \{\pm\infty\})$. Then ψ is *even* if
$$\psi(\omega) = \psi(-\omega) \qquad \text{a.s. } (P)$$
and ψ is *odd* if
$$\psi(\omega) = -\psi(-\omega) \quad \text{a.s. } (P).$$

(*d*) Let $\Psi : \Omega \to {}^*\mathbb{R}$ be internal; then Ψ is *even* if
$$\Psi(\omega) = \Psi(-\omega), \quad \text{for all } \omega$$
and Ψ is *odd* if
$$\Psi(\omega) = -\Psi(-\omega), \quad \text{for all } \omega.$$

The following is elementary.

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

LEMMA 5·3. *Let $\psi:\Omega\to\bar{\mathbb{R}}$ be Loeb-measurable.*

(i) *The following are equivalent:*

(*a*) *ψ is even;*

(*b*) *ψ has an even lifting Ψ;*

(*c*) *ψ is $\mathscr{F}_e$-measurable.*

(ii) *ψ is odd if and only if it has an odd lifting Ψ.*

Proof. (i) We have an internal probability space

$$(\Omega, \mathscr{E}, \nu_{\mathscr{E}})$$

where $\nu_{\mathscr{E}}$ is the counting measure ν restricted to $\mathscr{E}$. Notice that an internal function $\Psi:\Omega\to{}^*\mathbb{R}$ is even if and only if it is $\mathscr{E}$-measurable, so the equivalence $(b)\equiv(c)$ is just basic Loeb theory (see e.g. [7]), together with the observation that $\mathscr{F}_e$ and $\mathscr{F}$ have the same null sets.

Clearly $(b)\Rightarrow(a)$, since ${}^\circ\Psi$ is even if Ψ is; for $(a)\Rightarrow(b)$, simply take a lifting Ψ of ψ and if necessary adjust it on the set $\Omega_{-1}=\{\omega:\omega(0)=-1\}$ to ensure that Ψ is even.

(ii) is proved just like $(a)\Leftrightarrow(b)$ in (i). ∎

This lemma immediately gives:

COROLLARY 5·4. (*a*) *$\psi:\Omega\to\bar{\mathbb{R}}$ is even if and only if ψ has the form*

$$\psi(\omega)={}^\circ\sum_{2m\leq N}\sum_{\Delta^T_{2m}} F_{2m}(t_1,\ldots,t_{2m})\,\Delta B_{t_1}\ldots\Delta B_{t_{2m}}$$

(*b*) *Similarly for ψ odd, with $2m+1$ in place of $2m$.*

In particular, $\phi\in Z_n$ is even or odd according as n is even or odd.

Proof. Simply observe that $\Delta B_t(-\omega)=-\Delta B_t(\omega)$ is odd, and apply Theorem 4·1.

COROLLARY 5·5. $\mathscr{H}_e\subseteq\mathscr{F}_e$.

Our theorem that generalizes (5·1) is now

THEOREM 5·6. *If $\psi\in L^1(\Omega)$ is odd then*

$$E(\psi\,|\,\mathscr{F}_e)=0\quad a.s.$$

Proof. Let $\Psi:\Omega\to{}^*\mathbb{R}$ be an odd SL^1-lifting of ψ, and let $A\in\mathscr{E}$.

Let $A_i=\{\omega\in A:\omega(0)=i\}\,(i=\pm1)$. Then since A is even, $A_{-1}=\{-\omega:\omega\in A_1\}$ and so

$$\begin{aligned}\int_A\psi\,dP&\approx\sum_A\Psi(\omega)\,\Delta\omega\quad(\Delta\omega=|\Omega|^{-1})\\&=\sum_{A_1}\Psi(\omega)\,\Delta\omega+\sum_{A_{-1}}\Psi(\omega)\,\Delta\omega\\&=\sum_{A_1}\Psi(\omega)\,\Delta\omega+\sum_{A_1}\Psi(-\omega)\,\Delta\omega\\&=0\quad\text{since }\Psi\text{ is odd.}\end{aligned}$$

So $\int_A\psi\,dP=0$ for all $A\in\mathscr{E}$; since $\mathscr{F}_e=$ completion of $\sigma(\mathscr{E})$, this means that $E(\psi\,|\,\mathscr{F}_e)=0$ as required. ∎

The second author's research was supported by a grant from the S.E.R.C.

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press

REFERENCES

[1] S. Albeverio, J. E. Fenstad, R. Høegh-Krohn and T. Lindstrøm. *Nonstandard Methods in Stochastic Analysis and Mathematical Physics* (Academic Press, 1986).
[2] R. M. Anderson. A non-standard representation for Brownian motion and Itô integration. *Israel J. Math.* **25** (1976), 15–46.
[3] D. R. Bell. *The Malliavin Calculus*. Pitman Monographs and Surveys in Pure and Applied Math. vol. 34 (Longman, 1987).
[4] N. J. Cutland. Nonstandard measure theory and its applications. *Bull. London Math. Soc.* **15** (1983), 529–589.
[5] N. J. Cutland (editor). *Nonstandard Analysis and its Applications* (Cambridge University Press, 1988).
[6] K. Itô. Multiple Wiener integral. *J. Math. Soc. Japan* **3** (1951), 157–169.
[7] T. Lindstrøm. An invitation to nonstandard analysis, in [5], pp. 1–105.
[8] H. P. McKean. Geometry of differential space. *Ann. Probab.* **1** (1973), 197–206.
[9] D. Nualart, A. S. Ustunel and M. Zakai. On the moments of a multiple Wiener–Itô integral and the space induced by the polynomials of the integral. *Stochastics* **25** (1988), 233–240.
[10] D. W. Stroock. The Malliavin calculus and its applications. In *Stochastic Integrals*, Lecture Notes in Math. vol. 851 (Springer-Verlag, 1987), pp. 394–432.
[11] D. W. Stroock. Homogeneous chaos revisited. In *Séminaire de Probabilités* XXI, Lecture Notes in Math. vol. 1247 (Springer-Verlag, 1987), pp. 1–7.
[12] N. Wiener. The homogeneous chaos. *Amer. J. Math.* **60** (1938), 897–936.

https://doi.org/10.1017/S0305004100070420 Published online by Cambridge University Press