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Higher-dimensional foliated Mori theory

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Abstract

We develop some foundational results in a higher-dimensional foliated Mori theory, and show how these results can be used to prove a structure theorem for the Kleiman–Mori cone of curves in terms of the numerical properties of $K_{\mathcal{F}}$ for rank 2 foliations on threefolds. We also make progress toward realizing a minimal model program (MMP) for rank 2 foliations on threefolds.

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1. Introduction

We will always work over $\mathbb{C}$. Let X be a normal variety and let $\mathcal{F}$ be a foliation on X. We define $-c_1(\mathcal{F}) = K_{\mathcal{F}}$ to be the canonical divisor of the foliation. In recent years much work has been done understanding the birational geometry of the foliation in terms of $K_{\mathcal{F}}$ when the rank of $\mathcal{F}$ is 1, especially in the case of rank 1 foliations on surfaces; see for example [McQ05, McQ08, Bru99]. By contrast, the case of higher-rank foliations remains relatively unexplored. The goal of this paper is to begin developing the theory in the case of co-rank 1 foliations, especially in the case of threefolds.

An essential first step in understanding the birational geometry of a variety or foliation is a structure theorem on the closed cone of curves $\overline{NE}(X)$. Our first main result is the following foliated cone theorem.

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Theorem 1.1. *Let X be a projective $\mathbb{Q}$-factorial and kawamata log terminal (klt) threefold and $\mathcal{F}$ a co-rank 1 foliation with non-dicritical foliation singularities. Suppose that $(\mathcal{F}, \Delta)$ has canonical singularities. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{(K_{\mathcal{F}}+\Delta)\geqslant 0} + \sum \mathbb{R}_+[L_i]$$

where L_i is a rational curve tangent to $\mathcal{F}$ with $(K_{\mathcal{F}} + \Delta) \cdot L_i \geqslant -6$.

In particular, the $K_{\mathcal{F}} + \Delta$-negative extremal rays are locally discrete in the $(K_{\mathcal{F}} + \Delta) < 0$ portion of the cone.

This result is perhaps surprising and interesting in its own right. It is difficult to determine which line bundles on X can be realized as the canonical bundle of a foliation: this result places necessary conditions on such a line bundle which have a strong geometric flavor.

With the cone theorem in hand we then turn to the problem of constructing minimal models. That is, given a variety and a foliation $(X, \mathcal{F})$ is there a sequence of birational modifications that can be performed resulting in a model $(Y, \mathcal{G})$ with $K_{\mathcal{G}}$ nef?

Toward this we prove a contraction theorem. Given an extremal ray $R \subset \overline{NE}(X)$ we say that $c : X \to Y$ is a contraction of R provided $c(\Sigma)$ is a point if and only if $[\Sigma] \in R$.

Theorem 1.2. *Suppose X is a projective $\mathbb{Q}$-factorial and klt threefold. Suppose that $(\mathcal{F}, \Delta)$ has canonical, non-dicritical singularities and that $(\mathcal{F}, \Delta)$ is terminal along* $\operatorname{sing}(X)$.

Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray. Then there is a contraction of R which only contracts curves tangent to $\mathcal{F}$.

Unfortunately, we are not yet able to prove the existence of minimal models in full generality. The difficulty, as one might expect, is to prove the existence and termination of flips. We are able to show existence and termination in the smooth situation, which seems to already be useful for classification problems.

Theorem 1.3. *A minimal model program (MMP) for $\mathcal{F}$ exists if $\mathcal{F}$ is a smooth rank 2 foliation on a smooth threefold X.*

It is also possible to prove a toric foliated MMP.

Theorem 1.4. *Let $\mathcal{F}$ be a co-rank 1 toric foliation on a n-dimensional toric variety X. Suppose that $\mathcal{F}$ has canonical, non-dicritical singularities. Then there is an MMP for $\mathcal{F}$.*

We are also able to prove the following existence of flips result.

Theorem 1.5. *Let X be a $\mathbb{Q}$-factorial and klt threefold and $\mathcal{F}$ a co-rank 1 foliation with terminal singularities. Let $f : X \to Z$ be a flipping contraction, then the flip of f exists.*

As a quick application of the smooth MMP we get the following.

Corollary 1.6. *Let X be a smooth threefold and let $\mathcal{F}$ be a smooth rank 2 foliation on X with $K_{\mathcal{F}}$ pseudo-effective and $\nu(K_{\mathcal{F}}) = 0$. Then $(X, \mathcal{F})$ is birational to $(Y, \mathcal{G})$ where $(Y, \mathcal{G})$ is one of the following:*

(i) *up to a finite cover $\mathcal{G}$ is the product of a fibration in Calabi–Yau manifolds and a linear foliation on a torus;*
(ii) *$\mathcal{G}$ is transverse to a $\mathbb{P}^1$-bundle;*
(iii) *up to a finite cover $\mathcal{G}$ is induced by a fibration $X' \times C \to X$ where $c_1(X') = 0$.*

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We briefly explain the proofs of the cone and contraction theorems.

For simplicity, let us assume that X is smooth. Let R be an extremal ray of the cone of curve $\overline{NE}(X)$ with $R \cdot K_{\mathcal{F}} < 0$. Suppose that H_R is a supporting hyperplane of R.

H_R is a nef divisor on X. If $H_R^k = 0$ for some $k \leqslant n$ then we can show by a foliated bend and break result that through a general point of X there is a rational curve tangent to the foliation spanning R.

If $H_R^n \neq 0$, then we may take H_R to be effective, and so we see that R actually comes from a lower-dimensional subvariety S of X. The idea here is to proceed by induction on dimension. Unfortunately, the singularities which arise in this induction step are worse than the singularities of $(X, \mathcal{F})$. Indeed, as we will see the singularities in our induction are sometimes worse than log canonical. The bulk of our work is therefore to surmount these difficulties.

The strategy around this will be to do the following.

(i) First show that the extremal ray is spanned by a curve.

(ii) Then using a foliated subadjunction result (extending Kawamata's subadjunction result to the foliated case) we show that the extremal ray is spanned by a curve tangent to the foliation.

(iii) Finally we develop an algebraicity criterion for 2-dimensional leaves of foliations, which allows us to show that the extremal ray is spanned by a rational curve tangent to the foliation.

The subadjunction and algebraicity results alluded to are Theorem 4.5 and Lemma 5.8 respectively. An amusing corollary of our algebraicity criterion is the following.

COROLLARY 1.7. *Let $\mathcal{F}$ be a smooth rank 2 foliation on a smooth projective variety. Let C be a $K_{\mathcal{F}}$-negative curve tangent to $\mathcal{F}$. Let S be a germ of a leaf around C. Then:*

(i) *either C is rational;*

(ii) *or S is algebraic (i.e. its Zariski closure is a surface).*

For the contraction and flip theorem, the rough idea is to deduce the foliated statement by running a well-chosen log MMP. In proving the flip theorem we prove a version of Malgrange's theorem for singular varieties, which we think is of independent interest; see Lemma 9.8.

THEOREM 1.8. *Let $0 \in X$ be a threefold germ with a co-rank 1 foliation $\mathcal{F}$. Suppose X has log terminal singularities and that $\mathcal{F}$ has terminal singularities. Then $\mathcal{F}$ has a holomorphic first integral.*

We make one final remark: modern methods for proving the statements in the classical MMP are largely cohomological, i.e. utilizing various theorems on the vanishing and non-vanishing of cohomology groups. These are all false in the foliated context. The main challenge in this paper is to develop techniques for foliations which can replace the cohomological methods developed in the classical setting.

2. Set up and basic results

DEFINITION 2.1. Given a normal variety X a *foliation* $\mathcal{F}$ is a coherent saturated subsheaf of the tangent sheaf of X which is closed under the Lie bracket.

The *rank* of the foliation, $\mathrm{rk}(\mathcal{F})$, is its rank as a sheaf and its *co-rank* is $\dim(X) - \mathrm{rk}(\mathcal{F})$.

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The *singular locus* of the foliation is the locus where $\mathcal{F}$ fails to be a sub-bundle of T_X. Note that $\mathrm{sing}(\mathcal{F})$ has codimension at least 2.

The canonical divisor plays a central role in the birational geometry of foliations. We define it as follows.

Definition 2.2. Let U be the locus where X and $\mathcal{F}$ are smooth. We can associate a divisor to $\det(\mathcal{F}|_U)^*$, which gives a Weil divisor on all of X, denoted $K_{\mathcal{F}}$.

For the rest of this paper we will take $\mathcal{F}$ to be a co-rank 1 foliation over $\mathbb{C}$.

Definition 2.3. We say $W \subset X$ is *tangent* to $\mathcal{F}$ if the tangent space of W factors through $\mathcal{F}$ along $X - (\mathrm{sing}(\mathcal{F}) \cup \mathrm{sing}(X))$. Otherwise we say that W is *transverse* to the foliation.

If $\mathcal{F}$ factors through the tangent space of W, $\mathcal{F}|_W \to T_W \to T_X|_W$, we say that W is *invariant*.

2.1 1-forms and pulled back foliations

Definition 2.4. Let ω be a rational 1-form with $\omega \wedge d\omega = 0$. Then we can define a foliation by contraction. Namely, we take $\mathcal{F}$ to be the kernel of the pairing of ω with T_X. Thus, given a rank 1 coherent subsheaf of Ω_X, we can define a foliation by contraction.

On the other hand, given a foliation $\mathcal{F}$ we can define a subsheaf of Ω_X by taking the kernel of $\Omega_X \to \mathcal{F}^*$.

Let $\mathcal{F}$ be a co-rank 1 foliation on X, and suppose that it is defined by the rank 1 subsheaf of the cotangent sheaf $0 \to \mathcal{L} \to \Omega_X$.

Definition 2.5. Let $f : W \to X$. We have a morphism $df : f^*\Omega_X \to \Omega_W$. Assume that $f(W)$ is not tangent to $\mathcal{F}$ and that $f(W)$ is not contained in $\mathrm{sing}(\mathcal{F}) \cup \mathrm{sing}(X)$. Then $df(f^*\mathcal{L})$ is a rank 1 coherent subsheaf of Ω_W. Observe that if $\omega \in \mathcal{L}$ is an integrable 1-form then $df(\omega)$ is still integrable.

This gives a foliation $\mathcal{F}_W$, called the *pulled back* foliation.

When f is a closed immersion we will sometimes refer to it as the restricted foliation.

In general, even if $\mathcal{L}$ is a saturated subsheaf, $f^*\mathcal{L}$ might not be saturated.

Definition 2.6. Let $0 \to \mathcal{L} \to \Omega_X$ define a foliation, $\mathcal{F}$. We call the saturation of $\mathcal{L}$ in Ω_X the *conormal sheaf*, denoted $N^*_{\mathcal{F}}$.

On the smooth locus of X, $(N^*_{\mathcal{F}})|_{X^{sm}}$ is a line bundle represented by 1-forms with zero loci of codimension at least 2. We can therefore associate to $N^*_{\mathcal{F}}$ a well-defined Weil divisor. We will denote this divisor by $[N^*_{\mathcal{F}}]$.

Lemma 2.7. *We have the following equivalence of Weil divisors:* $K_X = K_{\mathcal{F}} + [N^*_{\mathcal{F}}]$.

2.2 Foliated pairs and foliation singularities

Frequently in birational geometry it is useful to consider pairs (X, Δ) where X is a normal variety, and Δ is a $\mathbb{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. By analogy we have the following definition.

Definition 2.8. A *foliated pair* $(\mathcal{F}, \Delta)$ is a pair of a foliation and a $\mathbb{Q}$-Weil ($\mathbb{R}$-Weil) divisor such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{Q}$-Cartier ($\mathbb{R}$-Cartier).

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Note also that we are typically interested only in the cases when $\Delta \geqslant 0$, although it simplifies some computations to allow Δ to have negative coefficients.

Given a birational morphism $\pi : \widetilde{X} \to X$ and a foliated pair $(\mathcal{F}, \Delta)$ on X let $\widetilde{\mathcal{F}}$ be the pulled back foliation on $\tilde{X}$ and $\pi_*^{-1}\Delta$ be the strict transform. We can write

$$K_{\widetilde{\mathcal{F}}} + \pi_*^{-1}\Delta = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a(E_i, \mathcal{F}, \Delta)E_i.$$

DEFINITION 2.9. We say that $(\mathcal{F}, \Delta)$ is *terminal, canonical, log terminal, log canonical* if $a(E_i, \mathcal{F}, \Delta) > 0$, $\geqslant 0$, $> -\epsilon(E_i)$, $\geqslant -\epsilon(E_i)$, respectively, where $\epsilon(D) = 0$ if D is invariant and 1 otherwise and where π varies across all birational morphisms.

If $(\mathcal{F}, \Delta)$ is log terminal and $\lfloor \Delta \rfloor = 0$ we say that $(\mathcal{F}, \Delta)$ is *foliated klt*.

Notice that these notions are well defined, i.e. $\epsilon(E)$ and $a(E, \mathcal{F}, \Delta)$ are independent of π.

Observe that in the case where $\mathcal{F} = T_X$ no exceptional divisor is invariant, i.e. $\epsilon(E) = 1$, and so this definition recovers the usual definitions of (log) terminal, (log) canonical.

We have the following nice characterization due to [McQ08, Corollary I.2.2.].

PROPOSITION 2.10. *Let $0 \in X$ be a surface germ with a terminal foliation $\mathcal{F}$. Then there exists a smooth foliation, $\mathcal{G}$, on a smooth surface, Y, and a cyclic quotient $Y \to X$ such that $\mathcal{F}$ is the quotient of $\mathcal{G}$ by this action.*

We also make note of the following easy fact.

LEMMA 2.11. *Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be a birational morphism. Write $\pi^*(K_{\mathcal{F}} + \Delta) = K_{\mathcal{G}} + \Gamma$. Then $a(E, \mathcal{F}, \Delta) = a(E, \mathcal{G}, \Gamma)$ for all E.*

Remark 2.12. Observe that if any component of $\operatorname{supp}(\Delta)$ is foliation invariant, then $(\mathcal{F}, \Delta)$ is not log canonical.

We will also make use of the class of simple foliation singularities.

DEFINITION 2.13. We say that $p \in X$ with X smooth is a *simple singularity* for $\mathcal{F}$ provided in formal coordinates around p we can write the defining 1-form for $\mathcal{F}$ in one of the following two forms, where $1 \leqslant r \leqslant n$.

(i) There are $\lambda_i \in \mathbb{C}^*$ such that

$$\omega = (x_1 \dots x_r)\left(\sum_{i=1}^{r} \lambda_i \frac{dx_i}{x_i}\right)$$

and if $\sum a_i\lambda_i = 0$ for some non-negative integers a_i then $a_i = 0$ for all i.

(ii) There is an integer $k \leqslant r$ such that

$$\omega = (x_1 \dots x_r)\left(\sum_{i=1}^{k} p_i \frac{dx_i}{x_i} + \psi(x_1^{p_1} \dots x_k^{p_k}) \sum_{i=2}^{r} \lambda_i \frac{dx_i}{x_i}\right)$$

where p_i are positive integers, without a common factor, $\psi(s)$ is a series which is not a unit, and $\lambda_i \in \mathbb{C}$ and if $\sum a_i\lambda_i = 0$ for some non-negative integers a_i then $a_i = 0$ for all i.

We say the integer r is the dimension-type of the singularity.

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Remark 2.14. A general hyperplane section of a simple singularity is again a simple singularity.

By Cano, [Can04], every foliation on a smooth threefold admits a resolution by blow ups centered in the singular locus of the foliation such that the transformed foliation has only simple singularities.

Using [AD13] it is easy to check the following.

Lemma 2.15. *Simple singularities (including smoothly foliated points) are canonical.*

The converse of this statement is false.

Example 2.16. Consider the germ of the foliation $(0 \in X, \mathcal{F})$ given by the degeneration of smooth surfaces to the cone over an elliptic curve. Consider the blow up π at the point 0 with exceptional divisor E and let $\mathcal{F}'$ be the transformed foliation. Observe that $\mathcal{F}'$ has simple singularities, and that E is invariant.

Write $K_{\mathcal{F}'} = \pi^* K_{\mathcal{F}} + aE$.

Denote by L the closure of a leaf in X passing through 0, and L' its strict transform. $K_{\mathcal{F}}|_L = K_L$, $K_{\mathcal{F}'}|_{L'} = K_{L'} + E|_{L'}$ and $K_{L'} = \pi^* K_L - E|_{L'}$.

From this we see that $K_{L'} + E|_{L'} = \pi^* K_L + aE|_{L'}$ and so $a = 0$, hence $\mathcal{F}$ is canonical. However, $\mathcal{F}$ is not simple since simple singularities in dimension $\geqslant 3$ are never isolated.

We will need to define one final type of foliation singularity.

Definition 2.17. Given a foliated pair $(X, \mathcal{F})$ we say that $\mathcal{F}$ has *non-dicritical* singularities if for any sequence of blow ups $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ such that X' is smooth and any $q \in X$ we have $\pi^{-1}(q)$ is tangent to the foliation.

Remark 2.18. Observe that non-dicriticality implies that if W is $\mathcal{F}$ invariant, then $\pi^{-1}(W)$ is $\mathcal{F}'$ invariant.

Definition 2.19. Given a germ $0 \in X$ with a foliation $\mathcal{F}$ such that 0 is a singular point for $\mathcal{F}$ we call a (formal) hypersurface germ $0 \in S$ a *(formal) separatrix* if it is invariant under $\mathcal{F}$.

Note that away from the singular locus of $\mathcal{F}$ a separatrix is in fact a leaf. Furthermore, being non-dicritical implies that there are only finitely many separatrices through a singular point.

Example 2.20. Consider the foliation on $\mathbb{C}^3$ defined by the 1-form

$$\omega = (x^m y - z^{m+1})\, dx + (y^m z - x^{m+1})\, dy + (z^m x - y^{m+1})\, dz$$

where $m \geqslant 2$. Blowing up the origin shows that this is a dicritical foliation. Observe that every line passing through the origin is tangent to this foliation; however, it admits no separatrices at the origin.

Example 2.21. Let $\lambda \in \mathbb{R}$. Consider the foliation $\mathcal{F}_\lambda$ on $\mathbb{C}^2$ generated by $x\partial_x + \lambda y \partial_y$. For $\lambda \in \mathbb{Q}_{>0}$ we can see that $\mathcal{F}_\lambda$ is dicritical, and otherwise is non-dicritical.

Example 2.22. Simple singularities are non-dicritical.

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Even for simple foliation singularities it is possible that there are separatrices which do not converge. However, as the following definition/result of [CC92] shows there is always at least one convergent separatrix along a simple foliation singularity of codimension 2.

DEFINITION 2.23. For a simple singularity of type (i), all separatrices are convergent.

For a simple singularity of type (ii), around a general point of a codimension 2 component of the singular locus we can write $\omega = py\,dx + qx\,dy + x\psi(x^p y^q)\lambda\,dy$. $x = 0$ is a convergent separatrix, called the *strong separatrix*.

DEFINITION 2.24. Suppose X is a normal variety and $\mathcal{F}$ is a co-rank 1 foliation with non-dicritical singularities.

We say $W \subset X$ (possibly contained in $\operatorname{sing}(X)$) is *tangent* to the foliation if for any birational morphism $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ and any (equivalently some) π-exceptional divisor E such that E dominates W we have that E is $\mathcal{F}'$-invariant.

We say $W \subset X$ (possibly contained in $\operatorname{sing}(X)$) is *transverse* to the foliation if for any birational morphism $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ and any (equivalently some) π-exceptional divisor E such that E dominates W we have that E is not $\mathcal{F}'$-invariant.

Notice that this definition agrees with the one given earlier if W is not contained in $\operatorname{sing}(X) \cup \operatorname{sing}(\mathcal{F})$.

2.3 Foliated MMP for surfaces

As mentioned earlier, McQuillan in [McQ08] proves the existence of a foliated MMP, namely the following.

THEOREM 2.25. *Let X be a smooth projective surface and $\mathcal{F}$ a foliation with canonical foliation singularities. Then, there is an MMP starting with X, namely a sequence of contractions of curves $\pi : X \to Y$ and a foliation $\mathcal{G}$ on Y, birationally equivalent to $\mathcal{F}$ such that either $K_{\mathcal{G}}$ is nef, or it is a $\mathbb{P}^1$-bundle over a curve.*

Observe that we can make the following modifications, implicit in [McQ08]; see for example [McQ08, Proposition III.2.1.].

COROLLARY 2.26. *Let $f : X \to U$ be a projective birational morphism of surfaces, and let $\mathcal{F}$ be foliation on X. Suppose X is smooth and $\mathcal{F}$ has canonical singularities. Let Δ be a divisor not containing any fibres of f. Then we can run the relative MMP, i.e. there is a birational map $g : X \to Y$ and $h : Y \to U$ and a foliation $\mathcal{G}$ on Y such that $K_{\mathcal{G}} + g_*\Delta$ is $h-$nef and $f = h \circ g$.*

Proof. If C is a $(K_{\mathcal{F}} + \Delta)$-negative curve contracted by f, then in fact $\Delta \cdot C \geqslant 0$ and so C is $K_{\mathcal{F}}$-negative. By the cone theorem for surface foliations, [McQ08, Corollary II.4.3], we see that C is an invariant rational curve, and following [McQ08, III.2] we can contract it to a point. Notice that the contracted space still maps down to U. Continuing inductively, and letting $(Y, \mathcal{G})$ be the output of this MMP we see that $K_{\mathcal{G}} + \Delta_Y$ is nef over U where Δ_Y is the pushforward of Δ to Y. □

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2.4 Foliated bend and break

We recall the following theorem due to [Miy87], [SB92, Theorem 9.0.2] or [BM01]

Theorem 2.27. *Let $(X, \mathcal{F})$ be a normal foliated variety of dimension n, and let $H_1, \dots, H_{n-1}$ be ample divisors. Let C be a general intersection of elements $D_i \in |m_i H_i|$ where $m_i \gg 0$. Suppose that $C \cdot K_{\mathcal{F}} < 0$. Then if A is an ample divisor through a general point of C there is a rational curve Σ tangent to $\mathcal{F}$ with*

$$A \cdot \Sigma \leqslant 2n \frac{A \cdot C}{-K_{\mathcal{F}} \cdot C}.$$

We make a minor modification of a lemma due to [KMM94].

Corollary 2.28 (Bend and Break). *Let X be a normal projective variety of dimension n. Let $\mathcal{F}$ be a foliation of rank r on X, and $\Delta \geqslant 0$. Let M be any nef divisor. Suppose that there are nef $\mathbb{R}$-divisors $D_1, \dots, D_n$ such that*

(1) $D_1 \cdot D_2 \cdot \ldots \cdot D_n = 0$;
(2) $-(K_{\mathcal{F}} + \Delta) \cdot D_2 \cdot \ldots \cdot D_n > 0$.

Then, through a general point of X there is a rational curve Σ with $D_1 \cdot \Sigma = 0$ and

$$M \cdot \Sigma \leqslant 2n \frac{M \cdot D_2 \cdot \ldots \cdot D_n}{-K_{\mathcal{F}} \cdot D_2 \cdot \ldots \cdot D_n}$$

and Σ is tangent to $\mathcal{F}$.

Proof. We can pick ample $\mathbb{Q}$-divisors $H_2, \dots, H_n$ sufficiently close to $D_2, \dots, D_n$ so that

$$-K_{\mathcal{F}} \cdot H_2 \cdot \ldots \cdot H_n > \Delta \cdot H_2 \cdot \ldots \cdot H_n \geqslant 0.$$

Pick $m_i \gg 0$ such that $m_i H_i$ is very ample, and let C be an intersection of general elements in $|m_i H_i|$. Then, we may take C to be contained in the smooth locus of both X and $\mathcal{F}$.

Then, apply the above theorem to give rational curves Σ_k tangent to the foliation with

$$\begin{aligned}(kD_1 + H) \cdot \Sigma_k &\leqslant 2n \frac{(kD_1 + H) \cdot m_2 H_2 \cdot \ldots \cdot m_n H_n}{-K_{\mathcal{F}} \cdot m_2 H_2 \cdot \ldots \cdot m_n H_n} \\ &= 2n \frac{(kD_1 + H) \cdot H_2 \cdot \ldots \cdot H_n}{-K_{\mathcal{F}} \cdot H_2 \cdot \ldots \cdot H_n}.\end{aligned}$$

As H_i approaches D_i, the right-hand side of the inequality approaches a bounded constant. Thus, as k varies, $\Sigma_k = \Sigma$ belongs to a bounded family, so for $k \gg 0$ we may take Σ to be fixed. Letting H approach M and letting k go to infinity gives our result. □

Remark 2.29. Observe that this result is totally independent of either the rank of the foliation or the dimension of the ambient variety. We recover the usual form of bend and break when we take the rank of the foliation to be $r = \dim(X)$.

3. Some adjunction results for foliations

Many of the results in this section are known for rank 1 foliations on surfaces, equivalently co-rank 1 foliations on surfaces. The statements (and proofs) for co-rank 1 foliations in general are similar, but since we could not find these results in the literature already we have decided to include them here.

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We begin with a simple lemma.

LEMMA 3.1. *Let $f : Y \to X$ be a morphism of normal varieties. Let $\mathcal{F}$ be a foliation on X. Suppose that $f(Y)$ is not tangent to $\mathcal{F}$ and that $f(Y)$ is not contained in* $\operatorname{sing}(X)$. *Let $\mathcal{F}_Y$ be the pulled back foliation. Suppose $K_X + \Delta_X$ is $\mathbb{R}$-Cartier and:*

(i) *either $N^*_{\mathcal{F}}$ is a line bundle (e.g. X is smooth);*

(ii) *or we have a morphism $f^*\Omega^{[1]}_X \to \Omega^{[1]}_Y$ between sheaves of reflexive differentials, and $(N^*_{\mathcal{F}})^{**}$ is a line bundle. Here $\Omega^{[1]}_X$ means $(\Omega^1_X)^{**}$.*

Then $K_{\mathcal{F}} + \Delta_X$ is $\mathbb{R}$-Cartier and

$$f^*(K_{\mathcal{F}} + \Delta_X) - K_{\mathcal{F}_Y} = f^*(K_X + \Delta_X) - K_Y + \Theta$$

where $\Theta \geqslant 0$.

Proof. By Lemma 2.7 we have the equality $K_X = K_{\mathcal{F}} + [N^*_{\mathcal{F}}]$ and by assumption $[N^*_{\mathcal{F}}]$ is a Cartier divisor. Thus $K_{\mathcal{F}} + \Delta_X = (K_X + \Delta_X) - [N^*_{\mathcal{F}}]$ and so $K_{\mathcal{F}} + \Delta_X$ is $\mathbb{R}$-Cartier.

It suffices to prove the stated equality outside of a codimension 2 subset on Y, and so we may assume that Y is smooth.

By definition in case (i) $N^*_{\mathcal{F}_Y}$ is the saturation of $f^*N^*_{\mathcal{F}}$ in Ω^1_Y (in case (ii) it is the saturation of $f^*((N^*_{\mathcal{F}})^{**})$ in Ω^1_Y). And so $f^*N^*_{\mathcal{F}} = N^*_{\mathcal{F}_Y} - \Theta$ (in case (ii) $f^*((N^*_{\mathcal{F}})^{**}) = N^*_{\mathcal{F}_Y} - \Theta$). Apply Lemma 2.7 to finish. □

Remark 3.2. Observe that if X is not klt the morphism $f^*\Omega^{[1]}_X \to \Omega^{[1]}_Y$ does not always exist.

Of particular interest are the cases where f is a closed immersion, f is a blow up or f is a fibration. In these cases, we get the following.

COROLLARY 3.3. *Let X be smooth.*

(1) *Let $\nu : D^\nu \to D \subset X$ be the normalization of a divisor transverse to the foliation; then $\nu^*(K_{\mathcal{F}} + D) = K_{\mathcal{F}_{D^\nu}} + \Theta$ where $\Theta \geqslant 0$. Furthermore, $\nu(\Theta)$ is either contained in* $\operatorname{sing}(D)$ *or is tangent to $\mathcal{F}$.*

(2) *The foliation discrepancy is less than or equal to the usual discrepancy.*

(3) *If the fibres of $f : Y \to X$ are all reduced, then $f^*K_{\mathcal{F}} - K_{\mathcal{F}_Y} = f^*K_X - K_Y$.*

Proof. The only part which does not follow from Lemma 3.1 is the claim in item (1) that $\nu(\Theta)$ is either contained in $\operatorname{sing}(D)$ or is tangent to $\mathcal{F}$. This is a local problem and so may be checked in a neighborhood of a general point of $\nu(\Theta)$ where $\mathcal{F}$ is defined by a 1-form ω.

Write $\nu^*(K_X + D) = K_{D^\nu} + \Delta$ where Δ is the usual different. Observe that Δ is supported on $\nu^{-1}(\operatorname{sing}(D))$.

Thus, we see that Θ is supported on the union of the zero locus of $\nu^*\omega$ and $\operatorname{supp}(\Delta)$. Let $C \subset D^\nu$ be a component of the zero locus of $\nu^*\omega$ so that $\nu(C)$ is not contained in $\operatorname{sing}(D)$ and let $i : \nu(C) \to X$ be the inclusion. Then observe that $i^*\omega = 0$ and so $\nu(C)$ is tangent to $\mathcal{F}$. □

The following is a more general version of foliation adjunction that we will need.

PROPOSITION 3.4. *Let $\mathcal{F}$ be a co-rank 1 foliation on a normal projective variety X, let S be a prime divisor transverse to the foliation, with normalization S^ν, and let $\mathcal{F}_{S^\nu}$ be the foliation restricted to S^ν.*

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Let Δ be an effective divisor such that S is not contained in the support of Δ. If $K_{\mathcal{F}}+\Delta+S$ is an $\mathbb{R}$-Cartier divisor, then

$$\nu^*(K_{\mathcal{F}}+\Delta+S)=K_{\mathcal{F}_{S^\nu}}+\Delta_{S^\nu}$$

where $\Delta_{S^\nu}\geqslant 0$.

Furthermore Δ_{S^ν} is supported on $\nu^{-1}(\operatorname{sing}(X)\cup\operatorname{sing}(S)\cup\operatorname{supp}(\Delta))$ and on centers tangent to the foliation.

Remark 3.5. Observe that we are not assuming that $K_X+\Delta+S$ is $\mathbb{Q}$-Cartier.

Proof. We construct Δ_{S^ν} as follows: pass to a log resolution $\pi: Y\to X$ of $(X,\Delta+S)$ and write $K_{\mathcal{F}_Y}+S'+\Delta'+E=\pi^*(K_{\mathcal{F}}+S+\Delta)$. Let $\sigma: S'\to S^\nu$ factor through $S'\to S$. We may apply Corollary 3.3 item (i) to $(\mathcal{F}_Y, S'+\Delta'+E)$ (since Y is smooth) and push forward along σ.

It remains to check that $\Delta_{S^\nu}\geqslant 0$. Using Corollary 2.26, the line of argument of the general adjunction statement in the K_X case works equally well in the foliated situation; see for example [Amb07, 9.2.1] or [Fuj11, 14.1]. For the reader's convenience we explain the argument here.

By taking general hyperplane cuts we may reduce to the case where X is a surface, $\mathcal{F}$ is a foliation by curves and S is a curve. Let $\mu: Y\to X$ be a log resolution of $(X,\Delta+S)$, i.e. Y is smooth and $\operatorname{exc}(\mu)\cup\mu_*^{-1}(\operatorname{supp}(\Delta+S))$ is a simple normal crossings (snc) divisor, and let $\mathcal{G}$ be the transformed foliation. Perhaps passing to a higher model we may assume that $\mathcal{G}$ has canonical singularities. Let Δ' denote the strict transform of Δ_Y and let S_Y denote the strict transform of S.

By Corollary 2.26 we may run a $K_{\mathcal{G}}+S_Y$-MMP over X terminating in $\pi:\widetilde{X}\to X$. We claim that this MMP only contracts curves E which are disjoint from the strict transform of S. We prove this by induction on the number of steps in the MMP, and so let $Y\to W$ be some intermediate step of the MMP and let S_W denote the strict transform of S and Δ_W the strict transform of Δ. Observe that by assumption W is smooth in a neighborhood of S_W so if E is any curve meeting S_W we have that $S_W\cdot E\geqslant 1$. On the other hand, the foliated MMP over X only contracts curves with $0>K_{\mathcal{F}_W}\cdot E\geqslant -1$. Thus if $(K_{\mathcal{F}_W}+\Delta_W+S_W)\cdot E<0$ we see that E must be disjoint from S_W.

Therefore we see that $\widetilde{X}$ is smooth in a neighborhood of $S_{\widetilde{X}}$ and moreover $S_{\widetilde{X}}$ is also smooth. Write

$$(K_{\mathcal{F}_{\widetilde{X}}}+\Delta_{\widetilde{X}}+S_{\widetilde{X}})+\Gamma=\pi^*(K_{\mathcal{F}}+\Delta+S)$$

where Γ is π-exceptional. Observe that, since $K_{\mathcal{F}_{\widetilde{X}}}+S_{\widetilde{X}}$ is π-nef and since $\Delta_{\widetilde{X}}$ is an effective divisor containing no π-exceptional divisor,

$$-\Gamma=(K_{\mathcal{F}_{\widetilde{X}}}+\Delta_{\widetilde{X}}+S_{\widetilde{X}})-\pi^*(K_{\mathcal{F}}+\Delta+S)$$

is π-nef and so the negativity lemma, [KM98, Lemma 3.38], applies to show that $\Gamma\geqslant 0$.

For ease of notation set $T=S_{\widetilde{X}}$ and observe that we have an isomorphism $T\to S^\nu$.

Write $(K_{\mathcal{F}_{\widetilde{X}}}+\Delta_{\widetilde{X}}+S_{\widetilde{X}}+\Gamma)|_T=K_{\mathcal{F}_T}+\Delta_T$ and by the fact that $\Gamma\geqslant 0$ and by Corollary 3.3 we see that $\Delta_T\geqslant 0$. On the other hand, by construction we see that $\Delta_T=\Delta_{S^\nu}$ and so we are done. □

Definition 3.6. We will refer to Δ_{S^ν} as the *foliated different*.

We also have a foliated Riemann–Hurwitz formula.

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PROPOSITION 3.7. *Let $\pi : Y \to X$ be a surjective, finite morphism of normal varieties. Let $\mathcal{F}$ be a co-rank 1 foliation on X, with $K_{\mathcal{F}}$ $\mathbb{Q}$-Cartier and let $\mathcal{F}_Y$ be the pulled back foliation. Then*

$$K_{\mathcal{F}_Y} = \pi^* K_{\mathcal{F}} + \sum \epsilon(D)(r_D - 1)D$$

where the sum is over divisors D with ramification index r_D.

Proof. This is proven in [Bru00, pp. 20–21] where X, Y are surfaces, but the proof works equally well for any co-rank 1 foliation. □

Remark 3.8. If the ramification of $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ is foliation invariant then $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}}$.

Later on we will need to compute the discrepancies of pairs $(\mathcal{F}, \Delta)$. The following two results will be useful in this regard.

COROLLARY 3.9. *Suppose X is klt and $\mathbb{Q}$-factorial and let $\mathcal{F}$ be a co-rank 1 foliation. Let Δ be an effective divisor. Let $\pi : Y \to X$ be a birational morphism which extracts divisors of usual discrepancy with respect to $(X, \Delta) \leqslant -1$. Then if π extracts E, the discrepancy of E with respect to $(\mathcal{F}, \Delta)$ is less than or equal to $-\epsilon(E)$ with strict inequality if $\epsilon(E) = 0$. In particular, π only extracts divisors of foliation discrepancy < 0.*

Remark 3.10. This result can be phrased as saying that the non-klt places of (X, Δ) are non-klt places of $(\mathcal{F}, \Delta)$.

Proof. The statement can be checked locally on X, so consider the following diagram.

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Here $f : X' \to X$ is the index 1 cover associated to $N^*_{\mathcal{F}}$; note f is étale in codimension 2. Denote by $\mathcal{F}'$ the foliation on X' and let Y' be the normalization of $X' \times_X Y$. Observe that g is finite.

Next, note $f^* K_X = K_{X'}$ and $f^* K_{\mathcal{F}} = K_{\mathcal{F}'}$. Write $\Delta' = f^* \Delta$.

Let E be a divisor contracted by π and let E' be a divisor contracted by π' such that $g(E') = E$; let r be the ramification index. Working around a general point of E, E' we may write

$$\begin{aligned} K_Y + \pi_*^{-1}\Delta &= \pi^*(K_X + \Delta) + aE, \\ K_{\mathcal{F}_Y} + \pi_*^{-1}\Delta &= \pi^*(K_{\mathcal{F}} + \Delta) + bE, \\ K_{Y'} + \pi_*'^{-1}\Delta' &= \pi'^*(K_{X'} + \Delta') + a'E', \\ K_{\mathcal{F}_{Y'}} + \pi_*'^{-1}\Delta' &= \pi'^*(K_{\mathcal{F}'} + \Delta') + b'E'. \end{aligned}$$

We have $(N^*_{\mathcal{F}'})^{**}$ is a line bundle sub-sheaf of $\Omega^{[1]}_{X'}$. Next, by [GKKP11, Theorem 4.3] we have a morphism

$$d\pi' : \pi'^{[*]}(N^*_{\mathcal{F}'})^{**} \to \Omega^{[1]}_{Y'}.$$

We apply Lemma 3.1 to see $b' \leqslant a'$.

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Next, by the Riemann–Hurwitz formula,

$$\begin{aligned} K_{Y'} + \pi_*'^{-1}\Delta' &= g^*(K_Y + \pi_*^{-1}\Delta) + (r-1)E' \\ &= g^*(\pi^*(K_X + \Delta) + aE) + (r-1)E'. \end{aligned}$$

Pulling back the other way around the diagram shows that

$$a' = ra + (r-1).$$

Likewise, the foliated Riemann–Hurwitz formula, Proposition 3.7, tells us that

$$\begin{aligned} K_{\mathcal{F}_{Y'}} + \pi_*'^{-1}\Delta' &= g^*(K_{\mathcal{F}_Y} + \pi_*^{-1}) + \epsilon(r-1)E' \\ &= g^*(\pi^*(K_{\mathcal{F}} + \Delta) + bE) + \epsilon(r-1)E' \end{aligned}$$

where $\epsilon = 0$ if E' is invariant and $= 1$ otherwise. Again, pulling back the other way around the diagram gives

$$b' = rb + \epsilon(E)(r-1).$$

Since $a \leqslant -1$, we get that $a' \leqslant -1$. And so $rb + \epsilon(E)(r-1) = b' \leqslant a' \leqslant -1$. This gives that $b \leqslant (-\epsilon(E)(r-1) - 1)/r \leqslant -\epsilon(E)$ with strict inequality if $\epsilon(E) = 0$. □

Lemma 3.11. *Let $\mathcal{F}$ be a co-rank 1 foliation with non-dicritical singularities on a $\mathbb{Q}$-factorial threefold X. Let $\pi : X' \to X$ be a birational morphism. Suppose Z is a center transverse to the foliation in the sense of Definition 2.24. Then the foliation discrepancy of a divisor E centered over Z is equal to the usual discrepancy.*

Proof. Perhaps passing to a higher resolution $\mu : X'' \to X'$ we may assume that $(X'', \mathrm{exc}(\pi \circ \mu))$ is log smooth. Let $\mathcal{G}$ be the foliation pulled back to X''. Perhaps shrinking around a general point of Z we may assume that every exceptional divisor dominates Z. Let E be one such divisor.

By non-dicriticality of $\mathcal{F}$ we see that if $\mathcal{H}$ is the foliation restricted to E then $\mathcal{H}$ is induced by the fibration $\sigma : E \to Z$. Let f be a general fibre of σ. Notice that $K_{\mathcal{H}} \cdot f = K_E \cdot f$ and so

$$(K_{\mathcal{G}} + E) \cdot f = K_{\mathcal{H}} \cdot f = K_E \cdot f = (K_{X''} + E) \cdot f$$

where the first equality follows from foliation adjunction, Corollary 3.3, and the third equality follows from usual adjunction, hence $K_{\mathcal{G}} \cdot f = K_{X''} \cdot f$.

Repeating this computation for every exceptional divisor E shows (perhaps shrinking X a bit more) that $K_{\mathcal{G}}$ and $K_{X''}$ are f-numerically equivalent, and so their discrepancies agree. □

4. Foliation sub-adjunction

In this section we prove a foliated version of sub-adjunction. We recall the definition of divisorial log terminal (dlt) and some related results.

Definition 4.1. A pair $(X, \Delta = \sum a_i \Delta_i)$ is called *divisorial log terminal (dlt)* if $0 \leqslant a_i \leqslant 1$ and there exists a log resolution $\pi : (Y, \Gamma) \to (X, \Delta)$ such that π only extracts divisors of discrepancy > -1.

We will need the following result due to Hacon on the existence of dlt models; see for example [Fuj11, Theorem 10.4].

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THEOREM 4.2. *Let X be a quasi projective variety, and B a boundary such that $K_X + B$ is $\mathbb{R}$-Cartier. One can construct a projective birational morphism $f : Y \to X$ where Y is normal and $\mathbb{Q}$-factorial. Furthermore, f only extracts divisors of discrepancy less than or equal to -1, and if we set $B_Y = f_*^{-1}B + \sum_{f\text{-exceptional}} E$, then (Y, B_Y) is dlt.*

LEMMA 4.3. *Let (X, Δ) be dlt and let $S_1, \dots, S_k$ be the irreducible components of $\lfloor \Delta \rfloor$.*

(1) *The pair (X, Δ) is log canonical.*
(2) *Each S_i is normal and if we write $(K_X + \Delta)|_{S_i} = K_{S_i} + \Delta_i$ then (S_i, Δ_i) is dlt.*
(3) *If $\lfloor \Delta \rfloor = 0$ then (X, Δ) is klt.*

Proof. The proof is standard; see for example [KM98]. □

DEFINITION 4.4. Given a pair (X, Δ) or $(\mathcal{F}, \Delta)$ we say that W is a *log canonical center* of (X or $\mathcal{F}, \Delta$) if (X or $\mathcal{F}, \Delta$) is log canonical above the generic point of W, and there is a divisor D of discrepancy $= -\epsilon(D)$ dominating W.

THEOREM 4.5. *Let X be a $\mathbb{Q}$-factorial and klt threefold and let $\mathcal{F}$ be a co-rank 1 foliation with non-dicritical singularities. Suppose that W is a log canonical center of $(\mathcal{F}, \Delta)$ with $\dim(W) = 1$ Furthermore, suppose that W is transverse to the foliation in the sense of Definition 2.24.*

Then $(K_{\mathcal{F}} + \Delta) \cdot W \geqslant 0$.

Proof. Let $\nu : W^\nu \to W$ be the normalization and let $G = \nu^*(K_{\mathcal{F}} + \Delta)$.

First, notice that since W is transverse if E is any divisor such that the center of E on X is W then E is transverse to the foliation and we have by Lemma 3.11 that $a(E, \mathcal{F}, \Delta) = a(E, X, \Delta)$. In particular, W is a log canonical center of (X, Δ).

Let $f : (Y, \mathcal{H}) \to (X, \mathcal{F})$ be a dlt modification of (X, Δ) and write

$$f^*(K_X + \Delta) = K_Y + \Gamma'$$

and

$$f^*(K_{\mathcal{F}} + \Delta) = K_{\mathcal{H}} + \Gamma.$$

Since f only extracts divisors of usual discrepancy less than or equal to -1, by Corollary 3.9 it only extracts divisors of foliation discrepancy less than or equal to 0, and so $\Gamma \geqslant 0$.

Observe that if E is a divisor then the coefficient of E in Γ (respectively Γ') is $-a(E, \mathcal{F}, \Delta)$ (respectively $-a(E, X, \Delta)$) and so by Lemma 3.11 we know that Γ and Γ' agree above the generic point of W.

Let $E \to W$ be a divisor dominating W which has coefficient 1 in Γ. Notice that E is transverse to $\mathcal{H}$, and if we write $\mathcal{H}_E$ for the foliation restricted to E we have that $\mathcal{H}_E$ is the foliation induced by $\sigma : E \to W^\nu$.

Write $(K_Y + \Gamma')|_E = K_E + \Theta'$ and $(K_{\mathcal{H}} + \Gamma)|_E = K_{\mathcal{H}_E} + \Theta$. Note that $\Theta, \Theta' \geqslant 0$ and that if $D \subset E$ is a divisor dominating W it is transverse to the foliation and therefore has the same coefficient in Θ and Θ' (by the construction of the different, Proposition 3.4, and Lemma 3.11). Thus $(\mathcal{H}_E, \Theta)$ is log canonical (lc) above the generic point of W^ν.

Suppose for sake of contradiction that $\deg(G) < 0$. Then $K_{\mathcal{H}_E} + \Theta = \sigma^* G$ is not nef and we can apply the cone theorem for surface foliations, see Theorem 6.3 below, to conclude that there is some rational curve tangent to $\mathcal{H}_E$ which is $K_{\mathcal{H}_E} + \Theta$-negative. However, if C is any curve tangent to $\mathcal{H}_E$, then

$$(K_{\mathcal{H}_E} + \Theta) \cdot C = \sigma^* G \cdot C = 0,$$

which contradicts the supposition. □

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Remark 4.6. This should be viewed as a foliated version of Kawamata's subadjunction, [Kaw98]. Indeed, with more work it is possible to prove foliated subadjunction for X and W of any dimension. We will only need the case where $\dim(X) = 3$ and $\dim(W) = 1$ to prove our main result and so have restricted our attention to this case.

Lemma 4.7. *Let X be a $\mathbb{Q}$-factorial and klt threefold, $\mathcal{F}$ a co-rank 1 foliation with non-dicritical singularities, and let S a surface transverse to the foliation. Suppose that $(\mathcal{F}, \Delta)$ has canonical singularities. Let R be an extremal ray of $\overline{NE}(X)$ such that $K_{\mathcal{F}} + \Delta$ and S are negative on R.*

Then R is spanned by the class of a curve C which is tangent to the foliation.

Proof. Let $\nu : S^{\nu} \to S$ be the normalization of S. Write $\nu^*(K_{\mathcal{F}} + \Delta + S) = K_{\mathcal{F}_{S^{\nu}}} + \Theta$. A straightforward computation (see for example [KM98, Propostion 5.46]) shows that the non-log canonical centers of $(\mathcal{F}_{S^{\nu}}, \Theta)$ are contained in non-log canonical centers of $(\mathcal{F}, \Delta + S)$.

Since $R \cdot S < 0$, there exists an extremal ray R' in $\overline{NE}(S)$ such that $\nu_* R' = R$ in $\overline{NE}(X)$.

By the cone theorem for surface foliations, see Theorem 6.3 below, R' is spanned by a curve C and so R is spanned by $\nu(C)$. Furthermore:

(i) either C is transverse to $\mathcal{F}_{S^{\nu}}$ and contained in the non-log canonical locus of $(\mathcal{F}_{S^{\nu}}, \Theta)$;
(ii) or C is tangent to $\mathcal{F}_{S^{\nu}}$.

In the first case observe that $\nu(C)$ is a non-log canonical center of $(\mathcal{F}, \Delta + S)$ transverse to $\mathcal{F}$.

Thus we may find $0 < \lambda \leqslant 1$ so that $\nu(C)$ is a log canonical center of $K_{\mathcal{F}} + \Delta + \lambda S$ transverse to $\mathcal{F}$. In fact, since $\nu(C)$ is not a log canonical center of $(\mathcal{F}, \Delta + S)$ we see that $\lambda < 1$. However, $(K_{\mathcal{F}} + \Delta + \lambda S) \cdot C < 0$, a contradiction of Theorem 4.5. Thus C is tangent to the foliation and so $\nu(C)$ is tangent to $\mathcal{F}$. □

5. $K_{\mathcal{F}}$-negative curves tangent to $\mathcal{F}$

Throughout this section we will assume X to be a threefold and $\mathcal{F}$ a co-rank 1 foliation. The object of this section is to show that if there exists a $K_{\mathcal{F}}$-negative curve tangent to the foliation, then there exists a rational $K_{\mathcal{F}}$-negative curve tangent to the foliation.

5.1 Existence of germs of leaves

Cano and Cerveau in [CC92] prove the following.

Theorem 5.1. *Let $(0 \in X, \mathcal{F})$ be the germ of a 3-dimensional complex manifold with a co-rank-1 foliation. Suppose that $\mathcal{F}$ has non-dicritical singularities. Let γ be a curve tangent to the foliation and not contained in* $\operatorname{sing}(\mathcal{F})$. *Then γ is contained in a unique convergent separatrix.*

As noted earlier there are examples of dicritical singularities with no separatrices, convergent or formal.

In what follows we adapt the techniques and ideas in [CC92] to work in the setting where X is singular.

We will use the following fact about simple foliation singularities found in [CC92, Proposition II.5.5].

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LEMMA 5.2. *Let $(0 \in X, \mathcal{F})$ be a foliated germ with simple foliation singularities. Let $Q_i \in \operatorname{sing}(\mathcal{F})$ and $Q_i \to 0$. Suppose that at each Q_i there is a convergent germ of a separatrix S_{Q_i} such that the S_{Q_i} agree on overlaps. Then there is a convergent germ of a separatrix at 0 which extends the S_{Q_i}.*

LEMMA 5.3. *Let X be smooth threefold, $\mathcal{F}$ a co-rank 1 foliation with simple singularities and E a compact $\mathcal{F}$-invariant divisor. Let γ be a germ of a curve tangent to $\mathcal{F}$ meeting E but not contained in E. Then there exists a neighborhood U of E and a closed $\mathcal{F}$-invariant hypersurface $S \subset U$ such that $\gamma \subset S$.*

Proof. The proof is essentially a small generalization of the proofs of [CC92, Lemma IV.1.4, Corollary IV.1.5].

Since γ is not contained in E we have $\gamma \cap E = \{P_1, \dots, P_n\}$. Moreover, if U is a sufficiently small neighborhood of E we see that $U \cap \gamma = \bigcup_i \gamma_i$ where $\gamma_i \cap E = P_i$ and $\gamma_i = \bigcup_j \gamma_{ij}$ where γ_{ij} is irreducible. It suffices to construct a separatrix containing each γ_{ij} and thus shrinking X about E we are free to assume that γ is irreducible and $\gamma \cap E = P$ is a single point.

Furthermore, possibly further shrinking X around E, by passing to a resolution, $\pi : X' \to X$, setting $E' = \pi^{-1}E$ and setting $\mathcal{F}'$ for the transformed foliation, we may assume that each point of $\operatorname{sing}(\mathcal{F}') \cap E'$ has at most one (formal) separatrix not contained in $E' \cup \operatorname{exc}(\pi)$.

Let $W = \operatorname{sing}(\mathcal{F}') \cap E'$, let V be the union of those irreducible components W' of W such that at a general point of W' there is exactly one separatrix not contained in $E' \cup \operatorname{exc}(\pi)$. Let γ' be the strict transform of γ and let $\gamma' \cap E' = P'$.

Observe that $P' \in \operatorname{sing}(\mathcal{F}') \cap E'$ and there is a separatrix of $\mathcal{F}'$, call it Σ, in a neighborhood of P' containing γ'. Since γ' is not contained in $E' \cup \operatorname{exc}(\pi)$ we see that Σ is not contained in $E' \cup \operatorname{exc}(\pi)$ and so $P' \in V$. Let V_0 be the connected component of V containing $\gamma' \cap E'$.

We let $\mathcal{A}$ be the locus of points $Q \in V_0$ such that:

(i) there exists an open set $Q \in U_Q$ and separatrix $Q \in S'_Q \subset U_Q$ such that S'_Q is not contained in $E' \cup \operatorname{exc}(\pi)$; and

(ii) for every $R \in E' - V_0$ there exists an open set W_R containing R such that $W_R \cap S'_Q = \emptyset$.

Observe that if $R \in E' - V_0$ and if $Q \in V_0$ and U_Q is a small open neighborhood of Q and $S'_Q \subset U_Q$ is a separatrix at Q then $S'_Q \cap E' \subset V_0$ by definition and so $R \notin S'_Q$.

Thus, it is easy to see that $\mathcal{A}$ is open in V_0. To see that $\mathcal{A}$ is closed in V_0 let Q_i be a sequence of points in $\mathcal{A}$ converging to $Q \in V_0$. By Lemma 5.2 we can find a separatrix S'_Q at Q agreeing with the S'_{Q_i} on overlaps, and therefore S'_Q satisfies (i) and (ii). By assumption $\gamma' \cap E' \in \mathcal{A}$ is non-empty, and so $\mathcal{A} = V_0$.

There exist finitely many P_i such that U_{P_i} cover V_0. For all $R \in E' - V_0$ there exists an open W_R disjoint from all the S'_{P_i}. The collection $\{W_R, U_{P_i}\}$ forms an open cover of a neighborhood of E', call it U. Let $S' = \cup S'_{P_i}$. Note that these separatrices agree on overlaps since if $Q \in U_{P_i} \cap U_{P_j} \cap V_0$ there is at most 1 separatrix at Q not contained in $E' \cup \operatorname{exc}(\pi)$ and S'_{P_i}, S'_{P_j} are both separatrices at Q not contained in $E' \cup \operatorname{exc}(\pi)$. Thus S' is a well-defined closed foliation invariant hypersurface in U.

Finally, since π is proper, by the proper mapping theorem we have that $V = \pi(U)$ is neighborhood of E and $S = \pi(S') \subset V$ is a $\mathcal{F}$-invariant hypersurface in V. □

COROLLARY 5.4. *Let C be a compact curve tangent to a co-rank 1 foliation with non-dicritical singularities on threefold X such that C is not contained in $\operatorname{sing}(\mathcal{F}) \cup \operatorname{sing}(X)$. Then there is a germ of an analytic surface containing C, call it S, such that S is tangent to the foliation.*

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Proof. First, pick a point $p \in C$ which is smooth point of $X, \mathcal{F}$ and C. Since $\mathcal{F}$ has rank 2 there is a smooth germ of a curve γ tangent to $\mathcal{F}$ meeting C at exactly p.

Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be a resolution of singularities of X and $\mathcal{F}$ so that Y is smooth and $\mathcal{G}$ has simple singularties. Passing to a higher resolution if needed we may assume that $\pi^{-1}(C)$ is a snc divisor, which must be $\mathcal{G}$ invariant.

Let γ' be the strict transform of γ. By Lemma 5.3 we may find S', the germ of an invariant hypersurface containing γ' in a small neighborhood U of $\pi^{-1}(C)$. By the proper mapping theorem $\pi(S') = S$ is a $\mathcal{F}$-invariant hypersurface containing C. □

Remark 5.5. Observe that in contrast to the smooth case where every non-dicritical singularity admits at least one convergent separatrix, if X is singular it is possible for there to be no separatrices (formal or otherwise) through a particular point $x \in X$. In the case of surfaces an example is given by considering the contraction of an elliptic Gorenstein leaf. In these cases, however, there are no germs of curves tangent to $\mathcal{F}$ passing through x. Intriguingly, these very same counter-examples are also intimately related to counter-examples to abundance for foliations.

Corollary 5.6. *Let X be a threefold, and suppose $\mathcal{F}$ is a co-rank 1 foliation on X with canonical and non-dicritical singularities and suppose that $C \subset \operatorname{sing}(\mathcal{F})$ is a compact curve that is not contained in $\operatorname{sing}(X)$. Furthermore, if $\mathcal{F}$ is simple at the generic point of C then we may choose S so that S agrees with the strong separatrix along C.*

Proof. Let H be a general hyperplane cut of X passing through C. Let $P = H \cap C$ and let $\mathcal{G}$ be the foliation restricted to H. By a result of Camacho and Sad, [Bru00, Theorem 3.3], there exists a germ of a curve, γ tangent to $\mathcal{G}$ passing through P. Moreover, if $\mathcal{F}$ has simple singularities at the generic point of C then we may take γ to be the strong separatrix at P. Pushing forward, we see that γ is a germ of a curve in X tangent to $\mathcal{F}$ and meeting C.

As above, let $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ be a resolution of singularities of X and $\mathcal{F}$ such that $\pi^{-1}(C)$ is a snc divisor. Since C is contained in the singular locus, we see that C is tangent to $\mathcal{F}$ and since $\mathcal{F}$ is non-dicritical every divisor on X' dominating C is invariant; see Definition 2.24. Thus $\pi^{-1}(C)$ is invariant.

By Lemma 5.3 there exists an extension of γ to an invariant hypersurface S' in a neighborhood of $\pi^{-1}(C)$. Taking $\pi(S')$ gives our desired separatrix. □

Corollary 5.7. *Let X be a threefold and suppose that $\mathcal{F}$ is a co-rank 1 foliation with canonical and non-dicritical singularities and suppose that $C \subset \operatorname{sing}(X)$ is a compact curve tangent to $\mathcal{F}$. Suppose that $\mathcal{F}$ is canonical at the generic point of C. Then there exists a germ of a separatrix S containing C.*

Proof. Since $\mathcal{F}$ is canonical, taking a general hyperplane H section meeting C we have, if $P = C \cap H$, that $\mathcal{F}_H$ is canonical near P. Applying the classification of canonical foliation singularities on surfaces, see [McQ08, Fact I.2.4], there exists a germ of a curve γ tangent to $\mathcal{F}$ meeting C transversely.

As above let $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ be a resolution of singularities of X and $\mathcal{F}$ and so that $\pi^{-1}(C)$ is a snc divisor. Since C is tangent to $\mathcal{F}$, $\pi^{-1}(C)$ is invariant, and arguing as in Corollary 5.4 we may produce our desired separatrix containing γ and our result follows. □

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5.2 Producing rational curves

We begin here with an algebraicity criterion.

LEMMA 5.8. *Let C be a compact curve, and S an analytic surface germ containing C, sitting inside a projective variety X and suppose that C is not contained in* $\operatorname{sing}(S)$. *Let $\nu : S^\nu \to S$ be the normalization of S and let $C' \subset S^\nu$ be the strict transform of C.*

Assume that $K_{S^\nu} + \Delta$ is $\mathbb{Q}$-Cartier and $(K_{S^\nu} + \Delta) \cdot C' < 0$, and that $\Delta \geqslant 0$ is a boundary along C'. Then either C is rational and $(K_{S^\nu} + \Delta) \cdot C \geqslant -2$ or S is algebraic, i.e. the Zariski closure of S is an algebraic surface.

Proof. Let Y be the Zariski closure of S with $K(Y)$ the field of rational functions on Y. The algebraicity of S follows if the transcendence degree of $K(Y)$ over $\mathbb{C}$ is 2.

Let $T \xrightarrow{f} S^\nu \xrightarrow{g} S$ be the minimal resolution of the normalization of S (perhaps after restricting S to a smaller neighborhood of C) and let $B \subset T$ be the strict transform of C. We have $K_T + \Delta_T = f^*(K_{S^\nu} + \Delta)$ where $\Delta_T \geqslant 0$.

Let $\mathfrak{T}$ denote the formal scheme given by the completion of T along B and let $K(\mathfrak{T})$ be the field of formal meromorphic functions on $\mathfrak{T}$. Notice that $K(\mathfrak{T})$ is a field extension of $K(Y)$, and so it suffices to bound the transcendence degree of $K(\mathfrak{T})$. Let $n : B' \to B$ be the normalization of B.

By assumption there exists a $t \geqslant 0$ such that $K_T + \Delta_T + tB = K_T + \Theta + B$ where $\Theta \geqslant 0$ and B is not contained in $\operatorname{supp}(\Theta)$. By adjunction $(K_T + \Theta + B) \cdot B = 2g(B') - 2 + d_B$, where $d_B \geqslant 0$.

If $n^*\mathcal{O}(B)$ is not ample, then the left-hand side of the equation is negative, hence B is rational, and $(K_T + \Delta_T) \cdot B \geqslant (K_T + \Theta + B) \cdot B \geqslant -2$.

On the other hand, if $n^*\mathcal{O}(B)$ is ample, the normal bundle of B in $\mathfrak{T}$ is ample, which by a result of Hartshorne, [Har68, Theorem 6.7], or by an observation of Bogomolov and McQuillan, [BM01, Fact 2.1.1], implies that $K(\mathfrak{T})$ has transcendence degree at most 2 over $\mathbb{C}$, and our result follows. □

As mentioned in the introduction this has the following immediate consequence.

COROLLARY 5.9. *Let $\mathcal{F}$ be a smooth rank 2 foliation on a smooth projective variety. Let C be a $K_{\mathcal{F}}$-negative curve tangent to $\mathcal{F}$. Let S be a germ of a leaf around C. Then:*

(i) *either C is rational;*
(ii) *or S is algebraic (i.e. its Zariski closure is a surface).*

In the following proof we will make use of the following definition.

DEFINITION 5.10. Given a reflexive sheaf L and a positive integer $q \leqslant \dim(X)$ a *Pfaff field* of rank q is a non-zero morphism $\Omega^q_X \to L$. Given a foliation $\mathcal{F}$ of rank q, by taking the qth wedge power of $\Omega^1_X \to \mathcal{F}^*$ we get a Pfaff field $\Omega^q_X \to \mathcal{O}(K_{\mathcal{F}})$ of rank q.

LEMMA 5.11. *Let X be threefold. Suppose that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier and $\mathcal{F}$ has only non-dicritical singularities.*

Let C be a compact curve tangent to the foliation such that either C is not contained in $\operatorname{sing}(X) \cup \operatorname{sing}(\mathcal{F})$ *or $\mathcal{F}$ is canonical at the generic point of C.*

Then there exists a germ of an analytic surface S such that C is contained in S, and S is foliation invariant.

If $\nu : S^\nu \to S$ is the normalization, then $\nu^* K_{\mathcal{F}} = K_{S^\nu} + \Delta$ where $\Delta \geqslant 0$.

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Proof. By Lemmas 5.4, 5.6, 5.7 we get the existence of the germ S containing C.

To prove our last statement, if $\Omega^2_X \to \mathcal{O}(K_{\mathcal{F}})$ is the Pfaff field associated to our foliation, since S is foliation invariant we have a morphism $(\Omega^2_S)^{\otimes m} \to \mathcal{O}_S(mK_{\mathcal{F}})$, where m is the Cartier index of $K_{\mathcal{F}}$. We can apply [AD14, Lemma 3.7] to see that this lifts to a map $(\Omega^2_{S^\nu})^{\otimes m} \to \nu^* O_S(mK_{\mathcal{F}})$. Observe that the lemma is proven in the case where S is a variety; however, the proof works just as well in the case where S is an analytic variety.

Thus, we have a non-zero map $\mathcal{O}(mK_{S^\nu}) \to \mathcal{O}(m\nu^* K_{\mathcal{F}})$ and our result follows. Observe that Δ is supported on the locus where this map fails to be surjective, which is contained within $\operatorname{sing}(X) \cup \operatorname{sing}(\mathcal{F})$. □

Example 5.12. In the case that X is smooth with simple singularities, the computation of Δ is easy. Notice that Δ is supported on $\nu^{-1}(\operatorname{sing}(\mathcal{F}))$, and if Z is a component of $\operatorname{sing}(\mathcal{F})$ and S is a strong separatrix along Z then the coefficient of Z in Δ is exactly 1. Otherwise the coefficient of Z is some positive integer k which depends on the analytic type of the singularity.

More generally, passing to a resolution $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$ of X, S and so that $\mathcal{F}'$ has simple singularities we can write

$$K_{\mathcal{F}'} + \Gamma = \pi^* K_{\mathcal{F}}$$

and

$$(K_{\mathcal{F}'} + \Gamma)|_{S'} = K_{S'} + \Delta'$$

where Δ' can be computed as above. If $\sigma : S' \to S^\nu$ is the induced morphism we have that $\Delta = \sigma_* \Delta'$.

Definition 5.13. McQuillan's classification of $\mathbb{Q}$-Gorenstein canonical surface foliation singularities, [McQ08, Fact I.2.4], implies that the underlying surface has at worst quotient singularities. Thus if $(\mathcal{F}, \Delta)$ has canonical singularities, we see that X has at worst quotient singularities in codimension 2. In this situation, if $C \subset \operatorname{sing}(X)$ we will say C is contained in $\operatorname{sing}(\mathcal{F})$ if around a general point of C there exists a quotient $q : (X', \mathcal{F}') \to (X, \mathcal{F})$ with X' smooth so that $q^{-1}(C) \subset \operatorname{sing}(\mathcal{F}')$.

We finish the section with our characterization of $(K_{\mathcal{F}} + \Delta)$-negative curves tangent to $\mathcal{F}$.

Lemma 5.14. *Let X be a projective $\mathbb{Q}$-factorial threefold and let $\mathcal{F}$ be a co-rank 1 foliation on X.*

Let C be a curve tangent to $\mathcal{F}$, with $(K_{\mathcal{F}} + \Delta) \cdot C < 0$. Suppose that $\mathcal{F}$ has non-dicritical singularities and that $(\mathcal{F}, \Delta)$ is canonical. Then, $[C] = \sum a_i [M_i] + \beta$ where $(K_{\mathcal{F}} + \Delta) \cdot \beta \geqslant 0$ and the M_i are rational curves tangent to the foliation with $0 > K_{\mathcal{F}} \cdot M_i \geqslant -4$.

Proof. Let S be the germ of a surface tangent to the foliation containing C whose existence is guaranteed by Lemma 5.11 and let $\nu : S^\nu \to S$ be the normalization. Again, by Lemma 5.11 we may write $\nu^*(K_{\mathcal{F}} + \Delta) = K_{S^\nu} + \Theta$.

First, suppose that $C \subset \operatorname{sing}(\mathcal{F})$. By McQuillan's classification of canonical surface singularities, around a general point of C we have a quotient $q : (X', \mathcal{F}') \to (X, \mathcal{F})$ where X' is smooth. Set $C' = q^{-1}(C)$. We can find a smooth separatrix around a general point of C', call it S', so that $K_{\mathcal{F}'}|_{S'} = K_{S'} + C + \Theta'$ where C' is not contained in $\operatorname{supp}(\Theta')$. We can therefore choose our separatrix S around C to be such that $S = q(S')$. A direct computation as in Example 5.12 shows that $\Theta = C + \Theta_0$ where C is not contained in $\operatorname{supp}(\Theta_0)$. Adjunction tells us that

$$0 > (K_{S^\nu} + C + \Theta_0) \cdot C \geqslant 2g(C) - 2$$

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and so C is rational. Moreover, since $(\mathcal{F}, \Delta)$ is canonical and $C \subset \operatorname{sing}(\mathcal{F})$ we see that C is not contained in $\operatorname{supp}(\Delta)$ and so $\Delta \cdot C \geqslant 0$ which gives us $(K_{\mathcal{F}} + \Delta) \cdot C \geqslant -2$.

Otherwise, since $(\mathcal{F}, \Delta)$ is canonical we see that Θ is a boundary along C. We can then apply our algebraicity criterion, Lemma 5.8, to see that either C is rational and of bounded negativity, or S is algebraic. In the latter case we can apply the usual cone theorem for surfaces.

Computing explicitly as in Example 5.12 we see that the non-log canonical locus of (S^ν, Θ) is supported on the singular locus of $\mathcal{F}$ and on a finite collection of points. The cone theorem for surfaces tells us that in $\overline{NE}(S^\nu)$ we can write $[C] = \sum a_i [L_i] + \beta$ where the L_i are curves contained in the non-log canonical locus of (S^ν, Θ) or are rational curves with $(K_{S^\nu} + \Theta) \cdot L_i = (K_{\mathcal{F}} + \Delta) \cdot L_i \geqslant -4$, and $(K_{\mathcal{F}} + \Delta) \cdot \beta \geqslant 0$.

Again, since the non-log canonical locus of (S^ν, Θ) is contained in $\operatorname{sing}(\mathcal{F})$, if L_i is $K_{\mathcal{F}}$-negative it must be rational. And so pushing forward to X gives our result. □

6. The cone theorem for surfaces

We will need the following extension of the foliated cone theorem to foliations with a boundary. In proving it we use the following definition and result from convex geometry.

DEFINITION 6.1. Let K be a convex cone containing no lines. A ray R of K is called exposed if there is a hyperplane meeting K exactly along R.

LEMMA 6.2. *If K is a closed convex cone containing no lines, then K is the closure of the subcone generated by the exposed rays.*

Proof. See [Roc70, Corollary 18.7.1]. □

THEOREM 6.3. *Let X be a normal projective surface, $\mathcal{F}$ a rank 1 foliation, and $\Delta = \sum a_i D_i$ an effective divisor. Suppose that $K_{\mathcal{F}} + \Delta$ is $\mathbb{R}$-Cartier. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}}+\Delta \geqslant 0} + Z_{-\infty} + \sum \mathbb{R}^+ [L_i]$$

where L_i are invariant rational curves with $(K_{\mathcal{F}} + \Delta) \cdot L_i \geqslant -4$, and $Z_{-\infty}$ is spanned by those D_i in $\operatorname{supp}(\Delta)$ with $a_i > \epsilon(D_i)$.

In particular, if H is ample, there are only finitely many curves with extremal rays R with $(K_{\mathcal{F}} + \Delta + H) \cdot R < 0$.

Proof. Let W denote the closure of the right-hand side of the desired equality.

Assume that W is strictly smaller than $\overline{NE}(X)$. Then, by Lemma 6.2, if H is a suitable ample divisor, if we choose t so that $H_R = K_{\mathcal{F}} + \Delta + tH$ is nef, it is zero precisely on one exposed extremal ray R, not contained in W.

We argue depending on $\nu(H_R)$. If $\nu(H_R) \leqslant 1$, then, we apply foliated bend and break lemma, Corollary 2.28, setting $D_i = H_R$ for $i \leqslant \nu(H_R) + 1$ and $D_i = H$ otherwise. Then, $D_1 \cdot D_2 = 0$ and $(K_{\mathcal{F}} + \Delta) \cdot D_2 = -tH \cdot D_2 < 0$. We can then conclude that through a general point of S there is a rational curve Σ with $D_1 \cdot \Sigma = H_R \cdot \Sigma = 0$, with

$$M \cdot \Sigma \leqslant 4 \frac{M \cdot D_2}{-K_{\mathcal{F}} \cdot D_2}$$

where M is any nef divisor.

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For $k \gg 0$ sufficiently large we see that $A := kH_R - (K_{\mathcal{F}} + \Delta)$ is ample and

$$A \cdot \Sigma \leqslant 4\frac{(kH_R - K_{\mathcal{F}} - \Delta) \cdot D_2}{-K_{\mathcal{F}} \cdot D_2} = 4\frac{-(K_{\mathcal{F}} + \Delta) \cdot D_2}{-K_{\mathcal{F}} \cdot D_2} \leqslant 4.$$

In particular, the extremal ray is spanned by the class $[\Sigma]$ and $A \cdot \Sigma = -(K_{\mathcal{F}} + \Delta) \cdot \Sigma \leqslant 4$.

If $\nu(H_R) = 2$, then, writing $H_R = A + E$ where A is ample and E is effective we see that $E \cdot R < 0$, and hence R is spanned by some component of E; call it C.

Write $E = rC + E'$ with $r > 0$. If Δ is a boundary along C, we see that there exists some $\alpha \geqslant 0$ such that $K_{\mathcal{F}} + \Delta + \alpha E = K_{\mathcal{F}} + \Delta' + C$ where $\Delta' \geqslant 0$ and C is not contained in $\operatorname{supp}(\Delta')$. However, we have $(K_{\mathcal{F}} + \Delta' + C) \cdot C < 0$ which is a contradiction of foliation adjunction, Proposition 3.4, if C is not invariant. Thus C must be invariant and so R is spanned by an invariant curve, C.

If C is not contained in Δ then $\nu^*(K_{\mathcal{F}} + \Delta) = K_{C^\nu} + \Theta \geqslant -2$ where $\Theta \geqslant 0$ and where $\nu : C^\nu \to C$ is the normalization. Thus, either $C \subset \operatorname{supp}(\Delta)$ or C^ν is rational and $(K_{\mathcal{F}} + \Delta) \cdot C \geqslant -2$. We see then that W and $\overline{NE}(X)$ coincide.

Standard arguments then apply to show that the right-hand side of our equality is already closed, and that the extremal rays are locally discrete. □

Remark 6.4. Observe that $Z_{-\infty}$ is in fact the contribution to the cone coming from the non-log canonical locus of (X, Δ).

7. The cone theorem for threefolds

With the work of the previous sections in hand, we are now in a position to give a proof of the foliated cone theorem. The argument is similar to the one used to prove the cone theorem for surfaces.

Theorem 7.1. *Let X be a projective $\mathbb{Q}$-factorial and klt threefold and $\mathcal{F}$ a co-rank 1 foliation with non-dicritical foliation singularities. Suppose that $(\mathcal{F}, \Delta)$ has canonical singularities. Then*

$$\overline{NE}(X) = \overline{NE}(X)_{K_{\mathcal{F}} + \Delta \geqslant 0} + \sum \mathbb{R}_+[L_i]$$

where L_i is a rational curve with $(K_{\mathcal{F}} + \Delta) \cdot L_i \geqslant -6$.

In particular, the $(K_{\mathcal{F}} + \Delta)$-negative extremal rays are locally discrete in the $(K_{\mathcal{F}} + \Delta)$-negative portion of the cone.

Proof. Choose a suitable ample divisor H so that $H_R = (K_{\mathcal{F}} + \Delta) + H$ is nef, and such that H_R is zero on precisely one exposed extremal ray, R. We argue based on the numerical dimension of H_R.

If $\nu = \nu(H_R) < 3$, as in the proof of Theorem 6.3, we see that R is spanned by a family of rational curves Σ_t tangent to $\mathcal{F}$, passing through a general point of X and with $\Sigma_t \cdot (K_{\mathcal{F}} + \Delta) \geqslant -6$ by applying Corollary 2.28 with $D_i = H_R$ for $1 \leqslant i \leqslant \nu(H_R) + 1$ and $D_i = H$ for $\nu(H_R) + 1 < i \leqslant \dim(X) = 3$.

So, suppose that $\nu(H_R) = 3$, i.e. H_R is big and nef. Taking $\epsilon > 0$ sufficiently small, we may take $K_{\mathcal{F}} + \Delta + (t - \epsilon)H$ to be a big $\mathbb{Q}$-divisor which is negative on R and so there exists some effective prime divisor D such that $D \cdot R < 0$. We have then that R comes from an extremal ray $R' \subset \overline{NE}(D^\nu)$ where $\nu : D^\nu \to D$ is the normalization of D.

Suppose first that D is invariant. By Lemma 5.11 we may write $\nu^*(K_{\mathcal{F}} + \Delta) = K_{D^\nu} + \Delta_{D^\nu}$ where $\Delta_{D^\nu} \geqslant 0$. The (classical) cone theorem for surfaces shows that R' is spanned by a K_{D^ν}-negative rational curve or it is spanned by a curve contained in $\operatorname{supp}(\Delta_{D^\nu})$. In either case, we see that R is spanned by a curve tangent to $\mathcal{F}$ in which case may apply Lemma 5.14 to conclude.

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Now suppose that D is transverse to the foliation. In this case, since R is $K_{\mathcal{F}} + \Delta$ and D-negative Lemma 4.7 applies to show that R is spanned by the class of a curve tangent to the foliation, in which case Lemma 5.14 applies again.

In both cases, R is spanned by the class of a rational curve C tangent to the foliation with $(K_{\mathcal{F}} + \Delta) \cdot C \geqslant -4$.

Our result then follows by standard arguments to show that the cone of curves indeed has the claimed structure. □

8. The contraction theorem

In this section we will prove a contraction result for $K_{\mathcal{F}} + \Delta$-negative extremal rays in the cone of curves.

DEFINITION 8.1. Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray in $\overline{NE}(X)$. By a *contraction* of R we mean a morphism $c_R : X \to Y$ between normal varieties with $c_{R*}\mathcal{O}_X = \mathcal{O}_Y$ and $c_R(\Sigma) = p$, a point, if and only if $[\Sigma] \in R$.

Our goal is to prove the following.

THEOREM 8.2. *Suppose X is a projective $\mathbb{Q}$-factorial and klt threefold. Suppose that $\mathcal{F}$ has non-dicritical singularities and that $(\mathcal{F}, \Delta)$ is canonical and is terminal along* $\operatorname{sing}(X)$.

Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray. Then there is a contraction of R which only contracts curves tangent to $\mathcal{F}$.

We will handle the cases of fibre, divisorial and flipping type contractions separately.

DEFINITION 8.3. Given an extremal ray $R \subset \overline{NE}(X)$ we define $\operatorname{loc}(R)$ to be all those points x such that there exists a curve C with $x \in C$ and $[C] \in R$.

LEMMA 8.4. *Let X be a $\mathbb{Q}$-factorial threefold. Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray. Then* $\operatorname{loc}(R)$ *is closed.*

Proof. Let H_R be a supporting hyperplane to R.

Suppose that $\nu(H_R) < 3$. As in the proof of the cone theorem, we see that R is spanned by a family of rational curves tangent to $\mathcal{F}$ and passing through a general point of X by applying Corollary 2.28 with $D_i = H_R$ for $1 \leqslant i \leqslant \nu(H_R) + 1$ and $D_i = H$ for $\nu(H_R) + 1 < i \leqslant \dim(X) = 3$ where H is an ample divisor.

Let B be the closed subscheme of the Hilbert scheme of X parametrizing those subvarieties tangent to $\mathcal{F}$; see [LPT18, § 2.3]. We may consider the following diagram as in [Kol91, § 4]:

$$\begin{array}{ccc} U & \xrightarrow{F} & X \\ \downarrow{\scriptstyle p} & & \\ Z & & \end{array}$$

where Z is the closed subscheme of B whose general element paramatrizes a rational curve tangent to $\mathcal{F}$ and spanning R. Let $p : U \to Z$ be the universal family over Z and $F : U \to X$ the evaluation map and observe that F is surjective.

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However, observe that if Σ is any curve contained in a fibre of p then $[F(\Sigma)] \in R$ and so $\mathrm{loc}(R) = X$.

Otherwise H_R is big and nef, and so there exists an irreducible effective divisor S with $R \cdot S < 0$, in particular $\mathrm{loc}(R) \subset S$. We can write $H_R = K_{\mathcal{F}} + \Delta + t\epsilon(S)S + A$ where A is ample and t is the log canonical threshold of $(\mathcal{F}, \Delta)$ along S. Write $G = H_R|_{S^\nu}$ where $S^\nu \to S$ is the normalization and observe that $G = K_{S^\nu} + \Theta + \nu^* A$ or $= K_{\mathcal{F}_{S^\nu}} + \Theta + \nu^* A$ depending on whether S^ν is invariant or not and where $\Theta \geqslant 0$.

If $\nu(G) = 2$ then $\mathrm{loc}(R)$ is a finite collection of curves. Otherwise, as above, we can apply bend and break, to produce through every point $x \in S$ a curve spanning R and so $\mathrm{loc}(R) = S$. □

Definition 8.5. Let $\mathrm{loc}(R) = Z$. If $\dim(Z) = 3$ we say that the contraction corresponding to R is of *fibre type*, if $\dim(Z) = 2$ we say that the contraction corresponding to R is of *divisorial type*, and if $\dim(Z) = 1$ we say that the contraction is of *flipping type*.

8.1 Preliminary computations

We collect here several computations that we will use repeatedly through this section.

Lemma 8.6. *Let X be a $\mathbb{Q}$-factorial and klt threefold. Suppose that $(X, \mathcal{F})$ has non-dicritical terminal foliation singularities. Then* $\mathrm{sing}(X)$ *is tangent to $\mathcal{F}$.*

Proof. Suppose not. Then there exists a curve $C \subset \mathrm{sing}(X)$ transverse to $\mathcal{F}$. Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be a resolution of singularities of X. Perhaps shrinking around a general point of C we may assume that π only extracts divisors which dominate C and which are transverse to the foliation. Furthermore, we may run a K_Y-MMP over X and by replacing Y by the output of this MMP we may assume that K_Y is π-nef.

On the other hand, since $\mathcal{F}$ is terminal, by the negativity lemma, [KM98, Lemma 3.38], we know that $K_{\mathcal{G}}$ is not π-nef, so let $f \subset Y$ be a curve mapping to a point with $K_{\mathcal{G}} \cdot f < 0$, and let $E \subset Y$ be a divisor with $E \cdot f < 0$.

Let $(E, \mathcal{H})$ be the restriction of $\mathcal{G}$ to E. By non-dicriticality, $\mathcal{H}$ is induced by the fibration $E \to C$. By foliation adjunction, Proposition 3.4, we can write $(K_{\mathcal{G}} + E)|_E = K_{\mathcal{H}} + \Delta$ where $\Delta \geqslant 0$. Since $K_{\mathcal{G}}, E$ are both Cartier and intersect f negatively we get that $-2 \leqslant K_{\mathcal{H}} \cdot f \leqslant -2$, and so in fact $\mathcal{H}$ is induced by a fibration in rational curves and $E \cdot f = -1$.

Thus, $K_E \cdot f = -2 = K_{\mathcal{H}} \cdot f$, which since $E \cdot f = -1$, implies that $K_Y \cdot f = -1$, a contradiction of the π-nefness of K_Y. □

Lemma 8.7. *Let X be a threefold and $\mathcal{F}$ a co-rank 1 foliation with non-dicritical terminal foliation singularities. Let H be a general hyperplane. Then $(H, \mathcal{F}_H)$ has terminal foliation singularities.*

Proof. The proof of the corresponding statement for varieties works equally well in this case. □

Corollary 8.8. *Let X be a threefold and $\mathcal{F}$ a terminal Gorenstein co-rank 1 foliation. Then $(X, \mathcal{F})$ has at most isolated singularities.*

Proof. Follows from the above lemma and the fact that Gorenstein terminal surface foliations are smooth foliations on smooth surfaces. □

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LEMMA 8.9. *Let X be a threefold and $\mathcal{F}$ a co-rank 1 foliation. Suppose that $(\mathcal{F}, \Delta)$ has canonical singularities and $(\mathcal{F}, \Delta)$ is terminal along* $\operatorname{sing}(X)$. *Let $D, D_1, \dots, D_n$ be a collection of $\mathcal{F}$-invariant divisors. Suppose that $D, D_1, \dots, D_n$ are $\mathbb{Q}$-Cartier. Let $D^\nu \to D$ be the normalization.*

Write $K_{\mathcal{F}}|_{D^\nu} = K_{D^\nu} + \Theta$ and $(K_X + D + \sum D_i)|_{D^\nu} = K_{D^\nu} + \Delta$. Then $\Theta \geqslant \Delta \geqslant 0$ with equality along those centers contained in $\operatorname{sing}(X)$.

In particular, if $C \subset D$ is not contained in $\operatorname{sing}(\mathcal{F})$, *then $K_{\mathcal{F}} \cdot C \geqslant (K_X + D + \sum D_i) \cdot C$.*

Proof. First, observe that since $\mathcal{F}$ is terminal along $\operatorname{sing}(X)$ we see that $\operatorname{sing}(\mathcal{F}) \cap \operatorname{sing}(X)$ does not contain any curve.

Write $\Theta = \sum a_i T_i + \sum b_j S_j$ and $\Delta = \sum c_i T_i + \sum d_j S_j$ where $T_i \subset \operatorname{sing}(\mathcal{F})$ and $S_j \subset \operatorname{sing}(X)$.

Notice that since $\mathcal{F}$ has canonical singularities, $D \cup D_1 \cup \cdots \cup D_n$ has normal crossings in codimension 2, [LPT18, Corollary 3.6]. Furthermore, since $\mathcal{F}$ is terminal along $\operatorname{sing}(X)$ any two invariant divisors cannot intersect along $\operatorname{sing}(X)$.

This gives us that $D_i|_D$ is a reduced divisor and that $c_k = 1$ for all k such that $T_k \subset \operatorname{supp}(\Delta)$. $D \cap D_i \subset \operatorname{sing}(\mathcal{F})$ for all i and so $\operatorname{supp}(\sum c_i T_i) \subset \operatorname{supp}(\sum a_i T_i)$.

We have that $a_i \geqslant 1$, with equality if and only if:

(i) either $\mathcal{F}$ is simple at the generic point of T_i and D is a strong separatrix along T_i;
(ii) or $\mathcal{F}$ is canonical but not simple at the generic point of T_i.

To see this we may cut by a generic hyperplane, and so we may assume that D is a curve and T_i is a point. If T_i is a simple singularity then the claim follows from Example 5.12. So suppose that T_i is canonical but not simple and let ω be a 1-form defining the foliation in a neighborhood of T_i. In this case by the classification of canonical surface foliation singularities, [McQ08, Fact I.2.4], we see that $\omega = px\,dy - (qy + \epsilon x)\,dx +$ higher-order terms where $\epsilon \in \{0, 1\}$ and $D = \{x = 0\}$ or $\{y = 0\}$. In either case $\omega|_D$ vanishes to order 1 at T_i and so $a_i = 1$. In particular, from this it follows that $a_i \geqslant c_i$ for all i.

To see that $d_j = b_j$ for all j we may cut by a generic hyperplane, and so we may assume that X is a surface and that D is a curve. The equality then follows from [McQ08, III.2.bis.1], for example.

Thus, for $C \neq T_i$ for all i, we have that $0 > (K_{D^\nu} + \Theta) \cdot C \geqslant (K_{D^\nu} + \Delta) \cdot C$. □

The next lemma guarantees that we only contract curves tangent to the foliation in the course of the MMP, which in turn implies that the singularities of our foliation stay non-dicritical.

LEMMA 8.10. *Let X be $\mathbb{Q}$-factorial and klt threefold and $\mathcal{F}$ be non-dicritical, co-rank 1 foliation, and suppose furthermore that $(\mathcal{F}, \Delta)$ is log canonical.*

Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray. Suppose that $[C] \in R$. Then C is tangent to the foliation.

Proof. Suppose for sake of contradiction that there exists a $C \subset X$ with $[C] \in R$ and C is transverse to $\mathcal{F}$. Let H_R be a supporting hyperplane to R.

Suppose first that $\nu(H_R) < \dim(X) = 3$. As in the proof of the cone theorem, or in the proof of Lemma 8.4 we see through a general point of X there is a rational curve tangent to $\mathcal{F}$ spanning R.

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Let B be the closed subscheme of the Hilbert scheme of X parametrizing those subvarieties tangent to $\mathcal{F}$. We may consider the following diagram as in [Kol91, § 4]:

$$\begin{array}{ccc} U & \xrightarrow{F} & X \\ \downarrow{\scriptstyle p} & & \\ Z & & \end{array}$$

where Z is the closed subscheme of B whose general element paramatrizes a rational curve tangent to $\mathcal{F}$ and spanning R. Let $p : U \to Z$ be the universal family over Z and $F : U \to X$ the evaluation map and observe that F is surjective. After taking general hyperplane sections of Z we may assume $\dim(Z) = 2$.

For $x \in Z$ let $V_x = \{y \in Z : F(p^{-1}(x)) \cap F(p^{-1}(y)) \neq \emptyset\}$ and observe that $V_x \subset Z$ is a closed subset. Moreover, by non-dicriticality of $\mathcal{F}$ we see that V_x is a strict subset of Z for all $x \in Z$. Indeed, if $\dim(V_x) = 2$ then $F(p^{-1}(x))$ would be a curve tangent to $\mathcal{F}$ meeting infinitely many leaves, which contradicts the supposition.

There are therefore two cases for general $x \in Z$:

(i) $\dim(V_x) = 0$; or
(ii) $\dim(V_x) = 1$.

Suppose that we are in case (ii). For $x \in Z$ a general point define $D_x = F_* p^*(V_x)$. Since $\mathcal{F}$ is non-dicritical we see that for all $y \in V_x$ that $F(p^{-1}(y))$ all must belong to the same leaf and thus D_x is in fact a $\mathcal{F}$-invariant divisor. Thus $\mathcal{F}$ admits a meromorphic first integral $\phi : X \dashrightarrow S$ where $\dim(S) = 1$.

We claim that ϕ is in fact a morphism. Let $\mu : \widetilde{X} \to X$ be a resolution of singularities of ϕ, let $\widetilde{\mathcal{F}}$ be the transformed foliation on $\widetilde{X}$, let $\tilde{\phi} : \widetilde{X} \to S$ be the resolved map and observe that $\tilde{\phi}$ is a first integral for $\widetilde{\mathcal{F}}$. Since $\mathcal{F}$ is non-dicritical then for all $x \in X$ we have that $\mu^{-1}(x)$ is tangent to $\widetilde{\mathcal{F}}$ and so we see that $\tilde{\phi}(\mu^{-1}(x))$ must be a single point $s \in S$. The rigidity lemma, [BS95, Lemma 4.1.13], then implies that $\tilde{\phi}$ descends to a morphism $X \to S$, namely ϕ itself.

By definition we have that if $y \in Z - V_x$ is general then $D_x \cap F(p^{-1}(y)) = \emptyset$ and hence $0 = D_x \cdot R = D_x \cdot C$. However, since C is transverse to the foliation and D_x is invariant, for a general x we must have that $C \cdot D_x > 0$, which contradicts the supposition.

So we are in case (i). We define $H' = F_* p^* H$ where H is a general ample divisor on Z. Observe that if $y \in Z$ is a general point that $H' \cap F(p^{-1}(y)) = \emptyset$ hence $0 = H' \cdot R = H' \cdot C$. Notice that $F^{-1}(C) \subset U$ must be a curve; otherwise, through a general point $P \in C$ there is a 1-dimensional family of curves tangent to $\mathcal{F}$ passing through P, and hence the leaf through P is algebraic. Arguing as in case (ii), we see that $\mathcal{F}$ would admit a first integral, which contradicts the assumption. Thus, we see that $p^* H \cdot F^{-1}(C) > 0$, and so $H' \cdot C > 0$, which contradicts the supposition.

So we have $\nu(H_R) = 3$ and so $H_R \sim_{\mathbb{Q}} A_0 + E_0$ where A_0 is ample and $E_0 \geqslant 0$. Since $H_R \cdot R = 0$ and $A_0 \cdot R > 0$ we see that $E_0 \cdot R < 0$ and so there is some irreducible component E of E_0 with $E \cdot R < 0$. Thus $E \cdot C < 0$ and so $C \subset E$. Since C is transverse to $\mathcal{F}$ we see that E is transverse to $\mathcal{F}$. We may then find $0 \leqslant s_0 \leqslant 1$ such that $\Delta + s_0 E = \Delta' + E$ where E is not contained in the support of Δ'.

Moreover, we know that $(\mathcal{F}, \Delta' + E = \Delta + s_0 E)$ is log terminal at the generic point of C. Indeed, suppose otherwise and let $0 \leqslant t \leqslant s_0$ be the log canonical threshold of $(\mathcal{F}, \Delta)$ with respect to E at the generic point of C. On one hand, since $E \cdot C < 0$ and $(K_{\mathcal{F}} + \Delta) \cdot C < 0$

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we have that $(K_{\mathcal{F}} + \Delta + tE) \cdot C < 0$. On the other hand foliation subadjunction, Theorem 4.5, implies that $(K_{\mathcal{F}} + \Delta + tE) \cdot C \geqslant 0$, which contradicts the supposition.

Writing $\nu : E^{\nu} \to E$ for the normalization map and $\mathcal{G}$ for the foliation induced on E^{ν} we have by Proposition 3.4, $K_{\mathcal{G}} + \Theta = \nu^*(K_{\mathcal{F}} + \Delta' + E)$ where $\Theta \geqslant 0$. A computation exactly as in [KM98, Propostion 5.46] shows that $(\mathcal{G}, \Theta)$ is log terminal at the generic point of C' where C' is the strict transform of C. Note that we also have

$$(K_{\mathcal{G}} + \Theta) \cdot C' < 0.$$

Letting $K_{\mathcal{F}} + \Delta' + E + A = H_R$ be a supporting hyperplane for R, with A ample, we have that $K_{\mathcal{G}} + \Theta + \nu^* A$ is a nef divisor, and $(K_{\mathcal{G}} + \Theta + (1-\epsilon)\nu^* A) \cdot C' < 0$ for $\epsilon > 0$.

We claim that $K_{\mathcal{G}} + \Theta + \nu^* A$ is not big. Indeed, suppose for sake of contradiction that $K_{\mathcal{G}} + \Theta + \nu^* A$ were big. Then for $\epsilon > 0$ sufficiently small we have that $K_{\mathcal{G}} + \Theta + (1-\epsilon)\nu^* A$ is big and $(K_{\mathcal{G}} + \Theta + (1-\epsilon)\nu^* A) \cdot C' < 0$ and so we may write $(K_{\mathcal{G}} + \Theta + (1-\epsilon)\nu^* A) \sim_{\mathbb{Q}} A' + G + aC'$ where A' is ample, G is an effective divisor whose support does not contain C' and $a > 0$.

Since $(\mathcal{G}, \Theta)$ is log terminal at the generic point of C' we may find $t_0 > 0$ so that $\Theta + t_0(A' + G + aC') = \Theta' + C'$ where the support of Θ' does not contain C'. On one hand, we see that $(K_{\mathcal{G}} + \Theta' + C') \cdot C' < 0$, but, on the other hand, foliation adjunction, Proposition 3.4, implies that $(K_{\mathcal{G}} + \Theta' + C') \cdot C' \geqslant 0$, and so we have a contradiction.

Thus $(K_{\mathcal{G}} + \Theta + \nu^* A)^2 = 0$ and so we may apply foliated bend and break, Corollary 2.28, as earlier in the proof to produce rational curves Σ through a general point of E^{ν} tangent to the foliation with $\nu^* H_R \cdot \Sigma = 0$ and hence the pushforward of these curves span the ray R.

By non-dicriticality of $\mathcal{F}$, we see that $\mathcal{G}$ is the foliation induced by a fibration in rational curves $E^{\nu} \to B$. Let f be a general fibre of this morphism. We have that $[f] \in R$. Thus $\nu^* H_R \cdot f = \nu^* H_R \cdot C' = 0$ and so we may apply Lemma 8.11 below to conclude that $\nu^* H_R$ is numerically trivial. However, this implies that $-(K_{\mathcal{G}} + \Theta)$ is numerically equivalent to $\nu^* A$. Since $\nu^* A$ is ample this implies that $-(K_{\mathcal{G}} + \Theta)$ is ample as well. Observe, moreover, that $\nu^* H_R$ being numerically trivial implies that if $\Sigma \subset E^{\nu}$ is a curve then $[\nu_* \Sigma] \in R$.

The foliated cone theorem for surfaces, Theorem 6.3, applies to $(\mathcal{G}, \Theta)$ to show that every extremal ray in $\overline{NE}(E^{\nu})$ is spanned by a curve tangent to the foliation or contained in the non-lc locus of $(\mathcal{G}, \Theta)$.

Let Σ be some curve contained in the non-lc locus of $(\mathcal{G}, \Theta)$. Then $\nu_* \Sigma$ is a non-lc center of $(\mathcal{F}, \Delta' + E = \Delta + s_0 E)$. By our above observations we know that $[\nu_* \Sigma] \in R$, and so:

(i) $E \cdot \nu_* \Sigma < 0$; and
(ii) $(K_{\mathcal{F}} + \Delta) \cdot \nu_* \Sigma < 0$.

Suppose for a contradiction that Σ is transverse to $\mathcal{G}$. Let $0 \leqslant t \leqslant s_0$ be the log canonical threshold of $(\mathcal{F}, \Delta)$ with respect to E at the generic point of $\nu_* \Sigma$. We see by inequalities (i) and (ii) above that $(K_{\mathcal{F}} + \Delta + tE) \cdot \nu_* \Sigma < 0$, but on the other hand foliation subadjunction, Theorem 4.5, tells us that $(K_{\mathcal{F}} + \Delta + tE) \cdot \nu_* \Sigma \geqslant 0$, which contradicts the supposition. Thus Σ is tangent to $\mathcal{G}$.

This implies that $\overline{NE}(E^{\nu})$ is spanned by fibres of $E^{\nu} \to B$, which is an impossibility. □

LEMMA 8.11. *Let $p : S \to B$ be a surjective morphism where S is a projective surface and B is a curve. Let $F \subset S$ be a general fibre of $p : S \to B$ and let $C \subset S$ be a curve which dominates B. Suppose that H is a nef divisor on S with $H \cdot F = H \cdot C = 0$. Then H is numerically trivial.*

Proof. Let $\mu : S' \to S$ be a resolution of singularities of S and let $p' : S' \to B$ be the induced fibration. Let C' denote the strict transform of C and F' denote the strict transform of F.

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Observe that C' still dominates B and F' is still a general fibre of p'. Moreover, μ^*H is numerically trivial if and only if H is so we may freely replace S by S' and so may assume that S is smooth.

For $k \gg 0$ observe that the divisor $D_k = C + kF$ is a big divisor and so we may write $D_k \sim_{\mathbb{Q}} A + E$ where A is ample and E is effective. Since H is nef and $H \cdot D_k = 0$ this implies $H \cdot E = H \cdot A = 0$. In particular, it has zero intersection with an ample divisor and is therefore numerically trivial. □

8.2 Fibre type contractions

Throughout this section we assume X is a projective $\mathbb{Q}$-factorial and klt threefold. Let $\mathcal{F}$ be a co-rank 1 foliation with non-dicritical singularities, $\Delta \geqslant 0$ and let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray with $\mathrm{loc}(R) = X$. Notice that in this case R is in fact $K_{\mathcal{F}}$-negative.

Suppose that $\mathrm{loc}(R) = X$. Let H_R be a supporting hyperplane of R. In the proof of the cone theorem we see that in this case $\nu(H_R) < 3$, and that there is a general complete intersection curve C such that $K_{\mathcal{F}} \cdot C < 0$.

Lemma 8.12. *Let $\mathcal{E} = \mathcal{F}|_C$.*

(i) *Either $\mathcal{E}$ is semi-stable and $\mathcal{F}$ is a fibration over a curve.*

(ii) *Or $\mathcal{E}$ is unstable, and either there is a foliation by rational curves tangent to $\mathcal{F}$ realizing $\mathcal{F}$ as the pullback of a surface foliation or $\mathcal{F}$ is a fibration over a curve.*

Proof. If $\mathcal{E}$ is semi-stable, then since $\det(\mathcal{E})$ is ample, $\mathcal{E}$ is ample as a vector bundle. [BM01] applies in this case.

Otherwise, let $\mathcal{L} \subset \mathcal{E}$ be a maximal destabilizing subbundle. By the Mehta–Ramanathan theorem this extends to a subsheaf $\mathcal{G} \subset \mathcal{F}$ which is a foliation by rational curves.

Consider the following diagram as in Lemma 8.10:

$$\begin{array}{ccc} U & \xrightarrow{F} & X \\ \downarrow{\scriptstyle p} & & \\ Z & & \end{array}$$

where Z is projective and where $p : U \to Z$ is a projective morphism whose general fibre is a rational curve tangent to $\mathcal{G}$.

For $z \in Z$ let $C_z := p^{-1}(z)$ and $D_z := F(C_z)$ so D_z is tangent to $\mathcal{G}$. Let D_{z_0} be a general curve, then:

(i) either D_{z_0} does not intersect any other D_z;

(ii) or D_{z_0} intersects infinitely many other D_z.

In case (i), $\mathcal{G}$ in fact defines a fibration over a surface. Since this fibration is tangent to $\mathcal{F}$, we see that $\mathcal{F}$ is in fact pulled back from this surface.

In case (ii), if D_z meets D_{z_0}, then since $\mathcal{F}$ is non-dicritical, D_z and D_{z_0} must (generically) belong to the same leaf. Since there are infinitely many D_z meeting D_{z_0}, the leaf containing D_{z_0} must in fact be algebraic. Moreover, we see that the leaf is covered by rational curves. In any case, the result follows. □

Theorem 8.13. *Set up as above. The contraction associated to R exists.*

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Proof. By our previous lemma we see that R is in fact K_X-negative; indeed, if C is a general curve which spans R then $K_{\mathcal{F}} \cdot C = K_{X/Z} \cdot C = K_X \cdot C$ where $\pi : X \to Z$ is the fibration guaranteed by the previous lemma. The existence of the contraction follows the corresponding statement about K_X-negative contractions. □

8.3 Divisorial contractions

Throughout this section we assume X is a projective $\mathbb{Q}$-factorial and klt threefold.

The idea behind constructing a divisorial contraction of a divisor D is to realize it as a $K_X + (1 - \epsilon(D))D$-contraction.

LEMMA 8.14. *Let X be as above and suppose that $\mathcal{F}$ is a co-rank 1 foliation with non-dicritical singularities and that $(\mathcal{F}, \Delta)$ is log canonical (log terminal). Let $D = \sum D_i$ be an invariant divisor. Then $(X, \Delta + D)$ is log canonical (log terminal).*

Proof. Let $\pi : (Y, \Delta' + D') \to (X, \Delta + D)$ be a log resolution of $(X, \Delta + D)$ where Δ' is the strict transform of Δ and D' is the strict transform of D. Perhaps passing to a higher resolution we may assume that $\mathcal{G}$ has simple (hence canonical) singularities where $\mathcal{G}$ is the pulled back foliation.

Write

$$K_{\mathcal{G}} + \Delta' + \sum E_j^1 = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a_i^0 E_i^0 + \sum a_j^1 E_j^1$$

and

$$K_Y + \Delta' + D' + \sum E_i^0 + \sum E_j^1 = \pi^*(K_X + \Delta + D) + \sum b_i^0 E_i^0 + \sum b_j^1 E_j^1$$

where E_i^0 are the invariant π-exceptional divisors and E_j^1 are the non-invariant π-exceptional divisors.

By assumption $a_i^0, a_i^1 \geqslant 0 (> 0)$ so if we can show that $b_k^\delta \geqslant a_k^\delta$ the result will follow.

By non-dicriticality the foliation restricted to each E_j^1 is exactly the fibration structure $E_j^1 \to \pi(E_j^1)$, and so if f_j is a general fibre of $\pi|_{E_j^1}$ we know that $(K_{\mathcal{G}} + E_j^1) \cdot f_j = (K_Y + E_j^1) \cdot f_j$ and so $K_{\mathcal{G}} \cdot f_j = K_Y \cdot f_j$. Notice also that $f_j \cdot (D' + \sum E_i^0) = 0$.

This computation and Lemma 8.9 show that

$$K_{\mathcal{G}} - \left(K_Y + D' + \sum E_i^0\right)$$

is π-nef away from finitely many curves and so the negativity lemma, [KM98, Lemma 3.38], applies to show that $a_k^\delta - b_k^\delta \leqslant 0$ for all δ, k and the result follows. □

LEMMA 8.15. *Let X be a $\mathbb{Q}$-factorial and klt threefold and $\mathcal{F}$ a non-dicritical co-rank 1 foliation and suppose that $(\mathcal{F}, \Delta)$ is canonical. Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray with $\operatorname{loc}(R) = D$.*

Suppose that D is transverse to the foliation. Then $K_{\mathcal{F}} \cdot R = K_X \cdot R$.

Proof. Let $\mathcal{G}$ be the foliation restricted to D^ν where $\nu : D^\nu \to D$ is the normalization.

Choose t so that $\Delta + tD = \Delta' + D$ where D is not contained in $\operatorname{supp}(\Delta')$ and write $\nu^*(K_{\mathcal{F}} + \Delta + tD) = K_{\mathcal{G}} + \Theta$. By non-dicriticality we see that $\mathcal{G}$ comes from a $\mathbb{P}^1$-fibration $D^\nu \to B$. If f is a general fibre of $D^\nu \to B$, which spans R, then $K_{\mathcal{G}} \cdot f = K_{D^\nu} \cdot f = -2$.

Thus, it suffices to show that if $\nu^*(K_X + \Delta + tD) = K_{D^\nu} + \Theta'$ then $\Theta' \cdot f = \Theta \cdot f$. We see that if $Y \to X$ is some high enough resolution then for centers transverse to the foliation by

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Lemma 3.11 the discrepancy with respect to $(\mathcal{F}, \Delta + tD)$ is exactly the discrepancy with respect to $(X, \Delta + tD)$ and so we see that, by construction of the different, Proposition 3.4, components of Θ and Θ' transverse to $\mathcal{G}$ must have the same coefficient.

Finally, observe that if $B \subset \operatorname{supp}(\Theta + \Theta')$ is a component tangent to $\mathcal{G}$ then $f \cdot B = 0$ and so we may conclude $\Theta \cdot f = \Theta' \cdot f$. □

Lemma 8.16. *Let X be a $\mathbb{Q}$-factorial and klt threefold and $\mathcal{F}$ a non-dicritical co-rank 1 foliation and suppose that $(\mathcal{F}, \Delta)$ is log canonical. Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray with $\dim(\operatorname{loc}(R)) = 2$. Then $\operatorname{loc}(R)$ is an irreducible divisor.*

Proof. Suppose for sake of contradiction that $\operatorname{loc}(R)$ contains two distinct irreducible components D_1 and D_2. We may assume without loss of generality that D_1 is a divisor. By definition for $i = 1, 2$ we may find a family of curves $\{C_t^i\}$ covering D_i such that $[C_t^i] \in R$.

Observe that $H_R - \epsilon A \sim_{\mathbb{Q}} E \geqslant 0$ where H_R is the supporting hyperplane to R, A is ample and $\epsilon > 0$ is sufficiently small. Thus there is some irreducible component E_0 of E with $R \cdot E_0 < 0$; in particular, $C_t^1 \cdot E_0 < 0$ and we see that $E_0 = D_1$ and so $C_t^1 \cdot D_1 < 0$. On the other hand, we have that $C_t^2 \cdot D_1 \geqslant 0$ but this is a contradiction of the fact that $[C_t^2] \in R$. □

Theorem 8.17. *Let X be a $\mathbb{Q}$-factorial and klt threefold. Suppose that $\mathcal{F}$ is a co-rank 1 foliation with non-dicritical singularities and $\Delta \geqslant 0$. Suppose furthermore that $(\mathcal{F}, \Delta)$ has canonical singularities and that $(\mathcal{F}, \Delta)$ is terminal along $\operatorname{sing}(X)$. Let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray and suppose that $\operatorname{loc}(R) = D$ a divisor. Then there exists a contraction of D.*

Proof. By Lemma 8.16 D is irreducible, thus it is either transverse to the foliation or is invariant by the foliation.

If D is transverse to the foliation, then by Lemma 8.15 we know that $(K_{\mathcal{F}} + \Delta) \cdot R = (K_X + \Delta) \cdot R$ and so R is in fact $K_X + \Delta$-negative. Otherwise, D is invariant and by Lemma 8.9 we see that R is $K_X + \Delta + D$-negative.

In either case, we see that R is a $K_X + \Delta + (1 - \epsilon(D))(1 - t)D$-negative extremal ray for $t > 0$ sufficiently small. By Lemma 8.14 we know that $(X, \Delta + (1 - \epsilon(D))(1 - t)D)$ is klt and so the contraction of R exists by [KM98, Theorem 3.7]. □

Corollary 8.18. *Let $c_R : X \to Y$ be the contraction constructed above. Then:*

(i) $\rho(X/Y) = 1$;
(ii) *Y is $\mathbb{Q}$-factorial and projective;*
(iii) *Y is klt.*

Proof. This is a consequence of standard facts about log contractions; see for instance [KM98, Theorem 3.7]. □

8.4 Flipping contractions

Throughout this section we assume X is a projective $\mathbb{Q}$-factorial and klt threefold. Let $\mathcal{F}$ a co-rank 1 foliation and let R be a $K_{\mathcal{F}} + \Delta$-negative extremal ray. Suppose $\operatorname{loc}(R)$ is 1-dimensional and let H_R be a supporting hyperplane to R.

In this section we show that a flipping contraction can be realized in the category of algebraic spaces. While we are able to show that this contraction (and the flip) can be realized in the category of projective spaces in some special cases, we are unable to deduce a complete flip theorem.

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For a Cartier divisor D let $\mathrm{Null}(D) = \{P : P \in V, V \cdot D^{\dim(V)} = 0\}$, and $BS(D)$ denotes the stable base locus of D, i.e. $\bigcap_{m \geqslant 0} bs(mD)$ where $bs(mD)$ is the base locus of mD. It is easy to see that $BS(D) = bs(mD)$ for m sufficiently large and divisible.

We will make use of the following result [CL14, Corollary 1].

LEMMA 8.19. *Let X be normal threefold and let L be big and nef. Let A be an ample divisor. Then, for all $\epsilon > 0$ sufficiently small, Null$(L) = BS(L - \epsilon A)$.*

As we have seen H_R is big and nef. By our next lemma $\mathrm{Null}(H_R)$ is a finite collection of curves.

LEMMA 8.20. *Set up as above. Let $S \subset X$ be a surface. Then $H_R^2 \cdot S > 0$.*

Proof. Suppose for sake of contradiction that there is some surface S such that $H_R^2 \cdot S = 0$.

Let $f : S^\nu \to S$ be the normalization of S.

Observe that $H_R|_S$ is nef and so it is pseudo-effective, and we proceed case by case on the numerical dimension of f^*H_R.

If $\nu(f^*H_R) = 0$ then H_R is zero on a moving curve, and hence is zero on infinitely many curves, which contradicts the supposition.

If $\nu(f^*H_R) = 1$ then write $H_R = K_{\mathcal{F}} + \Delta + A$ where A is ample. We have that $f^*(H_R)^2 = 0$, and that $f^*(H_R)$ has positive intersection with any ample divisor on S^ν (otherwise H_R would be zero on a moving curve).

Thus

$$f^*(K_{\mathcal{F}} + \Delta) \cdot f^*H_R = -f^*A \cdot f^*H_R < 0.$$

Perhaps rescaling H_R by a positive constant we may write $H_R = A' + D + S$ where A' is ample, and D is effective, and the support of D does not contain S. Then

$$f^*H_R \cdot f^*S = -f^*H_R \cdot f^*(A' + D) \leqslant -f^*H_R \cdot f^*A' < 0.$$

If S is $\mathcal{F}$ invariant, then $f^*(K_{\mathcal{F}} + \Delta) = K_{S^\nu} + \Theta$ where $\Theta \geqslant 0$.

We apply bend and break, Lemma 2.28, to $D_1 = D_2 = f^*H_R$. $D_1 \cdot D_2 = 0$ by supposition, and by our above computation $(K_{S^\nu} + \Theta) \cdot D_1 = f^*(K_{\mathcal{F}} + \Delta) \cdot f^*H_R < 0$. Thus, we get through a general point of S a rational curve Σ with $0 = D_2 \cdot \Sigma = H_R \cdot \Sigma$, which contradicts the supposition.

If S is not $\mathcal{F}$ invariant then choose $0 \leqslant t \leqslant 1$ so that $\Delta + tS = \Delta' + S$ where S is not contained in $\mathrm{supp}(\Delta')$. By foliation adjunction we see that $f^*(K_{\mathcal{F}} + \Delta + tS) = K_{\mathcal{F}_{S^\nu}} + \Theta$, $\Theta \geqslant 0$. Again, by our above computations we have that $(K_{\mathcal{F}_{S^\nu}} + \Theta) \cdot f^*H_R = f^*(K_{\mathcal{F}} + \Delta + tS) \cdot f^*H_R < 0$.

Again we apply bend and break with $D_1 = D_2 = f^*H_R$. $D_1 \cdot D_2 = 0$ by assumption, and $D_1 \cdot (K_{\mathcal{F}_{S^\nu}} + \Theta) < 0$. Again, through a general point of S we get a rational curve Σ tangent to the foliation with $0 = D_2 \cdot \Sigma = H_R \cdot \Sigma$, which contradicts the supposition.

If $\nu(f^*H_R) = 2$ then $f^*H_R^2 = H_R^2 \cdot S > 0$ and we are done. □

LEMMA 8.21. *Set up as above. Then* $\mathrm{loc}(R)$ *can be contracted in the category of algebraic spaces.*

Proof. By our previous lemma $\mathrm{Null}(H_R)$ is a finite collection of curves, each of which span R. Let A be an ample divisor and choose ϵ sufficiently small and m sufficiently large so that $\mathrm{Null}(H_R) = bs(m(H_R - \epsilon A)) = B$.

Let $g : Y \to X$ be a resolution of the base locus of $m(H_R - \epsilon A)$ so that we have $g^*(m(H_R - \epsilon A)) = M + F$ where M is semi-ample, F is effective and $g(F) = B$ and $g(\mathrm{exc}(g)) = B$.

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Let G be an effective divisor $\mathbb{Q}$-Cartier divisor supported on $\operatorname{exc}(g)$ such that $-G$ is g-ample (such a G exists because X is $\mathbb{Q}$-factorial).

Then for $1 \gg \delta > 0$ we have that $\epsilon g^*A - \delta G + M = A'$ is ample on Y and $(F + \delta G) + A' = mg^*H_R$. Since $g^*H_R|_{\operatorname{exc}(g)}$ is trivial, we see that $(F + \delta G)|_{\operatorname{exc}(g)} = -A'|_{\operatorname{exc}(g)}$ is anti-ample.

Let N be such that $N(F + \delta G) = D$ is an integral Cartier divisor. By construction we know that $\mathcal{O}_{D_{\mathrm{red}}}(-D)$ is ample. Recall that a line bundle L on a scheme Z is ample if and only if $L|_{Z_{\mathrm{red}}}$ is ample where Z_{red} is the reduction; see for example [Har77, III.ex 5.7.(b)].

Thus, $\mathcal{O}_D(-D)$ is ample and so D is a subscheme with an anti-ample normal bundle, and so it may be contracted to a point by [Art70, Theorem 6.2].

This contraction factors through g and gives a contraction $X \to Z$. By [Art70] this contraction may be taken in the category of algebraic spaces. □

Note that we have not proven that Z is projective. However, we do have the following criterion which will be useful later.

Lemma 8.22. *Set up as above. Let H_R be the supporting hyperplane to R. Assume that H_R descends to a $\mathbb{Q}$-Cartier divisor on Z, then Z is projective.*

Proof. By assumption, if f is the contraction, then $H_R = f^*M$. We claim that M is in fact ample. First M is nef and $M^3 > 0$. If C is any curve in Z then we also have $M \cdot C > 0$. By Lemma 8.20 if S is any surface then $M^2 \cdot S > 0$. Thus the Nakai–Moishezon criterion for ampleness applies to show that M is ample, and so Z is projective. □

Remark 8.23. Notice that we do not know if the flipping contraction is a $K_X + D$-contraction for some divisor D. Indeed, the natural choice for D, namely, the germ of a leaf around a flipping curve might not be a $\mathbb{Q}$-Cartier, even if X is (algebraically) $\mathbb{Q}$-factorial, and so a $K_X + D$-contraction does not even make sense. We will revisit this issue when constructing terminal flips.

9. The smooth MMP and the existence of terminal flips

In this section we first give a classification of flipping curves when X is a smooth threefold, and then use this to deduce the existence of an MMP starting with a smooth foliation on a smooth threefold.

Finally, we explain how to construct flips for terminal co-rank 1 foliations on threefolds.

9.1 Flipping curves on smooth varieties

We begin with an example showing that this case can really happen, see also [BP11] for some similar examples.

Example 9.1. Let $\phi : X_1 \dashrightarrow X_2$ be the threefold toric flop. We can realize X_i as an $\mathbb{A}^1$-bundle over $\mathbb{A}^2$ blown up at a point, with exceptional curve C_i. Let $\mathcal{G}$ be a foliation on $\mathbb{A}^2$ blown up at a point so that the exceptional curve is invariant, meets exactly two other invariant curves and $\mathcal{G}$ has canonical singularities.

Let $\mathcal{F}_2$ be the pull back of this foliation to X_2. Let $\mathcal{F}_1$ be the strict transform of $\mathcal{F}_2$ under ϕ^{-1}. C_1 is the flop of C_2, and observe that $C_1 \subset \operatorname{sing}(\mathcal{F}_1)$, and $\mathcal{F}_1$ has canonical singularities along C_1.

Let X_0 be the blow up of X_i along C_i, with exceptional divisor E, and $\mathcal{F}_0$ the transformed foliation on X_0. Let $\pi_i : X_0 \to X_i$. Let $\tilde{C}_1$ be a $\mathbb{P}^1$ sitting above C_1.

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Observe that $\mathcal{F}_0|_E = K_E + \Delta$ where Δ consists of three of the four torus invariant divisors on E. Thus, $K_{\mathcal{F}_0} \cdot \tilde{C}_1 = -1$ and since $\pi^* K_{\mathcal{F}_1} = K_{\mathcal{F}_0}$ we get that $K_{\mathcal{F}_1} \cdot C_1 = -1$.

Furthermore, we can check that $K_{\mathcal{F}_2} \cdot C_2 = 1$.

Thus, we see that C_1 is an isolated $K_{\mathcal{F}_1}$-negative extremal ray, and the flip of C_1 exists.

We will use the following local version of Reeb stability.

LEMMA 9.2. *Let L be a leaf of a foliation $\mathcal{F}$ on X and $K \subset L$ a compact subset. Suppose that K is simply connected. Then there is an open subset of X, $K \subset W \subset X$ and a holomorphic submersion $W \to U$ such that the leaves of $\mathcal{F}$ are given by the fibres of this map.*

Proof. The usual proof of Reeb stability, see for example [MM03, Theorem 2.9], works in this case. □

COROLLARY 9.3. *Let X be a smooth threefold and $\mathcal{F}$ be a co-rank foliation on X and suppose that $\mathcal{F}$ has simple singularities. Let C span an $K_{\mathcal{F}}$-negative extremal ray of flipping type. Then C is contained in* $\operatorname{sing}(\mathcal{F})$.

Proof. Suppose the contrary. We first claim that C is actually disjoint from the singular locus. Let S be a germ of a leaf containing C, then since C is a flipping curve we must have $C^2 < 0$ in S. Adjunction tell us that $(K_S + C) \cdot C = 2g(C) - 2 < 0$ and so C is a smooth rational curve and $K_S \cdot C = C^2 = -1$.

On the other hand, by Lemma 5.11 we know that $K_{\mathcal{F}}|_S = K_S + \Delta$ where Δ is a divisor with integer coefficients whose support is exactly $\operatorname{sing}(\mathcal{F}) \cap S$. Thus if $K_{\mathcal{F}} \cdot C < 0$ we must have $\Delta \cdot C = 0$ and so $C \cap \operatorname{sing}(\mathcal{F}) = \emptyset$.

Now apply local Reeb stability as above to see that C actually moves to nearby leaves, giving our contradiction.

J. V. Pereira has given the following alternative proof. Suppose as above that C is not contained in the singular locus. By restricting Bott's partial connection on the leaf to C and noting that $N_{C/S} = \mathcal{O}(-1)$ we see that $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}$. Thus, since C is a smooth rational curve, $H^0(C, N_{C/X}) \cong \mathbb{C}$ and $H^1(C, N_{C/X}) = 0$. This latter equality implies that the first order deformations of C are unobstructed and so give a deformation of C in X, i.e. C moves in X, which contradicts the supposition. □

COROLLARY 9.4. *Let X be a smooth threefold and $\mathcal{F}$ be a smooth co-rank 1 foliation on X. Let R is a $K_{\mathcal{F}}$-negative extremal ray and let H_R be a supporting hyperplane to R. Suppose $\nu(H_R) = 3$. Then* $\operatorname{loc}(R)$ *is a divisor transverse to the foliation.*

Moreover $\operatorname{loc}(R)$ *is a $\mathbb{P}^1$-bundle over a curve B and there is a morphism $\mu : X \to Y$ contracting* $\operatorname{loc}(R)$ *to B and where Y is smooth.*

Proof. By Corollary 9.3 we know that $\operatorname{loc}(R) = D$ must be divisor. Suppose for sake of contradiction that D is foliation invariant, then D is covered by rational curves which by Reeb stability can be moved into nearby leaves and therefore $\mathcal{F}$ is uniruled.

Thus D is transverse to the foliation and D is covered by curves tangent to the foliation and so if $\mathcal{G}$ is the foliation restricted to D we see that $\mathcal{G}$ gives a $\mathbb{P}^1$-fibration structure $D \to C$ on D which induces $\mathcal{G}$ where a general fibre spans R. In particular, we see that $K_{\mathcal{G}} \cdot f = K_D \cdot f < 0$ where f is a general fibre of $D \to C$.

As in the proof of Lemma 3.11 we see that this implies that R is in fact K_X-negative, and so by [Kol91, Theorem 1.1] we see that the fibration $D \to C$ is in fact a $\mathbb{P}^1$-bundle and that moreover we may contract D by a morphism to a smooth variety. □

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9.2 Running the smooth MMP

Theorem 9.5. *Let X be a smooth threefold and $\mathcal{F}$ a smooth co-rank 1 foliation. Then there is a foliated MMP for $(X, \mathcal{F})$.*

Proof. Let R be any $K_{\mathcal{F}}$-negative extremal ray. If $\operatorname{loc}(R) = X$ then the contraction exists by Theorem 8.13 and we stop the MMP.

Otherwise, by Corollary 9.4 we have that $\operatorname{loc}(R) = D$ is a divisor transverse to the foliation and we may contract D to a smooth curve C by a morphism $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ and where $\mathcal{G}$ is the transformed foliation. We claim that $\mathcal{G}$ is smooth.

Away from C this immediate, and since D is transverse to the foliation we must have that $\mathcal{G}$ is smooth at the generic point of C and so $\mathcal{G}$ has at worst isolated singularities along C. However, since π is just the blowing up in a smooth curve transverse to the foliation a direct computation shows that if $Q \in C$ is a singularity of $\mathcal{G}$ then $\pi^{-1}(Q)$ is a singularity of $\mathcal{F}$, an impossibility and so the claim is proven.

Thus, we can perform the contraction in the category of smooth foliations on smooth varieties, allowing us to proceed with the MMP. Since each contraction drops the Picard number by 1, this process must eventually terminate. □

Corollary 9.6. *Let X be a smooth threefold and let $\mathcal{F}$ be a smooth rank 2 foliation on X with $\nu(K_{\mathcal{F}}) = 0$. Then $(X, \mathcal{F})$ is birational to $(Y, \mathcal{G})$ where $(Y, \mathcal{G})$ is one of the following:*

(i) *up to a finite cover $\mathcal{G}$ is the product of a fibration in Calabi–Yau manifolds and a linear foliation on a torus;*
(ii) *$\mathcal{G}$ is transverse to a $\mathbb{P}^1$-bundle;*
(iii) *up to a finite cover $\mathcal{G}$ is induced by a fibration $X' \times C \to C$ where $c_1(X') = 0$.*

Proof. Run an MMP $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$. We have that $c_1(\mathcal{G}) = 0$, and so we may apply [Tou08, Theorem 1.2]. □

9.3 Existence of terminal flips

We will need to make use of the following generalization of Malgrange's theorem due to Cerveau and Lins Neto [CLN08, Corollary 1].

Lemma 9.7. *Let X be a germ of an analytic variety at $0 \in \mathbb{C}^N$ of dimension $n \geqslant 3$, and let $\mathcal{F}$ be a holomorphic foliation on $X^* = X - \operatorname{sing}(X)$. Suppose that:*

(1) *X is a complete intersection;*
(2) *$\dim(\operatorname{sing}(X)) \leqslant n - 3$;*
(3) *$\mathcal{F}$ is defined by a holomorphic 1-form ω on X^* such that $\dim(\operatorname{sing}(\omega)) \leqslant n - 3$.*

Then $\mathcal{F}$ has a holomorphic first integral.

The following proposition can also be viewed as being a version of Malgrange's theorem.

Lemma 9.8. *Let $0 \in X$ be a threefold germ with a co-rank 1 foliation $\mathcal{F}$. Suppose X has log terminal singularities and that $\mathcal{F}$ has terminal singularities. Then $\mathcal{F}$ has a holomorphic first integral.*

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Proof. Take a Galois quasi-étale cover with Galois group G (ramified only over $\mathrm{sing}(X)$) so that $K_X, K_{\mathcal{F}}$ are both Cartier, [GKP16, Theorem 1.10]. Call this cover $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$. Since Y is log terminal and K_Y is Cartier, $K_Y = \pi^* K_X$ and $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}}$, which implies that Y is canonical. Since π is étale in codimension 1, we see that $\mathcal{G}$ is terminal.

Next we claim that Y is actually terminal. Notice that $(N^*_{\mathcal{G}})^{**}$ is a line bundle being the difference of two Cartier divisors, and, since Y is log terminal, we have that the foliation discrepancies are less than or equal to the usual discrepancies (Lemma 3.1). Thus, since $\mathcal{G}$ is terminal, this immediately implies that Y is terminal.

Since Y is terminal and index 1, by [KM98, 5.38] it is a compound DuVal (cDV) hypersurface singularity, in particular Y is a complete intersection and $\mathrm{sing}(Y)$ is isolated.

Notice also that $\mathcal{G}$ is smooth away from $\mathrm{sing}(Y)$. We claim that $\mathcal{G}$ has a holomorphic first integral.

Observe that for any $0 \in \mathrm{sing}(Y)$, if we write $Y^* = Y - 0$, that $\mathcal{G}$ is defined by global a 1-form on Y^* near 0. Indeed, take any generator ω of $(N^*_{\mathcal{G}})^{**}$ around 0. Observe that since $\mathcal{G}$ is smooth away from 0 we have that $\mathrm{sing}(\omega) \subset \{0\}$.

Thus, Lemma 9.7 applies to show that $\mathcal{G}$ has a holomorphic first integral, i.e. there is a holomorphic fibration $f : Y \to \mathbb{C}$ whose fibres determine $\mathcal{G}$. Notice that f is in fact G-equivariant and so descends to a first integral $X \to \mathbb{C}$. □

COROLLARY 9.9. *Let X be a threefold with a co-rank 1 foliation $\mathcal{F}$. Suppose (X, Δ) has klt singularities for some divisor Δ, and that $\mathcal{F}$ has terminal singularities. Let C be curve tangent to the foliation and S a germ of a leaf containing C. Then S is $\mathbb{Q}$-Cartier.*

Proof. This can be checked analytic locally around points of C, so replace X by a germ around some point $p \in C$. In this case, by Lemma 9.8, there exists a holomorphic first integral $f : (p \in X) \to (0 \in \mathbb{C})$ and we have that $S = f^{-1}(0)$. The result follows. □

THEOREM 9.10. *Let X be a $\mathbb{Q}$-factorial klt threefold. Suppose that $\mathcal{F}$ is a co-rank 1 foliation with terminal singularities. Let $f : X \to Z$ be a flipping contraction. Then the flip exists.*

Proof. First, observe for all points $P \in X$, by Lemma 9.8 $\mathcal{F}$ admits (locally) a holomorphic first integral. In particular, $\mathcal{F}$ has non-dicritical singularities.

We may assume that we are working in the neighborhood of a connected component of $\mathrm{exc}(f)$; call it C. We have the following sequence:

$$0 \to \mathrm{Pic}(Z) \otimes \mathbb{Q} \xrightarrow{f^*} \mathrm{Pic}(X) \otimes \mathbb{Q} \to \mathbb{Q}$$

where the last arrow is given by intersecting with C. Thus, to prove $\rho(X/Z) = 1$ it suffices to show that if $M \cdot C = 0$ then $M = f^* M'$ for some M'.

By Lemma 8.10 we see that C is tangent to $\mathcal{F}$, and so by Lemma 5.11 we may find S, a germ of a separatrix, around C. Let $p = f(C)$. and let $V \subset Z$ be a small neighborhood around p so that S is defined on $W = f^{-1}(U)$. By Lemma 8.14 we know that $(W, S - \epsilon S)$ is klt for ϵ sufficiently small, and that $-(K_W + S)$ is f-ample. For $n \gg 0$ we have that $nM|_W - (K_W + (1-\epsilon)S)$ is f-ample, and so the relative analytic base point free theorem, see for example [KM98, Theorem 3.24] or [Nak87], applies to show that $M|_W$ is semi-ample over W. Hence for some sufficiently large n, since M is f-trivial, nM is pulled back from a Cartier divisor nM' on V. Since f is an isomorphism away from C, it is easy to extend nM' to a Cartier divisor on all of Z, and the result follows. In particular, Z is projective.

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Next, run a $K_W + (1-\epsilon)S$-MMP over V, and let $f' : (W', \mathcal{F}') \to V$ be the output of this MMP, and so $K_{W'} + S'$ is f'-nef, where S' is the strict transform of S.

Notice that since $\mathcal{F}$ has terminal singularities $K_W + S$ is numerically equivalent over V to $K_{\mathcal{F}}$, and so $K_{W'} + S'$ is numerically equivalent to $K_{\mathcal{F}'}$ over V. Thus we see $K_{\mathcal{F}'}$ is f'-nef. Furthermore $\mathcal{F}'$ is terminal. In general, $(W', \mathcal{F}')$ will not be flip, the flip will be the canonical model of $\mathcal{F}'$ over V.

We claim that we can construct this canonical model. We have S is f-numerically equivalent to $K_{\mathcal{F}} - K_X$. Since f is a contraction of Picard rank 1, this implies that $S =_{\text{num}} \lambda K_{\mathcal{F}}$ for some $\lambda \in \mathbb{Q}$, and so $S' =_{\text{num}} \lambda K_{\mathcal{F}'}$. In particular for $1 \gg \delta > 0$ and $m \gg 0$ we know that $mK_{\mathcal{F}'} + \delta S'$ is nef over V.

Thus for large enough m, and small enough δ, $mK_{\mathcal{F}'} - (K_{W'} + (1-\delta)S')$ is big and nef over V, and $(X', (1-\delta)S')$ is klt. Thus, we may apply the relative base point free theorem to conclude that $K_{\mathcal{F}'}$ is semi-ample over Z.

Let $\phi : (W', \mathcal{F}') \to (W^+, \mathcal{F}^+)$ be the corresponding map over Z. Since $W' \to Z$ is small, we see that ϕ is small, and since $K_{\mathcal{F}^+} = \phi_* K_{\mathcal{F}}$ we get that $K_{\mathcal{F}^+}$ is ample over V.

By [Art70], we can realize the flip $X \dashrightarrow X^+$ in the category of algebraic spaces, and since $K_{\mathcal{F}^+}$ is ample over Z we get that X^+ is in fact projective. □

10. Toric foliated MMP

Definition 10.1. Let X be a toric variety. Let $\mathcal{F}$ be a foliation on X. We say that $\mathcal{F}$ is toric provided that it is invariant under the torus action on X.

Lemma 10.2. *Let $\mathcal{F}$ be a co-rank 1 toric foliation. Then $K_{\mathcal{F}} = -\sum D_\tau$ where the sum is over all the torus invariant and non-$\mathcal{F}$-invariant divisors.*

Proof. By passing to a toric resolution $\pi : (X', \mathcal{F}') \to (X, \mathcal{F})$, and noting that the strict transform of a divisor D is torus and $\mathcal{F}'$-invariant if and only if it is torus and $\mathcal{F}$-invariant, we see that it suffices to prove the result on a resolution of X. Thus we may assume that X is smooth.

Observe that $\mathcal{F}$ is defined by a rational torus invariant 1-form ω. Working in torus coordinates $x_1, \dots, x_n$, we see that $\omega = \sum \lambda_i (dx_i/x_i)$ where $\lambda_i \in \mathbb{C}$. $\lambda_i \neq 0$ if and only if the divisor associated to $\{x_i = 0\}$ is foliation invariant. In particular ω has a pole of order 1 along a torus invariant divisor if and only if it is a $\mathcal{F}$ invariant divisor. Thus $N_{\mathcal{F}}$ is equivalent to the sum of the torus and foliation invariant divisors, and the result follows. □

Definition 10.3. Let σ be a cone in a fan Δ defining a toric variety. Let $D(\sigma)$ denote the closed subvariety corresponding to σ.

Remark 10.4. We note that if $\tau = \langle v_1, \dots, v_n \rangle$ is a full-dimensional cone in the fan defining X, then this argument in fact shows $D(v_i)$ is $\mathcal{F}$-invariant for some i.

Furthermore, if $w = \langle v_1, \dots, v_{n-1} \rangle$ is a codimension 1 cone in the fan, then $D(w)$ is tangent to $\mathcal{F}$ if and only if $D(v_i)$ is invariant for some i.

We also make the following simple observation.

Proposition 10.5. *Let $\mathcal{F}$ be a co-rank 1 toric foliation on a toric variety X. Suppose that $\mathcal{F}$ is defined by $\omega = \sum_{i=1}^n \lambda_i (dx_i/x_i)$. If $\lambda_i = 0$ for some i, then ω, and hence $\mathcal{F}$, is pulled back along some dominant rational map $f : X \dashrightarrow Y$ with $\dim(Y) < \dim(X)$. In particular, if $K_{\mathcal{F}}$ is not nef, then $\mathcal{F}$ is a pull back.*

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Proof. The first claim is easy.

To prove the second claim, suppose that $\mathcal{F}$ is not a pull back along a dominant rational map. This remains true after passing to a resolution of singularities of X. Let $\mathcal{F}'$ be the transformed foliation. Since $\mathcal{F}'$ is not a pull back we have that every torus invariant divisor is also $\mathcal{F}'$ invariant and so $K_{\mathcal{F}'}$ is in fact trivial. Pushing forward gives the result. □

We show that the cone theorem holds for co-rank 1 toric foliations in all dimensions. First we have the following result.

THEOREM 10.6. *Let $\mathcal{F}$ be co-rank 1 toric foliation with canonical singularities on a toric variety X. Let C be a curve in X, and $K_{\mathcal{F}} \cdot C < 0$, then $[C] = [M] + \alpha$ where M is a torus invariant curve tangent to the foliation, and α is a pseudo-effective class.*

Proof. By [Mat02], we can write

$$C = \sum_{\text{tangent to } \mathcal{F}} a_u D(u) + \sum_{\text{not tangent to } \mathcal{F}} b_w D(w)$$

where u, w run over the codimension 1 subcones of the fan.

We show that some a_u can be taken to be non-zero. Assume the contrary, that $a_u = 0$ for all u.

Since $D(w)$ is not tangent to the foliation, we have that if $w = \langle v_1, \dots, v_{n-1} \rangle$, then all the $D(v_i)$ are not foliation invariant.

In order to have $K_{\mathcal{F}} \cdot C < 0$, we must have $D(w) \cdot D(v_i) > 0$ for some w, v_i. Let τ, τ' be the two full dimensional cones which are spanned by w and v_n, v_{n+1} respectively. Then $\tau \cup \tau'$ must be concave along $\langle v_1, \dots, \hat{v_i}, \dots, v_{n-1} \rangle$.

Thus, there must be $\sigma_1, \dots, \sigma_r$ cones in our fan such that $\tau \cup \tau' \bigcup_{i=1}^{r} \sigma_i$ is a convex subcone of our fan. Furthermore, we know that both $D(v_n)$ and $D(v_{n+1})$ are foliation invariant. By [Mat02] $\langle v_1, \dots, \hat{v_i}, \dots, v_{n-1}, v_n \rangle$ or $\langle v_1, \dots, \hat{v_i}, \dots, v_{n-1}, v_{n+1} \rangle$ are in the same extremal ray as $D(w)$, and both correspond to torus invariant curve tangent to the foliation. □

COROLLARY 10.7. *Let $\mathcal{F}$ be a co-rank 1, toric foliation with canonical singularities on a toric variety X. Then, $\overline{NE}(X)_{K_{\mathcal{F}}<0} = \sum \mathbb{R}^+[M_i]$ where the M_i are torus invariant rational curves tangent to the foliation.*

LEMMA 10.8. *Let $\mathcal{F}$ be a co-rank 1 toric foliation with canonical non-dicritical singularities on a toric variety X. Let R be a $K_{\mathcal{F}}$-negative extremal ray. Then there is a contraction corresponding to this extremal ray, and falls into one of the following types:*

(i) *fibre-type contractions;*
(ii) *divisorial contractions; or*
(iii) *small contractions.*

Furthermore, if a subvariety Y is contracted, it is tangent to the foliation. In particular, if $\mathcal{F}$ has non-dicritical singularities, the resulting foliation will still have non-dicritical singularities.

Proof. We know that the contraction exists; what is unclear is if the curves being contracted are tangent to the foliation. By our cone theorem for toric foliations we know that some curve contracted is tangent to the foliation; however, it might be the case that there is a contracted curve transverse to the foliation.

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Suppose that $\pi : X \to Z$ is the contraction and Y is a general fibre. Suppose for sake of contradiction that Y is not tangent to the foliation. Then there is an induced foliation on Y, call it $K_{\mathcal{G}}$. It is standard that $\rho(Y) = 1$, and so $-K_{\mathcal{G}}$ is ample. However, by 10.5 we see that since $\mathcal{G}$ is birational to a pullback from a lower-dimensional variety, $\mathcal{G}$ must have dicritical singularities, implying that $\mathcal{F}$ does as well, which contradicts the supposition. Finally, if a general fibre is tangent to the foliation, then every fibre is. □

We now handle the flipping case.

Lemma 10.9. *Set up as above. In the case of a small contraction, the flip exists and no infinite sequence of flips exists. Furthermore, if $\mathcal{F}$ has canonical and non-dicritical singularities, then the flipped foliation, $\mathcal{F}^+$ does as well.*

Proof. The existence and termination of the flip can be seen by the fact that toric log flips exist and terminate. By the negativity lemma, $\mathcal{F}^+$ has canonical singularities.

What remains to be shown is the claim about the non-dicriticality of $\mathcal{F}^+$. Let S be the flipping locus of $\mathcal{F}$. Let $\pi : X \to Z$ be the flipping contraction. As noted, π only contracts curves tangent to the foliation.

Suppose for sake of contradiction that $\mathcal{F}^+$ was dicritical. Then, if we let $\mathcal{G}$ be the induced foliation on Z we must have that $\mathcal{G}$ is dicritical. Let E be a divisor sitting over Z which is not foliation invariant. Without loss of generality we may assume that E maps to a point in Z.

Let W be the center of E on X. Observe that W cannot be a divisor since $X \to Z$ is small, and since $\mathcal{F}$ is non-dicritical W cannot be a point or a curve tangent to $\mathcal{F}$. Thus W is a curve transverse to $\mathcal{F}$, but which is nevertheless contracted by π. This is our desired contradiction. □

Putting all this together we have the following.

Theorem 10.10. *Let $\mathcal{F}$ be a co-rank 1 toric foliation with canonical and non-dicritical foliation singularities on a toric variety X. Then the MMP for $\mathcal{F}$ exists, and ends either with a foliation where $K_{\mathcal{F}}$ is nef, or with a fibration $\pi : X \to Z$ and $\mathcal{F}$ is pulled back from a foliation on Z. Furthermore, only curves tangent to the foliation are contracted in this MMP.*

Proof. If $K_{\mathcal{F}}$ is not nef, there is an extremal ray on which $K_{\mathcal{F}}$ is negative. We can contract this ray resulting in:

(i) a fibration, in which case we stop; or
(ii) a divisorial contraction, in which case we repeat with the new variety; or
(iii) a flipping contraction, in which case we perform the flip.

We can have only finitely many steps of type (i) or (ii). Notice that the foliation discrepancy of some place will always increase under a flip and that there are only finitely many toric models which can be reached by a sequence of flips. Thus there can be no infinite sequence of flips. □

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