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Equidistribution results for geodesic flows

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Abstract. Using the works of Mañé [On the topological entropy of the geodesic flows. *J. Differential Geom.* **45** (1989), 74–93] and Paternain [Topological pressure for geodesic flows. *Ann. Sci. Éc. Norm. Supér.* (4) **33** (2000), 121–138] we study the distribution of geodesic arcs with respect to equilibrium states of the geodesic flow on a closed manifold, equipped with a $\mathcal{C}^\infty$ Riemannian metric. We prove large-deviation lower and upper bounds and a contraction principle for the geodesic flow in the space of probability measures of the unit tangent bundle. We deduce a way of approximating equilibrium states for continuous potentials.

1. *Introduction*

Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric. We study the distribution of geodesic arcs of M with respect to equilibrium states. We prove large-deviation lower and upper bounds for the geodesic flow in the space of probability measures of the unit tangent bundle. More precisely, we consider Lebesgue measures supported on a finite number of geodesic arcs and show that they define a process which satisfies a large-deviation principle with action function given by the topological pressure. As an application, we obtain equidistribution results describing the proportion of geodesic arcs which support Lebesgue measures close to equilibrium states. For potentials with a unique equilibrium state, we show that this proportion converges exponentially fast to one when the length of the geodesic arcs tends to infinity. We also prove a contraction principle for these probability measures, which is a large-deviation theorem with constraints. This work is based on two remarkable formulas due to Mañé [**10**] and Paternain [**12**] which characterize the topological entropy and pressure as a growth rate of the number of the geodesic arcs (see Theorems 8 and 9). On the other hand, the techniques of convex analysis and large deviations in [**5, 8**] were particulary useful for this work. We adapt and extend certain of the general arguments in [**8**] to our situation.

We have two important situations where the results of the paper apply. The first situation concerns the class of manifolds with negative curvature. In this case, it is well known that

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for any Hölder continuous potential there exists a unique equilibrium state. There are three well-known invariant measures in this setting: the Bowen–Margulis measure, which is the equilibrium state (a measure of maximal entropy) corresponding to constant potentials; the harmonic measure, which corresponds to the potential $(d/dt)|_{t=0}(K \circ \widetilde{\varphi}^t)$, where K is the Poisson kernel and $\widetilde{\varphi}^t$ the geodesic flow of $S\widetilde{M}$, $\widetilde{M}$ being the universal cover of the manifold M; and the Liouville measure, which is the equilibrium state of the potential $(d/dt)|_{t=0} \det(d\varphi_t|_{E^s})$, where E^s is the stable tangent bundle of SM (see [**3, 4**] for more details). For the 'Liouville potential', we obtain that the geodesic arcs are uniformly distributed with respect to the Liouville measure.

The second situation deals with the more general class of Riemannian manifolds of non-positive curvature which are of rank one. By a result of Knieper [**9**], we know that there exists a uniquely determined invariant measure of maximal entropy for the geodesic flow. But it is not yet known which class of potentials admits a unique equilibrium state and this question remains open for rank-one manifolds. However, our results give a way of approximating equilibrium states of the geodesic flow of such manifolds. Indeed, these results are applicable wherever Mañé's and Paternain's formulas hold.

2. *Main results*

2.1. *Preliminaries and notation.* Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric and $\phi : SM \to SM$ be the geodesic flow on the unit tangent bundle SM. We assume that M has volume one, $\int_M dx = 1$, where dx is the volume form induced by the Riemannian metric of M.

We denote by $\mathcal{P}(SM)$ the space of probability measures on SM equipped with the weak star topology. Let $\mathcal{P}_{\text{inv}}(SM)$ be the subset of $\mathcal{P}(SM)$ of invariant probability measures. Given a potential $F \in C_{\mathbb{R}}(SM)$, the topological pressure of F is the number defined by the variational principle [**16**],

$$P(F) = \sup_{m \in \mathcal{P}_{\text{inv}}(SM)} \left(h(m) + \int_{SM} F \, dm \right), \tag{1}$$

where $h(m)$ is the entropy of m. For $F = 0$ this reduces to

$$P(0) = \sup_{m \in \mathcal{P}_{\text{inv}}(SM)} h(m) := h_{\text{top}},$$

where h_{top} is the topological entropy of the geodesic flow.

An equilibrium state for F is a measure $m \in \mathcal{P}_{\text{inv}}(SM)$ which achieves the maximum in (1),

$$h(m) + \int_{SM} F \, dm = P(F).$$

We denote by $\mathcal{P}_e(F)$ the subset of $\mathcal{P}_{\text{inv}}(SM)$ of equilibrium states corresponding to F. By a result of Newhouse [**11**], since the metric is C^∞, the entropy map $m \to h(m)$ is upper semicontinuous. Then $h_{\text{top}} < \infty$ and, consequently, the set $\mathcal{P}_e(F)$ is a non-empty, closed, compact, convex subset of $\mathcal{P}(SM)$ [**16**].

We define the functional Q_F on $C_{\mathbb{R}}(SM)$ based on the potential F by

$$Q_F(\omega) := P(F + \omega) - P(F). \tag{2}$$

By definition, Q_F is continuous on continuous functions (see Lemma 1).

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For any probability measure μ on SM, we set

$$J_F(\mu) := \sup_{\omega}\left(\int \omega\, d\mu - Q_F(\omega)\right), \tag{3}$$

where the sup is taken over the space of continuous functions ω on SM. Observe that since $Q_F(0) = 0$, J_F is a non-negative functional and clearly lower semicontinuous.

By duality,

$$Q_F(\omega) = \sup_{\mu\in\mathcal{P}(SM)}\left(\int \omega\, d\mu - J_F(\mu)\right). \tag{4}$$

Given a continuous function ω, by the lower semicontinuity of J_F, there exists a probability measure μ_ω maximizing (4):

$$Q_F(\omega) = \int \omega\, d\mu_\omega - J_F(\mu_\omega).$$

For any set $E \subset \mathcal{P}(SM)$, put

$$J_F(E) := \inf_{\mu\in E} J_F(\mu).$$

Given x and y in M, we denote by $\gamma_{xy} : [0, l(\gamma_{xy})] \to M$ a unit-speed geodesic arc joining x to y with length $l(\gamma_{xy})$. For any $\delta > 0$ and $T > 0$, we set

$$G_{\delta,T}(x, y) := \{\gamma_{xy} : T - \delta < l(\gamma_{xy}) \le T\} \quad \text{and} \quad G_T(x, y) := \{\gamma_{xy} : l(\gamma_{xy}) \le T\}.$$

Recall from [**1**] that, given $T > 0$, the set of geodesic arcs with length up to and including T is finite and its cardinality is locally constant for an open full Lebesgue measure subset of $M \times M$ (see also [**12**, p. 53] for a proof using Sard's theorem). By a result of Burns and Gutkin [**2**], the growth of $\#G_T(x, y)$ and a positive topological entropy are related to the condition of 'insecurity' of the manifold.

The integral $\int_{\gamma_{xy}} \omega$ of a function ω over the geodesic arc γ_{xy} is defined by

$$\int_{\gamma_{xy}} \omega := \int_0^{l(\gamma_{xy})} \omega(\phi_s(\dot\gamma_{xy}(0)))\, ds.$$

Here $\dot\gamma_{xy}(0)$ is the initial condition of the geodesic γ_{xy}. The Lebesgue measure $\delta_{\gamma_{xy}}$ with support in γ_{xy} is now defined by

$$\int_{SM} \omega\, d\delta_{\gamma_{xy}} := \frac{1}{l(\gamma_{xy})}\int_{\gamma_{xy}} \omega.$$

We will write $G_{\delta,T}$ and G_T for simplicity, since the dependence in (x, y) will be always clear.

2.2. *The results.* Consider a real continuous potential F defined on SM. For any Borel subset E of $\mathcal{P}(SM)$, we set

$$\nu_T(E) := \frac{\int_{M\times M}(\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in E} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int_{M\times M}(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy}. \tag{5}$$

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This defines a process $(\nu_T)_{T>0}$ on the space $\mathcal{P}(SM)$. The first result gives large-deviation bounds for this process, namely

$$\limsup_{T\to+\infty} \frac{1}{T} \log \nu_T(K) \le -J_F(K),$$
$$\liminf_{T\to+\infty} \frac{1}{T} \log \nu_T(O) \ge -J_F(O),$$

respectively, for any closed subset K and open subset O of $\mathcal{P}(SM)$. The lower bound condition is proved under the following condition (see also [**5, 8**]).

Condition 1. Suppose that there exists a countable set $\{g_1, g_2, \ldots\}$ of continuous functions on SM which span a dense subset in $C_{\mathbb{R}}(SM)$ with respect to the topology of uniform convergence, $\|g_k\| = 1$ for all k, and such that, for all $n \ge 1$ and $\beta \in \mathbb{R}^n$, $F + \sum_{i=1}^n \beta_i g_i$ admits a unique equilibrium state, that is, an invariant probability measure realizing the sup (1).

It is well known that for any Hölder continuous function on a smooth compact manifold there exists a unique equilibrium state [**15**]. Thus, if $\{g_1, g_2, \ldots\}$ is a dense set of Hölder continuous functions, all the functions $F + \sum_{i=1}^n \beta_i g_i$ are Hölder for smooth potentials F and will satisfy Condition 1. On the other hand, as a consequence of the density of the set $\{g_1, g_2, \ldots\}$, we endow $\mathcal{P}(SM)$ with a metric which is compatible with the topology of weak convergence (see the proof of Theorem 1(2)).

THEOREM 1. *Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric and $F \in C_{\mathbb{R}}(SM)$. Then, for any $\delta > 0$:*

(1) *for any closed subset K of $\mathcal{P}(SM)$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in K} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \le -J_F(K);$$

(2) *under Condition 1, for any open subset O of $\mathcal{P}(SM)$,*

$$\liminf_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in O} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \ge -J_F(O).$$

A consequence of Theorem 1(1) is the following result which asserts that the proportion of geodesic arcs supporting a Lebesgue measure close to an element of $\{m : J_F(m) = 0\}$ is asymptotically equal to 1: $\lim_{T\to\infty} \nu_T(V) = 1$ and the convergence is exponential. If the potential admits a unique equilibrium state μ_F then the proportion of geodesic arcs supporting a Lebesgue measure close to μ_F is asymptotically equal to 1. Indeed, we will show in Lemma 2 below that the uniqueness of μ_F is equivalent to the fact that μ_F is the unique solution for the equation $J_F(m) = 0$.

COROLLARY 1. *For any open neighborhood V of $\{m : J_F(m) = 0\}$ in $\mathcal{P}(SM)$,*

$$\lim_{T\to+\infty} \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in V} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} = 1.$$

The convergence is exponential with speed $e^{-TJ(V^c)}$, where $V^c := \mathcal{P}(SM)\backslash V$.

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We know that large-deviation principles are preserved under continuous mapping; this is known as the contraction principle [**5**]. In the present case, the contraction principle reduces the preceding theorem to a finite-dimensional one. First, we need some notation.

For $g \in C_{\mathbb{R}^n}(SM)$ and $\alpha \in \mathbb{R}^n$, $n > 0$, set

$$\mathcal{P}_{g,\alpha}(SM) := \left\{ m \in \mathcal{P}(SM) : \int_{SM} g \, dm = \alpha \right\}.$$

We define the functional

$$J_F(\alpha;\, g) = \begin{cases} \inf(J_F(m) : m \in \mathcal{P}_{g,\alpha}(SM)) & \text{if } \mathcal{P}_{g,\alpha}(SM) \neq \emptyset, \\ +\infty & \text{if } \mathcal{P}_{g,\alpha}(SM) = \emptyset, \end{cases}$$

and, for any $E_n \subset \mathbb{R}^n$,

$$J_F(E_n;\, g) = \inf(J_F(\alpha;\, g) : \alpha \in E_n).$$

To simplify the notation, set $m(g) = \int g \, dm$ for $m \in \mathcal{P}(SM)$.

THEOREM 2. (Contraction principle) *Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric and $F \in C_{\mathbb{R}}(SM)$. Let $g = (g_1, g_2, \ldots, g_n) \in C_{\mathbb{R}^n}(SM)$. Then, for any $\delta > 0$:*

(1) *for any closed subset $K_n \subset \mathbb{R}^n$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}(g)\in K_n} e^{\int_{\gamma_{xy}} F}) \, dx \, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F}) \, dx \, dy} \leq -J_F(K_n;\, g);$$

(2) *if, for all $\beta \in \mathbb{R}^n$, $F + \sum_{i=1}^n \beta_i g_i$ has a unique equilibrium state, then for any open subset $O_n \subset \mathbb{R}^n$,*

$$\liminf_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}(g)\in O_n} e^{\int_{\gamma_{xy}} F}) \, dx \, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F}) \, dx \, dy} \geq -J_F(O_n;\, g).$$

Part (1) of Theorem 2 is a consequence of Theorem 1(1), by the continuity of the function g. Note that we do not assume in part (2) of Theorem 2 that g takes the form $g = (g_1, g_2, \ldots, g_n)$ as in Theorem 1. In other words, the conclusion of Theorem 2(2) holds for any continuous g such that, for all $\beta \in \mathbb{R}^n$, $F + \sum_{i=1}^n \beta_i g_i$ admits a unique equilibrium state.

For any $\delta > 0$ we define the probability measures on SM,

$$m_{\delta,T} := \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F} \cdot \delta_{\gamma_{xy}}) \, dx \, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F}) \, dx \, dy}. \tag{6}$$

As an application of Theorem 1(1) we prove the following theorem.

THEOREM 3. *Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric and $F \in C_{\mathbb{R}}(SM)$. For any $\delta > 0$, any weak limit m_∞ of $(m_{\delta,T})_T$ is an equilibrium state for the potential F, that is, m_∞ is invariant and satisfies*

$$h(m_\infty) + \int F \, dm_\infty = P(F).$$

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These theorems apply to the geodesic flow of a manifold of negative curvature and Hölder continuous potentials F. The following result is particulary important when the unique equilibrium state μ_F corresponding to F is the Bowen–Margulis measure or the harmonic measure (§1). In particular, if F is the 'Liouville potential' (§1) then this corollary says that the geodesic arcs are uniformly distributed.

COROLLARY 2. *Let M be a closed and connected manifold of negative curvature equipped with a C^∞ Riemannian metric. Suppose that the potential F is Hölder continuous and let μ_F be the unique corresponding equilibrium state. Then, for any $\delta > 0$, the measures $(m_{\delta,T})_T$ converge weakly to μ_F as $T \to +\infty$.*

Consider now the measure $\mu_{\max}$ of maximal entropy [**9**] of the geodesic flow of a rank-one manifold. Set

$$\mu_{\delta,T} := \frac{\int (\sum_{\gamma_{xy} \in G_{\delta,T}} \delta_{\gamma_{xy}})\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_{\delta,T}})\, dx\, dy}.$$

COROLLARY 3. *Let M be a closed and connected rank-one manifold equipped with a C^∞ Riemannian metric. Then, for any $\delta > 0$, the measures $(\mu_{\delta,T})_T$ converge weakly to $\mu_{\max}$ as $T \to +\infty$.*

3. *Constant and positive potentials*

We state in this section some results which are not a direct consequence of the previous ones.

3.1. *Positive potentials.*

Consider the probability measures defined on SM by

$$m_T := \frac{\int (\sum_{\gamma_{xy} \in G_T} e^{\int_{\gamma_{xy}} F} \delta_{\gamma_{xy}})\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy}.$$

THEOREM 4. *Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric and $F \in C_{\mathbb{R}}(SM)$. Assume that $P(F) > 0$. Then the weak limits of $(m_T)_T$ are equilibrium states corresponding to the potential F. If M has negative curvature then $(m_T)_T$ converges to the unique equilibrium state μ_F corresponding to F.*

Theorem 4 is a consequence of part (1) of the following theorem.

THEOREM 5. *Let M be a closed and connected manifold equipped with a C^∞ Riemannian metric and $F \in C_{\mathbb{R}}(SM)$.*

(1) *If $P(F) > 0$ then, for any closed subset K of $\mathcal{P}(SM)$ such that $J_F(K) \le P(F)$,*

$$\limsup_{T \to +\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy} \in G_T : \delta_{\gamma_{xy}} \in K} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \le -J_F(K).$$

(2) *Under Condition 1, for any open subset O of $\mathcal{P}(SM)$,*

$$\liminf_{T \to +\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy} \in G_T : \delta_{\gamma_{xy}} \in O} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \ge -J_F(O).$$

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Given F and any constant c, observe that $\mathcal{P}_e(F) = \mathcal{P}_e(F + c)$. On the other hand, we can find a constant $c > 0$ such that $F + c \geq 0$. Thus, up to a constant, we can always assume that $P(F) \geq 0$.

3.2. *Constant potentials.* We consider here constant potentials, equivalent to set $F \equiv 0$. Let J be the functional defined by (3) and corresponding to $F = 0$, namely $J(m) = \sup_\omega(\int \omega\, dm - Q(\omega))$ with $Q(\omega) = P(\omega) - h_{\text{top}}$.

THEOREM 6. *Suppose that M is a closed and connected manifold equipped with a C^∞ Riemannian metric. If M has no conjugate points then, for any $\delta > 0$:*

(1) *for any closed subset K of $\mathcal{P}(SM)$ and almost every (a.e.) $(x, y) \in M \times M$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \frac{\#\{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}} \in K\}}{\#G_{\delta,T}} \leq -J(K);$$

(2) *for a.e. $(x, y) \in M \times M$, the weak limits of*

$$\mu_{\delta,T}(x, y) := \frac{\sum_{\gamma_{xy}\in G_{\delta,T}} \delta_{\gamma_{xy}}}{\#G_{\delta,T}}$$

are measures of maximal entropy.

In part (1) of Theorem 6 (and Theorem 7 below), the set of points $(x, y) \in M \times M$ for which we have the upper bound depends on the given closed set K, while in part (2) it depends only on δ.

THEOREM 7. *Suppose that M is a closed and connected manifold equipped with a C^∞ Riemannian metric. If M has no conjugate points and $h_{\text{top}} > 0$ then:*

(1) *for any closed subset K of $\mathcal{P}(SM)$ such that $J(K) \leq h_{\text{top}}$ and a.e. $(x, y) \in M \times M$,*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \frac{\#\{\gamma_{xy} \in G_T : \delta_{\gamma_{xy}} \in K\}}{\#G_T} \leq -J(K);$$

(2) *for a.e. $(x, y) \in M \times M$, the weak limits of*

$$\mu_T(x, y) := \frac{\sum_{\gamma_{xy}\in G_T} \delta_{\gamma_{xy}}}{\#G_T}$$

are measures of maximal entropy.

4. *Proofs*

4.1. *Growth of geodesic arcs.* The following two theorems of Mañé [**10**] and Paternain [**12**] are the main tools in the proof of our results.

THEOREM 8. (Mañé [**10**]) *Let M be a closed and connected manifold equipped with a C^∞ metric.*

(1) *We have*

$$h_{\text{top}} = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} \#G_T(x, y)\, dx\, dy.$$

(2) *If M has no conjugate points then, for all (x, y),*

$$h_{\text{top}} = \lim_{T\to\infty} \frac{1}{T} \log \#G_T(x, y).$$

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(3) *Suppose that the metric is of class C^3 and M does not have conjugate points. Then for any $\delta > 0$ and all (x, y),*

$$h_{\text{top}} = \lim_{T\to\infty} \frac{1}{T} \log \#G_{\delta,T}(x, y).$$

THEOREM 9. (Paternain [**12**]) *Let M be a closed and connected manifold equipped with a C^∞ metric.*

(1) *For any $\delta > 0$,*

$$P(F) = \lim_{T\to+\infty} \frac{1}{T} \log \int_{M\times M} \left(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F} \right) dx\, dy.$$

(2) *If $P(F) \geq 0$, then*

$$P(F) = \lim_{T\to+\infty} \frac{1}{T} \log \int_{M\times M} \left(\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F} \right) dx\, dy.$$

For any invariant probability measure μ, we set

$$I(\mu) := P(F) - \left(h(\mu) + \int_{SM} F\, d\mu \right). \tag{7}$$

LEMMA 1.

(1) *The functional Q_F is convex and continuous on continuous functions. Furthermore, it is invariant by the geodesic flow, that is, $Q_F(\omega \circ \phi_t) = Q_F(\omega)$ for all t.*

(2) *$Q_F(\omega) = \sup_{\mu\in\mathcal{P}_{\text{inv}}(SM)} (\int \omega\, d\mu - I(\mu))$. In other words, the functionals I and J_F agree on invariant measures.*

(3) *Let G be a continuous function on SM and J_{F+G} the functional corresponding to Q_{F+G}. Then*

$$J_{F+G}(m) = J_F(m) + Q_F(G) - \int G\, dm.$$

Proof. Part (1) is a consequence of the convexity of the pressure function P and the variational principle (equation (1)) from which we can easily deduce that $|P(f) - P(g)| \leq \|f - g\|_\infty$ [**16**]. Part (2) follows from (7) and

$$\begin{aligned} \sup_{\mu\in\mathcal{P}_{\text{inv}}(SM)} \left(\int \omega\, d\mu - I(\mu) \right) &= \sup_{\mu\in\mathcal{P}_{\text{inv}}(SM)} \left(\int \omega\, d\mu - P(F) + h(\mu) + \int F\, d\mu \right) \\ &= P(F + \omega) - P(F) = Q_F(\omega). \end{aligned}$$

Part (3) results from the following equalities:

$$\begin{aligned} &\sup_m \left(\int \omega\, dm - J_F(m) - Q_F(G) + \int G\, dm \right) \\ &= \sup_m \left(\int (\omega + G)\, dm - J_F(m) \right) - Q_F(G) \\ &= Q_F(\omega + G) - Q_F(G) \\ &= P(F + \omega + G) - P(F + G) = Q_{F+G}(\omega). \end{aligned}$$

□

In the following, differentiability means Gâteaux differentiability for convex functions [**6, 7, 14**].

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LEMMA 2. *Let G be a continuous function on SM. The following statements are equivalent.*

(1) *There exists a unique probability measure m on SM such that $J_{F+G}(m) = 0$.*
(2) *There exists a unique probability measure m on SM such that $Q_F(G) = \int G\, dm - J_F(m)$.*
(3) *The functional $\omega \to Q_F(\omega)$ is differentiable at G.*
(4) *The functional $\omega \to P(F + \omega)$ is differentiable at G.*
(5) *There exists a unique equilibrium state for the potential $F + G$.*

Proof. Points (1) and (2) are equivalent by Lemma 1(3). By [**7**, Theorems 4.3.3 and 4.3.5], points (2) and (3) (and (4) and (5), respectively) are also equivalent (see also [**14**, Theorem 25.1, p. 242]). Now the equivalence between (3) and (4) follows from the definition (2) of Q_F. Observe that by Lemma 1(2), (1) implies (5) and, furthermore, the unique probability measure such that $J_{F+G}(m) = 0$ is necessarily the unique equilibrium state for $F + G$. □

In this paper, we apply this lemma to the special case where $G = \sum_{i=1}^{n} \beta_i g_i$ (see Condition 1). Thus the potential $F + \sum_{i=1}^{n} \beta_i g_i$ admits a unique equilibrium state if and only if $(\alpha_1, \ldots, \alpha_n) \to Q_F(\sum_{i=1}^{n} \alpha_i g_i)$ is differentiable at β. Consequently, by Lemma 2, if $G \equiv 0$, the uniqueness of m such that $J_F(m) = 0$ is equivalent to the uniqueness of the equilibrium state μ_F corresponding to F. We necessarily have $J_F(\mu_F) = 0$.

4.2. *Proof of Theorem 1(1).*

Proof. We have to prove that

$$\limsup_{T\to\infty} \frac{1}{T} \log \nu_T(K) \le -\inf_{m\in K} J_F(m) := -J_F(K).$$

Let $\epsilon > 0$. There exist a finite number of continuous functions $\omega_1, \ldots, \omega_l$ such that $K \subset \bigcup_{i=1}^{l} K_i$, where

$$K_i = \left\{ m \in \mathcal{P}(SM) : \int \omega_i\, dm - Q_F(\omega_i) > J_F(K) - \epsilon \right\}.$$

Put

$$\Gamma_i(x, y, T) := \{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}} \in K_i\}$$

and

$$Z_i(T) := \int_{M\times M} \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} F}\, dx\, dy.$$

Now $\nu_T(K) \le \sum_{i=1}^{l} \nu_T(K_i)$ and

$$\nu_T(K_i) = \frac{Z_i(T)}{\int_{M\times M} (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy}.$$

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Also

$$Z_i(T) \le \int_{M\times M} \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} F} e^{l(\gamma_{xy})(\int \omega_i d\delta_{\gamma_{xy}} - Q_F(\omega_i)-(J_F(K)-\epsilon))}\, dx\, dy.$$

Set

$$C := \sum_{i\le l} \sup(1,\, e^{-\delta(-Q_F(\omega_i)-(J_F(K)-\epsilon))}).$$

Thus, by taking into account the sign of $-Q_F(\omega_i) - (J_F(K) - \epsilon)$,

$$\begin{aligned} Z_i(T) &\le \int_{M\times M} \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}}(F+\omega_i)} e^{l(\gamma_{xy})(-Q_F(\omega_i)-(J_F(K)-\epsilon))}\, dx\, dy \\ &\le Ce^{T(-Q_F(\omega_i)-(J_F(K)-\epsilon))} \int_{M\times M} \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}}(F+\omega_i)}\, dx\, dy. \end{aligned}$$

For T sufficiently large, it follows from Theorem 9(1) that

$$\begin{aligned} \nu_T(K) &\le \sum_{i=1}^{l} \frac{Z_i(T)}{\int \sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F}\, dx\, dy} \\ &\le C\sum_{i=1}^{l} e^{T(P(F+\omega_i)+\epsilon)} e^{-T(P(F)-\epsilon)} e^{T(-Q(\omega_i)-(J_F(K)-\epsilon))} \\ &= Cle^{T(-J_F(K)+3\epsilon)}. \end{aligned}$$

Taking logarithms, dividing by T and taking the lim sup, one obtains

$$\limsup_{T\to\infty} \frac{1}{T} \log \nu_T(K) \le -J_F(K) + 3\epsilon.$$

Since ϵ is arbitrary, Theorem 1(1) follows. □

4.3. *Proof of Corollary 1.* It suffices to apply Theorem 1 to the closed set $K = \mathcal{P}(SM)\backslash V$, where V is an open neighborhood of $\{m : J_F(m) = 0\}$. We have $J_F(K) = J_F(m)$ for some $m \in K$ and $\nu_T(V) = 1 - \nu_T(K)$. For T sufficiently large,

$$1 \ge \nu_T(V) \ge 1 - e^{-TJ_F(m)}.$$

Since $J_F(m) > 0$, the corollary follows.

4.4. *Proof of Theorem 2(2).* Given a continuous function $g : SM \to \mathbb{R}^n$, we have

$$Q_F(\beta \cdot g) = \sup_{\alpha\in\mathbb{R}^n} (\beta\cdot\alpha - J_F(\alpha;\, g)), \tag{8}$$

where $\beta \cdot g = \sum_{i=1}^{n} \beta_i g_i$ for $\beta \in \mathbb{R}^n$. By duality,

$$J_F(\alpha;\, g) = \sup_{\beta\in\mathbb{R}^n} (\beta\cdot\alpha - Q_F(\beta\cdot g)). \tag{9}$$

The same ideas as in [**8**] together with the convex analysis techniques in [**14**] will produce two points α_ε and β_ε, with $J_F(\alpha_\varepsilon;\, g) < +\infty$, such that

$$Q_F(\beta_\varepsilon \cdot g) = \beta_\varepsilon \cdot \alpha_\varepsilon - J_F(\alpha_\varepsilon;\, g).$$

We repeat here the arguments with our notation for the convenience of the reader.

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If $J_F(O_n;\, g) = +\infty$ then there is nothing to do. Suppose, then, that $J_F(O_n;\, g) < +\infty$. Thus, given $\varepsilon > 0$, we can choose $\alpha_\varepsilon \in O_n$ with $J_F(\alpha_\varepsilon;\, g) < +\infty$ such that

$$J_F(O_n;\, g) > J_F(\alpha_\varepsilon;\, g) - \varepsilon. \tag{10}$$

In particular, $J_F(\alpha_\varepsilon;\, g) < +\infty$, that is, $\alpha_\varepsilon \in \mathrm{Dom}(J_F(\cdot\,;\, g))$.

Let $\mathrm{ri}(\mathrm{Dom}(J_F(\cdot\,;\, g)))$ (see [**14**, p. 44]) be the relative interior of the affine hull of the (convex) domain

$$\mathrm{Dom}(J_F(\cdot\,;\, g)) := \{\alpha : J_F(\alpha;\, g) < +\infty\}.$$

Then either $\mathrm{ri}(\mathrm{Dom}(J_F(\cdot\,;\, g))) \neq \emptyset$ or $\mathrm{Dom}(J_F(\cdot\,;\, g))$ consists of one point. In the first case, by [**14**, Corollary 7.5.1] for any α and any $\alpha' \in \mathrm{Dom}(J_F(\cdot\,;\, g))$, it follows from the convexity and the lower semicontinuity of $J_F(\cdot\,;\, g)$ that

$$\lim_{p\uparrow 1} J_F(p\alpha + (1-p)\alpha';\, g) = J_F(\alpha;\, g). \tag{11}$$

In particular, this is true for $\alpha = \alpha_\varepsilon$ and $\alpha' \in \mathrm{Dom}(J_F(\cdot\,;\, g))\backslash\{\alpha_\varepsilon\}$ (which exists by assumption). For p sufficiently close to 1, the resulting point $p\alpha + (1-p)\alpha'$ is an element of $\mathrm{ri}(\mathrm{Dom}(J_F(\cdot\,;\, g)))$ and, by (11), satisfies (10). So we can choose $\alpha_\varepsilon \in \mathrm{ri}(\mathrm{Dom}(J_F(\cdot;\, g)))$.

Thus, by [**14**, Theorems 23.4 and 23.5], there exists $\beta_\epsilon \in \mathbb{R}^n$ such that

$$Q_F(\beta_\epsilon \cdot g) = \beta_\epsilon \cdot \alpha_\epsilon - J_F(\alpha_\epsilon;\, g). \tag{12}$$

If $\mathrm{Dom}(J_F(\cdot\,;\, g))$ consists of one point then, by (8), equality (12) is satisfied for all $\beta \in \mathbb{R}^n$.

Consider now a small neighborhood of α_ε,

$$O_{n,r} := \{\alpha \in \mathbb{R}^n : |\alpha_\varepsilon - \alpha| \le r\},$$

such that $O_{n,r} \subset O_n$. Define, for any $E \subset \mathbb{R}^n$,

$$\Gamma_T(E) := \{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}}(g) \in E\},$$

$$Z_T(E) := \frac{\int (\sum_{\gamma_{xy}\in\Gamma_T(E)} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy}.$$

We have $Z_T(O_n) \ge Z_T(O_{n,r})$ and

$$\begin{aligned}
&\sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}} F} \\
&= e^{-T\beta_\varepsilon\cdot\alpha_\varepsilon} \sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}} F} e^{-T(\beta_\varepsilon\cdot(\delta_{\gamma_{xy}}(g)-\alpha_\varepsilon))} e^{T\beta_\varepsilon\cdot\delta_{\gamma_{xy}}(g)} \\
&\ge e^{-T\beta_\varepsilon\cdot\alpha_\varepsilon} e^{-r\|\beta_\varepsilon\|T} \sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}} F} e^{T\beta_\varepsilon\cdot\delta_{\gamma_{xy}}(g)} \\
&= e^{-T\beta_\varepsilon\cdot\alpha_\varepsilon} e^{-r\|\beta_\varepsilon\|T} \sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}} F} e^{l(\gamma_{xy})\beta_\varepsilon\cdot\delta_{\gamma_{xy}}(g)} e^{(T-l(\gamma_{xy}))\beta_\varepsilon\cdot\delta_{\gamma_{xy}}(g)}.
\end{aligned}$$

By the condition $0 \le T - l(\gamma_{xy}) \le \delta$,

$$e^{(T-l(\gamma_{xy}))\beta_\varepsilon\cdot\delta_{\gamma_{xy}}(g)} \ge e^{-\delta\|\beta_\varepsilon\cdot g\|_\infty}.$$

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Thus

$$\sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}} F} \geq e^{-T\beta_\varepsilon\cdot\alpha_\varepsilon} e^{-r\|\beta_\varepsilon\|T} e^{-\delta\|\beta_\varepsilon\cdot g\|_\infty} \sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)}.$$

Set

$$Z_T^\varepsilon(O_{n,r}) := \frac{\int(\sum_{\gamma_{xy}\in\Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)})\,dx\,dy}$$

and

$$Z_T(\beta_\varepsilon\cdot g) := \frac{1}{T}\log\frac{\int(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)})\,dx\,dy}{\int\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F}\,dx\,dy}.$$

Therefore,

$$\begin{aligned}\frac{1}{T}\log Z_T(O_{n,r}) &\\ \geq -r\|\beta_\varepsilon\| - \frac{\delta\|\beta_\varepsilon\cdot g\|_\infty}{T} + (Z_T(\beta_\varepsilon\cdot g) - \beta_\varepsilon\cdot\alpha_\varepsilon) + \frac{1}{T}\log Z_T^\varepsilon(O_{n,r}). &\quad (13)\end{aligned}$$

From Theorem 9(1) we get

$$\lim_{T\to\infty} Z_T(\beta_\varepsilon\cdot g) = P(F+\beta_\varepsilon\cdot g) - P(F) = Q_F(\beta_\varepsilon\cdot g).$$

Thus

$$\liminf_{T\to\infty}\frac{1}{T}\log Z_T(O_n) \geq -r\|\beta_\varepsilon\| + (Q_F(\beta_\varepsilon\cdot g) - \beta_\varepsilon\cdot\alpha_\varepsilon) + \lim_{T\to\infty}\frac{1}{T}\log Z_T^\varepsilon(O_{n,r}).$$

We will show that

$$\lim_{T\to\infty} Z_T^\varepsilon(O_{n,r}) = 1. \quad (14)$$

Let us see how to finish the proof using (14):

$$\begin{aligned}\liminf_{T\to\infty}\frac{1}{T}\log Z_T(O_n) &\geq -r\|\beta_\varepsilon\| + Q_F(\beta_\varepsilon\cdot g) - \beta_\varepsilon\cdot\alpha_\varepsilon\\ &= -r\|\beta_\varepsilon\| - J_F(\alpha_\varepsilon; g)\\ &\geq -r\|\beta_\varepsilon\| - J_F(O_n; g) - \varepsilon,\end{aligned}$$

for any $\varepsilon > 0$. Since $r > 0$ was chosen arbitrarily, we let $r \to 0$ and $\varepsilon \to 0$ and we get

$$\liminf_{T\to\infty}\frac{1}{T}\log Z_T(O_n) \geq -J_F(O_n; g),$$

which completes the proof Theorem 2(2).

It remains to show (14). Let $K_{n,r}$ be the complement set of $O_{n,r}$ in the image $g*(\mathcal{P}(SM))$ of $\mathcal{P}(SM)$ under the continuous map $g*: m \to g\cdot m$. We have $Z_T^\varepsilon(O_{n,r}) + Z_T^\varepsilon(K_{n,r}) = 1$. The goal is to show that $Z_T^\varepsilon(K_{n,r})$ decreases exponentially fast to zero as $T \to \infty$ using Theorem 1.

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Consider $J^\varepsilon := J_{F+\beta_\varepsilon \cdot g}$, the functional J corresponding to $Q^\varepsilon := Q_{F+\beta_\varepsilon \cdot g}$. We have

$$Q^\varepsilon(\omega) = P(F + \beta_\varepsilon \cdot g + \omega) - P(F + \beta_\varepsilon \cdot g)$$

and, by (3),

$$J^\varepsilon(m) = \sup_\omega \left(\int \omega \, dm - Q^\varepsilon(\omega) \right).$$

From Lemma 1(3) we get

$$J^\varepsilon(m) = J_F(m) + Q_F(\beta_\varepsilon \cdot g) - \int \beta_\varepsilon \cdot g \, dm$$

and

$$\inf_{m(g)=\alpha} J^\varepsilon(m) = \inf_{m(g)=\alpha} J_F(m) + Q_F(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha.$$

The set $K_{n,r}$ is compact in $\mathbb{R}^n$ and, by Theorem 1(1),

$$\limsup_{T\to\infty} \frac{1}{T} \log Z_T^\varepsilon(K_{n,r}) \le -J^\epsilon(K),$$

with $K := (g*)^{-1}(K_{n,r})$, which is a closed subset of $\mathcal{P}(SM)$. We have

$$J^\varepsilon(K) = \inf_{\alpha \in K_{n,r}} (J_F(\alpha;\, g) + Q_F(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha).$$

If $J^\varepsilon(K) = +\infty$ there is nothing to do and the result follows. The key point is to prove that $J^\varepsilon(K) > 0$. Set

$$J^\varepsilon(\alpha;\, g) := J_F(\alpha;\, g) + Q_F(\beta_\varepsilon \cdot g) - \beta_\varepsilon \cdot \alpha.$$

The functional J^ε is non-negative (since $Q^\varepsilon(0) = 0$), lower semicontinuous and then it achieves its minimum on compact sets. By (9), $J^\varepsilon(\alpha;\, g) \ge 0$; and by (12), $J^\varepsilon(\alpha_\varepsilon;\, g) = 0$. Recall that if $J^\varepsilon(\alpha;\, g) = 0$ for some $\alpha \in K_{n,r}$, then there corresponds to α an equilibrium state $m_\alpha \in K$ for the potential $F + \beta_\varepsilon \cdot g$ such that $m_\alpha(g) = \alpha$. The vector α_ε is the unique point realizing the minimum, that is, the unique solution for the equation $J^\varepsilon(\alpha;\, g) = 0$. Indeed, by Lemma 2, two different solutions will produce two distinct equilibrium states for the potential $F + \beta_\varepsilon \cdot g$, which contradicts Condition 1. Since $\alpha_\varepsilon \in O_{n,r}$, it follows that $J^\varepsilon(\alpha;\, g) > 0$ for $\alpha \in K_{n,r}$. On the other hand, the set $K_{n,r}$ being compact, by the lower semicontinuity of J^ε we have $J^\varepsilon(K) = \inf_{\alpha \in K_{n,r}} J^\varepsilon(\alpha;\, g) > 0$. Thus we have proved that

$$\limsup_{T\to\infty} \frac{1}{T} \log Z_T^\varepsilon(K_{n,r}) \le -J^\varepsilon(K) < 0,$$

from which (14) follows immediately.

4.5. *Proof of Theorem 1(2).* Let $O \subset \mathcal{P}(SM)$ be an open set and $\epsilon > 0$. Choose $m_\epsilon \in O$ such that

$$J_F(m_\epsilon) \le J_F(O) + \epsilon.$$

We endow the space $\mathcal{P}(SM)$ with a compatible topology generated by the distance given by

$$d(m, m') := \sum_{k=1}^{\infty} 2^{-k} |m(g_k) - m'(g_k)|,$$

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where the functions g_k were defined in §2.2. Following [**8**], we define

$$d_n(m, m') := \sum_{k=1}^{n} 2^{-k}|m(g_k) - m'(g_k)|.$$

Set $2r = \inf\{d(m, m_\epsilon) : m \in \mathcal{P}(SM)\backslash O\}$. We have $r > 0$, since $\mathcal{P}(SM)\backslash O$ is a compact subset of $\mathcal{P}(SM)$. Since, for all k, $\|g_k\| = 1$,

$$0 \le d(m, m') - d_n(m, m') \le 2^{-(n-1)}.$$

Thus, for n sufficiently large,

$$O_{\epsilon,r} := \{m \in \mathcal{P}(SM) : d_n(m, m_\epsilon) < r\} \subset O.$$

For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, write $\|\alpha\|_n = \sum_{k=1}^{n} 2^{-k}|\alpha_k|$. Set

$$\alpha_\epsilon := \left(\int g_1\, dm_\epsilon, \ldots, \int g_n\, dm_\epsilon\right) = m_\epsilon(g^{(n)})$$

and

$$O_{n,r} := \{\alpha \in \mathbb{R}^n : \|\alpha_\epsilon - \alpha\|_n < r\}.$$

Then $g^{(n)}(O_{\epsilon,r}) = O_{n,r} \cap g^{(n)}(\mathcal{P}(SM))$. From Theorem 2(2) we get

$$\begin{aligned}
&\liminf_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in O} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \\
&\quad \ge \liminf_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in O_{\epsilon,r}} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \\
&\quad = \liminf_{T\to+\infty} \frac{1}{T} \log \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}(g^{(n)})\in O_{n,r}} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \\
&\quad \ge -J_F(O_{n,r}; g) \\
&\quad \ge -J_F(\alpha_\epsilon; g) \ge -J_F(m_\epsilon) \ge -J_F(O) - \epsilon,
\end{aligned}$$

for any $\epsilon > 0$. This completes the proof of the main Theorem 1.

4.6. *Proof of Theorem 3.* First, observe that any weak limit of $m_{\delta,T}$ is invariant by the flow. Indeed, for any continuous function f,

$$\begin{aligned}
m_{\delta,T}(f \circ \phi_s) = m_{\delta,T}(f) &+ \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F} \delta^{(1)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \\
&- \frac{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F} \delta^{(2)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy},
\end{aligned}$$

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where

$$\delta^{(1)}_{\gamma_{xy}}(f) = \frac{1}{l(\gamma_{xy})} \int_{l(\gamma_{xy})}^{l(\gamma_{xy})+s} f(\phi_t(\dot{\gamma}_{xy}(0)))\, dt$$

and

$$\delta^{(2)}_{\gamma_{xy}}(f) = \frac{1}{l(\gamma_{xy})} \int_{0}^{s} f(\phi_t(\dot{\gamma}_{xy}(0)))\, dt.$$

It is clear that

$$|\delta^{(1)}_{\gamma_{xy}}(f)| \le \frac{Cs}{T-\delta} \quad \text{and} \quad |\delta^{(2)}_{\gamma_{xy}}(f)| \le \frac{Cs}{T-\delta},$$

where C is a constant depending only on f.

By Lemma 1(2) it remains to show that any weak limit m_∞ of $m_{\delta,T}$ satisfies $J_F(m_\infty) = 0$. Let $V \subset \mathcal{P}(SM)$ be a convex open neighborhood of the closed and convex set $E_0 := \{m \in \mathcal{P}(SM) : J_F(m) = 0\}$ and $\epsilon > 0$. We consider a finite open cover $(B_i(\epsilon))_{i \le N}$ of E_0 by balls of diameter ϵ all contained in V. Define the measures on SM:

$$m_T^V := \frac{\int (\sum_{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}} \in V} e^{\int_{\gamma_{xy}} F} \delta_{\gamma_{xy}})\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy}.$$

Decompose the set $U := \bigcup_{i=1}^N B_i(\epsilon)$ into a disjoint union

$$U = \bigcup_{j=1}^{N'} U_j^\epsilon,$$

where the sets U_j^ϵ are disjoint and contained in one of the balls $(B_i(\epsilon))_{i \le N}$. Then

$$E_0 \subset U \subset V.$$

We fix in each U_j^ϵ a probability measure m_j, $j \le N'$, and let m_0 be a probability measure distinct from those above (for example, take $m_0 \in V \backslash U$).

Define

$$\beta_T = \sum_{j=1}^{N'} \nu_T(U_j^\epsilon) m_j + (1 - \nu_T(U)) m_0. \tag{15}$$

We have $\sum_{j=1}^{N'} \nu_T(U_j^\epsilon) = \nu_T(U)$. The probability measure β_T lies in V since it is a convex combination of elements in the convex set V. Then $d(m_{\delta,T}, V) \le d(m_{\delta,T}, \beta_T)$. We will show that

$$d(m_{\delta,T}, \beta_T) \le \epsilon \nu_T(U) + 3\nu_T(U^c),$$

where $U^c = \mathcal{P}(SM) \backslash U$ which is closed.

By definition of $m_{\delta,T}$ and m_T^V and the fact that $U \subset V$,

$$\sum_{k \ge 1} 2^{-k} |m_{\delta,T}(g_k) - m_T^V(g_k)| \le \nu_T(U^c).$$

It remains to show that $d(m_T^V, \beta_T) \le \epsilon \nu_T(U) + \nu_T(U^c)$. We have, for all $k \ge 1$,

$$|m_T^V(g_k) - \beta_T(g_k)| \le A + B + C,$$

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where

$$A = \frac{\sum_{j=1}^{N'} \int (\sum_{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}} \in U_j^\epsilon} e^{\int_{\gamma_{xy}} F} |\delta_{\gamma_{xy}}(g_k) - m_j(g_k)|)\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy},$$

$$B = \frac{\int (\sum_{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}} \in V \setminus U} e^{\int_{\gamma_{xy}} F} \delta_{\gamma_{xy}}(g_k))\, dx\, dy}{\int (\sum_{\gamma_{xy} \in G_{\delta,T}} e^{\int_{\gamma_{xy}} F})\, dx\, dy},$$

$$C = |(1 - \nu_T(U)) m_0(g_k)|.$$

Thus, since $\|g_k\| = 1$ for all $k \geq 1$, by definition of ν_T (5) we get

$$\sum_{k \geq 1} 2^{-k} |m_T^V(g_k) - \beta_T(g_k)| \leq \epsilon \sum_{j=1}^{N'} \nu_T(U_j^\epsilon) + \nu_T(U^c) + (1 - \nu_T(U))$$
$$= \epsilon \nu_T(U) + 2\nu_T(U^c).$$

Finally, we obtain

$$d(m_{\delta,T}, \beta_T) \leq \epsilon \nu_T(U) + 3\nu_T(U^c).$$

This implies the desired inequality,

$$d(m_{\delta,T}, V) \leq \epsilon \nu_T(U) + 3\nu_T(U^c).$$

By Corollary 1, since U^c is closed, we know that $\lim_{T \to \infty} \nu_T(U) = 1$. Thus,

$$\limsup_{T \to \infty} d(m_{\delta,T}, V) \leq \epsilon \quad \text{for all } \epsilon > 0.$$

We conclude that $\limsup_{T \to \infty} d(m_{\delta,T}, V) = 0$. The neighborhood V of E_0 being arbitrary, this implies that all the weak limit measures of $m_{\delta,T}$ are contained in E_0. In particular, if E_0 is reduced to one measure μ, this shows that m_T converges to μ.

4.7. *Proof of Theorem 5.*

4.7.1. *Proof of part (1).* Let $\epsilon > 0$. There exist a finite number of continuous functions $\omega_1, \ldots, \omega_l$ such that $K \subset \bigcup_{i=1}^l K_i$, where

$$K_i = \left\{ m \in \mathcal{P}(SM) : \int \omega_i\, dm - Q_F(\omega_i) > J_F(K) - \epsilon \right\}.$$

We can suppose that all the ω_i are non-negative since, adding a constant $c > 0$, we have

$$\int (\omega_i + c)\, dm - Q_F(\omega_i + c) = \int \omega_i\, dm - Q_F(\omega_i).$$

Put

$$\Gamma_i(x, y, T) := \{\gamma_{xy} \in G_T : \delta_{\gamma_{xy}} \in K_i\}$$

and

$$Z_i(T) := \int \sum_{\gamma_{xy} \in \Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} F}\, dx\, dy.$$

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From the definition of $\Gamma_i(x,\, y,\, T)$ we get

$$Z_i(T) \le \int \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} F} e^{l(\gamma_{xy})(\int \omega_i d\delta_{\gamma_{xy}} - Q_F(\omega_i)-(J_F(K)-\epsilon))}\, dx\, dy.$$

We decompose the interval $[0,\, T]$ into subintervals $[T-(j+1)\delta,\, T-j\delta]$ and set

$$\Gamma_{ij}(x,\, y,\, T) := \{\gamma_{xy} : T-(j+1)\delta < l(\gamma_{xy}) \le T-j\delta,\, \delta_{\gamma_{xy}} \in K_i\}.$$

Since the functions ω_i were supposed to be non-negative, we have $Q_F(\omega_i) \ge 0$ and then $Q_F(\omega_i) + J_F(K) \ge 0$. Thus, from Theorem 9(1) and for T sufficiently large,

$$\begin{aligned} Z_i(T) &\le \int \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}}(F+\omega_i)} e^{l(\gamma_{xy})(-Q_F(\omega_i)-(J_F(K)-\epsilon))}\, dx\, dy \\ &\le \sum_j e^{(T-(j+1)\delta)(-Q_F(\omega_i)-(J_F(K)-\epsilon))} \int \sum_{\gamma_{xy}\in\Gamma_{ij}(x,y,T)} e^{\int_{\gamma_{xy}}(F+\omega_i)}\, dx\, dy \\ &\le \sum_j e^{(T-(j+1)\delta)(-Q_F(\omega_i)-(J_F(K)-\epsilon))} e^{(T-j\delta)(P(F+\omega_i)+\epsilon)} \\ &= \sum_j e^{(T-j\delta)(P(F)-J_F(K)+2\epsilon)} e^{-\delta(-Q_F(\omega_i)-(J_F(K)-\epsilon))}. \end{aligned}$$

We assumed that $J_F(K) \le P(F)$; then

$$C := \sum_j e^{-j\delta(P(F)-J_F(K)+2\epsilon)} < \infty.$$

Setting $\lambda_i := e^{-\delta(-Q_F(\omega_i)-(J_F(K)-\epsilon))}$, we get for T sufficiently large and by Theorem 9(2) $(P(F) > 0)$,

$$\begin{aligned} \nu_T(K) &\le \frac{\sum_{i=1}^{l} Z_i(x,\, y,\, T)}{\int \sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F}\, dx\, dy} \\ &\le \sum_{i=1}^{l} \sum_j e^{T(-J_F(K)+3\epsilon)} \lambda_i e^{-j\delta(P(F)-J_F(K)+2\epsilon)} \\ &= e^{T(-J_F(K)+3\epsilon)} C \sum_{i=1}^{l} \lambda_i. \end{aligned}$$

Taking logarithms, dividing by T and taking the lim sup, one obtains

$$\limsup_{T\to\infty} \frac{1}{T} \log \nu_T(K) \le -J_F(K) + 3\epsilon.$$

Since ϵ is arbitrary, Theorem 5(1) follows.

4.7.2. *Proof of part (2).* As for the proof of part (2) of Theorem 5, the proof will be a consequence of the contraction principle,

$$\liminf_{T\to\infty} \frac{\int (\sum_{\gamma_{xy}\in G_T:\delta_{\gamma_{xy}}(g)\in O_n} e^{\int_{\gamma_{xy}} F})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy} \ge -J_F(O_n;\, g).$$

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We proceed along the lines of the proof of Theorem 2(2) with the same notation. Let $\delta > 0$. We have

$$\frac{\int(\sum_{\gamma_{xy}\in G_T:\delta_{\gamma_{xy}}(g)\in O_n} e^{\int_{\gamma_{xy}} F})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\,dx\,dy} \geq \frac{\int(\sum_{\gamma_{xy}\in \Gamma_T(O_n)} e^{\int_{\gamma_{xy}} F})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\,dx\,dy},$$

where

$$\Gamma_T(O_n) := \{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}}(g) \in O_n\}.$$

Set

$$Z_T(O_n) := \frac{\int(\sum_{\gamma_{xy}\in \Gamma_T(O_n)} e^{\int_{\gamma_{xy}} F})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\,dx\,dy}.$$

Then $Z_T(O_n) \geq Z_T(O_{n,r})$ and (see (13))

$$\begin{aligned}\frac{1}{T}\log Z_T(O_n) &\geq \frac{1}{T}\log Z_T(O_{n,r})\\ &\geq -r\|\beta_\varepsilon\| - \frac{\delta\|\beta_\varepsilon\cdot g\|_\infty}{T} + (Z_T(\beta_\varepsilon\cdot g) - \beta_\varepsilon\cdot\alpha_\varepsilon) + \frac{1}{T}\log Z_T^\varepsilon(O_{n,r}),\end{aligned}$$

where we have set

$$Z_T^\varepsilon(O_{n,r}) := \frac{\int(\sum_{\gamma_{xy}\in \Gamma_T(O_{n,r})} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)})\,dx\,dy}$$

and

$$Z_T(\beta_\varepsilon\cdot g) := \frac{1}{T}\log\frac{\int(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}}(F+\beta_\varepsilon\cdot g)})\,dx\,dy}{\int\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F}\,dx\,dy}.$$

From Theorem 9(1) and (2) respectively,

$$\lim_{T\to\infty} Z_T(\beta_\varepsilon\cdot g) = P(F+\beta_\varepsilon\cdot g) - P(F) = Q_F(\beta_\varepsilon\cdot g).$$

We proved in (14) that $\lim_{T\to\infty} Z_T^\varepsilon(O_{n,r}) = 1$. Thus,

$$\begin{aligned}&\liminf_{T\to\infty}\frac{1}{T}\log\frac{\int(\sum_{\gamma_{xy}\in G_T:\delta_{\gamma_{xy}}(g)\in O_n} e^{\int_{\gamma_{xy}} F})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\,dx\,dy}\\ &\geq \liminf_{T\to\infty}\frac{1}{T}\log\frac{\int(\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}(g)\in O_n} e^{\int_{\gamma_{xy}} F})\,dx\,dy}{\int(\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\,dx\,dy}\\ &\geq -r\|\beta_\varepsilon\| + (Q_F(\beta_\varepsilon\cdot g) - \beta_\varepsilon\cdot\alpha_\varepsilon) + \lim_{T\to\infty}\frac{1}{T}\log Z_T^\varepsilon(O_{n,r})\\ &= -r\|\beta_\varepsilon\| - J_F(\alpha_\varepsilon; g)\\ &\geq -r\|\beta_\varepsilon\| - J_F(O_n; g) - \varepsilon,\end{aligned}$$

for any $\varepsilon > 0$. Since $r > 0$ is arbitrary, to finish the proof of the contraction principle, we let $r \to 0$ and $\varepsilon \to 0$.

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4.8. *Proof of Theorem 4.* We have to prove that the weak limits of

$$m_T := \frac{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F} \delta_{\gamma_{xy}})\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy}$$

are in $\mathcal{P}_e(F)$. To show that the weak limits of m_T are invariant, we proceed as in the proof of Theorem 3. Let f be a continuous function on SM. We have

$$m_T(f\circ\phi_s) = m_T(f) + \frac{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F} \delta^{(1)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy} - \frac{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F} \delta^{(2)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy},$$

where

$$\delta^{(1)}_{\gamma_{xy}}(f) = \frac{1}{l(\gamma_{xy})} \int_{l(\gamma_{xy})}^{l(\gamma_{xy})+s} f(\phi_t(\dot\gamma_{xy}(0)))\, dt$$

and

$$\delta^{(2)}_{\gamma_{xy}}(f) = \frac{1}{l(\gamma_{xy})} \int_0^{s} f(\phi_t(\dot\gamma_{xy}(0)))\, dt.$$

Consider now $T_0 > 0$ and decompose the set G_T into

$$G_T = G_{T_0} \cup G_{T_0,T},$$

where $G_{T_0,T} = \{\gamma_{xy} : T_0 < l(\gamma_{xy}) \le T\}$. For $i \in \{1, 2\}$,

$$\frac{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F} \delta^{(i)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy} = \frac{\int (\sum_{\gamma_{xy}\in G_{T_0}} e^{\int_{\gamma_{xy}} F} \delta^{(i)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy} + \frac{\int (\sum_{\gamma_{xy}\in G_{T_0,T}} e^{\int_{\gamma_{xy}} F} \delta^{(i)}_{\gamma_{xy}}(f))\, dx\, dy}{\int (\sum_{\gamma_{xy}\in G_T} e^{\int_{\gamma_{xy}} F})\, dx\, dy}.$$

For T_0 fixed, since we have assumed that $P(F) > 0$, by Theorem 9(2), the first term on the right-hand side of this equality goes to zero as $T \to +\infty$. The second term is bounded above by C/T_0, where C depends only on f. Thus, since T_0 is arbitrary the invariance of the weak limits follows.

It remains to show that $J_F(m_\infty) = 0$ for any weak limit m_∞ of m_T. For this and in order to follow the proof of Theorem 3, we must show that we are able to apply Theorem 5(1) to any open and convex neighborhood V of $E_0 := \{m \in \mathcal{P}(SM) : J_F(m) = 0\}$, $V \subset \mathcal{P}(SM)$. There are minor changes due to the conditions $P(F) > 0$ and $J_F(K) \le P(F)$ in Theorem 5(1) ($K = V^c$).

Let $\nu \in E_0$ and $\mu \in \mathcal{P}(SM)\backslash E_0$ and consider a convex sum of these two measures $m = \alpha\mu + \beta\nu$, where $\alpha + \beta = 1$ ($\alpha \neq 0$ and $\beta \neq 0$). Observe that $m \notin E_0$ ($J_F(\nu) = 0$ but $J_F(\mu) > 0$). From the convexity of J_F we get $J_F(m) \le \alpha J_F(\mu)$. Let $V \subset \mathcal{P}(SM)$

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be an open and convex small neighborhood of E_0 such that $m \in V^c := K$ (note that, in particular, for all other contained small neighborhoods V we will have $m \in V^c$). Therefore,

$$\inf_{k\in K} J_F(k) \le J_F(m) \le \alpha J_F(\mu). \tag{16}$$

Now, since we have assumed that $P(F) > 0$, for a sufficiently small $\alpha > 0$ we get, from (16),

$$J_F(K) := \inf_{k\in K} J_F(k) \le P(F).$$

We can then apply Theorem 5(1) to the closed sets $K = V^c$ and conclude with the proof of Theorem 3.

4.9. *Proof of Theorem 6.*

4.9.1. *Proof of part (1).* This is essentially the proof of Theorem 1(1) with some modifications since it is in part based on Theorem 8 and the following lemma (see [**10**, Lemma 4.3], [**13**, Lemma 3.33, p. 68]).

LEMMA 3. *Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $f_n : X \to (0, +\infty)$ be a sequence of integrable functions. Then, for μ a.e. $x \in X$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log f_n(x) \le \limsup_{n\to\infty} \frac{1}{n} \log \int_X f_n \, d\mu.$$

Set $F = 0$ and proceed as in the proof of Theorem 1(1) with the same notation. We have

$$\begin{aligned} Z_i(x, y, T) &\le \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} \omega_i} e^{l(\gamma_{xy})(-Q(\omega_i)-(J(K)-\epsilon))} \\ &\le C e^{T(-Q(\omega_i)-(J(K)-\epsilon))} \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} \omega_i} \end{aligned}$$

and

$$\nu_T(K) \le \sum_{i=1}^{l} \frac{Z_i(x, y, T)}{\#G_{\delta,T}}.$$

For all (x, y) and T sufficiently large (depending on (x, y)), it follows from Theorem 8(2) that

$$\#G_{\delta,T} \ge e^{T(h_{\text{top}}-\epsilon)}.$$

On the other hand, it follows from Lemma 3 above (which can be applied to continuous time) and Theorem 9(1) that

$$\begin{aligned} \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\gamma_{xy}\in\Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} \omega_i} &\le \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} \omega_i} \\ &\le \limsup_{T\to\infty} \frac{1}{T} \log \int \Bigg(\sum_{\gamma_{xy}\in G_{\delta,T}} e^{\int_{\gamma_{xy}} \omega_i} \Bigg) dx \, dy \\ &\le P(\omega_i), \end{aligned}$$

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for (x, y) in a subset B_i of $M \times M$ of full Lebesgue measure and $i \le l$. The set B_i can be taken independent from i for obvious reasons, and then we will say that the above inequalities hold for a.e. (x, y) in a set B. However, note that B depends on the set of functions $\{\omega_i, i \le l\}$, which means that B depends on K. The proof can be now achieved similarly: for $(x, y) \in B$ fixed and T sufficiently large (depending on (x, y)),

$$\begin{aligned} \nu_T(K) &\le \sum_{i=1}^{l} \frac{Z_i(x, y, T)}{\#G_{\delta,T}} \\ &\le C \sum_{i=1}^{l} e^{T(P(\omega_i)+\epsilon)} e^{-T(h_{\text{top}}-\epsilon)} e^{T(-Q(\omega_i)-(J(K)-\epsilon))} \\ &= Cle^{T(-J(K)+3\epsilon)}. \end{aligned}$$

Thus $\limsup_{T\to\infty}(1/T)\log \nu_T(K) \le -J(K)$.

4.9.2. *Proof of part (2).* Set

$$\nu_T(E) = \frac{\#\{\gamma_{xy} \in G_{\delta,T} : \delta_{\gamma_{xy}} \in E\}}{\#G_{\delta,T}}$$

and

$$\mu_{T,V}(x, y) := \frac{\sum_{\gamma_{xy}\in G_{\delta,T}:\delta_{\gamma_{xy}}\in V} \delta_{\gamma_{xy}}}{\#G_{\delta,T}}.$$

Thus, proceeding as in the proof of Theorem 3, we get, for any open and convex neighborhood V of $\{m : J(m) = 0\}$ and all (x, y),

$$d(\mu_T(x, y), V) \le \epsilon \nu_T(U) + 3\nu_T(U^c). \tag{17}$$

Let $(V_i)_{i\ge1}$ be a decreasing sequence of sets of type V such that $\bigcap_{i\ge1} V_i = \{m : J(m) = 0\}$. For (x, y) in a set B_{V_i} of full Lebesgue measure, we get from Theorem 6(1) and (17) that

$$\limsup_{T\to\infty} d(\mu_T(x, y), V_i) \le \epsilon,$$

for all $\epsilon > 0$. Thus

$$\limsup_{T\to\infty} d(\mu_T(x, y), V_i) = 0.$$

Therefore, if $\mu(x, y)$ is a weak limit of $\mu_{\delta,T}(x, y)$, $(x, y) \in B_{V_i}$, we will have $d(\mu(x, y), V_i) = 0$. Also, there exists a set B of full Lebesgue measure where we have $d(\mu(x, y), V_i) = 0$ for all $i \ge 1$. From this we deduce that $\mu(x, y) \in \{m : J(m) = 0\}$ for $(x, y) \in B$. On the other hand, following the proof of Theorem 3, the measures $\mu(x, y)$, $(x, y) \in B$, are invariant. Thus, for a.e. $(x, y) \in M \times M$, $\mu(x, y)$ is a measure of maximal entropy.

4.10. *Proof of Theorem 7.*

4.10.1. *Proof of part (1).* The proof adapts the arguments of the proof of Theorem 5(1). For any $E \subset \mathcal{P}(SM)$, set

$$\nu_T(E) = \frac{\#\{\gamma_{xy} \in G_T : \delta_{\gamma_{xy}} \in E\}}{\#G_T}.$$

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Let $\epsilon > 0$. There exist a finite number of continuous functions $\omega_1, \ldots, \omega_l$ such that $K \subset \bigcup_{i=1}^{l} K_i$, where

$$K_i = \left\{ m \in \mathcal{P}(SM) : \int \omega_i \, dm - Q(\omega_i) > J(K) - \epsilon \right\}.$$

Again we can suppose that all the ω_i are non-negative.

Put

$$\Gamma_i(x, y, T) := \{\gamma_{xy} \in G_T : \delta_{\gamma_{xy}} \in K_i\}$$

and

$$Z_i(x, y, T) := \#\Gamma_i(x, y, T).$$

From the definition of $\Gamma_i(x, y, T)$,

$$Z_i(x, y, T) \leq \sum_{\gamma_{xy} \in \Gamma_i(x,y,T)} e^{l(\gamma_{xy})(\int \omega_i d\delta_{\gamma_{xy}} - Q(\omega_i) - (J(K) - \epsilon))}.$$

Set

$$\Gamma_{ij}(x, y, T) := \{\gamma_{xy} : T - (j+1)\delta < l(\gamma_{xy}) \leq T - j\delta, \, \delta_{\gamma_{xy}} \in K_i\}.$$

Since the functions ω_i were supposed to be non-negative, we have $Q(\omega_i) \geq 0$ and then $Q(\omega_i) + J(K) \geq 0$. Thus, from Theorem 9(1) and for T sufficiently large,

$$\begin{aligned} Z_i(x, y, T) &\leq \sum_{\gamma_{xy} \in \Gamma_i(x,y,T)} e^{\int_{\gamma_{xy}} \omega_i} e^{l(\gamma_{xy})(-Q(\omega_i) - (J(K) - \epsilon))} \\ &\leq \sum_j e^{(T-(j+1)\delta)(-Q(\omega_i) - (J(K) - \epsilon))} \sum_{\gamma_{xy} \in \Gamma_{ij}(x,y,T)} e^{\int_{\gamma_{xy}} \omega_i} \\ &\leq \sum_j e^{(T-(j+1)\delta)(-Q(\omega_i) - (J(K) - \epsilon))} e^{(T - j\delta)(P(\omega_i) + \epsilon)} \\ &= \sum_j e^{(T - j\delta)(h_{\text{top}} - J(K) + 2\epsilon)} e^{-\delta(-Q(\omega_i) - (J(K) - \epsilon))}. \end{aligned}$$

We assumed that $J(K) \leq h_{\text{top}}$; then

$$C := \sum_j e^{-j\delta(h_{\text{top}} - J(K) + 2\epsilon)} < \infty.$$

Setting $\lambda_i := e^{-\delta(-Q(\omega_i) - (J(K) - \epsilon))}$, we get, for T sufficiently large and by Theorem 8(2),

$$\begin{aligned} \nu_T(K) &\leq \sum_{i=1}^{l} \frac{Z_i(x, y, T)}{\#G_T} \\ &\leq \sum_{i=1}^{l} \sum_j e^{T(-J(K) + 3\epsilon)} \lambda_i e^{-j\delta(h_{\text{top}} - J(K) + 2\epsilon)} \\ &= e^{T(-J(K) + 3\epsilon)} C \sum_{i=1}^{l} \lambda_i. \end{aligned}$$

Taking logarithms, dividing by T and taking the lim sup, one obtains

$$\limsup_{T \to \infty} \frac{1}{T} \log \nu_T(K) \leq -J(K) + 3\epsilon.$$

Since ϵ is arbitrary, Theorem 7(1) follows.

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4.10.2. *Proof of part (2).* The proof follows readily from the proof of Theorems 4, 6(2) and 7(1).

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