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A class of unique g-measures

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Abstract. We consider g-measures for the one-sided shift on a sequence space. We give a condition which guarantees that the g-measure is unique and has a Bernoulli natural extension.

1. *Introduction*

In this paper, we prove a theorem about the uniqueness and ergodic properties of a certain type of g-measure. We start by describing the context of the theorem; a more detailed account can be found in [**5**] or [**9**], for example.

Let S be a finite set with the discrete topology, $X = \prod_{i=0}^{\infty} S$ have the product topology, and let T be the shift on X; that is, $(Tx)_i = x_{i+1}$. (Here and later, x_i denotes the ith coordinate of x.) Let $\mathcal{B}$ denote the Borel σ-algebra on X. Given any Borel probability measure μ on X, there is a non-negative Borel function g_μ on X which determines the conditional probabilities $\mu(\cdot \mid T^{-1}\mathcal{B})$; specifically, $\mu(\{x \in X : x_0 = s\} \mid T^{-1}\mathcal{B})(x) = g_\mu(sx_1x_2\ldots)$ for all $s \in S$, or equivalently,

$$\int_X f\,d\mu = \int_X \sum_{y \in T^{-1}Tx} g_\mu(y) f(y)\,d\mu$$

for all $f \in L^1(\mu)$.

Let $\mathcal{G}_X$ be the set of all non-negative Borel functions g on X such that $\sum_{y \in T^{-1}x} g(y) = 1$ for all $x \in X$, and let $g \in \mathcal{G}_X$. A probability measure μ on X is a *g-measure* if μ is T-invariant and $g_\mu = g$. The term g-measure was introduced by Keane [**6**], who studied the question of uniqueness of g-measures when g is (strictly) positive and continuous; an equivalent notion, chains with infinite connections, was studied by Doeblin and Fortet in [**3**].

The assumption that g is continuous and positive guarantees the existence of at least one g-measure μ, and at the same time rules out any trivial examples of non-uniqueness that arise due to the fact that g_μ is only well-defined modulo sets of μ-measure 0, since positivity implies that any set of μ-measure 1 is dense, and so g_μ can be represented by at most one continuous function. However, this is not sufficient to guarantee uniqueness: Bramson and Kalikow [**2**] have given an example of a positive, continuous g for which there is more than one g-measure.

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A useful tool in the study of g-measures is the *Ruelle operator* $\mathcal{L}_g$, defined by

$$\mathcal{L}_g f(x) = \sum_{y \in T^{-1}x} g(y) f(y) \quad (g \in \mathcal{G}_X,\ f \in C(X),\ x \in X).$$

A probability measure μ is a g-measure if and only if μ is $\mathcal{L}_g$-invariant; that is, $\mu(\mathcal{L}_g f) = \mu(f)$ for all $f \in C(X)$. Hence, for continuous g, uniqueness of the g-measure is equivalent to the uniform convergence of $n^{-1}\sum_{i=0}^{n-1} \mathcal{L}_g^i f$ to a constant for all $f \in C(X)$; if there is a unique g-measure μ, it follows from the dominated convergence theorem that the constant is $\mu(f)$.

Most known conditions for uniqueness require some stronger form of continuity (although see [**5**] for an exception to this rule). For example, let

$$\mathrm{var}_n(f) = \sup\{f(x) - f(y) : x, y \in X,\ x_i = y_i\ \forall\ 0 \le i \le n-1\} \quad (f \in C(X)).$$

Then if $g \in \mathcal{G}_X$ is positive and $\sum \mathrm{var}_n(\log g) < \infty$, there is a unique g-measure μ; the natural extension of (T, μ) is Bernoulli, and $\mathcal{L}_g^n f \to \mu(f)$ uniformly for all $f \in C(X)$. This result is due to Walters [**9**] (and is proved in the more general case where T is a topologically mixing subshift of finite type). Berbee [**1**] obtained the same conclusions under the weaker assumption that

$$\sum_{n=1}^{\infty} \exp\left(-\sum_{i=1}^{n} \mathrm{var}_i(\log g)\right) = \infty.$$

Given a function f on X, let

$$v_n(f) = \sup\{f(x) - f(y) : x, y \in X,\ x_i = y_i\ \forall\ i \ne n\}.$$

We prove the following.

THEOREM 1.1. *Let $g \in \mathcal{G}_X$ be positive and continuous. If $\sum v_n(g) < \infty$, there is a unique g-measure μ. The natural extension of (T, μ) is Bernoulli, and $\mathcal{L}_g^n f \to \mu(f)$ uniformly as $n \to \infty$, for all $f \in C(X)$.*

For a type of example not covered by previously known results, consider the following. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of non-negative numbers, let $S = \{-1, 1\}$, and define $g \in \mathcal{G}_X$ by

$$g(x) = \exp H(x)\left\{\sum_{y \in T^{-1}Tx} \exp H(y)\right\}^{-1},$$

where

$$H(x) = x_0 \sum_{i=1}^{\infty} x_i a_i.$$

Then $\sum v_n(g) < \infty$ for any choice of $\sum a_n$; hence there is a unique g-measure. This result has a certain curiosity value when compared with the situation for $\widetilde{X} = \prod_{i=-\infty}^{\infty}\{-1, 1\}$, since g-measures on X are the analogue of stationary Gibbs states on $\widetilde{X}$ in the sense that both are defined in terms of probabilities conditioned on the configuration outside a finite set (see [**8**] for a detailed description). An example which has received

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considerable attention is the Ising ferromagnet. Dyson [**4**] gave a condition under which it has more than one stationary Gibbs state, and so it might be expected that a similar construction would give examples of non-unique g-measures. However, this leads to the above class of examples, which have unique g-measures.

2. *Preliminaries*

Throughout the rest of this paper, we assume that $S = \{0, \dots, k\}$ for some $k \geq 1$, and if X is a space of the form $\prod_{i=0}^{\infty} S$ for some S, then X is partially ordered in the usual way: $x \leq y$ if $x_i \leq y_i$ for all $i \geq 0$. If $\xi \in S^n$ then $[\xi]$ denotes the cylinder set $\{x \in X : x_i = \xi_i,\ 0 \leq i \leq n-1\}$ and ξx is the concatenation of ξ and x $(x \in X)$. If $s \in S$ and $1 \leq n \leq \infty$, then s^n is the n-tuple such that $(s^n)_i = s$ for all $0 \leq i \leq n-1$.

Note that if μ is a g-measure and $\xi \in S^n$, then

$$\mu([\xi]) = \int g(\xi x) g(T\xi x) \cdots g(T^{n-1}\xi x)\, d\mu(x).$$

It follows that if g is bounded away from 0 (or equivalently, from 1) then μ is non-atomic.

Given $g \in \mathcal{G}_X$, we define probability measures $P^n_{x,y}$ on $S^n \times S^n$ $(n \geq 1,\ x, y \in X)$ as follows. Let $P(\cdot \mid x, y)$ be the probability measure on $S \times S$ defined by

$$P(\xi \geq i, \eta \geq j \mid x, y) = \min\left\{ \sum_{p=i}^{k} g(px), \sum_{p=j}^{k} g(py) \right\} \quad ((\xi, \eta) \in S \times S).$$

Then

$$\begin{aligned} &P(\xi = i, \eta = j \mid x, y) \\ &= \ P(\xi \geq i, \eta \geq j \mid x, y) - P(\xi \geq i+1, \eta \geq j \mid x, y) \\ &\quad -P(\xi \geq i, \eta \geq j+1 \mid x, y) + P(\xi \geq i+1, \eta \geq j+1 \mid x, y). \end{aligned}$$

It is clear from the definition that the first term on the right-hand side is greater than or equal to both the second and third terms, and if $\sum_{p=j+1}^{k} g(py) \geq \sum_{p=i+1}^{k} g(px)$, then

$$P(\xi \geq i+1, \eta \geq j+1 \mid x, y) = P(\xi \geq i+1, \eta \geq j \mid x, y),$$

whereas if $\sum_{p=j+1}^{k} g(py) \leq \sum_{p=i+1}^{k} g(px)$, then

$$P(\xi \geq i+1, \eta \geq j+1 \mid x, y) = P(\xi \geq i, \eta \geq j+1 \mid x, y).$$

Hence, $P(\xi = i, \eta = j \mid x, y) \geq 0$, and since $P(\xi \geq 0, \eta \geq 0 \mid x, y) = 1$, it follows that $P(\cdot \mid x, y)$ is indeed a probability measure. Now let $P^1_{x,y} = P(\cdot \mid x, y)$ and, for $n > 1$, let

$$P^n_{x,y}(\xi, \eta) = \prod_{i=0}^{n-1} P(\xi_i, \eta_i \mid \xi_{i+1} \dots \xi_{n-1} x, \eta_{i+1} \dots \eta_{n-1} y) \quad ((\xi, \eta) \in S^n \times S^n).$$

Note that $P^1_{x,y}(\xi \geq i) = \sum_{p=i}^{k} g(px)$ and $P^1_{x,y}(\eta \geq j) = \sum_{p=j}^{k} g(py)$; therefore,

$$P^1_{x,y}(\xi = i) = g(ix) \quad \text{and} \quad P^1_{x,y}(\eta = j) = g(jy).$$

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It follows that if $f \in C(X)$, then

$$\int f(\xi x)\, dP^n_{x,y}(\xi, \eta) = \mathcal{L}^n_g f(x) \tag{2.1}$$

and

$$\int f(\eta y)\, dP^n_{x,y}(\xi, \eta) = \mathcal{L}^n_g f(y). \tag{2.2}$$

The case $n = 1$ is easily checked, and this, together with the definition of $P^n_{x,y}$, implies that

$$\int f(\xi x)\, dP^n_{x,y}(\xi, \eta) = \int \mathcal{L}_g f(\xi' x)\, dP^{n-1}_{x,y}(\xi', \eta');$$

hence, (2.1) follows by induction. A similar calculation gives (2.2).

A $g \in \mathcal{G}_X$ is said to be *attractive* if $\sum_{p=i}^k g(px)$ is an increasing function of x for all $1 \le i \le k$; let $\mathcal{F}_X$ denote the set of all such g. If $g \in \mathcal{F}_X$ and $x \ge y$, then $P^n_{x,y}$ is supported on $\{(\xi, \eta) \in S^n \times S^n : \xi \ge \eta\}$. This is because if $i < j$, then

$$P(\xi \ge i, \eta \ge j \mid x, y) = \sum_{p=j}^k g(py) = P(\xi \ge i+1, \eta \ge j \mid x, y),$$

and so $P((i, j) \mid x, y) = 0$. We note the following results, which were proved in [**5**]. (In [**5**] it is assumed that g is bounded away from 0, but this is not needed for the proofs of these results.)

LEMMA 2.1. *If $g \in \mathcal{F}_X$ and $f \in C(X)$ is increasing, then $\mathcal{L}^n_g f$ is increasing for all $n \ge 1$.*

LEMMA 2.2. *Let $g \in \mathcal{F}_X$ be continuous, and let $\chi_j = \mathbf{1}\{x \in X : x_0 \ge j\}$ $(1 \le j \le k)$. If $\mathcal{L}^n_g \chi_j(k^\infty) - \mathcal{L}^n_g \chi_j(0^\infty) \to 0$ as $n \to \infty$ for all $1 \le j \le k$, then there is a unique g-measure μ, and $\mathcal{L}^n_g f \to \mu(f)$ uniformly as $n \to \infty$ for every $f \in C(X)$.*

It follows from Lemma 2.1 that if $g \in \mathcal{F}_X$ is continuous, then $\nu(f) = \lim_{n\to\infty} \mathcal{L}^n_g f(k^\infty)$ $(f \in C(X))$ defines a g-measure.

3. *Ergodic properties*

In this section, we give a sufficient condition for a g-measure to have a Bernoulli natural extension. If $g \in \mathcal{G}_X$, then $P^n_{x,y}$ denotes the corresponding couplings described in §2, and we define

$$\sigma_j(x, y) = \sum_{p=j}^k (g(px) - g(py)) \quad (x, y \in X,\ 0 \le j \le k).$$

PROPOSITION 3.1. *Let $g \in \mathcal{G}_X$. If*

$$\frac{1}{n} \sum_{i=1}^n \int |\sigma_j(\xi x, \eta y)|\, dP^i_{x,y}(\xi, \eta) \to 0$$

as $n \to \infty$, for all $x, y \in X$, $1 \le j \le k$, and μ is a non-atomic g-measure, then the natural extension of (T, μ) is Bernoulli.

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Proof. Let $(\widetilde{X}, \widetilde{\mathcal{B}}, \tilde{\mu}, \widetilde{T})$ be the natural extension of $(X, \mathcal{B}, \mu, T)$. Let α denote the ordered partition of X into the sets $A_j = \{x \in X : x_0 = j\}, 0 \le j \le k$, and let $\tilde{\alpha}$ denote its extension. For $0 \le p \le q$, let $\alpha_p^q = \bigvee_{i=p}^q T^{-i}\alpha$. If B is a non-trivial set in $\mathcal{B}$, let μ_B denote the measure

$$\mu_B(f) = \frac{1}{\mu(B)} \int_B f \, d\mu$$

on $X \cap B$, and if $\gamma = \{C_1, \ldots, C_r\}$ is an ordered partition of X, let γ/B denote the partition $\{C_i \cap B\}_{i=1}^r$ of $X \cap B$. It is enough to show that $\tilde{\alpha}$ is a very weak Bernoulli partition of $\widetilde{X}$ (see [7]). Since $\tilde{\mu}$ is $\widetilde{T}$-invariant, this is equivalent to the following: given $\varepsilon > 0$, there exists an integer n and subsets G_m of α_n^{m+n} $(m \ge 0)$ such that $\mu(\bigcup_{B \in G_m} B) > 1 - \varepsilon$ and, for each $B \in G_m$, there is a Lebesgue space (M, λ) and isomorphisms $\phi : (X, \mu) \to (M, \lambda)$, $\psi : (X \cap B, \mu_B) \to (M, \lambda)$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} d(\phi(T^{-i}\alpha), \psi(T^{-i}\alpha/B)) < \varepsilon,$$

where d is the Hamming distance; that is, if $\beta = \{B_1, \ldots, B_r\}, \gamma = \{C_1, \ldots, C_r\}$ are two ordered partitions of M, then

$$d(\beta, \gamma) = \sum_{i=1}^r \lambda(B_i \triangle C_i).$$

Given n and non-trivial $B \in \alpha_n^{m+n}$, we define a probability measure π_B^n on $S^n \times S^n$ by

$$\pi_B^n(\xi, \eta) = \iint P^n_{x, T^n y}(\xi, \eta) \, d\mu(x) \, d\mu_B(y) \quad ((\xi, \eta) \in S^n \times S^n).$$

Then if $\zeta \in S^n$ is fixed, it follows from (2.1) and (2.2) that

$$\pi_B^n\{(\xi, \eta) : \xi = \zeta\} = \int \mathcal{L}_g^n \mathbf{1}[\zeta](x) \, d\mu(x) = \mu([\zeta]), \tag{3.1}$$

and

$$\pi_B^n\{(\xi, \eta) : \eta = \zeta\} = \int \mathcal{L}_g^n \mathbf{1}[\zeta](T^n y) \, d\mu_B(y) = \mu_B([\zeta]). \tag{3.2}$$

Let (M, λ) be the unit interval with Lebesgue measure, and divide M into intervals $I(\xi, \eta)$ of length $\pi_B^n(\xi, \eta)$. All sets in α_0^{n-1} have the form $C_\xi = [\xi]$ for some $\xi \in S^n$. Using (3.1), we can partition each C_ξ into sets $\{C_{\xi,\eta} : \eta \in S^n\}$ such that

$$\mu(C_{\xi,\eta}) = \pi_B^n(\xi, \eta).$$

Define

$$\phi(C_{\xi,\eta}) = I(\xi, \eta). \tag{3.3}$$

This defines a measure-preserving map from the partition $\{C_{\xi,\eta}\}$ of X onto a partition of M, which can be extended to an isomorphism $\phi : X \to M$, since any two continuous Lebesgue spaces are isomorphic.

Similarly, all sets in α_0^{n-1}/B have the form $C'_\eta = [\eta] \cap B$ for some $\eta \in S^n$. Using (3.2), we can partition each C'_η into sets $\{C'_{\xi,\eta} : \xi \in S^n\}$ such that

$$\mu_B(C'_{\xi,\eta}) = \pi_B^n(\xi, \eta).$$

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Define

$$\psi(C'_{\xi,\eta}) = I(\xi,\eta). \tag{3.4}$$

As before, this can be extended to an isomorphism $\psi : X \cap B \to M$.

Consider $\phi(T^{-i}A_j)\triangle\psi(T^{-i}A_j \cap B)$. It follows from (3.3) and (3.4) that

$$I(\xi,\eta) \subseteq \phi(T^{-i}A_j)\triangle\psi(T^{-i}A_j \cap B)$$

if and only if $\xi_i = j$ and $\eta_i \neq j$, or vice-versa. Therefore,

$$\begin{aligned}\lambda(\phi(T^{-i}A_j)\triangle\psi(T^{-i}A_j \cap B)) &= \pi_B^n(\{\xi_i = j\}\triangle\{\eta_i = j\}) \\ &= \iint P^n_{x,T^n y}(\{\xi_i = j\}\triangle\{\eta_i = j\})\, d\mu(x)\, d\mu_B(y),\end{aligned}$$

and so,

$$\begin{aligned}&\frac{1}{n}\sum_{i=0}^{n-1} d(\phi(T^{-i}\alpha), \psi(T^{-i}\alpha/B)) \\ &\quad = \iint \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{k} P^n_{x,T^n y}(\{\xi_i = j\}\triangle\{\eta_i = j\})\, d\mu(x)\, d\mu_B(y).\end{aligned}$$

Now,

$$\begin{aligned}P(\{\xi_0 = j, \eta_0 > j\} \cup \{\xi_0 > j, \eta_0 = j\} \mid x, y) &\leq \left|\sum_{p=j+1}^{k} (g(px) - g(py))\right| \\ &= |\sigma_{j+1}(x,y)|\end{aligned}$$

and

$$\begin{aligned}P(\{\xi_0 = j, \eta_0 < j\} \cup \{\xi_0 < j, \eta_0 = j\} \mid x, y) &\leq \left|\sum_{p=j}^{k} (g(px) - g(py))\right| \\ &= |\sigma_j(x,y)|; \end{aligned} \tag{3.5}$$

hence,

$$\begin{aligned}P^n_{x,y}(\{\xi_i = j\}\triangle\{\eta_i = j\}) &= P^{n-i}_{x,y}(\{\xi_0 = j\}\triangle\{\eta_0 = j\}) \\ &= \int P(\{\xi_0 = j\}\triangle\{\eta_0 = j\} \mid \xi' x, \eta' y)\, dP^{n-i-1}_{x,y}(\xi', \eta') \\ &\leq \int (|\sigma_j(\xi' x, \eta' y)| + |\sigma_{j+1}(\xi' x, \eta' y)|)\, dP^{n-i-1}_{x,y}(\xi', \eta').\end{aligned}$$

(For $i = n-1$, we have $P^n_{x,y}(\{\xi_{n-1} = j\}\triangle\{\eta_{n-1} = j\}) \leq |\sigma_j(x,y)| + |\sigma_{j+1}(x,y)|$, and we take integration with respect to $P^0_{x,y}$ to mean evaluation at (x,y).) Choose n such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\int |\sigma_j(\xi x, \eta y)|\, dP^i_{x,y}(\xi,\eta) < \varepsilon$$

for all $x, y \in X$, $1 \leq j \leq k$. Then

$$\begin{aligned}\sum_{i=0}^{n-1}\sum_{j=0}^{k} P^n_{x,y}(\{\xi_i = j\}\triangle\{\eta_i = j\}) &\leq \sum_{i=0}^{n-1}\sum_{j=1}^{k}\int 2|\sigma_j(\xi x, \eta y)|\, dP^{n-i-1}_{x,y}(\xi,\eta) \\ &< 2nk\varepsilon\end{aligned}$$

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for all $x, y \in X$. Therefore,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(\phi(T^{-i}\alpha), \psi(T^{-i}\alpha/B)) < 2k\varepsilon.$$

Since ε can be arbitrarily small, this proves the result. □

4. *Proof of Theorem 1.1*

Given a positive, continuous $g \in \mathcal{G}_X$ such that $\sum v_n(g) < \infty$, let $P^n_{x,y}$ be the corresponding couplings. There is an $\varepsilon > 0$ such that $P^1_{x,y}(\xi \neq \eta) \leq 1 - \varepsilon$ for all $x, y \in X$. Let

$$a_n = \sup\left\{\sum_{j=1}^{k} |\sigma_j(x, y)| : x_i = y_i \ \forall\ i \neq n\right\},$$

where the σ_j are as in §3. Then $\sum a_n < \infty$; choose m such that $\sum_{n=m}^{\infty} a_n \leq 1 - \varepsilon$.

Let $Z = \prod_{i=0}^{\infty}\{0, 1\}$; we define $\delta: X \times X \to Z$ by

$$(\delta(x, y))_i = \begin{cases} 1, & x_i \neq y_i, \\ 0, & x_i = y_i, \end{cases}$$

and $f_n, \bar{f}_n \in C(Z)$ $(n \geq 1)$ by $f_n = \mathbf{1}\{z_i = 1 \text{ for some } 0 \leq i \leq n-1\}$, and $\bar{f}_n = \mathbf{1}[0^n]$. Let $\gamma \in \mathcal{F}_Z$ be defined by

$$\begin{aligned} \gamma(1z) &= (1-\varepsilon) f_m(z) + \bar{f}_m(z) s(z), \\ \gamma(0z) &= 1 - \gamma(1z), \end{aligned}$$

where

$$s(z) = \sum_{i=m}^{\infty} z_i a_i.$$

Then γ is continuous, and $0 \leq \gamma(1z) \leq 1 - \varepsilon$ for all $z \in Z$; moreover, it follows from (3.5) that

$$P^1_{x,y}(\xi \neq \eta) \leq \gamma(1\delta(x, y))$$

for all $x, y \in X$.

LEMMA 4.1. *Let $\phi \in C(X \times X)$, and let $f \in C(Z)$ be increasing. If $\phi(x, y) \leq f(\delta(x, y))$ for all $x, y \in X$, then*

$$\int \phi(\xi x, \eta y)\, dP^n_{x,y}(\xi, \eta) \leq \mathcal{L}^n_\gamma f(\delta(x, y))$$

for all $x, y \in X$, $n \geq 1$.

Proof. For $n = 1$,

$$\begin{aligned} \int \phi(\xi x, \eta y)\, dP^1_{x,y}(\xi, \eta) &= \left\{\sum_{\xi \neq \eta} + \sum_{\xi = \eta}\right\} P^1_{x,y}(\xi, \eta)\phi(\xi x, \eta y) \\ &\leq P^1_{x,y}(\xi \neq \eta) f(1\delta(x, y)) + P^1_{x,y}(\xi = \eta) f(0\delta(x, y)) \end{aligned}$$

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$$\begin{aligned} &= P^1_{x,y}(\xi \neq \eta) f(1\delta(x,y)) \\ &\quad +(\gamma(1\delta(x,y)) - P^1_{x,y}(\xi \neq \eta)) f(0\delta(x,y)) \\ &\quad +\gamma(0\delta(x,y)) f(0\delta(x,y)) \\ &\leq \gamma(1\delta(x,y)) f(1\delta(x,y)) + \gamma(0\delta(x,y)) f(0\delta(x,y)) \\ &= \mathcal{L}_\gamma f(\delta(x,y)), \end{aligned}$$

since f is increasing. Now assume the result holds for $n-1$. Since $\mathcal{L}_\gamma f$ is also increasing,

$$\begin{aligned} \int \phi(\xi x, \eta y)\, dP^n_{x,y}(\xi,\eta) &= \int \left(\int \phi(\xi_0 \xi' x, \eta_0 \eta' y)\, dP^1_{\xi' x, \eta' y}(\xi_0, \eta_0) \right) dP^{n-1}_{x,y}(\xi', \eta') \\ &\leq \mathcal{L}^{n-1}_\gamma (\mathcal{L}_\gamma f)(\delta(x,y)) \\ &= \mathcal{L}^n_\gamma f(\delta(x,y)). \end{aligned}$$

Hence, the result is true for all $n \geq 1$. □

PROPOSITION 4.2. *Let $f \in C(Z)$. Then $\mathcal{L}^n_\gamma f \to f(0^\infty)$ uniformly, as $n \to \infty$.*

Proof. Since $\gamma \in \mathcal{F}_Z$, it is enough to show that $\lim_{n\to\infty} \mathcal{L}^n_\gamma f_1(1^\infty) = 0$. Let ν be the γ-measure defined by $\nu(f) = \lim_{n\to\infty} \mathcal{L}^n_\gamma f(1^\infty)$ $(f \in C(Z))$. Then

$$\begin{aligned} \nu(f_m) &= \sum_{i=1}^m \nu([0^{i-1}1]) \\ &= \sum_{i=1}^m \int \mathcal{L}^i_\gamma \mathbf{1}[0^{i-1}1]\, d\nu, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^i_\gamma \mathbf{1}[0^{i-1}1](z) &= \gamma(1z) \prod_{j=1}^{i-1} \gamma(0^j 1z) \\ &= ((1-\varepsilon) f_m(z) + \bar{f}_m(z) s(z)) \varepsilon^{i-1}. \end{aligned}$$

Therefore,

$$\nu(f_m) = (1-\varepsilon^m)(\nu(f_m) + (1-\varepsilon)^{-1} \nu(\bar{f}_m s)).$$

Now,

$$\begin{aligned} \nu(\bar{f}_m s) &= \sum_{i=m}^\infty a_i \int z_i \bar{f}_m(z)\, d\nu(z) \\ &= \sum_{i=m}^\infty a_i \int z_i E(\bar{f}_m \mid T^{-i}\mathcal{B})(z)\, d\nu(z) \end{aligned}$$

and

$$E(\bar{f}_m \mid T^{-i}\mathcal{B})(z) = \sum_{w \in T^{-i}T^i z} \left(\prod_{j=0}^{i-1} \gamma(T^j w) \right) \bar{f}_m(w).$$

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If $z_i \neq 0$ and $w \in T^{-i}T^i z$ is such that $\bar{f}_m(w) \neq 0$, then w contains at least m consecutive 0's before the first 1, which occurs at or before the ith coordinate. Therefore, $E(\bar{f}_m \mid T^{-i}\mathcal{B})(z) \leq \varepsilon^m$ if $z_i \neq 0$. Hence,

$$\begin{aligned} \nu(\bar{f}_m s) &\leq \sum_{i=m}^{\infty} a_i \varepsilon^m \nu(f_1) \\ &\leq (1-\varepsilon)\varepsilon^m \nu(f_m), \end{aligned}$$

and so

$$\begin{aligned} \nu(f_m) &\leq (1-\varepsilon^m)(\nu(f_m) + \varepsilon^m \nu(f_m)) \\ &= (1-\varepsilon^{2m})\nu(f_m). \end{aligned}$$

It follows that $\nu(f_m) = 0$, and hence $\nu(f_1) = 0$. □

Given $f \in C(X)$, define $\tilde{f} \in C(Z)$ by $\tilde{f}(z) = \sup\{f(x) - f(y) : \delta(x, y) \leq z\}$. Then $\tilde{f}$ is increasing, and by Lemma 4.1 and Proposition 4.2,

$$\begin{aligned} \mathcal{L}_g^n f(x) - \mathcal{L}_g^n f(y) &= \int (f(\xi x) - f(\eta y))\, dP_{x,y}^n(\xi, \eta) \\ &\leq \mathcal{L}_\gamma^n \tilde{f}(\delta(x, y)) \\ &\to 0 \end{aligned}$$

uniformly in x and y as $n \to \infty$. It follows that $\mathcal{L}_g^n f$ converges uniformly to a constant $\mu(f)$, where μ is the unique g-measure. (Note, since g is positive, μ is non-atomic.) Now define $\tilde{\sigma}_j \in C(Z)$ $(1 \leq j \leq k)$ by $\tilde{\sigma}_j(z) = \sup\{|\sigma_j(x, y)| : \delta(x, y) = z\}$. Then by Lemma 4.1 and Proposition 4.2 again,

$$\begin{aligned} \int |\sigma_j(\xi x, \eta y)|\, dP_{x,y}^n(\xi, \eta) &\leq \mathcal{L}_\gamma^n \tilde{\sigma}_j(\delta(x, y)) \\ &\to 0 \end{aligned}$$

as $n \to \infty$, for all $x, y \in X$, $1 \leq j \leq k$. Therefore, by Proposition 3.1, the natural extension of (T, μ) is Bernoulli.

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REFERENCES

[1] H. Berbee. Chains with infinite connections: uniqueness and Markov representation. *Probab. Th. Rel. Fields* **76** (1987), 243–253.

[2] M. Bramson and S. Kalikow. Nonuniqueness in g-functions. *Israel J. Math.* **84** (1993), 153–160.

[3] W. Doeblin and R. Fortet. Sur les chaînes à liaisons complète. *Bull. Soc. Math. France* **65** (1937), 132–148.

[4] F. J. Dyson. Existence of a phase-transition in a one-dimensional Ising ferromagnet. *Comm. Math. Phys.* **12** (1969), 91–107.

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[5] P. Hulse. Uniqueness and ergodic properties of attractive g-measures. *Ergod. Th. & Dynam. Sys.* **11** (1991), 65–77.
[6] M. Keane. Strongly mixing g-measures. *Invent. Math.* **16** (1972), 309–324.
[7] D. S. Ornstein. *Ergodic Theory, Randomness and Dynamical Systems*. Yale University Press, New Haven, 1974.
[8] D. Ruelle. *Thermodynamic Formalism (Encyclopaedia of Mathematics and its Applications 5)*. Addison-Wesley, Reading, 1978.
[9] P. Walters. Ruelle's operator theorem and g-measures. *Trans. Amer. Math. Soc.* **214** (1975), 375–387.

https://doi.org/10.1017/S0143385797086422 Published online by Cambridge University Press