J. Inst. Math. Jussieu (2026), **25**(3), 1393–1439
doi:10.1017/S1474748025101473

FLOWS ON UNIFORM ROE ALGEBRAS

BRUNO DE MENDONÇA BRAGA[1], RUY EXEL[2] AND ALCIDES BUSS[3]

[1] *IMPA, Brazil*
(demendoncabraga@gmail.com)
(https://sites.google.com/view/demendoncabraga)
[2] *Universidade Federal de Santa Catarina, Brazil*
(ruyexel@gmail.com)
(http://www.mtm.ufsc.br/~exel/)
[3] *Universidade Federal de Santa Catarina, Brazil*
(alcides.buss@ufsc.br)
(http://www.mtm.ufsc.br/~alcides/)

(Received 29 April 2025; revised 21 November 2025; accepted 24 November 2025; first published online 10 April 2026)

Abstract For a uniformly locally finite metric space (X,d), we investigate *coarse* flows on its uniform Roe algebra $\mathrm{C}^*_u(X)$, defined as one-parameter groups of automorphisms whose differentiable elements include all partial isometries arising from partial translations on X. We first show that any flow σ on $\mathrm{C}^*_u(X)$ corresponds to a (possibly unbounded) self-adjoint operator h on $\ell_2(X)$ such that $\sigma_t(a) = e^{ith}ae^{-ith}$ for all $t \in \mathbb{R}$, allowing us to focus on operators h that generate flows on $\mathrm{C}^*_u(X)$.

Assuming Yu's property A, we prove that a self-adjoint operator h on $\ell_2(X)$ induces a coarse flow on $\mathrm{C}^*_u(X)$ if and only if h can be expressed as $h = a + d$, where $a \in \mathrm{C}^*_u(X)$ and d is a diagonal operator with entries forming a coarse function on X. We further study cocycle equivalence and cocycle perturbations of coarse flows, showing that, under property A, any coarse flow is a cocycle perturbation of a diagonal flow. Finally, for self-adjoint operators h and k that induce coarse flows on $\mathrm{C}^*_u(X)$, we characterize conditions under which the associated flows are either cocycle perturbations of each other or cocycle conjugate to each other. In particular, if $h - k$ is bounded, then the flow induced by h is a cocycle perturbation of the flow induced by k.

1. Introduction

Given a (uniformly locally finite) metric space X, its uniform Roe algebra $\mathrm{C}^*_u(X)$ is a C*-algebra which captures the large-scale geometry of X in C*-algebraic terms (see Subsection 1.2 for its precise definition). These algebras were introduced by J. Roe in the

B. M. Braga was partially supported by FAPERJ, grant E-26/200.167/2023, by CNPq, grant 303571/2022-5, and by Serrapilheira, grant R-2501-51476.
A. Buss and R. Exel were partially supported by CNPq, Brazil.

Keywords: uniform Roe algebras; flows on C*-algebras; noncommutative dynamics; cocycle perturbations; automorphisms

Mathematics Subject Classification: Primary 46L55
Secondary 46L05

© The Author(s), 2026. Published by Cambridge University Press.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

study of the index theory of elliptic operators on noncompact manifolds ([Roe88, Roe93]) and are of interest to researchers working on the coarse Baum-Connes and Novikov conjectures ([Yu00]). Recently, mathematical physicists have started using uniform Roe algebras in the study of topological insulators (see [Kub17, EM19, Jon21, LT21, Bou22]).

Motivated by mathematical physics, [BE25] initiated the treatment of flows and KMS states on uniform Roe algebras. In a nutshell, the aforementioned article identified (and analysed) a naturally occurring class of flows on uniform Roe algebras: given a uniformly locally finite metric space (X,d) and a map $h\colon X \to \mathbb{R}$, we can interpret h as a multiplication (unbounded) operator on $\ell_2(X)$ — the Hilbert space of square-summable functions $X \to \mathbb{C}$. Throughout these notes, we use the convention that an *operator* on a Hilbert space H means a *not necessarily bounded* operator defined on a vector *subspace* of H. A multiplication operator induced by such $h\colon X \to \mathbb{R}$ is clearly self-adjoint and diagonal with respect to the canonical orthonormal basis $(\delta_x)_{x\in X}$ of $\ell_2(X)$. In particular, e^{ith} is a well-defined unitary operator on $\ell_2(X)$ which is also diagonal and the formula

$$\sigma_{h,t}(a) = e^{ith}ae^{-ith}, \tag{1.1.1}$$

for $t \in \mathbb{R}$ and $a \in \mathrm{C}^*_u(X)$, defines a *pre-flow* (i.e., a not necessarily continuous action) σ_h of $\mathbb{R}$ on $\mathrm{C}^*_u(X)$ by automorphisms. Moreover, the large-scale geometric properties of h may be used to characterize when σ_h is strongly continuous, in which case we call it a *flow*. Recall that a map $h = (h_x)_{x\in X}\colon X \to \mathbb{R}$ is *coarse* if, for all $r > 0$, there is $s > 0$, such that

$$d(x,z) \leq r \quad \text{implies} \quad |h_x - h_z| \leq s$$

for all $x,z \in X$.[1] We then have the following:

Proposition 1.1.2 [BE25, Proposition 2.1]. *Let X be a uniformly locally finite metric space and σ_h be given by a map $h\colon X \to \mathbb{R}$ as in (1.1.1). Then σ_h is a flow on $\mathrm{C}^*_u(X)$ if and only if $h\colon X \to \mathbb{R}$ is coarse.*

Flows of this sort are called *diagonal flows*. The goal of the present paper is to extend the study of flows on uniform Roe algebras outside the class of diagonal flows. For that, we develop a theory of *noncommutative coarse maps* or, equivalently, of *coarse operators*, and study cocycle perturbation and cocycle conjugacy for flows on uniform Roe algebras.

One of the main reasons justifying one's interest in cocycle equivalence for flows is that, when two flows are cocycle conjugate to each other, then the fields of KMS states are isomorphic ([Kis00, Proposition 2.1]). So, in particular, all of the information about phase transitions may be passed from one flow to the other. Since a lot is now known about KMS states for diagonal flows ([BE25]), the results herein extend that knowledge to a vastly larger class of flows.

We now describe the main findings of this paper in more detail.

[1] The terms *bornologous* and *controlled* are often used in the literature instead of *coarse*.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

1.2. Uniform Roe algebras

All metric spaces considered herein will be not only discrete but also uniformly locally finite. Recall that a metric space X is *uniformly locally finite* (abbreviated as *u.l.f.* from now on) if

$$\sup_{x\in X}|B_r(x)|<\infty \quad \text{for all} \quad r>0,$$

where $|B_r(x)|$ denotes the number of elements of the closed ball in X of radius r centred at x. In this case, X is necessarily countable and the metric topology is discrete.

Given a u.l.f. metric space (X,d), we identify $\ell_\infty(X)$ — the C*-algebra of bounded, complex valued maps on X — with the algebra of diagonal operators on $\ell_2(X)$ in the canonical way. The uniform Roe algebra of X is then the C*-subalgebra of $\mathcal{B}(\ell_2(X))$[2] generated by the partial isometries on $\ell_2(X)$ given by partial translations of X. Precisely, the notion of a translation function on a vector space is adapted to arbitrary metric spaces as follows: a partial bijection $f\colon \operatorname{dom}(f)\subseteq X\to \operatorname{ran}(f)\subseteq X$ is said to be a *partial translation of* X if

$$\sup_{x\in\operatorname{dom}(f)} d(x,f(x))<\infty.$$

Any such f induces a sort of *partial shift* operator v_f on $\ell_2(X)$ determined by

$$v_f\delta_x=\begin{cases}\delta_{f(x)}, & \text{if } x\in\operatorname{dom}(f),\\ 0, & \text{if } x\notin\operatorname{dom}(f).\end{cases}$$

We can now make the definition of the uniform Roe algebra of a u.l.f. metric space precise:

Definition 1.2.1. The *uniform Roe algebra* of a u.l.f. metric space X, denoted $\mathrm{C}^*_u(X)$, is the C*-subalgebra of $\mathcal{B}(\ell_2(X))$ generated by all v_f, where f is a partial translation of X.

While we will only deal with u.l.f. spaces here, one can define uniform Roe algebras for more general spaces. This is, for instance, done in [WY20, MV24].

If A is any subset of X, observe that the orthogonal projection p_A from $\ell_2(X)$ onto $\ell_2(A)\subseteq\ell_2(X)$ coincides with the operator v_f, where f is the identity function on $\operatorname{dom}(f):=A$. As the set of all such operators spans a dense linear subspace of $\ell_\infty(X)$, we see that $\ell_\infty(X)\subseteq\mathrm{C}^*_u(X)$.

Alternatively, the uniform Roe algebra of X can be defined as the norm closure of the set of all bounded operators on $\ell_2(X)$ with *finite propagation*, where the *propagation* of an operator $a\in\mathcal{B}(\ell_2(X))$ is given by

$$\operatorname{prop}(a)=\sup\{d(x,y)\ :\ \langle a\delta_y,\delta_x\rangle\neq 0\}. \tag{1.2.2}$$

1.3. Flows on uniform Roe algebras

With the present subsection we begin our detailed study of flows.

[2]Given a Hilbert space H, $\mathcal{B}(H)$ denotes the space of all bounded operators on H.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Definition 1.3.1. Given a C*-algebra A, by a *pre-flow* on A, also known as a *one-parameter group of automorphisms* of A, we shall mean a group homomorphism

$$\sigma\colon \mathbb{R} \to \mathrm{Aut(A)},$$

from the additive group of real numbers to the group of *-automorphisms of A. If the pre-flow σ is *strongly continuous* in the sense that $t \in \mathbb{R} \mapsto \sigma_t(a) \in A$ is continuous for all $a \in A$, then σ is called a *flow*.[3]

We start our study of flows on uniform Roe algebras by showing that every such flow is completely determined by a self-adjoint operator on $\ell_2(X)$. Precisely, let H be a Hilbert space and h be a self-adjoint operator on H. Continuous functional calculus gives us that e^{ih} is a unitary operator on H. Hence, each such operator induces a pre-flow σ_h on $\mathcal{B}(H)$ given by

$$\sigma_{h,t}(a) = e^{ith} a e^{-ith} \tag{1.3.2}$$

for all $t \in \mathbb{R}$, and all $a \in \mathcal{B}(H)$. It is well known that any weakly continuous flow on $\mathcal{B}(H)$ is of this form for some self-adjoint operator h on H (see [BR87, Example 3.2.35]). In Subsection 2.2, we employ this result to prove the following:

Proposition 1.3.3. *Let H be a separable Hilbert space, $A \subseteq \mathcal{B}(H)$ be a C*-algebra containing all compact operators, and σ be a flow on A. Then there is a self-adjoint operator h on H such that*

$$\sigma_t(a) = e^{ith} a e^{-ith}$$

for all $t \in \mathbb{R}$, and all $a \in A$.

Proposition 1.3.3 may be well known to experts, but for lack of a precise citing source and since this will play a very important role in our understanding of flows on uniform Roe algebras, we present a detailed proof in Subsection 2.2 below.

Since a uniform Roe algebra always contains the compacts, Proposition 1.3.3 applies to the flows that we are interested in. Therefore it suffices to study flows on uniform Roe algebras arising from self-adjoint operators on $\ell_2(X)$ via the formula (1.3.2). This problem splits naturally into two parts:

Question 1.3.4. Let h be a self-adjoint operator on $\ell_2(X)$.

(I) Under which conditions can we guarantee that $\sigma_{h,t}(\mathrm{C}^*_u(X)) = \mathrm{C}^*_u(X)$ for all $t \in \mathbb{R}$?

(II) Assuming that the above holds, when is σ_h strongly continuous on $\mathrm{C}^*_u(X)$? In other words, when is a pre-flow on $\mathrm{C}^*_u(X)$ of the form σ_h, an actual flow?

To deal with Question 1.3.4(II), we start with a detailed analysis of coarse maps in Section 3 which culminates in a noncommutative version of coarseness for self-adjoint operators on $\ell_2(X)$ and a notion of coarseness for flows. Precisely:

[3]Since the terminology "pre-flow" is not standard, we emphasize here that a pre-flow is simply a *not necessarily strongly continuous flow*.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Definition 1.3.5. Let X be a u.l.f. metric space and let h be a closed operator on $\ell_2(X)$.[4] We call h a *coarse operator* if the domain of such h is *invariant under partial translations* (see Definition 3.1.2), meaning

$$v_f(\mathrm{dom}(h)) \subseteq \mathrm{dom}(h),$$

and the commutator[5] of h and v_f satisfies

$$\|[h, v_f]\| < \infty$$

for all partial translations f on X.

We should remark that the natural domain of the commutator operator $[h, v_f] = hv_f - v_f h$ is $\mathrm{dom}(h)$, because $\mathrm{dom}(h)$ is invariant under v_f by hypothesis. Also, the invariance of $\mathrm{dom}(h)$ under the v_f easily implies that $c_{00}(X)$[6] is contained in the domain of h, and in fact $c_{00}(X)$ will be shown to be a core for h, in the sense that the graph of the restriction of h to $c_{00}(X)$ is dense in the graph of h. In Theorem 4.1.15, we will show even more, namely that every coarse operator is in fact *admissible* in the sense of Definition 1.3.8, below, a crucial tool for several of the arguments in our main proofs.

Definition 1.3.6. Let X be a u.l.f. metric space and σ be a flow on $\mathrm{C}^*_u(X)$. We call σ a *coarse flow* if

$$t \in \mathbb{R} \mapsto \sigma_{h,t}(v_f) \in \mathrm{C}^*_u(X)$$

is differentiable for all partial translations f of X.

A few words are in place here to justify the definitions above. First, as will be seen in Proposition 3.1.5 below, a map $h\colon X \to \mathbb{R}$ is coarse if and only if its interpretation as a diagonal operator on $\ell_2(X)$ is a coarse operator. Moreover, any flow on a uniform Roe algebra induced by a map $h\colon X \to \mathbb{R}$ is automatically a coarse flow (see Proposition 3.1.7). The study of coarse flows and of coarse operators is then a natural generalization of the commutative diagonal scenario.

With the definitions above in hand, we obtain the following noncommutative version of Proposition 1.1.2; which deals with Question 1.3.4(II) above.

Proposition 1.3.7. *Let X be a u.l.f. metric space and σ be a pre-flow on $\mathrm{C}^*_u(X)$. Then σ is a coarse flow if and only if it is induced by a coarse operator h on $\ell_2(X)$.*

While, for the case of diagonal flows, there is no reason to worry about Question 1.3.4(I), the treatment of this question is substantially more subtle in the noncommutative case. For starters, we must introduce the *unbounded part of* $\mathrm{C}^*_u(X)$ to the picture.

[4]Recall, a *closed operator* on a Hilbert space is a partially defined, not necessarily bounded, operator whose graph is closed.

[5]Throughout, $[a,b]$ denotes the commutator $ab - ba$.

[6]Here $c_{00}(X)$ denotes the vector subspace of $\ell_2(X)$ of finitely supported vectors.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Definition 1.3.8. Let X be a u.l.f. metric space and h be an operator on $\ell_2(X)$. We call h *admissible* if it is closed, $c_{00}(X) \subseteq \mathrm{dom}(h)$, and

$$h\xi = \sum_{y\in X} \langle \xi, \delta_y\rangle h\delta_y$$

for all $\xi \in \mathrm{dom}(h)$. We say that h has *finite propagation* if it is admissible and

$$\sup\{d(x,y) \ : \ \langle h\delta_x, \delta_y\rangle \neq 0\} < \infty.$$

We emphasize here that, whenever we say that an operator has finite propagation, it is assumed that we are speaking of an admissible operator, or else the condition on matrix entries above might not mean much. We note that this definition generalizes the usual concept of finite propagation for bounded operators to the unbounded setting.

We next introduce the *unbounded part* of $\mathrm{C}^*_u(X)$, a concept which will, as the name suggests, involve unbounded operators and, crucially, the difference $h - h'$, of two unbounded operators h and h'. This is a notoriously touchy issue not least because h and h' may have very different domains. For this reason, except for rare occasions, we shall only dare to speak of $h - h'$ when there is a common core to h and h'. Since every admissible operator admits $c_{00}(X)$ as a core, taking the difference of such operators will not involve any risks.

Definition 1.3.9. Let X be a u.l.f. metric space and h be an admissible operator on $\ell_2(X)$. We say that h is in the *unbounded part of* $\mathrm{C}^*_u(X)$ if, for all $\varepsilon > 0$, there is a finite propagation operator h' on $\ell_2(X)$ such that $h - h'$ (defined on its natural domain $\mathrm{dom}(h) \cap \mathrm{dom}(h')$) is a bounded operator with norm at most ε. We denote the set of all such operators by $\mathrm{C}^*_{u,unb}(X)$.[7]

Notice that any operator on $\ell_2(X)$ with finite propagation is in $\mathrm{C}^*_{u,unb}(X)$. In particular, any diagonal operator, regardless of being given by a coarse map, is in $\mathrm{C}^*_{u,unb}(X)$. On the other hand, the intersection of $\mathcal{B}(\ell_2(X))$ with $\mathrm{C}^*_{u,unb}(X)$ is precisely $\mathrm{C}^*_u(X)$.

Another way of thinking of $\mathrm{C}^*_{u,unb}(X)$ is to consider the set $\mathcal{L}_{adm}(\ell_2(X))$ formed by all admissible operators, equipped with the (possibly infinite) metric

$$d(h,k) = \|(h-k)\restriction_{c_{00}(X)}\|.$$

Within it one has the subset $\mathrm{C}^*_{u,unb}[X]$ formed by the operators of finite propagation, and then $\mathrm{C}^*_{u,unb}(X)$ turns out to be precisely the closure of $\mathrm{C}^*_{u,unb}[X]$ in $\mathcal{L}_{adm}(\ell_2(X))$.

With the purpose of keeping this introduction to a minimum, we will now restrict our results to metric spaces with G. Yu's property A: this is a fairly general geometric property imposed on the u.l.f. metric space X which is often seen as a sort of "geometric amenability" (see Subsection 2.3 for a precise definition), which is equivalent to nuclearity of $\mathrm{C}^*_u(X)$. For more general results which do not depend on property A, we simply refer the reader to the appropriate statements in the paper. For the specialists, we would like

[7] In these notes, we consider $\mathrm{C}^*_{u,unb}(X)$ simply as a set and do not attempt to view it with any algebraic or topological structure.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

to say that the use of property A is needed so that the uniform Roe algebra of X equals its *quasi-local algebra* (see Subsection 2.3 for definitions).

For u.l.f. metric spaces with property A, we completely characterize the self-adjoint operators which give rise to coarse flows on uniform Roe algebras. Precisely, we obtain the following.

Theorem 1.3.10. *Let X be a u.l.f. metric space with property A, h be a self-adjoint operator on $\ell_2(X)$, and σ_h be the pre-flow on $\mathcal{B}(\ell_2(X))$ induced by h. The following are equivalent.*

1. *σ_h is a coarse flow on $\mathrm{C}^*_u(X)$.*
2. *h is a coarse operator and it belongs to $\mathrm{C}^*_{u,unb}(X)$.*
3. *The map $E(h)\colon X \to \mathbb{R}$ given by*

$$E(h)_x = \langle h\delta_x, \delta_x\rangle \quad \text{for all} \quad x \in X$$

is coarse and, interpreting $E(h)$ as a diagonal operator, $h - E(h)$ is a bounded operator in $\mathrm{C}^*_u(X)$.[8]

The main ingredient for the implication (2)⇒(1) is Theorem 7.1.3, which shows that for any such h, the bounded operator e^{ih} is in the quasi-local algebra of X for all $t \in \mathbb{R}$. On the other hand, the implication (1)⇒(2) of Theorem 1.3.10 is always valid regardless of property A. This is obtained as Theorem 7.1.12 below and its proof uses an adaptation to the unbounded setting of the machinery developed in [LW20] to show that bounded derivations on uniform Roe algebras are always inner.

1.4. Cocycle equivalence of flows

Once Items (I) and (II) of Question 1.3.4 are dealt with, we move to classifying when coarse flows are, in some sense, equivalent:

Question 1.4.1. Let h and k be self-adjoint operators on $\ell_2(X)$ inducing flows σ_h and σ_k on $\mathrm{C}^*_u(X)$.

(III) What relation must be satisfied between h and k so that the flows σ_h and σ_k are "morally" the same?

This will be formalized in terms of *cocycle equivalence* between flows. For this, the notion of cocycle will be essential. Recall that we have the following well-known equivalence relations among flows on C*-algebras:

Definition 1.4.2. Let A be a unital C*-algebra, σ be a pre-flow on A, and $(u_t)_{t\in\mathbb{R}}$ be a norm continuous family of unitary operators in A.

1. We say that $(u_t)_{t\in\mathbb{R}}$ is a *cocycle for* σ if

$$u_{t+s} = u_t\sigma_t(u_s) \text{ for all } \ t,s \in \mathbb{R}.$$

[8]The notation "$E(h)$" is an allusion to the canonical conditional expectation from $\mathcal{B}(\ell_2(X)) \to \ell_\infty(X)$, the formula of which is given analogously.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

2. If $(u_t)_{t\in\mathbb{R}}$ is a cocycle for σ, then the expression $\rho_t(a) = u_t\sigma_t(a)u_t^*$, for $a \in A$ and $t \in \mathbb{R}$, defines a pre-flow ρ on A. We call ρ a *cocycle perturbation of* σ.
3. If, moreover, there exists a family $(u_t)_{t\in\mathbb{R}}$, as above, that is differentiable at 0, then ρ is called an *inner perturbation of* σ.[9]
4. We say that σ is *cocycle conjugate to* ρ if there is an automorphism τ of A such that ρ is a cocycle perturbation of $\tau\circ\sigma\circ\tau^{-1}$, where the later is the pre-flow given by $t \in \mathbb{R} \mapsto \tau\circ\sigma_t\circ\tau^{-1} \in \mathrm{Aut}(A)$.

All these notions define equivalence relations on the set of pre-flows on a given C*-algebra and relate to each other as in the diagram below. Moreover, if a pre-flow is equivalent under any of these notions to an actual flow, then it must be a flow as well.

$$\text{Inner perturbation} \longrightarrow \text{Cocycle perturbation} \longrightarrow \text{Cocycle conjugacy}.$$

For metric spaces with property A, we are capable of showing that every coarse flow is, up to cocycle perturbation, a diagonal flow. Moreover, this is more than an "existence theorem" as the diagonal cocycle perturbation can be actually computed in a fairly natural way. As diagonal flows are much easier to deal with, this can drastically reduce any analysis needed when working with coarse flows. In particular, this allows the machinery developed in [BE25] for the study of KMS states of diagonal flows to be applied to more general flows. Precisely, we prove the following.

Theorem 1.4.3. *Let X be a u.l.f. metric space with property A, and let h be a coarse operator on $\ell_2(X)$ such that*

$$e^{ith}\mathrm{C}_u^*(X)e^{-ith} = \mathrm{C}_u^*(X)$$

for all $t \in \mathbb{R}$. Then the flow σ_h on $\mathrm{C}_u^*(X)$ induced by h is a cocycle perturbation of a diagonal flow. Indeed, σ_h is a cocycle perturbation of the flow $\sigma_{E(h)}$, where $E(h)\colon X \to \mathbb{R}$ is the coarse map given by

$$x \in X \mapsto \langle h\delta_x, \delta_x\rangle \in \mathbb{R} \quad \text{for all} \quad x \in X.$$

Besides reducing the problem to the diagonal case, we also characterize when self-adjoint operators on $\ell_2(X)$ give rise to "equivalent" flows. For the commutative case, no geometric condition on X is needed, and we obtain not only a classification of flows, but also of pre-flows (notice here that the property of being a flow is preserved even under the weakest form of equivalence given by cocycle conjugacy):

Theorem 1.4.4. *Let (X,d) be a u.l.f. metric space, and let h and k be real-valued maps on X. The following are equivalent:*

1. *The maps h and k are close, i.e., $\sup_{x\in X} d(h_x,k_x) < \infty$,*
2. *The pre-flows σ_h and σ_k are inner perturbations of each other.*
3. *The pre-flows σ_h and σ_k are cocycle perturbations of each other.*

[9] If this is the case, $(u_t)_{t\in\mathbb{R}}$ is actually differentiable everywhere, and we refer the reader to [Kis00, Section 1] for further details.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Notice that the property in Theorem 1.4.4(1) above is equivalent to the difference operator $h-k$, given by the interpretation of the maps $h,k\colon X\to\mathbb{R}$ as operators on $\ell_2(X)$, being a bounded operator.

In order to obtain Theorems 1.4.3 and 1.4.4, we need yet another important ingredient which is interesting on its own:

Theorem 1.4.5. *Let X be a u.l.f. metric space with property A, and let h be a self-adjoint operator on $\ell_2(X)$ such that σ_h induces a flow on $\mathrm{C}^*_u(X)$. Then $e^{ith}\in\mathrm{C}^*_u(X)$ for all $t\in\mathbb{R}$.*

In fact, as the reader will see in Section 6, this is not a result about flows, but merely about strongly continuous paths of automorphisms. The proof of this result is quite technical and it makes use of three ingredients: (1) basics of K-theory, (2) techniques developed in [WW20] for the study of Cartan subalgebras of uniform Roe algebras, and (3) techniques developed for the solution of the rigidity problem of uniform Roe algebras ([BF21, BBF$^+$22]).

For the general noncommutative case, we must once again restrict ourselves to metric spaces with property A. Our main results are the following.

Theorem 1.4.6. *Let X be a u.l.f. metric space with property A, and let h and k be coarse operators on $\ell_2(X)$ inducing flows σ_h and σ_k on $\mathrm{C}^*_u(X)$. If $h-k$ is bounded on $c_{00}(X)$, then the flows σ_h and σ_k on $\mathrm{C}^*_u(X)$ are cocycle perturbations of each other.*

Theorem 1.4.7. *Let X be a u.l.f. metric space with property A, and let h and k be coarse, self-adjoint operators on $\ell_2(X)$, whose induced flows on $\mathrm{C}^*_u(X)$ are respectively denoted by σ_h and σ_k. The following are equivalent:*

1. *There is a unitary $u\in\mathrm{C}^*_u(X)$ such that $\mathrm{dom}(h)=\mathrm{dom}(uku^*)$, and $h-uku^*$ is bounded on this common domain.*
2. *The flows σ_h and σ_k are cocycle conjugate to each other.*

Finally, in Section 9, we show that, without the condition of property A, and working only with pre-flows, the results above do not hold. More precisely, we show that in any expander graph, one can find a natural pre-flow on its uniform Roe algebra which is not a cocycle perturbation of a diagonal flow (see Theorem 9.1.5).

2. Preliminaries

In this section, we recall the basic tools that we will need about unbounded operators, one-parameter unitary groups, flows on algebras of operators, and coarse geometry. For a detailed treatment of these, we refer the reader to [Ped89, Chapter 5], [Rud91, Chapter 13] or [BR87] for the first topics, and to [Roe03, NY12] for the last one. Besides setting up terminology, we also refine a well-known result on weakly continuous flows on $\mathcal{B}(H)$ [BR87, Example 3.2.35] to guarantee that any flow on a uniform Roe algebra is determined by an unbounded operator (Proposition 1.3.3).

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

2.1. Unbounded Operators on Hilbert spaces

Let H be a Hilbert space. From now on, the term *operator on* H will refer to a (not necessarily bounded) linear function

$$h\colon \operatorname{dom}(h) \subseteq H \to H,$$

where $\operatorname{dom}(h)$ is a vector subspace of H. The unbounded operators of interest in this paper will be in general self-adjoint. In particular, these operators are automatically closed and densely defined, i.e., $\operatorname{dom}(h)$ is dense in H. Addition, subtraction and multiplication of operators are defined in the obvious ways taking the necessary precautions regarding domains; precisely,

$$\operatorname{dom}(h \pm k) = \operatorname{dom}(h) \cap \operatorname{dom}(k)$$

and

$$\operatorname{dom}(kh) = \{\xi \in \operatorname{dom}(h) \ : \ h\xi \in \operatorname{dom}(k)\}.$$

The space of all (globally defined) bounded operators on H is denoted by $\mathcal{B}(H)$ and its ideal of compact operators by $\mathcal{K}(H)$.

Given the fundamental role played by the matrix representation of operators on $\ell_2(X)$ in the study of $\mathrm{C}^*_u(X)$, e.g., in the definition of the notion of propagation, it is natural that unbounded operators also be seen from the point of view of matrices. However one cannot expect to make sense out of the matrix entries

$$h_{x,y} = \langle h\delta_y, \delta_x \rangle$$

of an unbounded operator h on $\ell_2(X)$, since such operators are only densely defined, and there is no reason for δ_y to lie in its domain. In fact, even if all basis vectors δ_y belong to $\operatorname{dom}(h)$, the values of h on these vectors might not necessarily determine h outside $c_{00}(X)$. After all, the fact that h is discontinuous, as a rule, renders the density of $c_{00}(X)$ within $\ell_2(X)$ useless for the purpose of inferring the value of $h(\xi)$, when ξ is not in $c_{00}(X)$.

The operators that will play important roles in this work will therefore all be *admissible* in the following sense:

Definition 2.1.1 (cf. Definition 1.3.8). Let X be any set. An operator h on $\ell_2(X)$ will be called *admissible* if it is closed, $c_{00}(X) \subseteq \operatorname{dom}(h)$, and

$$h\xi = \sum_{y \in X} \langle \xi, \delta_y \rangle h\delta_y$$

for all $\xi \in \operatorname{dom}(h)$.

Under the above conditions, the coordinates of $h\xi$ relative to each basis vector δ_x are then given by

$$\langle h\xi, \delta_x \rangle = \sum_{y \in X} \langle \xi, \delta_y \rangle \langle h\delta_y, \delta_x \rangle = \sum_{y \in X} h_{x,y}\xi_y,$$

where $h_{x,y} = \langle h\delta_y, \delta_x \rangle$, and $\xi_y = \langle \xi, \delta_y \rangle$. In other words, the matrix representation of h holds, not only for vectors in $c_{00}(X)$, but also for every vector ξ in the domain of definition

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

of h. Furthermore, given any pair $(\xi,h\xi)$ in the graph of h, we may write $\xi = \lim_F \xi_F$, where

$$\xi_F = \sum_{y\in F} \langle \xi,\delta_y\rangle \delta_y,$$

the limit taken for F running in the directed set of all finite subsets $F \subseteq X$, and then the fact that h is admissible implies that $\big(\xi_F,h(\xi_F)\big)$ converges to $(h\xi,\xi)$, which is to say that $c_{00}(X)$ is a core for h.

Admissibility is a very useful property since it implies that $c_{00}(X)$ is not only a core, but it is actually a *'great core'* in the sense that we can choose our approximating sequences in the most natural way. Precisely, firstly recall that a subspace $D \subseteq \mathrm{dom}(h)$ is called a *core for* h if the graph of $h \restriction D$ is dense in the graph of h. In other words, if $\xi \in \mathrm{dom}(h)$, then there is a sequence $(\xi_n)_n$ in D, such that $\xi = \lim_n \xi_n$, and $h\xi = \lim_n h\xi_n$. Admissibility says that the sequence $(\xi_n)_n$ can be simply taken to be the truncations of ξ with respect to the canonical basis of $\ell_2(X)$.

We finish this subsection with a useful elementary result.

Proposition 2.1.2. *Let X be a u.l.f. metric space and let h and k be admissible operators on $\ell_2(X)$. The following holds.*

(a) $\|h-k\| = \|(h-k)\restriction_{c_{00}(X)}\|$.[10]

(b) *If $h-k$ is bounded on $c_{00}(X)$, then* $\mathrm{dom}(h) = \mathrm{dom}(k)$.

Proof. The first point is an easy consequence of the fact that $c_{00}(X)$ is a common core for h and k, so we leave the details for the reader.

As for the second point, let $\xi \in \mathrm{dom}(h)$ and let us show that $\xi \in \mathrm{dom}(k)$. For this choose an increasing sequence $(F_n)_n$ of finite subsets of X, whose union coincides with X, and for each n, let ξ_n be the projection of ξ on $\ell_2(F_n)$. Then clearly $\xi_n \to \xi$ and, since h is admissible, also $h\xi_n \to h\xi$. Noticing that $(\xi_n)_n$ is a sequence in $c_{00}(X)$, and that $h-k$ is bounded on this subspace, we have that $((h-k)\xi_n)_n$ is a Cauchy sequence, hence so is $(k\xi_n)_n$. As k is closed, this shows that $\xi \in \mathrm{dom}(k)$. This proves $\mathrm{dom}(h) \subseteq \mathrm{dom}(k)$ and, by symmetry of the hypothesis, the inclusion $\mathrm{dom}(k) \subseteq \mathrm{dom}(h)$ follows analogously. □

2.2. Flows and one-parameter groups

Definition 2.2.1. Let H be a Hilbert space and $(u_t)_{t\in\mathbb{R}}$ be a family of unitary operators in $\mathcal{B}(H)$. We call $(u_t)_{t\in\mathbb{R}}$ a *one-parameter unitary group on* H if

$$u_{t+s} = u_t u_s \quad \text{for all} \quad t,s \in \mathbb{R}.$$

If, moreover,

$$t \in \mathbb{R} \mapsto u_t\xi \in H$$

is continuous for all $\xi \in H$, we say that $(u_t)_{t\in\mathbb{R}}$ is *strongly continuous*.

[10] Here $\|h\| := \sup\{\|h(\xi)\| : \|\xi\| \leq 1, \xi \in \mathrm{dom}(h)\}$ denotes the possibly infinite norm of the (unbounded) operator h.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Given a self-adjoint operator h on a Hilbert space H, the spectral theorem allows us to define e^{ih}, a bounded operator on H. Moreover,

$$t \in \mathbb{R} \mapsto e^{ith} \in \mathcal{B}(H)$$

is a strongly continuous one-parameter unitary group and

$$\lim_{t\to 0} \frac{e^{ith}\xi - \xi}{t} = ih\xi \tag{2.2.2}$$

for all $\xi \in \mathrm{dom}(h)$. Conversely, if $\xi \in H$ is such that the limit in (2.2.2) exists, then necessarily $\xi \in \mathrm{dom}(h)$, and the identity in (2.2.2) holds (see [Ped89, Proposition 5.3.13]). Moreover, by Stone's theorem (see [Ped89, Theorem 5.3.15]), every strongly continuous one-parameter unitary group $t \in \mathbb{R} \mapsto u_t \in \mathcal{B}(H)$ is of the above form, that is, $u_t = e^{ith}$ for all $t \in \mathbb{R}$, for a unique self-adjoint operator h on H.

We now show that any flow on a uniform Roe algebra is given by conjugating by a one-parameter unitary group; this is a refinement of [BR87, Example 3.2.35] for our setting. This will be essential in our study of flows. Parts of the proof of Proposition 1.3.3 below are well known to experts, but we provide details for clarity and for the non-experts' convenience.

Proof of Proposition 1.3.3. We start by noticing that $\mathcal{K}(H)$ is invariant under σ_t. For that, observe that $\mathcal{K}(H)$ is the unique minimal ideal of A. Indeed, if J is another nonzero ideal of A, we have

$$J \cap \mathcal{K}(H) = J \cdot \mathcal{K}(H) \neq \{0\},$$

since $\mathcal{K}(H)$ is an essential ideal. As $\mathcal{K}(H)$ is a simple C*-algebra, it follows that $J \cap \mathcal{K}(H) = \mathcal{K}(H)$, whence $\mathcal{K}(H) \subseteq J$. As $\mathcal{K}(H)$ admits the above characterization in precise algebraic terms, we conclude that $\mathcal{K}(H)$ is invariant under automorphisms of A; in particular, $\sigma_t(\mathcal{K}(H)) = \mathcal{K}(H)$ for every $t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, let τ_t be the restriction of σ_t to $\mathcal{K}(H)$, so that τ becomes a flow on $\mathcal{K}(H)$. Observing that $\mathcal{B}(H)$ is the multiplier algebra of $\mathcal{K}(H)$, each τ_t extends canonically to an automorphism ρ_t of $\mathcal{B}(H)$, and it is not difficult to show that $\rho_t\rho_s = \rho_{t+s}$ for all $t,s \in \mathbb{R}$, i.e., that ρ is a pre-flow on $\mathcal{B}(H)$. Observe that, for $t \in \mathbb{R}$, $a \in A$, and $k \in \mathcal{K}(H)$, we have

$$\rho_t(a)k = \rho_t(a\rho_{-t}(k)) = \tau_t(a\tau_{-t}(k)) = \sigma_t(a\sigma_{-t}(k)) = \sigma_t(a)k.$$

So, $\rho_t(a) = \sigma_t(a)$. Hence, ρ extends σ from A to $\mathcal{B}(H)$. Let us now show that ρ is a weakly continuous flow on $\mathcal{B}(H)$. For this, pick any vector ξ in H, and let p_ξ be the projection onto the one-dimensional space spanned by ξ. For every bounded operator b on H we then have that

$$\rho_t(b)\xi = \rho_t(b)p_\xi\xi = \rho_t\big(b\rho_{-t}(p_\xi)\big)\xi = \sigma_t\big(b\sigma_{-t}(p_\xi)\big)\xi,$$

because both p_ξ and $b\rho_{-t}(p_\xi)$ are compact operators, and hence belong to A. Using that σ is a (continuous) flow on A, we see that $\rho_t(b)\xi$ is continuous as a function of t, meaning that ρ is strongly continuous, and hence also weakly continuous.

Now, [BR87, Example 3.2.35] shows that a weakly continuous flow on $\mathcal{B}(H)$ must be of the desired form, so the result follows.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

2.3. Coarse geometry

Although it has already been briefly presented in the introduction, we now properly define the morphisms and equivalences in the coarse category. Given metric spaces (X,d_X) and (Y,d_Y), a map $f\colon X\to Y$ is called *coarse* if for all $r>0$,

$$\sup_{d_X(x,z)\leq r} d_Y(f(x),f(z))<\infty.$$

Maps $f,g\colon X\to Y$ are called *close* if

$$\sup_{x\in X} d_Y(f(x),g(x))<\infty.$$

If $f\colon X\to Y$ is coarse and there is a coarse $g\colon Y\to X$ such that $g\circ f$ and $f\circ g$ are close to id_X and id_Y, respectively, then f is called a *coarse equivalence*.

The uniform Roe algebra has a very natural dense *-subalgebra which will be useful for us. Precisely, the algebraic counterpart of $\mathrm{C}^*_u(X)$, denoted $\mathrm{C}^*_u[X]$, is the *-subalgebra of $\mathcal{B}(\ell_2(X))$ generated by all av_f, where $a\in\ell_\infty(X)$, and f is a partial translation of X. Unless X is finite, $\mathrm{C}^*_u[X]$ will never be closed.

As already made clear in the introduction, the metric spaces of interest to us will be *uniformly locally finite*, or *u.l.f.* for short (see Subsection 1.2 for definition). One of the great advantages of such metric spaces is that an X-by-X matrix of complex numbers with finite propagation induces a bounded operator on $\ell_2(X)$ if and only if its entries are uniformly bounded. For further reference, we isolate this as a proposition below (see [Roe03, Lemma 4.27]).

Proposition 2.3.1. *Let X be a u.l.f. metric space. Then, for all $r>0$, if $A=[a_{x,y}]_{x,y\in X}$ is an X-by-X matrix of complex numbers with*

$$\sup_{x,y\in X}|a_{xy}|<\infty,$$

and $\mathrm{prop}(A)\leq r$, then A canonically induces a bounded operator a on $\ell_2(X)$, such that $\langle a\delta_y,\delta_x\rangle=a_{xy}$ for all $x,y\in X$. Moreover, there is a positive constant M, depending only on X and r, such that

$$\|a\|\leq M\sup_{x,y\in X}|a_{xy}|.$$

An important class of metric spaces in coarse geometry is the one of u.l.f. metric spaces with property A, introduced by G. Yu in [Yu00]. Although the definition of property A will not be used per se, we quickly recall it here since this property will play a major role throughout these notes.

Definition 2.3.2. Let (X,d) be a u.l.f. metric space. We say that X has *property A* if, for all $\varepsilon,r>0$, there is a function $\xi\colon X\to\partial B_{\ell_2(X)}$[11] such that

1. $\|\xi_x-\xi_y\|\leq\varepsilon$ for all $x,y\in X$ with $d(x,y)\leq r$, and
2. $\sup\{d(x,y)\ :\ \langle\xi_x,\delta_y\rangle\neq 0\}<\infty$.

[11]Here $\partial B_{\ell_2(X)}$ denotes the unit sphere of $\ell_2(X)$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

It is well known that a u.l.f. metric space X has property A if and only if $\mathrm{C}^*_u(X)$ is a nuclear C*-algebra (see [BO08, Theorem 5.5.7]).

Besides the uniform Roe algebra of X, the quasi-local algebra of X will play an important role in these notes.

Definition 2.3.3. Let (X,d) be a u.l.f. metric space. The *quasi-local algebra of* X, denoted by $\mathrm{C}^*_{ql}(X)$, is the C*-subalgebra of $\mathcal{B}(\ell_2(X))$ consisting of all operators $a \in \mathcal{B}(\ell_2(X))$ such that, for all $\varepsilon > 0$, there is $r > 0$, such that for all $A,B \subseteq X$,

$$d(A,B) > r \;\text{ implies }\; \|p_A a p_B\| \leq \varepsilon.$$[12]

Equivalently, a bounded operator a lies in $\mathrm{C}^*_{ql}(X)$, if and only if

$$\lim_{r\to\infty} \sup_{d(A,B)>r} \|p_A a p_B\| = 0.$$

While it is immediate that $\mathrm{C}^*_u(X) \subseteq \mathrm{C}^*_{ql}(X)$ for any u.l.f. metric space X, it was an important open problem whether $\mathrm{C}^*_u(X)$ and $\mathrm{C}^*_{ql}(X)$ coincide. J. Špakula and J. Zhang proved that this is indeed the case for u.l.f. metric spaces with property A (see [ŠZ20, Theorem 3.3]). Very recently, N. Ozawa has shown that these algebras are never the same as long as X contains a coarse copy of an expander graph (see [Oza23, Corollary C]).

3. Diagonal flows

The goal of this section is to set the ground needed to define a noncommutative version of coarseness in the next section. More precisely, we characterize coarseness of a real-valued function on a metric space X in terms which allow us to define "coarseness" for an arbitrary self-adjoint operator on $\ell_2(X)$ (Proposition 3.1.5). We also pin down a property satisfied by flows induced by maps $X \to \mathbb{R}$ which will be needed in our study of more general flows induced by self-adjoint operators on $\ell_2(X)$ (Proposition 3.1.7).

Our motivation for wanting to characterize coarseness of maps $X \to \mathbb{R}$ comes from the fact that coarse maps are precisely the ones which induce flows on $\mathrm{C}^*_u(X)$ in a canonical way; which in turn was intensively studied in [BE25]. Let us start by explaining this carefully. An arbitrary function $h\colon X \to \mathbb{R}$ canonically induces a self-adjoint operator on $\ell_2(X)$ by multiplication. By abuse of notation, we denote this operator by h as well. Precisely, h is the operator with

$$\mathrm{dom}(h) = \left\{ \xi \in \ell_2(X) \;:\; \sum_{x\in X} |h_x \xi_x|^2 < \infty \right\},$$

and given by the formula

$$h\xi = (h_x \xi_x)_{x\in X}$$

for all $\xi \in \mathrm{dom}(h)$. Letting $c_{00}(X)$ denote the subspace of $\ell_2(X)$ formed by the vectors with finitely many nonzero coordinates, it is clear that $c_{00}(X) \subseteq \mathrm{dom}(h)$, so $\mathrm{dom}(h)$ is

[12] Recall, as stated in the introduction, that p_A is the orthogonal projection from $\ell_2(X)$ to $\ell_2(A)$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

dense in $\ell_2(X)$. See Proposition 3.1.4 below for a proof that h is actually an admissible operator. Moreover, h defines a bounded operator on $\ell_2(X)$ if and only if $h\colon X \to \mathbb{R}$ is bounded. Hence, the identification above of functions $X \to \mathbb{R}$ with possibly unbounded self-adjoint operators on $\ell_2(X)$ is compatible with the usual identification of $\ell_\infty(X)$ as the bounded multiplication operators on $\ell_2(X)$, i.e., the diagonal operators with respect to the canonical orthonormal basis $(\delta_x)_{x\in X}$ of $\ell_2(X)$.

Convention 3.1.1. In order to avoid any confusion stemming from the abuse of notation of denoting by the same symbol — h usually — both a map $X \to \mathbb{R}$ and its interpretation as an unbounded operator on $\ell_2(X)$, we write $h = (h_x)_{x\in X}$ when h is being seen as a function $h\colon x \in X \mapsto h_x \in \mathbb{R}$. If h is being interpreted as an operator on $\ell_2(X)$, its evaluation at a vector $\xi \in \ell_2(X)$ is written as $h\xi$. Also, $\mathrm{dom}(h)$ will always refer to the domain of h as an operator on $\ell_2(X)$ (the domain of h as a function $X \to \mathbb{R}$ is simply X and this will not be a point of relevance in these notes).

Suppose that (X,d) is a u.l.f. metric space and that $h\colon X \to \mathbb{R}$ is a map. Then, e^{ih} may be seen as a bounded operator on $\ell_2(X)$ with propagation zero, so, $e^{ih} \in \mathrm{C}^*_u(X)$. In particular, $e^{ith}ae^{-ith} \in \mathrm{C}^*_u(X)$ for all $t \in \mathbb{R}$, and all $a \in \mathrm{C}^*_u(X)$, so the map

$$\sigma_{h,t}\colon a \in \mathrm{C}^*_u(X) \mapsto e^{ith}ae^{-ith} \in \mathrm{C}^*_u(X)$$

is an automorphism of $\mathrm{C}^*_u(X)$, and it can be easily shown to define a pre-flow σ_h on $\mathrm{C}^*_u(X)$. As seen in Proposition 1.1.2, σ_h is a flow if and only if the map $h\colon X \to \mathbb{R}$ is coarse, in which case we call σ_h a *diagonal flow*.

Proposition 1.1.2 is one of our hints to finding an appropriate operator version of coarseness. Before that, we must notice that the domain of an operator induced by a coarse map $X \to \mathbb{R}$ is invariant with respect to the coarse geometry of X. We introduce the following definition.

Definition 3.1.2. Let $E \subseteq \ell_2(X)$ be a linear subspace. Given a partial translation f of X, we say that E is *f-invariant* if $v_f(E) \subseteq E$. If E is f-invariant for all partial translations f of X, we say that E is *invariant under partial translations*.

Remark 3.1.3. Notice that if the domain of a self-adjoint operator h on $\ell_2(X)$ is invariant under partial translations, then $\mathrm{dom}(h)$ automatically contains $c_{00}(X)$. Indeed, as h is self-adjoint, its domain is densely defined and, in particular, non-zero. Pick a non-zero $\xi \in \mathrm{dom}(h)$ and $x_0 \in X$ such that the x_0-coordinate of ξ is non-zero. Then, given $x \in X$ and letting $f_x\colon \{x_0\} \to \{x\}$ be the partial translation taking x_0 to x, we have that $v_{f_x}\xi$ is in $\mathrm{dom}(h)$ for all $x \in X$. As $v_{f_x}\xi$ is a non-zero vector supported on x and $x \in X$ was arbitrary, we conclude that $\mathrm{dom}(h)$ contains $c_{00}(X)$. This will be used throughout these notes without further mention. The fact that $c_{00}(X) \subseteq \mathrm{dom}(h)$ does not necessarily mean that h is admissible, although most of the operators in this work will possess this property.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

For simplicity, we introduce the following (almost) self-explaining terminology: given a partial translation f of X, and $r \geq 0$, we say that f is a *partial r-translation* if

$$\sup_{x \in \mathrm{dom}(f)} d(f(x),x) \leq r.$$

Proposition 3.1.4. *Let X be a u.l.f. metric space and let $h\colon X \to \mathbb{R}$ be any function. Then, seen as a multiplication operator, h is admissible. Moreover, if h is a coarse function, then $\mathrm{dom}(h)$, the domain of h as an operator on $\ell_2(X)$, is invariant under partial translations.*

Proof. It is evident that $\mathrm{dom}(h)$ contains $c_{00}(X)$, and we leave it for the reader to prove the standard fact that h is a closed operator. On the other hand, given $\xi \in \mathrm{dom}(h)$, we have that

$$h\xi = (h_x\xi_x)_{x\in X} = \sum_{x\in X} h_x\xi_x\delta_x = \sum_{x\in X} \langle \xi,\delta_x\rangle h\delta_x,$$

so h is admissible. In order to show that $\mathrm{dom}(h)$ is invariant under partial translations, suppose that $h\colon X \to \mathbb{R}$ is coarse and that $\xi = (\xi_x)_{x\in X}$ is in the domain of h, seen as an operator on $\ell_2(X)$. So,

$$\|h\xi\|^2 = \sum_{x\in X} |h_x\xi_x|^2 < \infty.$$

Let f be a partial translation of X. As h is coarse,

$$s := \sup_{x\in \mathrm{dom}(f)} |h_{f(x)} - h_x| < \infty.$$

Then, by Minkowski's inequality, we have

$$\begin{aligned}\left(\sum_{x\in\mathrm{dom}(f)} |h_{f(x)}\xi_x|^2\right)^{1/2} &\leq \left(\sum_{x\in\mathrm{dom}(f)} |\xi_x|^2\cdot|h_{f(x)}-h_x|^2\right)^{1/2} \\ &\quad + \left(\sum_{x\in\mathrm{dom}(f)} |h_x\xi_x|^2\right)^{1/2} \\ &\leq s\|\xi\| + \|h\xi\| \\ &< \infty.\end{aligned}$$

This shows that $v_f\xi$ is in the domain of h and we are done. □

The following result will later motivate our notion of a coarse operator.

Proposition 3.1.5. *Let X be u.l.f. metric space and $h\colon X \to \mathbb{R}$ be a map. The following are equivalent.*

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

1. *The map* $h\colon X \to \mathbb{R}$ *is coarse.*
2. *The domain of* h*, seen as an operator on* $\ell_2(X)$*, is invariant under partial translations and, for all* $r > 0$*,*
$$\sup_f \|[h, v_f]\| < \infty,$$
where the supremum is taken over the set of all partial r*-translations* f *of* X*.*
3. *The domain of* h *is invariant under partial translations and*
$$\|[h, v_f]\| < \infty$$
for all partial translations f *of* X*.*

Proof. (1)⇒(2): Assume $h\colon X \to \mathbb{R}$ is coarse. By Proposition 3.1.4, $\mathrm{dom}(h)$ is invariant under partial translations. Fix $r > 0$. As $h\colon X \to \mathbb{R}$ is coarse, there is $s > 0$ such that $|h_x - h_y| \leq s$ for all $x, y \in X$ with $d(x,y) \leq r$. Hence, if f is a partial r-translation of X, then $|h_{f(x)} - h_x| \leq s$ for all $x \in \mathrm{dom}(f)$. Since $[h, v_f](\delta_x) = (h_{f(x)} - h_x)\delta_{f(x)}$ for all $x \in \mathrm{dom}(f)$, it follows that

$$\|[h, v_f]\| \leq \sup_{x \in \mathrm{dom}(f)} |h_{f(x)} - h_x| \leq s.$$

(2)⇒(3): This is immediate.

(3)⇒(1): Suppose $h\colon X \to \mathbb{R}$ is not coarse. So, there is $r > 0$ and sequences $(x_n)_n$ and $(y_n)_n$ in X such that $d(x_n, y_n) \leq r$ for all $n \in \mathbb{N}$ and

$$\lim_n |h_{x_n} - h_{y_n}| = \infty. \tag{3.1.6}$$

As X is u.l.f., going to subsequences if necessary, we can assume that $(x_n)_n$ and $(y_n)_n$ are both sequences of distinct elements of X. Therefore, letting

$$A = \{x_n \ : \ n \in \mathbb{N}\} \quad \text{and} \quad B = \{y_n \ : \ n \in \mathbb{N}\},$$

the map $f\colon A \to B$ given by $f(x_n) = y_n$ for all $n \in \mathbb{N}$ defines a partial translation, a partial r-translation to be precise. However, we have

$$\|[h, v_f]\| \geq \|[h, v_f]\delta_{x_n}\| = |h_{f(x_n)} - h_{x_n}|$$

for all $n \in \mathbb{N}$. By (3.1.6), the latter goes to infinity. But this contradicts the hypothesis. □

Before finishing this subsection, we now pin down an important property satisfied by diagonal flows on uniform Roe algebras.

Proposition 3.1.7. *Let* (X,d) *be a u.l.f. metric space and let* $h\colon X \to \mathbb{R}$ *be a map inducing the flow* σ_h *on* $\mathrm{C}^*_u(X)$*. For each partial translation* f *of* X*, the map*

$$t \in \mathbb{R} \mapsto \sigma_{h,t}(v_f) \in \mathrm{C}^*_u(X)$$

is differentiable at zero with derivative equal to $i[h, v_f]$*. Therefore the map*

$$t \in \mathbb{R} \mapsto \sigma_{h,t}(a) \in \mathrm{C}^*_u(X) \tag{3.1.8}$$

is also differentiable at zero for all a *in* $\mathrm{C}^*_u[X]$*, with derivative equal to* $i[h, a]$*.*

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Proof. As σ_h is a flow, h must be coarse (Proposition 1.1.2). Let us first show that the first statement indeed implies the second. Since $\mathrm{C}^*_u[X]$ is the linear span of av_f, for $a \in \ell_\infty(X)$, and f a partial translation of X, it is enough to notice that the map (3.1.8) is differentiable at zero for elements of the form av_f. Finally, as $\sigma_{h,t}(av_f) = a\sigma_{h,t}(v_f)$, we see that the second statement indeed follows from the first one.

Fix now a partial translation f of X. An elementary calculation shows that

$$\sigma_{h,t}(v_f)\delta_x = e^{it(h(f(x))-h(x))}v_f\delta_x, \tag{3.1.9}$$

if x is in $\mathrm{dom}(f)$, while $\sigma_{h,t}(v_f)\delta_x = 0$ otherwise. Denoting by $h\circ f - h$ the function on X given by

$$(h\circ f - h)(x) = \begin{cases} h(f(x)) - h(x), & \text{if } x \in \mathrm{dom}(f), \\ 0, & \text{otherwise,} \end{cases}$$

it follows from (3.1.9) that $\sigma_{h,t}(v_f) = v_f e^{it(h\circ f - h)}$. Observing that $h\circ f - h$ is a bounded function, thanks to h being coarse, we deduce that $\sigma_{h,t}(v_f)$ is differentiable as a function of the variable t, and that its derivative at zero coincides with $iv_f(h\circ f - h)$. Another elementary calculation shows that

$$v_f(h\circ f - h) = [h, v_f],$$

from where the conclusion follows. □

Remark 3.1.10. Proposition 3.1.7 can be stated in terms of the infinitesimal generator of the flow σ. Indeed, the *infinitesimal generator* of a flow σ on a C*-algebra A is the operator δ on A given by

$$\delta(a) = \lim_{t\to 0} \frac{\sigma_t(a) - a}{t}$$

for all $a \in A$ for which the limit above exists. With this terminology and considering a flow σ on a uniform Roe algebra $\mathrm{C}^*_u(X)$, Proposition 3.1.7 says that $\mathrm{C}^*_u[X]$ is contained in the domain of δ, and that $\delta(a) = i[h,a]$ for all a in $\mathrm{C}^*_u[X]$. We refer the reader to [BR87, Chapter 3] for more on that.

4. Coarse operators on $\ell_2(X)$

We will now abstract some of the contents of Section 3 to develop the theory of *noncommutative coarse functions* or, more precisely, of *coarse operators* on $\ell_2(X)$ for a given u.l.f. metric space X. The following is our main definition and it adapts Proposition 3.1.5 to the noncommutative case of self-adjoint operators on $\ell_2(X)$.

Definition 4.1.1 (Definition 1.3.5)**.** Let X be a u.l.f. metric space and h be a closed operator on $\ell_2(X)$. We call h a *coarse operator* if $\mathrm{dom}(h)$ is invariant under partial translations, and

$$\|[h, v_f]\| < \infty$$

for all partial translations f on X.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

We should stress that the above definition does not require h to be admissible; however this property will instead be *proved* later in Theorem 4.1.15.

Example 4.1.2. Bounded operators are obviously coarse operators.

Example 4.1.3. By Proposition 3.1.5, a map $h\colon X \to \mathbb{R}$ is coarse if and only if its interpretation as an operator on $\ell_2(X)$ is a coarse operator. For this reason, self-adjoint coarse operators on $\ell_2(X)$ should be thought of as noncommutative versions of coarse maps $X \to \mathbb{R}$.

Just as in Proposition 3.1.5, a more quantitative version of coarseness holds for operators. Precisely:

Proposition 4.1.4. *Let X be a u.l.f. metric space and h be a closed operator on $\ell_2(X)$, the domain of which is invariant under partial translations. Then h is a coarse operator if and only if, for all $r > 0$,*

$$\sup_f \|[h, v_f]\| < \infty,$$

where the supremum is taken over the set of all partial r-translations f of X.

Proof. The "if" part being immediate, we move on to the "only if" direction. Assuming that h is a coarse operator we therefore suppose, by way of contradiction, that the supremum in the statement is infinite.

Claim 4.1.5. The supremum in the statement is infinite, even if taken over the set of all partial r-translations f of X, with *finite domain*.

Proof. To see this, we fix $M > 0$, and we choose a partial r-translation g (which may or may not have finite domain), such that $\|[h, v_g]\| > M$. Since $[h, v_g]$ is bounded on $\mathrm{dom}(h)$, by hypothesis, and since $c_{00}(X)$ is dense in $\mathrm{dom}(h)$, we may pick a unit vector ξ in $c_{00}(X)$, such that

$$M < \|[h, v_g]\xi\| = \|hv_g\xi - v_g h\xi\|. \tag{4.1.6}$$

Fixing $\varepsilon > 0$, let C be a finite subset of X such that

$$\xi = p_C\xi, \quad \text{and} \quad \|h\xi - p_C h\xi\| < \varepsilon.$$

Letting f be the restriction of g to $\mathrm{dom}(g) \cap C$, we then have that $v_f = v_g p_C$. Moreover

$$v_g\xi = v_f\xi, \quad \text{and} \quad \|v_g h\xi - v_f h\xi\| < \varepsilon.$$

So, upon substituting g for f, the right-hand side of (4.1.6) changes by at most ε, so the inequality is preserved as long as we choose ε sufficiently small. □

Slightly changing tack, we now make the following:

Claim 4.1.7. If $C \subseteq X$ is any finite set, then the supremum in the statement is *finite*, if taken over the set of all r-partial translations f with $\mathrm{dom}(f) \subseteq C$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Proof. Indeed, given such an r-partial translation f, we have that

$$\begin{aligned}\|[h,v_f]\| &= \|hv_f - v_f h\| \\ &\leq \|hv_f\| + \|v_f h\| \\ &\leq \|hp_{B_r(C)}\| + \|p_C h\|,\end{aligned}$$

the last inequality due to the fact that $\mathrm{dom}(f) \subseteq C$, and hence that $\mathrm{ran}(f) \subseteq B_r(C)$, the ball of radius r around C, because r-partial translations are not allowed to move points more than r apart. As h is necessarily bounded on the finite-dimensional space $\ell_2(B_r(C))$, $\|hp_{B_r(C)}\|$ is finite. In order to estimate $\|p_C h\|$, notice that

$$\|p_C h\| \leq \|p_C hp_C\| + \|p_C hp_{X\setminus C}\| \leq \|hp_C\| + \|p_C hp_{X\setminus C}\|.$$

As before, $\|hp_C\|$ is finite and, since $p_C hp_{X\setminus C} = [h,p_{X\setminus C}]p_{X\setminus C}$, the fact that h is coarse implies that $\|p_C hp_{X\setminus C}\|$ is also finite. This completes the proof of the claim. □

Our next claim is a strengthening of Claim 4.1.7.

Claim 4.1.8. If $C \subseteq X$ is any finite set, then the supremum in the statement is *infinite*, even if taken over the set of all partial r-translations f of X, with finite domain, and such that both the domain of f and the range of f (henceforth denoted $\mathrm{ran}(f)$) are contained in $X\setminus C$.

Proof. Given $M > 0$, let N be the supremum referred to in Claim 4.1.7 relative to the finite set $B_r(C)$, and, using Claim 4.1.5, choose some r-partial translation g, with finite domain, and such that $\|[h,v_g]\| > M + N$. Letting g_1 and g_2 denote the restrictions of g to $\mathrm{dom}(g) \cap B_r(C)$, and $\mathrm{dom}(g)\setminus B_r(C)$, respectively, we see that $v_g = v_{g_1} + v_{g_2}$, so

$$\|[h,v_{g_2}]\| = \|[h,v_g - v_{g_1}]\| \geq \|[h,v_g]\| - \|[h,v_{g_1}]\| > M + N - N = M,$$

Recalling that $\mathrm{dom}(g_2) \subseteq X\setminus B_r(C)$, we have that $\mathrm{ran}(g_2) \subseteq X\setminus C$, for the same reason observed above that r-partial translations are not allowed to move points more than r apart. □

Claim 4.1.9. There is a sequence $(f_n)_n$ of partial r-translations of X such that the domains of the f_n are finite and pairwise disjoint, the same holding for their ranges, and such that $\lim_{n\to\infty} \|[h,v_{f_n}]\| = \infty$.

Proof. As the proof of this claim is an easy consequence of Claim 4.1.8, we leave it for the reader. □

From now on we fix a sequence $(f_n)_n$ as above.

Claim 4.1.10. For every finite subset $C \subseteq X$, one has that

$$\lim_{n\to\infty} \|p_{X\setminus C}[h,v_{f_n}]p_{X\setminus C}\| = \infty.$$

Proof. In order to verify this claim, observe that $\ell_2(C)$ is a finite dimensional space, hence $p_C h$ and hp_C are bounded operators on the domain of h, say with

$$\|p_C h\|,\ \|hp_C\| \leq M_C.$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Therefore, for every n, we have that

$$\|[h,v_{f_n}]p_C\| = \|hv_{f_n}p_C\| + \|v_{f_n}hp_C\| \leq M_{B_r(C)} + M_C,$$

and similarly $\|p_C[h,v_{f_n}]\| \leq M_{B_r(C)} + M_C$. Therefore

$$\begin{aligned}\|p_{X\setminus C}[h,v_{f_n}]p_{X\setminus C}\| &= \|(1-p_C)[h,v_{f_n}](1-p_C)\| \\ &\geq \|[h,v_{f_n}]\| - \|[h,v_{f_n}]p_C\| \\ &\quad - \|p_C[h,v_{f_n}]\| - \|p_C[h,v_{f_n}]p_C\| \\ &\geq \|[h,v_{f_n}]\| - 3(M_{B_r(C)} + M_C),\end{aligned}$$

from where the claim follows. □

Claim 4.1.11. Upon passing to a subsequence of $(f_n)_n$, there is a sequence $(X_n)_n$ of finite and pairwise disjoint subsets of X, such that both $\mathrm{dom}(f_n)$ and $\mathrm{ran}(f_n)$ are contained in X_n, and

$$\lim_{n\to\infty} \|p_{X_n}[h,v_{f_n}]p_{X_n}\| = \infty.$$

Proof. The proof of this claim is an easy application of Claim 4.1.10, so we leave it for the reader. □

In order to conclude the proof, we let $(f_n)_n$ be as in Claim 4.1.10, and we consider the r-partial translation defined on $\bigsqcup_{n\in\mathbb{N}} \mathrm{dom}(f_n)$ to be a common extension of all of the f_n.

Observing that, for $n \neq m$, one has that $\mathrm{dom}(f_m) \subseteq X_m$, and that X_m and X_n are disjoint, it follows that $v_{f_m}p_{X_n} = 0$, and similarly that $p_{X_n}v_{f_m} = 0$, so

$$p_{X_n}[h,v_f]p_{X_n} = p_{X_n}[h,v_{f_n}]p_{X_n},$$

whence, for every n,

$$\|[h,v_f]\| \geq \|p_{X_n}[h,v_f]p_{X_n}\| = \|p_{X_n}[h,v_{f_n}]p_{X_n}\|.$$

Observing that the right hand side goes to ∞, as $n \to \infty$, by Claim 4.1.11, we get a contradiction with the assumption that $[h,v_f]$ is a bounded operator. □

We must now understand what the vital properties satisfied by coarse operators are and, on the other hand, which properties are enough to guarantee coarseness of a given self-adjoint operator on $\ell_2(X)$. The next proposition deals with the former. For that, we introduce the *conditional expectation* for certain unbounded operators on $\ell_2(X)$.

Definition 4.1.12. Consider a set X and any operator h on $\ell_2(X)$ such that $c_{00}(X) \subseteq \mathrm{dom}(h)$. We define a map $E(h)\colon X \to \mathbb{R}$ by letting

$$E(h)_x = \langle h\delta_x, \delta_x\rangle \text{ for all } x \in X.$$

We identify the function $E(h)$ defined above with an operator on $\ell_2(X)$ as usual. Under this identification, it is clear that E, when restricted to the bounded operators on $\ell_2(X)$, is simply the canonical conditional expectation $\mathcal{B}(\ell_2(X)) \to \ell_\infty(X)$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Proposition 4.1.13. *Let X be a u.l.f. metric space and h be a coarse operator on $\ell_2(X)$. Then:*

1. *The map $E(h)$ given by Definition 4.1.12 is coarse.*
2. *For each $A \subseteq X$, $p_{X\setminus A}hp_A$ is bounded and*

$$\sup\{\|p_{X\setminus A}hp_A\| \ : \ A \subseteq X\} < \infty.$$

In particular, as an X-by-X matrix, the coordinates of h off the main diagonal are uniformly bounded.

Proof. (1) Suppose $E(h)$ is not coarse. Then, there are $r > 0$, and sequences $(x_n)_n$ and $(y_n)_n$ in X such that

$$\lim_n |E(h)_{x_n} - E(h)_{y_n}| = \infty \tag{4.1.14}$$

and $d(x_n,y_n) \leq r$ for all $n \in \mathbb{N}$. As X is u.l.f., going to subsequences if necessary, we can assume that $(x_n)_n$ and $(y_n)_n$ are sequences of distinct elements. Let $A = \{x_n \ : \ n \in \mathbb{N}\}$, $B = \{y_n \ : \ n \in \mathbb{N}\}$, and $f\colon A \to B$ be the partial translation given by $f(x_n) = y_n$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\|[h,v_f]\| &\geq \|[h,v_f]\delta_{x_n}\| \\
&= \left\| \sum_{z\in X} \langle h\delta_{y_n},\delta_z\rangle\delta_z - \langle v_f h\delta_{x_n},\delta_z\rangle\delta_z \right\| \\
&\geq |\langle h\delta_{y_n},\delta_{y_n}\rangle - \langle h\delta_{x_n},\delta_{x_n}\rangle| \\
&= |E(h)_{y_n} - E(h)_{x_n}|
\end{aligned}$$

for all $n \in \mathbb{N}$. By (4.1.14), this shows that $[h,v_f]$ is unbounded on $\mathrm{dom}(h)$, which contradicts the fact that h is a coarse operator.

(2) As h is a coarse operator, Proposition 4.1.4 gives $s > 0$, such that

$$\|[h,p_A]\| \leq s$$

for all $A \subseteq X$. Since $p_{X\setminus A}hp_A = [h,p_A]p_A$, the result follows. □

We can now prove that coarse operators are automatically admissible, hence allowing us to understand how these operators act not only on $c_{00}(X)$ but on all of its domain.

Theorem 4.1.15. *Let X be a u.l.f. metric space and h be a coarse operator on $\ell_2(X)$. If $\xi \in \mathrm{dom}(X)$, then*

$$h\xi = \sum_{x\in X} \langle \xi,\delta_x\rangle h\delta_x.$$

Consequently, h is admissible.

Proof. As h is a coarse operator, Proposition 4.1.13(2) implies that

$$M = \sup\{\|p_{X\setminus A}hp_A\| \ : \ A \subseteq X\}$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

is finite. Suppose towards a contradiction that the theorem fails and let $\xi \in \mathrm{dom}(h)$ be an offender. As h is a closed operator, the series in the statement of the theorem not only fails to coincide with $h\xi$, but it does not even converge. Consequently that series also fails to satisfy Cauchy's criterion for the summability of a series, whence, letting $\xi_F = \sum_{x\in F}\langle \xi, \delta_x\rangle \delta_x$ for each $F \subseteq X$, we can find $\varepsilon > 0$, and a sequence $(F(n))_n$ of finite disjoint subsets of X such that

$$\|h\xi_{F(n)}\| \geq \varepsilon \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.1.16}$$

As $\lim_n \xi_{F(n)} = 0$, going to a subsequence if necessary, we can assume that $\zeta = \sum_{n\in\mathbb{N}} \xi_{F(n)}$ has norm at most $\varepsilon/(3M)$. In particular, (4.1.16) can be improved as follows:

$$\|p_{F(n)}h\xi_{F(n)}\| \geq \|h\xi_{F(n)}\| - \|p_{X\setminus F(m)}hp_{F(n)}\xi_{F(n)}\| \geq \frac{2\varepsilon}{3}$$

for all $n \in \mathbb{N}$. Since h is a coarse operator, its domain is invariant under partial translations and, in particular, invariant under the projections p_A, $A \subseteq X$. So, $\zeta \in \mathrm{dom}(h)$, since ζ is the image of ξ under such projection. However,

$$\begin{aligned}\|p_{F(n)}h\zeta\| &\geq \|p_{F(n)}h\xi_{F(n)}\| - \left\| p_{F(n)}hp_{X\setminus F(n)}\left(\sum_{m\in\mathbb{N}\setminus\{n\}} \xi_{F(m)}\right)\right\| \\ &\geq \frac{\varepsilon}{3}\end{aligned}$$

for all $n \in \mathbb{N}$. As $(F(n))_n$ are disjoint sets, this shows that $h\zeta$ has infinite norm, a contradiction. □

Corollary 4.1.17. *Let X be a u.l.f. metric space, h and k be coarse operators on $\ell_2(X)$. If $h-k$ is bounded, then* $\mathrm{dom}(h) = \mathrm{dom}(k)$.

Proof. This is an immediate consequence of Proposition 2.1.2 and Theorem 4.1.15. □

We finish this section with a corollary which will be fundamental in many of the forthcoming proofs.

Corollary 4.1.18. *Let X be a u.l.f. metric space and h be a coarse operator on $\ell_2(X)$. If there is an operator h' on $\ell_2(X)$ with finite propagation, and such that $h-h'$ is bounded, then $h - E(h)$ is bounded. In particular, if h has finite propagation, then $h - E(h)$ is bounded.*

Proof. Suppose there is an operator h' on $\ell_2(X)$ with finite propagation so that $h-h'$ is bounded. In particular, $E(h-h') = E(h) - E(h')$ is also bounded. Recall that, as X is u.l.f., an operator w on $\ell_2(X)$ with finite propagation is bounded if and only if $\sup_{x,y\in X} |\langle w\delta_x, \delta_y\rangle| < \infty$ (Proposition 2.3.1). By Proposition 4.1.13(2) above,

$$\sup\{|\langle h\delta_x, \delta_y\rangle| \ : \ x,y \in X \text{ with } x \neq y\} < \infty,$$

so, since $h-h'$ is bounded, we have

$$\sup\{|\langle h'\delta_x, \delta_y\rangle| \ : \ x,y \in X \text{ with } x \neq y\} < \infty.$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Hence,

$$\sup_{x,y\in X} |\langle (h' - E(h'))\delta_x, \delta_y\rangle| < \infty.$$

As h' has finite propagation, so has $h' - E(h')$. So, $h' - E(h')$ is bounded. Finally, as the equality

$$h - E(h) = h - h' + h' - E(h') + E(h') - E(h)$$

holds on $c_{00}(X)$, it follows that $h - E(h)$ is bounded on $c_{00}(X)$. Therefore, since both h and $E(h)$ are coarse operators (for $E(h)$, this is a consequence of Propositions 3.1.5 and 4.1.13(2)), Theorem 4.1.15 implies that $h - E(h)$ is bounded on all of its domain. □

5. Flows given by self-adjoint operators on $\ell_2(X)$

In this short section, we continue the study of Section 3 and extend it to the case of flows on uniform Roe algebras given by self-adjoint operators using the machinery of Section 4. In view of Proposition 3.1.7, we start by introducing a geometric condition on our flows which will help us to better tame the self-adjoint operators inducing them.

Definition 5.1.1 (Definition 1.3.6). Let X be a u.l.f. metric space and σ be a flow on $\mathrm{C}^*_u(X)$. We call σ a *coarse flow* if

$$t \in \mathbb{R} \mapsto \sigma_t(v_f) \in \mathrm{C}^*_u(X)$$

is differentiable for all partial translations f of X.

Example 5.1.2. By Proposition 3.1.7, every flow on $\mathrm{C}^*_u(X)$ induced by a (necessarily coarse) map $X \to \mathbb{R}$ is coarse.

Our next goal is to prove Proposition 1.3.7, stated in the introduction, characterizing self-adjoint operators defining coarse flows on $\mathrm{C}^*_u(X)$.

Proof of Proposition 1.3.7. This proposition is simply an interpretation of [BR87, Proposition 3.2.55] to our setting. Precisely, suppose σ is a coarse flow, i.e., σ is a flow and

$$t \in \mathbb{R} \mapsto \sigma_t(v_f) \in \mathrm{C}^*_u(X)$$

is differentiable at zero for all partial translations f on X.[13] Since flows on $\mathrm{C}^*_u(X)$ are implemented by self-adjoint operators, we must have $\sigma = \sigma_h$ for some self-adjoint h on $\ell_2(X)$ (Proposition 1.3.3). The implication (1)⇒(3) of [BR87, Proposition 3.2.55] then says that $\mathrm{dom}(h)$ is invariant under partial translations and that $[h, v_f]$ is bounded on $\mathrm{dom}(h)$, i.e., h is a coarse operator. Regarding our usage of [BR87, Proposition 3.2.55] here, we point out that this result says that there is a *core* of h which is invariant by it and in which the restriction of $[h, v_f]$ to it is bounded. But, as noticed in the first paragraph

[13] In fact, this automatically implies strong continuity of σ because the v_f span a dense subspace of $\mathrm{C}^*_u(X)$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

of the proof of [BR87, Proposition 3.2.55], closure arguments imply that one can take the core to be the whole $\mathrm{dom}(h)$.

Conversely, suppose that h is a coarse operator and fix a partial translation f of X. Then $\mathrm{dom}(h)$ is f-invariant and $[h,v_f]$ is bounded on $\mathrm{dom}(h)$. We are then in the setting of item (3) of [BR87, Proposition 3.2.55] and the implication (3)⇒(1) therein says that

$$t \in \mathbb{R} \mapsto \sigma_{h,t}(v_f) \in \mathrm{C}^*_u(X)$$

is differentiable, and hence continuous at zero. As the v_f span a dense subset of $\mathrm{C}^*_u(X)$, by definition, we deduce that σ_h is a flow, which is coarse by what was said just above.

Remark 5.1.3. Since all elements of $\mathrm{C}^*_u(X)$ are coarse operators (Example 4.1.2), Proposition 1.3.7 ensures that σ_h is a coarse flow on $\mathrm{C}^*_u(X)$ for every $h \in \mathrm{C}^*_u(X)$. It is worth noting, however, that by choosing h appropriately one may obtain flows that do not leave the algebraic subalgebra $\mathrm{C}^*_u[X]$ of $\mathrm{C}^*_u(X)$ invariant. Indeed, if X is infinite and $x \in X$, one can choose h such that $\sigma_{h,1}$ maps the projection onto $\mathrm{span}\{\delta_x\}$ to the projection onto $\mathrm{span}\{\xi\}$ for some ξ with infinite support. Clearly, this latter projection does not belong to $\mathrm{C}^*_u[X]$.

We isolate a simple fact below for further reference.

Corollary 5.1.4. *Let X be a u.l.f. metric space and h be a self-adjoint operator on $\ell_2(X)$ such that σ_h is a coarse flow on $\mathrm{C}^*_u(X)$. Then, for all partial translations f of X, the derivative of*

$$t \in \mathbb{R} \mapsto \sigma_{h,t}(v_f) \in \mathrm{C}^*_u(X)$$

at zero is $i[h,v_f]$.

Proof. As in the proof of Proposition 1.3.7, this is given by [BR87, Proposition 3.2.55]. □

We finish this section showing that one can always find flows which are not coarse (as long as X is infinite).

Proposition 5.1.5. *Let X be a u.l.f. metric space with infinitely many points. Then there exists a flow on $\mathrm{C}^*_u(X)$ which is not coarse.*

Proof. Fix $x_0 \in X$ and let $h(x) = d(x,x_0)$ for all $x \in X$. So, $h\colon X \to \mathbb{R}$ is a coarse map and σ_h is a diagonal flow on $\mathrm{C}^*_u(X)$ (Proposition 1.1.2). Clearly, σ_{uhu*} is also a flow on $\mathrm{C}^*_u(X)$ for all unitaries u in $\mathrm{C}^*_u(X)$. Let us show that, if the unitary $u \in \mathrm{C}^*_u(X)$ is properly chosen, then σ_{uhu^*} is not coarse. For that, notice that, since h is unbounded, we can pick a vector $\xi \in \ell_2(X)$ with $\|\xi\| = 1$ such that $\xi \notin \mathrm{dom}(h)$. Moreover, without loss of generality, ξ can be picked so that $\|\delta_{x_0} - \xi\|$ is arbitrarily small. Let $p = p_{x_0}$ and let q be the rank 1 projection onto the span of ξ. We can then assume that $\|p - q\|$ is smaller than 1, so there is a unitary u in $\mathrm{C}^*_u(X)$ such that $uqu^* = p$. Since $u\xi$ equals $\lambda\delta_{x_0}$ for some λ in the unitary circle, replacing u by $\bar{\lambda}u$, we can assume that $u\xi = \delta_{x_0}$. Hence, since ξ is not in the domain of h, it follows that δ_{x_0} is not in the domain of h'.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

We can now conclude the proof. If $\sigma_{h'}$ were to be coarse, the operator h' would be a coarse operator (Proposition 1.3.7). However, coarse operators have domains which are invariant under partial translations and, in particular, contain $c_{00}(X)$. Since δ_{x_0} is not in $\mathrm{dom}(h')$, it follows that $\sigma_{h'}$ cannot be a coarse flow. □

Remark 5.1.6. Notice that, even though Proposition 5.1.5 shows that non-coarse flows exist, the examples constructed therein are still very "tame". Indeed, they are given by unitary conjugation of a (diagonal) coarse operator h on $\ell_2(X)$. In particular, the analysis of such flows, including for instance of their KMS states, reduces to the scenario of coarse (diagonal) operators; which we know how to deal with (see [BE25]).

6. Automorphisms in the connected component of $\mathrm{id}_{\mathrm{C}^*_u(X)}$

Section 5 has dealt with Question 1.3.4(II), namely the question of when a pre-flow on a uniform Roe algebra is an actual flow and, for coarse flows, this is completely characterized in terms of the coarseness of the self-adjoint operator h inducing σ (see Proposition 1.3.7). In this section and the following one, we will focus on Question 1.3.4(I), that is, the question of when a self-adjoint operator h on $\ell_2(X)$ defines a pre-flow on $\mathrm{C}^*_u(X)$. For that, we will now take a step back and study not flows per se, but strongly continuous paths of operators on uniform Roe algebras.

Definition 6.1.1. Let X be a u.l.f. metric space and σ be an automorphism of $\mathrm{C}^*_u(X)$. We say that σ *is in the connected component of the identity on* $\mathrm{C}^*_u(X)$ if there is a path $(\sigma_t)_{t\in[0,1]}$ of automorphisms of $\mathrm{C}^*_u(X)$ which is *strongly continuous* in the sense that

$$t \in \mathbb{R} \mapsto \sigma_t(a) \in \mathrm{C}^*_u(X)$$

is continuous for all $a \in \mathrm{C}^*_u(X)$, and such that $\sigma_0 = \mathrm{id}_{\mathrm{C}^*_u(X)}$ and $\sigma_1 = \sigma$.

If X is a u.l.f. metric space, any automorphism σ of $\mathrm{C}^*_u(X)$ is given by a unitary u in $\mathcal{B}(\ell_2(X))$ by conjugation, i.e., $\sigma = \mathrm{Ad}(u)$ (see [ŠW13, Lemma 3.1]). In this section, we show that, if the automorphism σ is in the connected component of the identity on $\mathrm{C}^*_u(X)$, then u belongs to $\mathrm{C}^*_u(X)$, as long as X has property A. Precisely, the following is the main result of this section:

Theorem 6.1.2. *Let X be a u.l.f. metric space with property A, and let σ be an automorphism of $\mathrm{C}^*_u(X)$ lying in the connected component of the identity on $\mathrm{C}^*_u(X)$. If w is any unitary operator on $\ell_2(X)$ such that $\sigma = \mathrm{Ad}(w)$, then $w \in \mathrm{C}^*_u(X)$.*[14]

Most of our uses of Theorem 6.1.2 will be through Theorem 1.4.5, which we now prove as an immediate corollary of Theorem 6.1.2.

Proof of Theorem 1.4.5. Observing that each $\sigma_{h,t}$ clearly lies in the connected component of the identity (Definition 6.1.1), and also that $\sigma_{h,t} = \mathrm{Ad}(e^{ith})$, it follows from Theorem 6.1.2 that $e^{ith} \in \mathrm{C}^*_u(X)$.

[14] Note that, as mentioned above, the existence of such a w is guaranteed by [ŠW13, Lemma 3.1].

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Our method to obtain Theorem 6.1.2 heavily depends on techniques developed in [WW20]. Precisely, the following is the theorem therein whose proof will be essential for our approach to flows on uniform Roe algebras.

Theorem 6.1.3 [WW20, Theorem E]. *Let X be a u.l.f. metric space with property A and σ be an automorphism of $\mathrm{C}^*_u(X)$. Then there is a unitary $w \in \mathrm{C}^*_u(X)$ such that $w^*\sigma(\ell_\infty(X))w = \ell_\infty(X)$.*

We now explain some of the specifics of how Theorem 6.1.3 is obtained in order to isolate a lemma which will guide us in our proof of Theorem 6.1.2. For that, let $\sigma\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(X)$ be an automorphism and let $f\colon X \to X$ be a map such that

$$\inf_{x\in X} \left\|\sigma(e_{xx})\delta_{f(x)}\right\| > 0. \tag{6.1.4}$$

It is well known in the theory of rigidity of uniform Roe algebras that any map satisfying (6.1.4) must be a coarse equivalence (this was first obtained in [ŠW13, Proof of Theorem 4.1], for a more citable result, see [BF21, Theorem 4.12]). Suppose now that there is a bijection $g\colon X \to X$ which is close to f.[15] In particular, g is also a coarse equivalence and we can define a unitary $v_g\colon \ell_2(X) \to \ell_2(X)$ by letting

$$v_g\delta_x = \delta_{g(x)} \quad \text{for all} \quad x \in X. \tag{6.1.5}$$

As shown in [ŠW13, Lemma 3.1], the automorphism σ is implemented by a unitary on $\ell_2(X)$, so fix a unitary $u \in \mathcal{B}(\ell_2(X))$ such that $\sigma = \mathrm{Ad}(u)$. A simple inspection of the proof of Theorem 6.1.3 tells us that, if X has property A, then

$$w = uv_g^* \in \mathrm{C}^*_u(X)$$

(this is detailed in [WW20, Page 983]).

We summarize the discussion above in a lemma:

Lemma 6.1.6 [WW20, Proof of Theoorem E]. *Let X be a u.l.f. metric space with property A, σ be an automorphism of $\mathrm{C}^*_u(X)$ implemented by a unitary $u \in \mathcal{B}(\ell_2(X))$, $f\colon X \to X$ be a map satisfying (6.1.4), and $g\colon X \to X$ be a bijection close to f. Then, letting v_g^* be the unitary given by (6.1.5), the unitary $w = uv_g$ belongs to $\mathrm{C}^*_u(X)$.*[16]

We can now clearly state our approach towards Theorem 6.1.2. Precisely, the following is our main technical result.

Theorem 6.1.7. *Let X be a u.l.f. metric space and σ be an automorphism of $\mathrm{C}^*_u(X)$ in the connected component of the identity on $\mathrm{C}^*_u(X)$. Then, for all $t \in \mathbb{R}$, there is a coarse equivalence $f_t\colon X \to X$ such that f_t is close to the identity and*

$$\inf_{x\in X} \left\|\sigma_t(e_{xx})\delta_{f_t(x)}\right\| > 0.$$

We start the proof of Theorem 6.1.7 with a proposition.

[15] It is not yet known if such bijection always exists. As of now, this is known to be the case if X has property A ([WW20, Corollary 6.13]), if X is non-amenable ([WW20, Theorem 5.1]), or if X is an expander graph ([BBF+24, Theorems 1.2 and 1.4]).

[16] Notice that w in Lemma 6.1.6 is precisely w in Theorem 6.1.3.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Proposition 6.1.8. *Let X be a u.l.f. metric space, σ be an automorphism of $\mathrm{C}^*_u(X)$, and $f\colon X \to X$ be a map such that*

$$\delta = \inf_{x\in X} \left\|\sigma(e_{xx})\delta_{f(x)}\right\| > 0.$$

Suppose that for all $A \subseteq X$, there is a unitary $u \in \mathrm{C}^*_u(X)$ such that $\sigma(p_A) = up_Au^*$. Then f is close to the identity.

Remark 6.1.9. Proposition 6.1.8 can also be derived from [MV24, Theorem 10.2.1]. Given the length and general scope of [MV24], we have chosen to retain our own proof, which may be more accessible to the general reader. We are grateful to the referee for bringing this to our attention.

Before proving Proposition 6.1.8, we must recall the notion of equi-approximable families in $\mathrm{C}^*_u(X)$ and a criterion for equi-approximability obtained in [BF21]; this will also play an important role in Section 7.

Definition 6.1.10. Let X be a u.l.f. metric space.

1. Given $\varepsilon, r > 0$, we say that $a \in \mathrm{C}^*_u(X)$ is *ε-r-approximable* if there is $b \in \mathrm{C}^*_u(X)$, with $\mathrm{prop}(b) \leq r$, such that $\|a - b\| \leq \varepsilon$.
2. A subset $S \subseteq \mathrm{C}^*_u(X)$ is *equi-approximable* if for all $\varepsilon > 0$, there is $r > 0$, such that every $a \in S$ is ε-r-approximable.

Lemma 6.1.11 [BF21, Lemma 4.9]. *Let X be a u.l.f. metric space and $(a_n)_n$ be a family of operators such that $a_M = \mathrm{SOT}\text{-}\sum_{n\in M} a_n \in \mathrm{C}^*_u(X)$ for all $M \subseteq \mathbb{N}$. Then the family $(a_M)_{M\subseteq\mathbb{N}}$ is equi-approximable.*

Proof of Proposition 6.1.8. Suppose towards a contradiction that $f\colon X \to X$ is not close to the identity. Then there is a sequence $(x_n)_n$ in X such that

$$\lim_{n\in\mathbb{N}} d(x_n, f(x_n)) = \infty.$$

Let $A = \bigcup_{n\in\mathbb{N}}\{x_n\}$. By hypothesis, there is a unitary $u \in \mathrm{C}^*_u(X)$ such that $\sigma(p_A) = up_Au^*$. As automorphisms of uniform Roe algebras are compact preserving and strongly continuous (see [ŠW13, Lemma 3.1]), going to a subsequence if necessary, we can assume that

$$\|e_{x_nx_n}u^*\sigma(e_{x_kx_k})u\| \leq 2^{-k-1}$$

for all distinct $k, n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\|e_{x_nx_n}u^*\sigma(e_{x_nx_n})u\| &\geq \|e_{x_nx_n}u^*\sigma(p_A)u\| - \sum_{k\neq n}\|e_{x_nx_n}u^*\sigma(e_{x_kx_k})u\| \qquad (6.1.12)\\ &\geq \|e_{x_nx_n}p_A\| - \frac{1}{2}\\ &\geq \frac{1}{2}\end{aligned}$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

for all $n \in \mathbb{N}$. Since $e_{x_n x_n}$, $u^*\sigma(e_{x_n x_n})$ and $e_{f(x_n)f(x_n)}$ have rank 1, we have that

$$\begin{aligned}\left\|e_{x_n x_n} u^*\sigma(e_{x_n x_n}) e_{f(x_n)f(x_n)}\right\| &= \|e_{x_n x_n} u^*\sigma(e_{x_n x_n})\| \left\|u^*\sigma(e_{x_n x_n}) e_{f(x_n)f(x_n)}\right\| \\ &= \|e_{x_n x_n} u^*\sigma(e_{x_n x_n}) u\| \left\|\sigma(e_{x_n x_n}) e_{f(x_n)f(x_n)}\right\|\end{aligned}$$

for all $n \in \mathbb{N}$ — this is a simple computation and it can be found, for instance, in [BFV21, Lemma 6.5]. Therefore, it follows from (6.1.12) and our choice of f that

$$\left\|e_{x_n x_n} u^*\sigma(e_{x_n x_n}) e_{f(x_n)f(x_n)}\right\| \geq \frac{\delta}{2} \tag{6.1.13}$$

for all $n \in \mathbb{N}$.

As σ is an automorphism, we know from [BF21, Theorem 4.4] that there is $r > 0$ such that each $\sigma(e_{x_n x_n})$ is $\delta/3$-r-approximable. Since $u \in \mathrm{C}^*_u(X)$, replacing r by a larger number if necessary, we can suppose that each $u^*\sigma(e_{x_n x_n})$ is $\delta/3$-r-approximable. However, since $\lim_{n\in\mathbb{N}} d(x_n, f(x_n)) = \infty$, this contradicts (6.1.13). So, f must be close to the identity of X.

Proof of Theorem 6.1.7. Fix $t \in \mathbb{R}$. By [BBF+22, Corollary 3.3], there is a map $f_t \colon X \to X$ such that

$$\inf_{x\in X} \left\|\sigma_t(e_{xx})\delta_{f_t(x)}\right\| > 0.$$

As σ is in the connected component of the identity, there is a continuous path of projections in $\mathrm{C}^*_u(X)$ starting in p_A and ending in $\sigma(p_A)$. Therefore, it follows from [Mur90, Lemma 7.5.4] that σ satisfies the hypothesis of Proposition 6.1.8. The result is now an immediate consequence of Proposition 6.1.8.

Proof of Theorem 6.1.2. By Theorem 6.1.7, there is a coarse equivalence $f \colon X \to X$ satisfying (6.1.4) which is close to the identity on X. Since the identity $X \to X$ is a bijection, applying Lemma 6.1.6 to $g = \mathrm{id}_X$, we obtain that the unitary u implementing σ must belong to $\mathrm{C}^*_u(X)$ as desired.

7. The unbounded part of $\mathrm{C}^*_u(X)$

If a flow on $\mathrm{C}^*_u(X)$ is given by a self-adjoint operator h on $\ell_2(X)$, then $e^{ith} \in \mathrm{C}^*_u(X)$ for all $t \in \mathbb{R}$, by Theorem 1.4.5. It is then important to understand which self-adjoint operators on $\ell_2(X)$ have this property. For that, we introduce the *unbounded part of* $\mathrm{C}^*_u(X)$.

Definition 7.1.1 (Definition 1.3.9). Let X be a u.l.f. metric space and h be an admissible operator on $\ell_2(X)$. We say that h is in the *unbounded part of* $\mathrm{C}^*_u(X)$ if, for all $\varepsilon > 0$, there is an operator h' on $\ell_2(X)$ with finite propagation and such that $h - h'$ is a bounded operator with norm at most ε. We denote the set of all such operators by $\mathrm{C}^*_{u,unb}(X)$.

Example 7.1.2. Clearly $\mathrm{C}^*_u(X) = \mathcal{B}(\ell_2(X)) \cap \mathrm{C}^*_{u,unb}(X)$. Moreover, if h is an operator on $\ell_2(X)$ with finite propagation, it is immediate that $h \in \mathrm{C}^*_{u,unb}(X)$. In particular, if $h \colon X \to \mathbb{R}$ is any map, then its interpretation as a diagonal operator on $\ell_2(X)$ is in $\mathrm{C}^*_{u,unb}(X)$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Theorem 7.1.3. *Let X be a u.l.f. metric space and h be a self-adjoint operator in $\mathrm{C}^*_{u,unb}(X)$ such that*

$$\sup\{|\langle h\delta_y,\delta_x\rangle| \ : \ x,y\in X,\ x\neq y\}<\infty. \tag{7.1.4}$$

Then $e^{ih}\in \mathrm{C}^*_{ql}(X)$. In particular, if h is coarse, then $e^{ih}\in \mathrm{C}^*_{ql}(X)$.

The proof of Theorem 7.1.3 is based on a test to guarantee an operator to be in $\mathrm{C}^*_{ql}(X)$ based on Higson functions which was developed in [ŠZ20]. We recall this concept now.

Definition 7.1.5. Let X be a u.l.f. metric space and $f\colon X\to\mathbb{C}$ be a function. We say that f is *Higson* if, for all $\varepsilon>0$ and all $r>0$, there is a finite set $F\subseteq X$ such that, for all $x,y\in X\backslash F$,

$$d(x,y)\leq r \ \text{ implies } \ |f_x-f_y|\leq\varepsilon.$$

Proof of Theorem 7.1.3. By [ŠZ20, Theorem 3.3], an operator $a\in\mathcal{B}(\ell_2(X))$ is in $\mathrm{C}^*_{ql}(X)$ if and only if $[a,f]$ is compact for all Higson functions $f\colon X\to\mathbb{C}$. Fix such f and let us show that $[e^{ih},f]$ is compact. This is equivalent to showing that $e^{ih}fe^{-ih}-f$ is compact, so, if $\pi\colon \mathrm{C}^*_u(X)\to \mathrm{C}^*_u(X)/\mathcal{K}(\ell_2(X))$ is the canonical quotient map, we must show that $\pi(e^{ih}fe^{-ih})=\pi(f)$. For that, we prove that the map

$$t\in\mathbb{R}\mapsto \pi(e^{ith}fe^{-ith})\in \mathrm{C}^*_u(X)/\mathcal{K}(\ell_2(X)) \tag{7.1.6}$$

is constant. Since this map equals $\pi(f)$ for $t=0$, the result will follow.

Claim 7.1.7. If h' is an operator on $\ell_2(X)$ with finite propagation and such that $h-h'$ is bounded, then $[h',f]$ is bounded and compact.

Proof. As $h-h'$ is bounded on $c_{00}(X)$ and h satisfies (7.1.4), this implies that

$$\sup\{\langle h'\delta_y,\delta_x\rangle \ : \ x,y\in X \text{ with } x\neq y\}<\infty.$$

As h' has finite propagation and

$$\langle [h',f]\delta_y,\delta_x\rangle=(f_y-f_x)\langle h'\delta_y,\delta_x\rangle \tag{7.1.8}$$

for all $x,y\in X$, the X-by-X matrix representation of $[h',f]$ has finite propagation and its coordinates are uniformly bounded. Since X is u.l.f., Proposition 2.3.1 implies that the X-by-X matrix $\big[\langle[h',f]\delta_y,\delta_x\rangle\big]_{x,y\in X}$ canonically induces a bounded operator on $\ell_2(X)$. Being a finite propagation operator, h' is also assumed to be admissible. Based on the fact that f is a diagonal operator, one can easily prove that $c_{00}(X)$ is a core for $[h',f]$. Therefore, since $[h',f]$ and the bounded operator induced by its matrix in the canonical basis of $\ell_2(X)$ clearly coincide on $c_{00}(X)$, it follows that $[h',f]$ is itself bounded.

In order to show that $[h',f]$ is compact, it is enough to notice that

$$\lim_{x,y\to\infty}\langle[h',f]\delta_y,\delta_x\rangle=0$$

(here we use again that $[h',f]$ has finite propagation). But this follows immediately from (7.1.8) since f is Higson and $\mathrm{prop}(h')<\infty$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Claim 7.1.9. $[h,f]$ is bounded and compact.

Proof. By hypothesis, for every $\varepsilon > 0$, there is an operator h' with finite propagation such that $h - h'$ is bounded and $\|h - h'\| < e$. Since the equality

$$[h,f] = [h-h',f] + [h',f] \tag{7.1.10}$$

holds on $c_{00}(X)$, we have that

$$\big\|[h,f]\big\|_0 \leq \big\|[h-h',f]\big\|_0 + \big\|[h',f]\big\|_0 \leq 2\varepsilon\|f\| + \big\|[h',f]\big\|_0 < \infty,$$

where by $\|\cdot\|_0$ we mean the norm of the restriction of the relevant operator to $c_{00}(X)$. Again, since f is diagonal, $c_{00}(X)$ is a core for $[h,f]$, so the above inequality implies that $[h,f]$ is bounded on its whole domain, as required. In addition,

$$\big\|[h,f] - [h',f]\big\|_0 = \big\|[h-h',f]\big\|_0 \leq 2\varepsilon\|f\|,$$

which means that $[h,f]$ may be approximated arbitrarily closely by compact operators, therefore itself being compact. □

By Claim 7.1.9, $[h,f]$ is bounded. Therefore, [BR87, Proposition 3.2.55] implies that the map

$$t \in \mathbb{R} \mapsto e^{ith} f e^{-ith} \in \mathcal{B}(\ell_2(X)) \tag{7.1.11}$$

is differentiable at zero and its derivative equals $i[h,f]$. Moreover, since σ_h is a group action, this implies that (7.1.11) is differentiable at all $t \in \mathbb{R}$ with derivative $\sigma_{h,t}(i[h,f])$. As $[h,f]$ is compact (Claim 7.1.9) and as the canonical quotient map $\pi\colon \mathrm{C}^*_u(X) \to \mathrm{C}^*_u(X)/\mathcal{K}(\ell_2(X))$ is linear, we conclude that (7.1.6) is differentiable at all $t \in \mathbb{R}$ with derivative 0, hence a constant function. Consequently

$$\pi(e^{ih} f e^{-ih}) = \pi(f),$$

whence $[e^{ih}, f]$ is compact, proving the required criterion for e^{ih} to belong to $\mathrm{C}^*_{ql}(X)$.

For the last statement, notice that, by Proposition 4.1.13, any coarse operator satisfies (7.1.4). □

Theorem 7.1.12. *Let X be a u.l.f. metric space and let h be a coarse operator inducing a pre-flow on $\mathrm{C}^*_u(X)$, i.e.,*

$$e^{ith}\mathrm{C}^*_u(X)e^{-ith} = \mathrm{C}^*_u(X)$$

for all $t \in \mathbb{R}$. Then $h \in \mathrm{C}^*_{u,unb}(X)$. In particular, for E as in Definition 4.1.12, $h - E(h)$ is bounded.

The proof of Theorem 7.1.12 is based on averaging operators over amenable groups and it closely follows ideas in [LW20]. Before presenting its proof, we start by setting up the territory to use the methods therein. Given a set X, consider the group $G = \{-1,1\}^X$, where the multiplication on G is given by multiplication coordinate-wise. Although G is

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

often equipped with the product topology, we shall view it as a discrete group. Being a group, G has a canonical action on $\ell_\infty(G)$ by left-multiplication given by letting

$$(g\cdot a)_h = a_{g^{-1}h}$$

for all $a=(a_g)_{g\in G}\in\ell_\infty(G)$ and all $g,h\in G$. As G is abelian, it is also amenable and we can fix an invariant state $\varphi\colon \ell_\infty(G)\to\mathbb{C}$, meaning that

$$\varphi(g\cdot a)=\varphi(a)$$

for all $a\in\ell_\infty(G)$, and all $g\in G$, where $g\cdot a$ refers to the natural action of G on $\ell^\infty(G)$. Any given state on $\ell^\infty(G)$ may be described as the integral against a certain *finitely additive measure* on G, but our use of this theory will be limited to adopting their notation

$$\int_G a_g d\varphi(g),$$

to mean nothing more than $\varphi(a)$, for $a=(a_g)_{g\in G}$ in $\ell_\infty(G)$. By definition, the underlying set of G, namely $\{-1,1\}^X$, consists of all functions from X to $\{-1,1\}$, and hence G is, technically speaking, a subset of $\ell^\infty(X)$. Moreover, the inclusion map turns out to be a unitary representation of G in $\ell^\infty(X)$, and hence also in $\mathrm{C}^*_u(X)$. We denote it by

$$\pi\colon G\hookrightarrow\ell^\infty(X)\subseteq\mathrm{C}^*_u(X). \tag{7.1.13}$$

The next lemma is well-known and its proof is a straightforward computation, so we omit the details.

Lemma 7.1.14. *Let X be a set and E be as in Definition 4.1.12. Let h be an operator on $\ell_2(X)$ such that $c_{00}(X)\subseteq\mathrm{dom}(h)$. Then*

$$\langle E(h)\delta_y,\delta_x\rangle=\int_G\langle\pi(\varepsilon)^*h\pi(\varepsilon)\delta_y,\delta_x\rangle d\varphi(\varepsilon)$$

for all $x,y\in X$.

Given a set X, $\mathcal{L}_1(\ell_2(X))$ denotes the trace-class operators on $\ell_2(X)$ and $\mathrm{Tr}\colon\mathcal{L}_1(\ell_2(X))\to\mathbb{C}$ denotes the canonical trace on $\mathcal{L}_1(\ell_2(X))$. We identify $\mathcal{B}(\ell_2(X))$ with the dual of $\mathcal{L}_1(\ell_2(X))$ in the canonical way, i.e., the duality is given by

$$a(b)=\mathrm{Tr}(ab) \tag{7.1.15}$$

for all $a\in\mathcal{B}(\ell_2(X))$, and all $b\in\mathcal{L}_1(\ell_2(X))$. For each $a=(a_g)_{g\in G}\in\ell_\infty(G,\mathcal{B}(\ell_2(X)))$,[17] we can define a functional

$$\psi_a\colon b\in\mathcal{L}_1(\ell_2(X))\mapsto\varphi((a_g(b))_{g\in G})\in\mathbb{C}.$$

Under the isometric identification above of $\mathcal{B}(\ell_2(X))$ with the dual of $\mathcal{L}_1(\ell_2(X))$, we consider each ψ_a as an element in $\mathcal{B}(\ell_2(X))$. We can then define the operator

$$\Psi\colon a\in\ell_\infty(G,\mathcal{B}(\ell_2(X)))\mapsto\psi_a\in\mathcal{B}(\ell_2(X)). \tag{7.1.16}$$

[17] Here $\ell_\infty(G,\mathcal{B}(\ell_2(X)))$ denotes the C*-algebra of bounded functions $G\to\mathcal{B}(\ell_2(X))$ endowed with the supremum norm.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

As φ is contractive, it is clear that Ψ is also contractive. Inasmuch as $\varphi\big((a_g)_g\big)$ may be thought of as an *average* of the bounded family of *scalars* a_g, one may think of $\Psi\big((a_g)_g\big)$ as an *average* of the bounded family of *operators* a_g. The following lemma will be important below.

Lemma 7.1.17 [LW20, Lemma 3.3]**.** *Let X be a u.l.f. metric space, $G = \{-1,1\}^X$, and*

$$\Psi\colon \ell_\infty(G,\mathcal{B}(\ell_2(X))) \to \mathcal{B}(\ell_2(X))$$

be as in (7.1.16). If $r > 0$, and $(a_\varepsilon)_{\varepsilon\in G} \in \ell_\infty(G,\mathrm{C}^*_u(X))$ is such that $\mathrm{prop}(a_\varepsilon) \leq r$ for all $\varepsilon \in G$, then, $\Psi((a_\varepsilon)_{\varepsilon\in G})$ has propagation at most r.

Proof of Theorem 7.1.12. Let σ_h be the pre-flow defined by h, which is actually a coarse flow thanks to Proposition 1.3.7. By Proposition 4.1.4, there is $s > 0$ such that

$$\|[h,p_A]\| \leq s$$

for all $A \subseteq X$. Hence, letting π be the inclusion of $G = \{-1,1\}^X$ into $\ell_\infty(X)$ (see (7.1.13)), it follows that the family $([h,\pi(\varepsilon)])_{\varepsilon\in G}$ is uniformly bounded. Moreover, $[h,\pi(\varepsilon)]$ lies in $\mathrm{C}^*_u(X)$, for every ε in G. To see this, recall that each $[h,\pi(\varepsilon)]$ is given by the derivative of σ_h:

$$i[h,\pi(\varepsilon)] = \lim_{t\to 0} \frac{\sigma_{h,t}(\pi(\varepsilon)) - \pi(\varepsilon)}{t}$$

(see Corollary 5.1.4). So,

$$([h,\pi(\varepsilon)])_{\varepsilon\in G} \in \ell_\infty(G,\mathrm{C}^*_u(X)). \tag{7.1.18}$$

Notice that this family is also equi-approximable. For that, observe that

$$\text{SOT-}\sum_{x\in A}[h,p_{\{x\}}] = [h,p_A] \in \mathrm{C}^*_u(X)$$

for all $A \subseteq X$. Lemma 6.1.11 then implies that $([h,\pi(\varepsilon)])_{\varepsilon\in G}$ is equi-approximable.

Fix an arbitrary $\theta > 0$, and let us show that there is an operator h' on $\ell_2(X)$, with finite propagation, and such that $h - h'$ is a bounded operator with norm at most θ. For that, using the equi-approximability of the family $([h,\pi(\varepsilon)])_{\varepsilon\in G}$, pick $r > 0$, such that $[h,\pi(\varepsilon)]$ is θ-r-approximable for all $\varepsilon \in G$. Since each $\pi(\varepsilon)$ has propagation zero, $\pi(\varepsilon)^*[h,\pi(\varepsilon)]$ is also θ-r-approximable for all $\varepsilon \in G$. For each $\varepsilon \in G$, pick $b_\varepsilon \in \mathrm{C}^*_u(X)$ with propagation at most r and such that

$$c_\varepsilon = \pi(\varepsilon)^*[h,\pi(\varepsilon)] - b_\varepsilon \tag{7.1.19}$$

has norm at most θ. By (7.1.18), $([h,\pi(\varepsilon)])_{\varepsilon\in G}$ is bounded, so $(b_\varepsilon)_{\varepsilon\in G}$ is bounded as well. Then, letting Ψ be as in (7.1.16), we can define

$$w = \Psi((\pi(\varepsilon)^*[h,\pi(\varepsilon)])_{\varepsilon\in G}),\ b = \Psi((b_\varepsilon)_{\varepsilon\in G}),\ \text{ and }\ c = \Psi((c_\varepsilon)_{\varepsilon\in G}).$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Since each b_ε has propagation at most r, Lemma 7.1.17 gives that the same holds for b. As each c_ε has norm at most θ, so has c. Also, since we have (7.1.19) for all $\varepsilon \in \{-1,1\}^X$, it follows that

$$c = w - b. \tag{7.1.20}$$

Claim 7.1.21. The operator $w+h$ has propagation zero.

Proof. First notice that, as w is bounded and h is a coarse operator, $w+h$ is also a coarse operator. Let now $x,y \in X$ be distinct and let us compute $\langle w\delta_y, \delta_x\rangle$. Letting $e_{y,x}$ be the partial isometry of rank 1 such that $e_{y,x}\delta_x = \delta_y$, we have

$$\langle w\delta_y, \delta_x\rangle = \mathrm{Tr}(we_{y,x}) = w(e_{y,x}),$$

where the last equality is given by (7.1.15). Therefore, by the formula of Ψ, φ, and using the duality relation in (7.1.15) again, we have

$$\begin{aligned}\langle w\delta_y, \delta_x\rangle &= \Psi((\pi(\varepsilon)^*[h,\pi(\varepsilon)])_{\varepsilon\in G})(e_{y,x}) \\ &= \int_{\{-1,1\}^X} \mathrm{Tr}(\pi(\varepsilon)^*[h,\pi(\varepsilon)]e_{y,x})d\varphi(\varepsilon).\end{aligned}$$

Notice that, even though h may not be bounded, the fact that $c_{00}(X) \subseteq \mathrm{dom}(h)$ implies that $he_{y,x}$ is well defined and bounded. Therefore, we can write

$$\pi(\varepsilon)^*[h,\pi(\varepsilon)]e_{y,x} = \pi(\varepsilon)^*h\pi(\varepsilon)e_{y,x} - he_{y,x}$$

and it follows that

$$\begin{aligned}\langle w\delta_y, \delta_x\rangle &= \int_{\{-1,1\}^X} \mathrm{Tr}(\pi(\varepsilon)^*h\pi(\varepsilon)e_{y,x}) - \mathrm{Tr}(he_{y,x})d\varphi(\varepsilon) \\ &= \int_{\{-1,1\}^X} \langle \pi(\varepsilon)^*h\pi(\varepsilon)\delta_y, \delta_x\rangle d\varphi(\varepsilon) - \langle h\delta_y, \delta_x\rangle.\end{aligned}$$

Since x and y are distinct and $\langle E(h)\delta_y, \delta_x\rangle = 0$, Lemma 7.1.14 implies that the first term in the right-hand side above is zero. This shows that $\langle w\delta_y, \delta_x\rangle = -\langle h\delta_y, \delta_x\rangle$ and we conclude that

$$\langle (w+h)\delta_y, \delta_x\rangle = 0.$$

Since $x,y \in X$ were arbitrary distinct elements of X, this shows that $w+h$ has propagation zero as desired.

We can finally define h' by letting

$$h' = w + h - b.$$

First notice that, as b is bounded, and $w+h$ is a coarse operator, h' is also a coarse operator. Moreover, since $w+h$ and b actually have finite propagation, so does h'. By (7.1.20), we have that $h - h' = -c$. So, $h-h'$ is a bounded operator with norm at most θ. As $\theta > 0$ was arbitrary, this shows that $h \in \mathrm{C}^*_{u,unb}(X)$.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

For the second statement, notice that the first part of the theorem implies that h satisfies the hypothesis of Corollary 4.1.18. The result is then a consequence of this proposition. □

Proof of Theorem 1.3.10. (2)⇒(1): By [ŠZ20, Theorem 3.3], since X has property A, we have $\mathrm{C}^*_u(X) = \mathrm{C}^*_{ql}(X)$. Hence, Theorem 7.1.3 gives that $\sigma_{h,t}(a) \in \mathrm{C}^*_u(X)$ for all $a \in \mathrm{C}^*_u(X)$ and, by Proposition 1.3.7, σ_h is indeed a coarse flow.

(1)⇒(2): By Proposition 1.3.7, h is a coarse operator and, by Theorem 7.1.12, $h \in \mathrm{C}^*_{u,unb}(X)$.

(1)⇒(3): By Theorem 7.1.12, h is in $\mathrm{C}^*_{u,unb}(X)$ and $h - E(h)$ is bounded. As h is in $\mathrm{C}^*_{u,unb}(X)$, for each $\varepsilon > 0$, there is an operator h' with finite propagation such that $h - h'$ has norm at most ε. As h is a coarse operator (Proposition 1.3.7), the supremum of the off-diagonal coordinates of h is finite, hence the same is valid for h'. In particular, as $h' - E(h')$ has finite propagation, it follows that $h' - E(h')$ is bounded (Proposition 2.3.1). Since

$$\|h - E(h) - (h' - E(h'))\| \leq \|h - h'\| + \|E(h) - E(h')\| \leq 2\varepsilon$$

and $\varepsilon > 0$ was arbitrary, this shows that $h - E(h)$ is in $\mathrm{C}^*_u(X)$. As h is a coarse operator, it follows from Proposition 4.1.13 that $E(h)\colon X \to \mathbb{R}$ is a coarse map.

(3)⇒(2): As $h = h - E(h) + E(h)$ and $h - E(h)$ is assumed to be a bounded operator, the domain of h is the one of $E(h)$. Hence, as we are also assuming here that $E(h)\colon X \to \mathbb{R}$ is a coarse map, it follows that the domain of h is invariant under partial translations (see Proposition 3.1.4). Moreover, since $h - E(h)$ is bounded and $E(h)$ is coarse, it follows from Proposition 3.1.5 that

$$[h, v_f] = [h - E(h), v_f] + [E(h), v_f]$$

is bounded for all partial translations f of X. So, h is a coarse operator. Moreover, as $h - E(h)$ is not only bounded but belongs to $\mathrm{C}^*_u(X)$, it is immediate that h is in $\mathrm{C}^*_{u,unb}(X)$.

8. Cocycle perturbation and equivalence of flows on uniform Roe algebras

There are natural equivalence relations on the set of flows on a given C*-algebra: *inner perturbation*, *cocycle perturbation*, and *cocycle conjugacy* (see Definition 1.4.2 above). In this section, we investigate these notions in the context of uniform Roe algebras. More precisely, given a u.l.f. metric space X, we aim to better understand the relationship between self-adjoint operators h and k on $\ell_2(X)$ which originate "equivalent" flows σ_h and σ_k and vice versa. In case of diagonal flows, this is completely described in Theorem 1.4.4. For more general coarse operators h on $\ell_2(X)$ generating coarse flows on $\mathrm{C}^*_u(X)$, we obtain that σ_h is always a cocycle perturbation of a diagonal flow as long as X has property A (Theorem 1.4.3). Finally, Theorem 1.4.7 characterizes when coarse operators give rise to cocycle conjugate flows for spaces with property A; in other words, this is a noncommutative version of Theorem 1.4.4 under the presence of property A.

We begin with a general fact about self-adjoint operators and the unitary groups they generate.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Lemma 8.1.1. *Let H be a separable Hilbert space, let h and k be self-adjoint operators on H, and consider the map*

$$w_{h,k} : t \in \mathbb{R} \mapsto e^{ith}e^{-itk} \in \mathcal{B}(H).$$

Then, regarding the following statements, each one implies the next:

(a) *$w_{h,k}$ is differentiable at zero in the norm topology of $\mathcal{B}(H)$.*

(b) *$w_{h,k}$ is strongly differentiable at zero*[18] .

(c) *$\mathrm{dom}(h) = \mathrm{dom}(k)$, and $h - k$ is bounded on that common domain.*

(d) *H contains a dense subspace D, such that $D \subseteq \mathrm{dom}(h) \cap \mathrm{dom}(k)$, $h - k$ is bounded on D, and D is invariant under e^{itk} for all t in $\mathbb{R}$.*

(e) *$w_{h,k}$ is Lipschitz continuous.*

Proof. (a)⇒(b). Obvious

(b)⇒(c). By hypothesis, the formula

$$\Delta(\xi) = \lim_{t\to 0} \frac{e^{ith}e^{-itk}\xi - \xi}{t}$$

provides a well defined linear operator on H. Moreover, as the operators

$$\Delta_t \colon \xi \in H \mapsto \frac{e^{ith}e^{-itk}\xi - \xi}{t} \in H$$

are bounded for all $t > 0$, and $(\Delta_{1/n})_n$ converges pointwise to Δ, the Banach–Steinhauss Theorem implies that Δ is a bounded operator. In addition, given $\xi \in \mathrm{dom}(k)$, we have for all $t \in \mathbb{R}$ that

$$e^{ith}\xi - \xi = e^{ith}\big(\xi - e^{-itk}\xi\big) + e^{ith}e^{-itk}\xi - \xi. \tag{8.1.2}$$

Since $(e^{ith})_t$ are uniformly bounded in norm and since $e^{ith}\xi \to \xi$, as $t \to 0$, for all $\xi \in H$, we have that

$$e^{ith}\frac{(\xi - e^{-itk}\xi)}{t} = e^{ith}ik\xi - e^{ith}\left(\frac{e^{-itk}\xi - \xi}{t} + itk\right) \underset{t\to 0}{\to} ik\xi.$$

So, dividing (8.1.2) by t throughout, and taking the limit of the above as $t \to 0$, we get

$$\frac{de^{ith}\xi}{dt}\Big|_{t=0} = ik\xi + \Delta\xi.$$

Based on [Ped89, Proposition 5.3.13], the conclusions are twofold: firstly, the existence of the derivative in the left-hand side implies that $\xi \in \mathrm{dom}(h)$, whence $\mathrm{dom}(k) \subseteq \mathrm{dom}(h)$, and, secondly, we deduce that

$$ih\xi = ik\xi + \Delta\xi.$$

Since the hypothesis is symmetric relative to h and k, we see that $\mathrm{dom}(k) = \mathrm{dom}(h)$, and moreover $h - k = i^{-1}\Delta$, on that common domain, proving that $h - k$ is bounded.

[18] That is, the map $t \mapsto e^{ith}e^{-itk}\xi$ is differentiable at zero for every ξ in H.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

(c)⇒(d). It suffices to take $D = \mathrm{dom}(k)$.

(d)⇒(e). Given $t_0, s \in \mathbb{R}$, observe that

$$\begin{aligned} e^{i(t_0+s)h}e^{-i(t_0+s)k} - e^{it_0h}e^{-it_0k} &= e^{it_0h}\big(e^{ish}e^{-isk} - 1\big)e^{-it_0k} \\ &= e^{it_0h}\Big(e^{ish}\big(e^{-isk} - 1\big) + e^{ish} - 1\Big)e^{-it_0k}. \end{aligned}$$

Dividing by s throughout, and taking the limit as $s \to 0$, we get,

$$\frac{de^{ith}e^{-itk}\xi}{dt}\Big|_{t=t_0} = ie^{it_0h}(h-k)e^{-it_0k}\xi,$$

for any $\xi \in D$. Incidentally, it might be worth observing that D is invariant under e^{-it_0k}, by hypothesis, or else the above expression might not make sense.

Letting M be the norm of the restriction of $h-k$ to D, we deduce from the intermediate value theorem that

$$\big\|e^{ith}e^{-itk}\xi - e^{ish}e^{-isk}\xi\big\| \leq M\,|t-s|\,\|\xi\|,$$

for every t and s in $\mathbb{R}$. As D is dense in H, we see that the above in fact holds for every ξ in H, from where the conclusion follows. □

The following result will be the basis for the proofs of Theorems 1.4.3, 1.4.4 and 1.4.6. Recall, a C*-algebra A represented in some $\mathcal{B}(H)$ is *irreducible* if $\{0\}$ and H are the only subspaces of H which are invariant with respect to all elements in A.

Proposition 8.1.3. *Let H be a separable Hilbert space and let $A \subseteq \mathcal{B}(H)$ be a unital C*-algebra. Given self-adjoint operators h and k on H, suppose that*

$$e^{ith}Ae^{-ith} = A, \quad \text{and} \quad e^{itk}Ae^{-itk} = A$$

for all $t \in \mathbb{R}$, and denote by σ_h and σ_k the associated pre-flows on A defined by h and k, respectively. Also consider the map

$$w_{h,k} : t \in \mathbb{R} \mapsto e^{ith}e^{-itk} \in \mathcal{B}(H).$$

Then the following statements hold:

(a) *If A is irreducible and σ_h is a cocycle perturbation of σ_k, then $w_{h,k}$ is norm continuous.*

(b) *If A is irreducible and σ_h is an inner perturbation of σ_k, then $\mathrm{dom}(h) = \mathrm{dom}(k)$ and $h-k$ is bounded, provided there is at least one nonzero vector[19] belonging to both $\mathrm{dom}(h)$ and $\mathrm{dom}(k)$.*

Assuming that $w_{h,k}(t)$ lies in A for all t in $\mathbb{R}$, we also have the following:

(c) *If $w_{h,k}$ is norm continuous, then σ_k is a cocycle perturbation of σ_h.*

(d) *If $w_{h,k}$ is differentiable at zero, then σ_k is an inner perturbation of σ_h.*

(e) *If H contains a dense subspace D, as in Lemma 8.1.1(d), in particular such that $h-k$ is bounded on D, then σ_k is a cocycle perturbation of σ_h.*

[19] Please see Proposition 8.1.7 for a special case in which this somewhat annoying hypothesis is not needed.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

(f) *If* $\mathrm{dom}(h) = \mathrm{dom}(k)$*, and* $h - k$ *is bounded on this common domain, then* σ_k *is a cocycle perturbation of* σ_h*.*

(g) *If* h *and* k *strongly commute*[20]*, and* H *contains a dense subspace* $D \subseteq \mathrm{dom}(h) \cap \mathrm{dom}(k)$ *such that* $h - k$ *is bounded on* D*, then* σ_k *is an inner perturbation of* σ_h*.*

Proof. (a) Let $(u_t)_{t\in\mathbb{R}}$ be a cocycle for σ_k such that

$$\sigma_{h,t} = \mathrm{Ad}(u_t) \circ \sigma_{k,t}$$

for all $t \in \mathbb{R}$. Unfolding definitions, this says that

$$e^{ith} a e^{-ith} = u_t e^{itk} a e^{-itk} u_t^*$$

for all $a \in A$, and all $t \in \mathbb{R}$. Rewriting the above as

$$a e^{-ith} u_t e^{itk} = e^{-ith} u_t e^{itk} a,$$

we see that

$$\lambda_t := e^{-ith} u_t e^{itk} \tag{8.1.4}$$

lies in the commutant of A. Since A is irreducible, we have that $A' = \mathbb{C} \cdot \mathrm{id}_H$, so λ_t is a scalar multiple of the identity operator, clearly with absolute value one. As the right-hand side of (8.1.4) is strongly continuous, $(\lambda_t)_{t\in\mathbb{R}}$ is continuous. In addition, since

$$w_{h,k} = e^{ith} e^{-itk} = \bar{\lambda}_t u_t$$

for all $t \in \mathbb{R}$, and since $(u_t)_{t\in\mathbb{R}}$ is norm continuous, we conclude that $w_{h,k}$ is norm continuous as desired.

(b) Let u_t be a cocycle as in (a), which we may now assume to be differentiable at zero.

We first observe that $u_0 = 1$, a fact that can easily be checked by plugging in $t = s = 0$ in the cocycle condition in Definition 1.4.2(1).

As we are under a special case of (a), we may re-run the argument given there up to (8.1.4) and then, picking any nonzero vector $\xi \in \mathrm{dom}(h) \cap \mathrm{dom}(k)$, we have that

$$\begin{aligned} \frac{d\lambda_t\xi}{dt}\Big|_{t=0} &= \lim_{t\to 0} \frac{e^{-ith} u_t e^{itk}\xi - \xi}{t} \\ &= \lim_{t\to 0} \left(e^{-ith} u_t \frac{e^{itk}\xi - \xi}{t} + e^{-ith} \frac{u_t\xi - \xi}{t} + \frac{e^{-ith}\xi - \xi}{t} \right) \\ &= \frac{du_t\xi}{dt}\Big|_{t=0} + i(k-h)\xi, \end{aligned} \tag{8.1.5}$$

[20] We shall say that two self-adjoint operators h and k strongly commute if there exists a projection-valued measure P on some measurable space Ω, and two measurable, real-valued functions f and g on Ω, such that $h = \int_\Omega f(\omega)\, dP(\omega)$, and $k = \int_\Omega g(\omega)\, dP(\omega)$. Very roughly, this means that h and k are "simultaneously diagonalizable". In particular, note that h and k necessarily admit a common core, namely $\bigcup_{n\in\mathbb{N}} P(U_n)H$, where $U_n = \{\omega \in \Omega : |f(\omega)| + |g(\omega)| \leq n\}$. There are various alternative characterizations of strong commutativity, such as requiring that e^{ith} commutes with e^{isk}, for every $t, s \in \mathbb{R}$, but nevertheless all such characterizations essentially express that there is an abelian von Neumann algebra to which both h and k are affiliated.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

from where we deduce that the map $t \in \mathbb{R} \mapsto \lambda_t \in \mathbb{C}$ is differentiable at zero. Since

$$w_{h,k} = e^{ith}e^{-itk} = \bar{\lambda}_t u_t,$$

as already observed, this shows that $w_{h,k}$ is differentiable at zero. So the conclusion follows from the implication (a)⇒(c) of Lemma 8.1.1.

(c) For each $t \in \mathbb{R}$, let $u_t = w_{h,k}(t)^* = e^{itk}e^{-ith}$. Then $u_t \in A$, for all $t \in \mathbb{R}$, and

$$\begin{aligned} u_t\sigma_{h,t}(u_s) &= e^{itk}e^{-ith}e^{ith}e^{isk}e^{-ish}e^{-ith} \\ &= e^{itk}e^{isk}e^{-ish}e^{-ith} \\ &= e^{i(t+s)k}e^{-i(t+s)h} \\ &= u_{t+s}. \end{aligned}$$

Since $(u_t)_{t\in\mathbb{R}}$ is norm continuous, it is a cocycle for σ_h. Hence, as $\sigma_{k,t} = u_t\sigma_{h,t}u_t^*$, for all $t \in \mathbb{R}$, σ_k is a cocycle perturbation of σ_h.

(d) Follows immediately, as above, under the extra assumption of differentiability of $w_{h,k}$ at zero.

(e) Thanks to the implication (d)⇒(e) of Lemma 8.1.1, we have that $w_{g,h}$ is continuous, so the conclusion follows from (c), above.

(f) This follows immediately from (e) upon taking $D = \mathrm{dom}(h) = \mathrm{dom}(k)$.

(g) Choose any real number $M > \|(h-k)\restriction_D\|$, and let Ω, P, f and g be as in Footnote 21. We then claim that $f-g$ is essentially bounded by M, in the sense that $P\big(\{\omega \in \Omega : |f(\omega)-g(\omega)| \geq M\}\big) = 0$. Assuming by contradiction that this is not so, and using the σ-additivity of P, we can find $N > 0$, such that $P(U) \neq 0$, where

$$U = \big\{\omega \in \Omega : |f(\omega)-g(\omega)| \geq M,\ |f(\omega)| \leq N, \text{ and } |g(\omega)| \leq N\big\}.$$

Choosing any unit vector in the range of $P(U)$, and taking a sequence of unit vectors $(\xi_n)_n$ in D converging to ξ, we get

$$\begin{aligned} M &\leq \left\| \int_\Omega f(\omega)-g(\omega)\, dP(\omega)\xi \right\| \\ &= \|h\xi - k\xi\| \\ &= \|hP_U\xi - kP_U\xi\| \\ &= \|P_Uh\xi - P_Uk\xi\| \\ &= \lim_{n\to\infty} \|P_Uh\xi_n - P_Uk\xi_n\| \\ &\leq \big\|(h-k)\restriction_D\big\|, \end{aligned}$$

where we have used that P_Uh and P_Uk are bounded operators (with norm at most N) in order to bring the above limit into play. This is a contradiction, hence proving that $f-g$ is essentially bounded. So $h-k$ is a bounded operator on its whole domain of definition and, since

$$w_{h,k} = e^{ith}e^{-itk} = e^{it(h-k)},$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

we deduce that $w_{h,k}$ is differentiable everywhere, so the conclusion finally follows from (d). □

Aiming at a proof of Theorem 1.4.7, let us now move to a slightly different situation, assuming, for the remainder of this section, that we are given a separable Hilbert space H and a unital C*-algebra $A \subseteq \mathcal{B}(H)$ containing the compacts. Also let h be a self-adjoint operator on H such that

$$e^{ith} A e^{-ith} = A$$

for all $t \in \mathbb{R}$, and denote by σ_h the associated pre-flow on A. Unlike in the previous situation, we will now assume that σ_h is actually strongly continuous, that is, a flow on A.

We shall denote by δ_h the associated infinitesimal generator of σ_h, given by

$$\delta_h(a) = \frac{d\sigma_{h,t}(a)}{dt}\Big|_{t=0},$$

whose domain of course consists of those elements a in A for which the derivative exists.

Our next result shows that membership to $\mathrm{dom}(\delta_h)$ is closely related to membership to $\mathrm{dom}(h)$.

Lemma 8.1.6. *For every nonzero vector ξ in H, a necessary and sufficient condition for ξ to belong to* $\mathrm{dom}(h)$ *is that the orthogonal projection onto the one-dimensional space spanned by ξ, henceforth denoted p_ξ, lies in* $\mathrm{dom}(\delta_h)$.

Proof. By [BR87, Proposition 3.2.55], we have that p_ξ belongs to $\mathrm{dom}(\delta_h)$ if and only if there exists a core D for h, such that $p_\xi(D) \subseteq D$, and $[h,p_\xi]$ is a bounded operator on D.

Thus, assuming that $p_\xi \in \mathrm{dom}(\delta_h)$, and choosing a core D, as above, observe that D cannot be contained in the kernel of p_ξ because D is dense in H. So $p_\xi(D)$ is a nontrivial vector space contained in D, whence $\xi \in D \subseteq \mathrm{dom}(h)$. Conversely, assuming that ξ lies in $\mathrm{dom}(h)$, and taking the largest possible core $D = \mathrm{dom}(h)$, it is evident that D is invariant under p_ξ, while, for every η in D, we have that

$$[h,p_\xi]\eta = hp_\xi\eta - p_\xi h\eta = h\big(\langle\eta,\xi\rangle\xi\big) - \langle h\eta,\xi\rangle\xi = \langle\eta,\xi\rangle h\xi - \langle\eta,h\xi\rangle\xi,$$

which is evidently a bounded operator on D as a function of the variable η. So p_ξ lies in $\mathrm{dom}(\delta_h)$ by the first paragraph in this proof. □

Lemma 8.1.7. *Let H be a separable Hilbert space and let $A \subseteq \mathcal{B}(H)$ be a unital* C*-*algebra containing the compacts. Also let h and k be self-adjoint operators on H inducing flows σ_h and σ_k on A, as above. If σ_k is an inner-perturbation of σ_h, then* $\mathrm{dom}(h) = \mathrm{dom}(k)$.

Proof. By definition there exists a cocycle $(u_t)_{t\in\mathbb{R}}$ for σ_h, which is differentiable at zero, and such that

$$\sigma_{k,t}(a) = u_t \sigma_{h,t}(a) u_t^*,$$

for every $t \in \mathbb{R}$, and every $a \in A$. Letting δ_h and δ_k be the corresponding infinitesimal generators, the fact that u_t is differentiable at zero clearly implies that $\mathrm{dom}(\delta_h) \subseteq$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

$\mathrm{dom}(\delta_k)$, so Lemma 8.1.6 implies that $\mathrm{dom}(h) \subseteq \mathrm{dom}(k)$. Multiplying the equation displayed above by u_t^* on the left, and by u_t on the right, and applying a similar reasoning, then gives the reverse inclusion $\mathrm{dom}(h) \supseteq \mathrm{dom}(k)$. □

8.2. Applications

We now apply the machinery developed above to prove our main results, already stated in the introduction, classifying coarse flows on uniform Roe algebras of u.l.f metric spaces with property A. Our results are twofold: (1) we show that all coarse flows are cocycle perturbations of diagonal flows (Theorem 1.4.3) and (2) we characterize when two given flows are cocycle equivalent to each other in terms of the coarse operators generating them (Theorem 1.4.7). For diagonal flows, we also characterize both inner and cocycle perturbation (Theorem 1.4.4).

Proof of Theorem 1.4.6. Since σ_h and σ_k are flows, Theorem 6.1.2 guarantees that e^{ith} and e^{itk} lie in $\mathrm{C}_u^*(X)$ for all $t \in \mathbb{R}$, so

$$w_{h,k}(t) = e^{ith}e^{-itk} \in \mathrm{C}_u^*(X)$$

for all $t \in \mathbb{R}$. As both h and k are coarse operators and $h-k$ is bounded, Corollary 4.1.17 implies that $\mathrm{dom}(h) = \mathrm{dom}(k)$. Therefore, Proposition 8.1.3(f) implies that σ_h and σ_k are cocycle perturbations of each other.

We can now prove Theorem 1.4.4, which summarizes our results for the commutative case, i.e., for diagonal flows on uniform Roe algebras.

Proof of Theorem 1.4.4. (1)⇒(2): It is immediate that the diagonal operators h and k, naturally associated to the given functions h and k, strongly commute. Moreover, given that the functions h and k are close, the difference operator $h-k$ is clearly bounded on $c_{00}(X)$. In order to be able to apply Proposition 8.1.3(g), and hence to finish the proof, we need to prove that $w_{h,k}(t)$ lies in $\mathrm{C}_u^*(X)$, for every $t \in \mathbb{R}$, but this is obvious once we observe that the bounded operators e^{ith} and e^{itk} have propagation zero.

The implication (2) ⇒(3) is obvious. For (3) ⇒(1) notice that, since σ_h and σ_k are cocycle perturbations of each other, then Proposition 8.1.3(a) implies that $w_{h,k}$ is norm continuous, and as we are in the commutative case, we have that

$$w_{h,k} = e^{ith}e^{-itk} = e^{it(h-k)}.$$

In addition, by [Rud91, Theorem 13.36], a unitary group is norm continuous if and only if its generator, presently $h-k$, is a bounded operator, which clearly means that h and k are close as maps from X to $\mathbb{R}$.

Proof of Theorem 1.4.3. The conclusion will be reached upon applying Proposition 8.1.3(e) to h and $E(h)$, but first we must provide a dense subspace D as required there. For this we observe that $D := c_{00}(X)$ is contained in the domain of h, because h is coarse, and hence admissible by Theorem 4.1.15. D is clearly also contained in the domain of the diagonal operator $E(h)$, and it is easy to see that the one-parameter unitary group generated by this operator leaves D invariant.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

Finally, observing that $h - E(h)$ is bounded on D by Theorem 7.1.12, we see that we are under the hypothesis of Proposition 8.1.3(e), from where the conclusion follows.

Proof of Theorem 1.4.7. (1)⇒(2): Suppose that $v \in \mathrm{C}^*_u(X)$ is a unitary element satisfying the conditions in Theorem 1.4.7(1). Letting $k' = vkv^*$, it is easy to see that

$$\sigma_{k'} = \mathrm{Ad}(v) \circ \sigma_k \circ \mathrm{Ad}(v^*),$$

so $\sigma_{k'}$ is clearly also a flow on $\mathrm{C}^*_u(X)$. By Theorem 1.4.5, we then deduce that

$$w_{h,k'}(t) := e^{ith}e^{-itk'} \in \mathrm{C}^*_u(X)$$

for all $t \in \mathbb{R}$. So Proposition 8.1.3(f) implies that $\sigma_{k'}$ is a *cocycle perturbation* of σ_h, whence σ_k is *cocycle conjugate* to σ_h.

(2)⇒(1): Suppose that σ_h and σ_k are cocycle conjugate to each other. By [Kis00, Lemma 1.1 and Corollary 1.3], this is equivalent to the existence of a unitary $v \in \mathrm{C}^*_u(X)$, and a family $(u_t)_{t\in\mathbb{R}}$ of unitaries in $\mathrm{C}^*_u(X)$ such that

1. $t \in \mathbb{R} \mapsto u_t \in \mathrm{C}^*_u(X)$ is differentiable everywhere,
2. $(u_t)_{t\in\mathbb{R}}$ is a cocycle for $\mathrm{Ad}(v) \circ \sigma_k \circ \mathrm{Ad}(v^*)$, and
3. $\sigma_{h,t}(a) = u_t v \sigma_{k,t}(v^*av)v^*u_t^*$ for all $t \in \mathbb{R}$ and all $a \in \mathrm{C}^*_u(X)$.

Since the spectral theorem for self-adjoint operators gives

$$e^{ivkv^*} = ve^{ik}v^*,$$

item (3) can be replaced by

3'. $\sigma_{h,t}(a) = u_t \sigma_{vkv^*,t}(a)u_t^*$ for all $t \in \mathbb{R}$ and all $a \in \mathrm{C}^*_u(X)$.

In other words, σ_h is an inner perturbation of σ_{vkv^*}. As $\mathrm{C}^*_u(X)$ contains the compacts, we may employ Lemma 8.1.7 to conclude that $\mathrm{dom}(h) = \mathrm{dom}(vkv^*)$, so the conclusion follows from Proposition 8.1.3(b).

9. Pre-flows given by SOT-summable projections

In this subsection, we identify a class of naturally occurring pre-flows on uniform Roe algebras which generalizes the diagonal pre-flows given by functions $X \to \mathbb{R}$ (Proposition 9.1.3). Our main result here provides an example of a pre-flow which is not a cocycle perturbation of a diagonal flow (Theorem 9.1.5), which goes in the opposite direction of Theorem 1.4.3.

Assumption 9.1.1. Let X be a u.l.f. metric space, $w\colon \mathbb{N} \to \mathbb{R}$ be a map, and $(p_n)_n$ be orthogonal projections on $\ell_2(X)$ such that

$$p_M = \text{SOT-}\sum_{n\in M} p_n \in \mathrm{C}^*_u(X)$$

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

for all $M \subseteq \mathbb{N}$. Let then, for each $n \in \mathbb{N}$,

$$h_n = w(n)p_n \quad \text{and} \quad h = \text{SOT-}\sum_n h_n.$$

So, h is a self-adjoint operator on $\ell_2(X)$.

Example 9.1.2. Any pre-flow σ given by a map $X \to \mathbb{R}$ is of the form σ_h for some self-adjoint operator on $\ell_2(X)$ defined as above. Indeed, consider an arbitrary function $h\colon X \to \mathbb{R}$. For each $x \in X$, let $p_x = p_{\{x\}}$ and $h'_x = h(x)p_x$. It is clear that

$$p_M = \text{SOT-}\sum_{x\in M} p_x = p_M \in \mathrm{C}^*_u(X)$$

for all $M \subseteq X$. Then, letting

$$h' = \text{SOT-}\sum_n h'_n,$$

it is immediate that, as self-adjoint operators on $\ell_2(X)$, $h = h'$. So, $\sigma_h = \sigma'_h$ as desired.

The previous example shows that a self-adjoint operator as in Assumption 9.1.1 can be seen as a sort of generalized function $X \to \mathbb{R}$. In the beginning of Section 3, we emphasize the (trivial) fact that if $h\colon X \to \mathbb{R}$ is an arbitrary map, then σ_h is a pre-flow. The next proposition shows that the same holds for arbitrary self-adjoint operators h as in Assumption 9.1.1.

Proposition 9.1.3. *In the setting of Assumption 9.1.1, σ_h defines a pre-flow on $\mathrm{C}^*_u(X)$.*

Proof. This is a simple consequence of the orthogonality of the sequence of projections $(p_n)_n$. Indeed, this implies

$$e^{ith} = \mathrm{id}_{\ell_2(X)} - p_{\mathbb{N}} + \text{SOT-}\sum_n e^{ith_n}p_n,$$

Moreover, since each h_n is a bounded operator, their exponentials can be easily computed and we have

$$e^{ith_n}p_n = \sum_{k=0}^{\infty} \frac{i^k t^k h_n^k}{k!} p_n = \sum_{k=0}^{\infty} \frac{i^k t^k w(n)^k p_n^k}{k!} p_n = p_n + (e^{itw(n)} - 1)p_n \tag{9.1.4}$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. It then follows that

$$e^{ith} = \mathrm{id}_{\ell_2(X)} + \text{SOT-}\sum_n (e^{itw(n)} - 1)p_n$$

for all $t \in \mathbb{R}$ and the hypothesis on the strong convergence of sums of $(p_n)_n$ implies that $e^{ith} \in \mathrm{C}^*_u(X)$ for all $t \in \mathbb{R}$. We conclude that $\sigma_{h,t}(\mathrm{C}^*_u(X)) \subseteq \mathrm{C}^*_u(X)$ for all $t \in \mathbb{R}$, i.e., σ_h is a pre-flow on $\mathrm{C}^*_u(X)$. □

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

The next result deals with expander graphs. We refer the reader to [NY12, Chapter 5.6] for precise definitions. In here, we simply say that if $(X_n)_n$ is a sequence of expander graphs and $X = \bigsqcup_n X_n$ is the coarse disjoint union of them, then

$$p_M = \text{SOT-}\sum_{n\in M} p_n \in \mathrm{C}^*_u(X)$$

for all $M \subseteq \mathbb{N}$, where for each $n \in \mathbb{N}$, p_n is the rank-one projection of $\ell_2(X)$ onto the vector subspace of $\ell_2(X_n) \subseteq \ell_2(X)$ of vectors with constant coordinates; i.e.,

$$p_n = \begin{bmatrix} \frac{1}{|X_n|} & \cdots & \frac{1}{|X_n|} \\ \vdots & \ddots & \vdots \\ \frac{1}{|X_n|} & \cdots & \frac{1}{|X_n|} \end{bmatrix}.$$

Theorem 9.1.5. *Let $X = \bigsqcup_n X_n$ be the coarse disjoint union of expander graphs and $w\colon \mathbb{N} \to \mathbb{R}$ be a map. For each $n \in \mathbb{N}$, let p_n be the averaging projection on $\mathcal{B}(\ell_2(X_n))$. Let $h_n = w(n)p_n$ for all $n \in \mathbb{N}$ and $h = \text{SOT-}\sum_n h_n$. The following holds.*

1. *σ_h is a pre-flow on $\mathrm{C}^*_u(X)$.*
2. *If w is unbounded, σ_h is not a flow.*
3. *If w is unbounded, the pre-flow σ_h is not a cocycle perturbation of any diagonal pre-flow on $\mathrm{C}^*_u(X)$.*

Proof. (1) Since $X = \bigsqcup_n X_n$ is the coarse disjoint union of expander graphs, we know that $\sum_{n\in M} p_n$ is in $\mathrm{C}^*_u(X)$ for all $M \subseteq \mathbb{N}$ (see [HLS02]). So, Proposition 9.1.3 guarantees that σ_h is a pre-flow on $\mathrm{C}^*_u(X)$.

(2) Suppose now that w is unbounded and let us notice that σ_h is not strongly continuous. For each $n \in \mathbb{N}$, pick $A_n \subseteq X_n$ such that $|A_n| = |X_n|/2$.[21] Then, letting $A = \bigsqcup_n A_n$, we have

$$\begin{aligned} \left\| e^{ith} p_A e^{-ith} - p_A \right\| &= \sup_n \left\| e^{ith_n} p_{A_n} e^{-ith_n} - p_{A_n} \right\| \\ &= \sup_n \left\| e^{ith_n} p_{A_n} - p_{A_n} e^{ith_n} \right\| \end{aligned}$$

for all $t \in \mathbb{R}$. Since, for each $n \in \mathbb{N}$,

$$e^{ith_n} p_{X_n} = p_{X_n} + (e^{itw(n)} - 1)p_n$$

(cf. (9.1.4)), it follows that

$$\left\| e^{ith_n} p_{A_n} - v p_{A_n} e^{ith_n} \right\| = |e^{itw(n)} - 1| \cdot \|p_n p_{A_n} - p_{A_n} p_n\|$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. In order to estimate the above, let $\mathbb{1}_{A_n}$ denote the $|A_n|$-by-$|X_n \backslash A_n|$ matrix with 1's in all its entries; so, $\mathbb{1}^*_{A_n}$ is the $|X_n \backslash A_n|$-by-$|A_n|$ matrix with 1's

[21] We are assuming here that each X_n has even cardinality. If this is now the case, the argument will follow analogously by taking $|A_n| = \lceil |X_n|/2 \rceil$ instead.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

in all its entries. Hence, writing each X_n in an order starting with elements in A_n and later in $X_n \setminus A_n$, we can write

$$(p_n v_{A_n} - v_{A_n} p_n) \restriction \ell_2(X_n) = \frac{1}{|X_n|} \begin{bmatrix} 0 & -\mathbb{1}_{A_n}^* \\ \mathbb{1}_{A_n} & 0 \end{bmatrix}.$$

As the operator above has norm $1/2$, this shows that

$$\left\| e^{ith_n} p_{A_n} - v p_{A_n} e^{ith_n} \right\| = \frac{1}{2} |e^{itw(n)} - 1|$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$.

The conclusion of the previous paragraph is that

$$\left\| e^{ith} p_A e^{-ith} - p_A \right\| = \frac{1}{2} \sup_n |e^{itw(n)} - 1|$$

for all $t \in \mathbb{R}$. As w is unbounded, regardless of how small $\delta > 0$ is, there is always $t \in \mathbb{R}$ with $|t| \leq \delta$ such that $|e^{itw(n)} - 1| = 2$. This shows that

$$t \in \mathbb{R} \mapsto \sigma_h(p_A) \in \mathrm{C}_u^*(X)$$

is not continuous; so, σ_h is not a flow.

(3) By Proposition 8.1.3(a) it is enough to show that if $k \colon X \to \mathbb{R}$ is a map, then $(e^{ith} e^{-itk})_{t \in \mathbb{R}}$ is not norm continuous. Fix such k and for each $n \in \mathbb{N}$ let $k_n = k \restriction X_n$. Then, letting $(A_n)_n$ and $\mathbb{1}_{A_n}$ be as above, we have

$$\begin{aligned} \left\| p_{A_n} (e^{ith_n} e^{-itg_n} - \mathrm{id}_{\ell_2(X_n)}) p_{X \setminus A_n} \right\| &= |e^{itw(n)} - 1| \cdot \left\| p_{A_n} p_n p_{X_n \setminus A_n} \right\| \\ &= \frac{|e^{itw(n)} - 1|}{|X_n|} \cdot \left\| \begin{bmatrix} 0 & 0 \\ \mathbb{1}_{A_n} & 0 \end{bmatrix} \right\| \\ &= \frac{|e^{itw(n)} - 1|}{2} \end{aligned}$$

for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$. Since,

$$\left\| e^{ith} e^{-itk} - \mathrm{id}_{\ell_2(X)} \right\| = \sup_n \left\| e^{ith_n} e^{-itk_n} - \mathrm{id}_{\ell_2(X_n)} \right\|$$

and w is unbounded, this shows that $(e^{ith} e^{-itk})_{t \in \mathbb{R}}$ does not converge to $\mathrm{id}_{\ell_2(X)}$ in norm as $t \to 0$. □

Acknowledgements. We would like to express our thanks to Stuart White for some very inspiring and visionary conversations during a visit of the third-named author to Oxford, when some conjectures we attempted to solve here were first sketched.

Competing interests. The authors declare none.

Data availability statement. No new data were created or analysed in this study; data sharing is not applicable.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

References

[BBF+22] Baudier F, Braga BM, Farah I, Khukhro A, Vignati A and Willett R (2022) Uniform Roe algebras of uniformly locally finite metric spaces are rigid. *Invent. Math.* **230**(3), 1071–1100.

[BBF+24] Baudier F, Braga BM, Farah I, Vignati A and Willett R (2024) Coarse equivalence versus bijective coarse equivalence of expander graphs, *Math. Z.* 307(3), Paper No. 44, 24 pp.

[BE25] Braga BM and Exel R (2025) KMS states on uniform Roe algebras, *Canadian Mathematical Communications*. 1, e1. doi: 10.4153/S2976859425100003.

[BF21] Braga BM and Farah I (2021) On the rigidity of uniform Roe algebras over uniformly locally finite coarse spaces. *Trans. Amer. Math. Soc.* **374**(2), 1007–1040.

[BFV21] Braga BM, Farah I and Vignati A (2021) Uniform Roe coronas. *Adv. Math.* **389**, Paper No. 107886, 35.

[BO08] Brown NP and Ozawa N (2008) *C*-algebras and Finite-dimensional Approximations, Graduate Studies in Mathematics*, vol. 88. Providence, RI: American Mathematical Society.

[Bou22] Bourne C (2022) Locally equivalent quasifree states and index theory. *J. Phys. A* **55**(10), Paper No. 104004, 38.

[BR87] Bratteli O and Robinson D (1987) *Operator Algebras and Quantum Statistical Mechanics. 1*, second ed., Texts and Monographs in Physics. New York: Springer-Verlag, C^*- and W^*-algebras, symmetry groups, decomposition of states. MR 887100.

[EM19] Ewert E and Meyer R (2019) Coarse geometry and topological phases. *Comm. Math. Phys.* **366**(3), 1069–1098.

[HLS02] Higson N, Lafforgue V and Skandalis G (2002) Counterexamples to the Baum–Connes conjecture, *Geom. Funct. Anal.* **12**(2), 330–354.

[Jon21] Jones C (2021) Remarks on anomalous symmetries of C*-algebras. *Comm. Math. Phys.* **388**(1), 385–417.

[Kis00] Kishimoto A (2000) Locally representable one-parameter automorphism groups of AF algebras and KMS states. *Rep. Math. Phys.* **45**(3), 333–356. MR 1772604.

[Kub17] Kubota Y (2017) Controlled topological phases and bulk-edge correspondence. *Comm. Math. Phys.* **349**(2), 493–525.

[LT21] Ludewig M and Thiang G (2021) Gaplessness of Landau Hamiltonians on hyperbolic half-planes via coarse geometry. *Comm. Math. Phys.* **386**(1), 87–106.

[LW20] Lorentz M and Willett R (2020) Bounded derivations on uniform Roe algebras. *Rocky Mt. J. Math.* **50**(5), 1747–1758.

[Mur90] Murphy G (1990) *C*-algebras and Operator Theory*. Boston, MA: Academic Press, Inc.

[MV24] Martínez D and Vigolo F (2024) *A rigidity framework for Roe-like algebras*, arXiv e-prints, arXiv:2403.13624.

[NY12] Nowak P and Yu G (2012) *Large Scale Geometry*, EMS Textbooks in Mathematics. Zürich: European Mathematical Society (EMS), MR 2986138.

[Oza23] Ozawa N (2023) *Embeddings of matrix algebras into uniform Roe algebras and quasi-local algebras*, arXiv e-prints, arXiv:2310.03677.

[Ped89] Pedersen GK (1989) *Analysis Now*, Graduate Texts in Mathematics, vol. 118. New York, NY: Springer-Verlag.

[Roe88] Roe J (1988) An index theorem on open manifolds. I, II. *J. Differential Geom.* **27**(1), 87–113, 115–136.

[Roe93] Roe J (1993) Coarse cohomology and index theory on complete Riemannian manifolds. *Mem. Amer. Math. Soc.* **104**(497), x+90.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press

[Roe03] Roe J (2003) *Lectures on Coarse Geometry*, University Lecture Series, vol. 31. Providence, RI: American Mathematical Society. viii+175 pp. MR2007488.

[Rud91] Rudin W (1991) *Functional Analysis*. McGraw-Hill Science, Engineering & Mathematics.

[ŠW13] Špakula J and Willett R (2013) On rigidity of Roe algebras. *Adv. Math.* **249**, 289–310.

[ŠZ20] Špakula J and Zhang J (2020) Quasi-locality and property A. *J. Funct. Anal.* **278**(1), 108299, 25.

[WW20] White S and Willett R (2020) Cartan subalgebras of uniform Roe algebras. *Group. Geom. Dyn.* **14**, 949–989.

[WY20] Willett R and Yu G (2020) *Higher Index Theory*, Cambridge Studies in Advanced Mathematics, vol. 189. Cambridge: Cambridge University Press, MR 4411373.

[Yu00] Yu G (2000) The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139**(1), 201–240.

https://doi.org/10.1017/S1474748025101473 Published online by Cambridge University Press