SELF-DUALITY AND $\mathscr{A}$-MANIFOLDS

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ABSTRACT

The paper gives new examples of $\mathscr{A}$-structures on certain $SO(3)$-principal fibre bundles.

0. *Introduction*

The class of $\mathscr{A}$-manifolds (for a definition see [**5**]) is an important class of Riemannian manifolds which has appeared in a natural way during the investigation of spaces with volume-preserving local geodesic symmetries (see [**1, 2, 9, 11**]). Until recently all the known examples of proper $\mathscr{A}$-manifolds were locally homogeneous. In [**7**] we explicitly constructed a family of locally non-homogeneous $\mathscr{A}$-manifolds on a circle bundle P over a Kähler–Einstein manifold M giving an answer to the problem stated in [**1**, p. 451, 16.56i]. This family is parameterised by a real number $c > 0$ and every manifold has a Ricci tensor with two constant eigenvalues. In the present paper we consider a related problem and give examples of locally non-homogeneous $\mathscr{A}$-manifolds with two constant eigenvalues of $SO(3)$-bundles over self-dual Einstein manifolds. We also characterise the Ricci tensor of the standard Riemannian structure on the principal $SO(4)$-bundle of frames over Einstein four-manifolds.

1. *Preliminaries*

Let (M, g) be an oriented Riemannian manifold, $\dim M = 4$ and $P = P(M, SO(4))$ be the principal fibre bundle of oriented orthonormal frames of TM. The bundle of 2-forms over M splits as the Whitney sum $\wedge^2 M = \wedge^+ M \oplus \wedge^- M$, $\wedge^+ M$, $\wedge^- M$ being the eigenspaces bundles of the Hodge star operator $* \in \mathrm{End}(\wedge^2 M)$. The Weyl conformal tensor $W \in \mathrm{End}(\wedge^2 M)$ leaves $\wedge^+ M$, $\wedge^- M$ invariant and the restrictions W^+, W^- of W to $\wedge^+ M$, $\wedge^- M$ may be considered as $(0,4)$ tensors operating trivially on $\wedge^- M$, $\wedge^+ M$ respectively. The oriented four-manifold (M, g) is called a self-dual (respectively anti-self-dual) Riemannian manifold if $W^- = 0$ (respectively $W^+ = 0\}$. by P^- we mean a principal $SO(3)$-bundle of oriented orthogonal bases $\{\omega, \eta, \gamma\} \subset \wedge^- M_x$, $x \in M$ of $\wedge^- M$ such that $g(\omega, \omega) = g(\eta, \eta) = g(\gamma, \gamma) = 2$. We have a homomorphism of principal fibre bundles $F: P \longrightarrow P^-$ defined by (see [**3**])

$$F(e_1, e_2, e_3, e_4) = (e_1 \wedge e_2 - e_3 \wedge e_4,\ e_1 \wedge e_3 - e_4 \wedge e_2,\ e_1 \wedge e_4 - e_2 \wedge e_3)$$

which is equivariant under the homomorphism $\Phi_-: SO(4) \longrightarrow SO(3)$ where $\Phi_\pm$ are defined by $\Phi_\pm(A) = \rho(A)_\pm$ ($C_\pm = C|_{\wedge^\pm \mathbb{R}^4}$) and $\rho: SO(4) \longrightarrow SO(\wedge^2 \mathbb{R}^4)$ is defined on

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simple 2-vectors by $\rho(A)(e_i \wedge e_j) = Ae_i \wedge Ae_j$. Since $\rho(A)* = *\rho(A)$, it is clear that $\rho(A)$ leaves invariant $\wedge^{\pm}\mathbb{R}^4$ and $\rho(A)_{\pm}$ are the corresponding endomorphisms of $\wedge^{\pm}$ respectively. The homomorphisms $\Phi_{\pm}$ induce the Lie algebras homomorphisms $\phi_{\pm}$: $\mathfrak{so}(4) \longrightarrow \mathfrak{so}(3)$ and $\mathfrak{so}(4) = \ker\phi_+ \oplus \ker\phi_-$ is the simple sum of ideals both isomorphic to $\mathfrak{so}(3)$. If we identify the Lie algebra $\mathfrak{so}(4)$ with $\wedge^2\mathbb{R}^4$ then $\ker\phi_- = \wedge^+\mathbb{R}^4$. Note that

$$\wedge^2\mathbb{R}^4 = \wedge^+\mathbb{R}^4 \oplus \wedge^-\mathbb{R}^4. \tag{1.1}$$

On $\wedge^2\mathbb{R}^4$ we have the standard metric given by $\langle e_i \wedge e_j, e_k \wedge e_s\rangle = \delta^i_k\delta^j_s$ for $i<j$, $k<s$ where by $\{e_1, e_2, e_3, e_4\}$ we denote the standard orthonormal basis of $\mathbb{R}^4$. If we furnish $\mathfrak{so}(4)$ with the metric $\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(X \circ Y)$ then the identification $\mathfrak{so}(4) = \wedge^2\mathbb{R}^4$ is an isometry. We shall denote by v_+, v_- or v^+, v^- the components of v with respect to decomposition (1.1). By $\mathscr{X}(M)$ we denote the Lie algebra of all local vector fields on M. If D is a vector bundle over M then by $\Gamma(D)$ we denote the set of all local sections of D. We also write $\mathscr{A}^k(M) = \Gamma(\wedge^k TM^*)$.

Let $(P, \bar{g}), (M, g)$ be Riemannian spaces. A Riemannian submersion $p: P \longrightarrow M$ (see [**9**]) is a submersion which preserves lengths of horizontal vectors. We shall denote by V the distribution of vertical vectors ($X \in V$ if X is tangent to a fibre $p^{-1}(x)$ for a certain $x \in M$) and by H the horizontal distribution which is an orthogonal complement of V. Let $\mathscr{H}$ and $\mathscr{V}$ denote the projections of the tangent space TM of M onto the subspaces H, V of horizontal and vertical vectors, respectively. By T and A we denote the O'Neill's tensors defined as follows (see [**9**]):

$$T_X Y = \mathscr{H}(\nabla_{\mathscr{V}X}\mathscr{V}Y) + \mathscr{V}(\nabla_{\mathscr{V}X}\mathscr{H}Y)$$
$$A_X Y = \mathscr{V}(\nabla_{\mathscr{H}X}\mathscr{H}Y) + \mathscr{H}(\nabla_{\mathscr{H}X}\mathscr{V}Y).$$

If $\overline{\nabla}$ is a Levi–Civita connection of (M, g) and $X, Y \in \mathscr{X}(M)$ then

$$(\overline{\nabla}_X Y)^* + \tfrac{1}{2}\mathscr{V}[X^*, Y^*] = \nabla_{X^*} Y^* \tag{1.2}$$

where by X^* we denote the horizontal lift of X. Finally, let us recall that a Riemannian manifold (M, g) is called an $\mathscr{A}$-manifold (see [**5**]) if its Ricci tensor ρ satisfies a condition $\nabla_X\rho(X, X) = 0$ for all $X \in TM$. An $\mathscr{A}$-manifold M is called proper if $\nabla\rho \neq 0$ and M is irreducible.

2. *$\mathscr{A}$-tensors with two constant eigenvalues*

By an $\mathscr{A}$-tensor on a Riemannian space (M, g) with Levi–Civita connection ∇ we mean a $(1,1)$ tensor S defined on M which satisfies the following conditions:

(1) $g(SX, Y) = g(X, SY)$ for every $X, Y \in \mathscr{X}(M)$.

(2) $g(\nabla S(X, X), X) = 0$ for every $X \in \mathscr{X}(M)$.

In [**7**] we proved the following three propositions.

PROPOSITION 1. *Let (M, g) be a Riemannian manifold and $S \in \mathrm{End}(TM)$ be a self-adjoint tensor ($g(SX, Y) = g(SY, X)$) with two constant eigenvalues $\lambda, \mu \in \mathbb{R}$. Then S is an $\mathscr{A}$-tensor if and only if*

$$\nabla S(X, X) = 0$$

for all $X \in D_\lambda = \ker(S - \lambda I)$ and all $X \in D_\mu = \ker(S - \mu I)$.

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PROPOSITION 2. *Let S be as above and assume that D_λ is integrable. Then D_λ is autoparallel.*

As a consequence we get (see [7, Theorem 2.1]) the following proposition.

PROPOSITION 3. *An $\mathscr{A}$-tensor with two constant eigenvalues λ, μ is parallel (that is, $\nabla S = 0$) if and only if both distributions D_λ, D_μ are integrable.*

Proof. Note that $TM = D_\lambda \oplus D_\mu$ and $D_\lambda \perp D_\mu$. Hence if both distributions D_λ, D_μ are autoparallel they are parallel, which means that $\nabla S = 0$.

Now we prove the following fact which will be useful in the sequel.

PROPOSITION 4. *Let (M, g) be a Riemannian space and $G \subset \mathrm{Iso}(M)$ be a Lie group of isometries of M such that the dimension of the orbit passing through x, $\mathrm{orb}_G(x)$, of the action of G is the same for all $x \in M$. Let $\mathfrak{g}$ be a Lie algebra of the group G and if $A \in \mathfrak{g}$ then by $A^+ \in \mathscr{X}(M)$ let us denote the Killing vector field generated by A. Let us define for two different real numbers $\lambda, \mu \in \mathbb{R}$ a tensor $S \in \mathrm{End}(TM)$ by*

$$SA^+ = \lambda A^+ \text{ if } A \in \mathfrak{g}, \quad SX = \mu X \text{ if } X \in H$$

where by H we denote the distribution $H = \{H \in TM : X \perp \mathrm{orb}_G(\pi(X))\}$ where $\pi : TM \longrightarrow M$ is the standard projection. Then $S \in \mathscr{A}$ if and only if the orbits of the action of G are totally geodesic submanifolds of M. The tensor S is parallel if and only if the distribution H is integrable.

Proof. Let A^+ be a Killing vector field corresponding to $A \in \mathfrak{g}$. Let $x \in M$ be an arbitrary point of M and let us write $N = \mathrm{orb}_G(x)$. By α we denote the second fundamental form of a submanifold N. If $\overline{\nabla}$ is the induced Levi–Civita connection of N and ∇ is the Levi–Civita connection of (M, g) then

$$\nabla_X Y = \overline{\nabla}_X Y + \alpha(X, Y)$$

for all $X, Y \in \mathscr{X}(N)$. Hence we obtain

$$\begin{aligned} \nabla S(A^+, A^+)_x &= (\lambda I - S)(\nabla_{A^+} A^+)_x \\ &= (\lambda I - S)(\overline{\nabla}_{A^+} A^+)_x + (\lambda I - S)(\alpha(A^+, A^+)_x) \\ &= (\mu - \lambda)(\alpha(A^+, A^+)_x). \end{aligned}$$

Thus $\nabla S(A^+, A^+) = 0$ on N if and only if $\alpha = 0$. Let us assume now that $X \in \Gamma(H)$ which means that

$$g(X, A^+) = 0 \tag{2.1}$$

for all $A \in \mathfrak{g}$. Since A^+ is a Killing vector field we have

$$g(X, \nabla_X A^+) = 0. \tag{2.2}$$

From (2.1) and (2.2) it follows that $g(\nabla_X X, A^+) = 0$ for all $X \in \Gamma(H)$ and all $A \in \mathfrak{g}$, which means that $\nabla_X X \in \Gamma(H)$ if $X \in \Gamma(H)$. Consequently for all $X \in \Gamma(H)$ we get $\nabla S(X, X) = (\mu I - S)(\nabla_X X) = 0$ and our proposition follows from Proposition 1.

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3. *The structure of an $\mathscr{A}$-manifold on a principal $SO(3)$ bundle over a self-dual Einstein four-manifold*

Let $P = P(M, G)$ be a principal fibre bundle over a Riemannian manifold (M, g) with compact, connected, semi-simple structural Lie group G. Let us denote by $\mathfrak{g}$ a Lie algebra of G. Suppose that $\omega \in \mathscr{A}^1(P) \otimes \mathfrak{g}$ is a connection form on P. Let us define a metric g_c on P by the formula

$$g_c = c^2 B(\omega(X), \omega(Y)) + p^* g(X, Y) \tag{3.1}$$

where $-B(X, Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$ is the Killing form of $\mathfrak{g}$.

PROPOSITION 5. *The group G is a group of isometries of (P, g_c). The orbits of the action of G on P are totally geodesic submanifolds of P isometric to the group G with biinvariant metric given by $c^2 B$.*

Proof. Let us note that

$$B(\operatorname{ad}_a X, \operatorname{ad}_a Y) = B(X, Y) \tag{3.2}$$

for all $a \in G$. Hence $R_a^* g_c = g_c$ where by R_a we denote the action of $a \in G$ on P. It follows that G acts on (P, g_c) by isometries. For every $A \in \mathfrak{g}$ denote by A^+ the Killing vector field generated by a one-parameter group of transformations R_{a_t}, where a_t is the one-parameter group of A. We have

$$g_c(A^+, A^+) = c^2 B(A, A). \tag{3.3}$$

From (3.3) it follows that A^+ is the Killing vector field of constant length. Hence for every $p \in P$ a curve $t \to R_{a_t}(p)$ is a geodesic. Thus the orbits of G are totally geodesic submanifolds of P. It is also clear that the map

$$G \ni a \longrightarrow R_a(p) \in \operatorname{orb}_G(p)$$

is an isometry of $(G, c^2 B)$ on $\operatorname{orb}_G(p)$ with the induced submanifold's metric.

Let us recall that a Riemannian manifold (M, g) with a curvature tensor R is said to have a harmonic curvature if $\delta R = 0$ (or equivalently if the Ricci endomorphism $S \in \operatorname{End}(TM)$ is a Codazzi tensor which means that $\nabla S(X, Y) = \nabla S(Y, X)$ for all $X, Y \in \mathscr{X}(M)$). Now let P be the principal $SO(n)$ fibre bundle of oriented frames over (M, g). Let us assume that ω is a connection form associated with the Levi–Civita connection $\overline{\nabla}$ of the metric g. Define a metric g_c on P by (3.1). By $H = \{X \in TP: \omega(X) = 0\}$ we denote the horizontal and by V the vertical distribution of P. By $R(\bar{R})$ we shall denote the curvature tensor of (P, g_c) $((M, g))$, respectively. Then we have the following proposition.

PROPOSITION 6. *The Ricci endomorphism of (P, g_c) leaves invariant the horizontal and vertical distributions of P if and only if (M, g) has harmonic curvature.*

Proof. We shall show that $\rho(\xi, X) = 0$ for every vertical field $\xi \in \Gamma(V)$ and every horizontal field $X \in \Gamma(H)$. From Proposition 5 it follows that the O'Neill [**10**] tensor $T = 0$. If $X, Y \in \Gamma(H)$ are horizontal vector fields on P then $A_X Y$ is a vertical vector field such that (see [**10**])

$$\omega(A_X Y) = -\tfrac{1}{2}\Omega(X, Y). \tag{3.4}$$

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Let $\{E_1, \dots, E_m, B_1, \dots, B_n\}$ be an orthonormal frame on P where $B_i = B(e_i)$ are the fundamental horizontal fields (see [**8**]) and E_i are vertical vector fields. Then from Proposition 5 it follows that

$$\rho(\xi, X) = \sum_{i=1}^{n} R(B_i, \xi, X, B_i). \tag{3.5}$$

The O'Neill formulae give us

$$g_c(R(B_i, X)\, B_i, \xi) = g_c(\nabla_{B_i} A(X, B_i), \xi) \tag{3.6}$$

where by ∇ we denote the Levi–Civita connection of the metric g_c. Let us assume that $\xi = A^+$ where $A \in \mathfrak{so}(n)$. From (3.6) we get

$$\begin{aligned} R(B_i, \xi, X, B_i) = {} & B_i\, g_c(A(X, B_i), \xi) - g_c(A(\nabla_{B_i} X, B_i), \xi) \\ & - g_c(A(X, \nabla_{B_i} B_i), \xi) - g_c(A(X, B_i), \nabla_{B_i} \xi). \end{aligned} \tag{3.7}$$

Note that $\nabla_{B_i} \xi = \nabla_{\xi} B_i + [B_i, \xi]$ and $[B_i, A^+] = -B(Ae_i) \in H$. Hence $\mathscr{V}(\nabla_{B_i} \xi) = \mathscr{V}(\nabla_{\xi} B_i) = T(\xi, B_i) = 0$. It is easy to verify using the Koszul formula that (note that $[B(\tau), B(\eta)]$ is vertical and $[A^+, B(\eta)] = B(A\eta)$ for $A \in \mathfrak{so}(n)$ and $\eta, \tau \in \mathbb{R}^n$, see [**8**])

$$\nabla_{B(\tau)} B(\xi) = \tfrac{1}{2}[B(\tau), B(\xi)]. \tag{3.8}$$

We can assume that $X = B$ is a fundamental horizontal vector field. Hence (3.4), (3.7) and (3.8) imply that

$$R(B_i, B, \xi, B_i) = \tfrac{1}{2}c^2\, B_i \langle \Omega(B_i, B), \omega(\xi) \rangle = \tfrac{1}{2}c^2 \langle B_i\, \Omega(B_i, B), \omega(\xi) \rangle$$

and consequently

$$\rho(\xi, B) = \tfrac{1}{2}c^2 \left\langle \sum_{i=1}^{n} B_i\, \Omega(B_i, B), \omega(\xi) \right\rangle. \tag{3.9}$$

Let us recall that for $u = (\bar{e}_1, \dots, \bar{e}_n) \subset T_x M$ we have a formula

$$\overline{\nabla}_{\bar{e}_i} \bar{R}(\bar{e}_i, dp(B))\, Z = u((B_i\, \Omega(B_i, B))\, (u^{-1} Z)). \tag{3.10}$$

Note that Ω is a horizontal form; hence in view of (3.8) we get

$$B_i\, \Omega(B_i, B) = \nabla_{B_i} \Omega(B_i, B).$$

Hence taking account of (3.10) we obtain

$$\overline{\nabla}_X \bar{R}(Y, Z)\, W = u((\nabla_{X^*} \Omega(Y^*, Z^*))\, (u^{-1} W)) \tag{3.11}$$

where $*$ denotes the horizontal lift to (P, g_c). Thus $\rho(\xi, B) = 0$ if and only if $\sum_i \overline{\nabla}_{\bar{e}_i} \bar{R}(\bar{e}_i, \cdot) = 0$ which is equivalent to $\delta \bar{R} = 0$.

Now we use an elementary lemma.

LEMMA 1. *Let S be a self-adjoint endomorphism of a Euclidean vector space (V, g) and $V = V_1 \oplus V_2$ where $V_1 \perp V_2$. Then S preserves subspaces V_i, $i = 1, 2$ if and only if $g(Su, v) = 0$ for all $u \in V_1$ and all $v \in V_2$.*

From Lemma 1 and the equality $\rho(\xi, X) = 0$ for all $\xi \in \Gamma(V)$ and all $X \in \Gamma(H)$ we infer that $SV \subset V$ and $SH \subset H$. This proves our proposition.

Let us assume that (M, g) is an Einstein four-manifold. Then $*R = R*$ and $R = R_+ + R_-$ where we treat a curvature tensor R of (M, g) as an endomorphism of the

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vector bundle $\wedge^2 TM = \wedge M$ and $R_\pm \in \mathrm{End}(\wedge^\pm M)$. The operators $R_\pm$ are self-adjoint with respect to the metric g on $\wedge M$. By $a_i; b_i$ $i = 1,2,3$ we denote the eigenfunction of R_+ and R_- respectively. Let us consider an isomorphism of vector spaces defined by

$$\alpha: \mathfrak{so}(4) \longrightarrow \wedge^2 \mathbb{R}^4, \quad \alpha(A) = \sum_{i<j} A^i_j e_i \wedge e_j$$

where e_i is the standard orthonormal basis of $\mathbb{R}^4$. On $\mathfrak{so}(4)$ we consider a biinvariant metric $\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}\, X \circ Y = \sum_{i<j} X^i_j Y^i_j$. Then α is an isometry of $(\mathfrak{so}(4), \langle,\rangle)$ on $\wedge^2\mathbb{R}^4$ furnished with the standard metric also denoted by $\langle,\rangle$ defined on the basis $e_i \wedge e_j$, $i<j$ by $\langle e_i \wedge e_j, e_p \wedge e_q\rangle = \delta^i_p \delta^j_q$. Note that $\ker \phi_- = \mathfrak{so}(4)_+ = \{X \in \mathfrak{so}(4): \alpha(X) \in \wedge^+\mathbb{R}^4\}$ and $\ker \phi_+ = \mathfrak{so}_-(4) = \{X \in \mathfrak{so}(4): \alpha(X) \in \wedge^- \mathbb{R}^4\}$.

PROPOSITION 7. *Let (M, g) be an oriented Einstein four-manifold and $p: P \longrightarrow M$ $(P = P(M, SO(4)))$ be the principle bundle of oriented frames over M. Let $\omega \in \mathscr{A}^1(P) \otimes \mathfrak{so}(4)$ be the connection form of the Levi–Civita connection of (M, g). If we define on P a metric g_c by the formula*

$$g_c(X, Y) = -\tfrac{1}{2}c^2\,\mathrm{tr}(\omega(X) \circ (\omega(Y)) + p^* g \tag{3.12}$$

then the Ricci tensor of (P, g_c) preserves the horizontal and vertical distributions of P and the Ricci tensor ρ of P has the following eigenfunctions: $\frac{1}{2}c^2 a_i^2 + c^{-2}$, $\frac{1}{2}c^2 b_i^2 + c^{-2}$, $\frac{1}{4}\tau_* - \frac{5}{32}c^2 \|R\|^2$.

Proof. We shall use the results from [**10**]. Since the fibres of a Riemannian submersion $p: P \longrightarrow M$ are totally geodesic (see Proposition 5) the O'Neill tensor $T = 0$. Since we have a decomposition $\wedge^2 TM = \wedge^+ M \oplus \wedge^- M$ every element v of $\wedge^2 TM$ can be represented as $v = v^+ + v^-$ where $v^\pm \in \wedge^\pm M$. If $X, Y \in \Gamma(H)$ are horizontal vector fields then

$$\omega(A(X, Y)) = -\tfrac{1}{2}\Omega(X, Y)$$

where Ω is the curvature form of the connection Γ defined by ω. Hence (we follow [**10**]) for a vertical vector field $V \in \Gamma(V)$ we have

$$\begin{aligned} g_c(A(X, V), Y) &= -g_c(V, A(X, Y)) \\ &= \tfrac{1}{2}c^2\langle \omega(V), \Omega(X, Y)\rangle \\ &= \tfrac{1}{2}c^{2+}(V), \Omega^+(X, Y)\rangle + \tfrac{1}{2}c^2\langle \omega^-(V), \Omega^-(X, Y)\rangle. \end{aligned}$$

Let us recall here that for $U, V \in \mathfrak{so}(4)$ we denote $\langle U, V\rangle = -\frac{1}{2}\mathrm{tr}\, U \circ V$. For a frame $u = (f_1, f_2, f_3, f_4) \in P$ we define a map

$$\alpha_u: \mathfrak{so}(4) \longrightarrow \wedge^2 T_{p(u)} M$$

by $\alpha_u(A) = u(\alpha(A)) = \sum_{i<j} A^i_j f_i \wedge f_j$. Then for $A, B \in \mathfrak{so}(4)$ we have $\langle A, B\rangle = g(\alpha_u(A), \alpha_u(B))$. Consequently we get (we write $X_* = p(X)$, $Y_* = p(Y)$)

$$\begin{aligned} g_c(A(X, V), Y)_u &= \tfrac{1}{2}c^2 g(R(\alpha_u(\omega(V))), X_* \wedge Y_*) \\ &= \tfrac{1}{2}c^2 g(R_-(\alpha_u(\omega(V))^-), X_* \wedge Y_*) + \tfrac{1}{2}c^2 g(R_+(\alpha_u(\omega(V))^+), X_* \wedge Y_*). \end{aligned}$$

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Note that $SO(4)$ with the metric $-\frac{1}{2}\mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ is locally isometric to $SO(3)\times SO(3) = \mathbb{RP}^3 \times \mathbb{RP}^3$ where $\mathbb{RP}^3$ has its standard metric of constant curvature $\frac{1}{2}$. Let $u \in P$ be any point of P and $A_i \in \mathfrak{so}(4)_+$, $C_i \in \mathfrak{so}(4)_-$, $i = 1,2,3$ be orthonormal bases of $\wedge^{\pm}$ respectively such that $R_+ \alpha_u(A_i) = a_i \alpha_u(A_i)$ and $R_- \alpha_u(C_i) = b_i \alpha_u(C_i)$. We can assume that

$$A_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4),\quad A_2 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3 + e_1 \wedge e_4),$$

$$A_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_4 \wedge e_2),\quad C_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4),$$

$$C_2 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3 - e_1 \wedge e_4),\quad C_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_4 \wedge e_2)$$

for a certain orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$. By $\{Y_1, Y_2, Y_3, Y_4\}$ we denote an orthonormal basis of $T_{p(u)}M$. Let X, Y be horizontal vector fields on P such that $X \perp Y$ and $\|X\| = \|Y\| = 1$ and write $X_* = p(X)$, $Y_* = p(Y)$. Note that $[A_i, A_j] = \pm\sqrt{2}A_k$ if $i,j,k \in \{1,2,3\}$ are different numbers and $g_c(A_i^+, A_i^+) = g_c(C_i^+, C_i^+) = c^2$. We have

$$\begin{aligned}\|A_X A_i^+\|^2 &= \sum_j g_c(A_X A_i, Y_j^*)^2\\ &= \tfrac{1}{4}c^4 \sum_j g(R\alpha_u(A_i), X_* \wedge Y_j)^2\\ &= \tfrac{1}{4}c^4 \sum_j g(R_+\alpha_u(A_i), X_* \wedge Y_j)^2\\ &= \tfrac{1}{4}c^4 a_i^2 \sum_j g(\alpha_u(A_i), (X_* \wedge Y_j)_+)^2.\end{aligned}$$

It is clear that if $\|X \wedge Y\| = 1$ then $\|X \wedge Y^{\pm}\| = 1/\sqrt{2}$. We can assume that $X_* = Y_1$. Note that the set $\{Y_1 \wedge Y_2^+, Y_1 \wedge Y_3^+, Y_1 \wedge Y_4^+\}$ is the basis of the space $\wedge^+ T_{p(u)}M$. Thus we obtain

$$\frac{\|A_X A_i^+\|^2}{\|A_i^+\|^2\,\|X\|^2} = \tfrac{1}{8}c^2 a_i^2.$$

We also have

$$\|A_X Y\|^2 = \tfrac{1}{4}c^2\|\Omega(X,Y)\|^2 = \tfrac{1}{4}c^2\|R(X_*, Y_*)\|^2.$$

Since $w_u(A_i^+) = A_i$ and $\omega_u(C_i^+) = C_i$ we obtain from formulae in [**10**]

$$K(P_{A_i^+ A_j^+}) = K(P_{C_i^+ C_j^+}) = \frac{1}{2c^2}$$

$$K(P_{X A_i^+}) = \tfrac{1}{8}a_i^2 c^2$$

$$K(P_{X C_i^+}) = \tfrac{1}{8}b_i^2 c^2$$

$$K(P_{XY}) = K_*(P_{X_* Y_*}) - \tfrac{3}{4}c^2\|R(X_*, Y_*)\|^2.$$

Hence we get

$$\rho(A_i^+, A_i^+) = c^{-2} + 4\tfrac{1}{8}a_i^2 c^2 = c^{-2} + \tfrac{1}{2}a_i^2 c^2.$$

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Analogously

$$\rho(C_i^+, C_i^+) = c^{-2} + \tfrac{1}{2} b_i^2 c^2.$$

Let us assume that X is a horizontal vector such that $\|X\| = 1$. Since $\|R\|^2 = 4(\sum_i a_i^2 + \sum_i b_i^2)$ we obtain

$$\sum \|R(X_*, Y_j)\|^2 = \tfrac{1}{4}\|R\|^2. \tag{3.13}$$

Note that one can also obtain formula (3.13) using the Singer–Thorpe basis (see, for example, [**11**, p. 218]). Hence

$$\rho(X, X) = \rho_*(X_*, X_*) - \tfrac{3}{4}c^2 \frac{\|R\|^2}{4} + \tfrac{1}{8}c^2 (\sum a_i^2 + \sum b_i^2).$$

Thus we get

$$\rho(X, X) = \tfrac{1}{4}\tau_* - \tfrac{3}{16}c^2 \|R\|^2 + \tfrac{1}{32}c^2 \|R\|^2 = \tfrac{1}{4}\tau_* - \tfrac{5}{32}c^2 \|R\|^2.$$

Let us note that if vectors X, Y are horizontal and W, V are vertical then

$$R(X, V, W, Y) = g_c(\nabla_V A(X, Y), W) + g_c(A_X V, A_Y W).$$

It follows that $R(X, V, W, X) = g_c(A_X V, A_X W)$. Let us denote by B_i the fundamental horizontal vector fields. Then we obtain

$$\begin{aligned}
\sum_i R(B_i, V, W, B_i) &= \sum_i g_c(A(B_i, V), A(B_i, W)) \\
&= \sum_{i,j} g_c(A(B_i, V), B_j)\, g_c(A(B_i, V), B_j) \\
&= \tfrac{1}{4}c^4 \sum_{i,j} \langle \Omega(B_i, B_j), \omega(V) \rangle \langle \Omega(B_i, B_j), \omega(W) \rangle \\
&= \tfrac{1}{4}c^4 \sum_{i,j} R(p(B_i) \wedge p(B_j), \alpha_u(\omega(V)))\, R(p(B_i) \wedge p(B_j), \alpha_u(\omega(W))).
\end{aligned} \tag{3.14}$$

Let us assume that $R\alpha_u(\omega(V)) = k_V \alpha_u(\omega(V))$ and $R\alpha_u(\omega(W)) = k_W \alpha_u(\omega(W))$. Then we infer from (3.14) that

$$\begin{aligned}
\sum_i R(B_i, V, W, B_i) &= \frac{c^4}{4} \sum_{i,j} k_V k_W\, g(B_i \wedge B_j, \alpha_u(\omega(V)))\, g(B_i \wedge B_j, \alpha_u(\omega(W))) \\
&= \frac{c^4}{2} k_V k_W\, g_c(\alpha_u \omega(V), \alpha_u \omega(W)) = \frac{c^2}{2} k_V k_W\, g_c(V, W).
\end{aligned}$$

As the mapping $\Phi = (\Phi^+, \Phi^-) : SO(4) \longrightarrow SO(3) \times SO(3)$ is covered by a map $\bar{\Phi}$: $\mathrm{Spin}(4) \longrightarrow S^3 \times S^3$ and the fibres are totally geodesic it is clear that the Ricci tensor $\bar{\rho}$ of the fibres is given by

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$$\bar{\rho}(U, V) = \frac{1}{c^2}(\langle U^+, V^+\rangle + \langle U^-, V^-\rangle)$$

where $U, V \in \mathfrak{so}(4)$ are left-invariant vector fields on $SO(4)$. Consequently $\rho(W, V) = 0$ if $W \perp V$ and $\alpha_u(\omega(V)), \alpha_u(\omega(W))$ are eigenvectors of R. It follows that $(A_i^+)_u, (C_i^+)_u$ are eigenvectors of ρ_u which completes the proof.

COROLLARY 1. *The principal bundle of oriented orthonormal frames over a compact Einstein four-manifold with positive scalar curvature admits a* (*standard*) *Riemannian structure with positive Ricci curvature.*

Proof. Let us take

$$0 < c < \inf \frac{\overline{8\tau_*}}{5\|R\|^2}.$$

PROPOSITION 8. *Let an oriented Riemannian four-manifold* (M, g) *be an Einstein self-dual space with nonzero scalar curvature* τ_*. *Then the principal fibre bundle* $p: P^- \longrightarrow M$ *of oriented normed and orthogonal bases of the vector bundle* $\wedge^- M$ *of anti-self-dual forms over* M *admits a one-parameter family* P_c^- *of proper* $\mathscr{A}$*-structures on* P^- *such that* $p: P_c^- \longrightarrow M$ *is a Riemannian submersion with fibres homothetic to* $\mathbb{RP}^3 = SO(3)$ *with the standard metric of constant curvature.*

Proof. Let $F: P \longrightarrow P^-$ be a homomorphism of principal fibre bundles defined by (see [**3**])

$$F(e_1, e_2, e_3, e_4) = (e_1 \wedge e_2 - e_3 \wedge e_4,\ e_1 \wedge e_3 - e_4 \wedge e_2,\ e_1 \wedge e_4 - e_2 \wedge e_3).$$

F is equivariant under a homomorphism $\Phi_-: SO(4) \longrightarrow SO(3)$ of Lie groups. The connection Γ on P given by ω induces a connection Γ^- on P^- with a connection from ω^- defined by $F^*\omega^- = \phi_- \omega$. Note that $B_{\mathfrak{so}(4)}(X^-, Y^-) = B_{\mathfrak{so}(3)}(\phi_- X, \phi_- Y)$ where $B_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$. We shall identify the Lie algebra of the structural group of P^- with $\mathfrak{so}(4)_-$. Then ϕ_- is just a projection $\phi_-: \mathfrak{so}(4) \longrightarrow \mathrm{so}(4)_-$ with respect to the decomposition $\mathfrak{so}(4) = \mathfrak{so}(4)_- \oplus \mathfrak{so}(4)_+$. The curvature form of Γ^- is Ω^- such that $F^*\Omega^- = \phi_- \Omega$. Let us define a metric g_c on P^- by

$$g_c = -\tfrac{1}{2}c^2 \operatorname{tr}(\omega^-(X) \circ \omega^-(Y)) + p^* g(X, Y).$$

A natural projection $p: P^- \longrightarrow M$ is a Riemannian submersion of $P_c^- = (P^-, g_c)$ onto (M, g). From Proposition 5 it follows that O'Neill's tensor $T = 0$ (the fibres of p are totally geodesic and homothetic to $SO(3)$). To compute the tensor A let us recall that if $X, Y \in T_{F(u)} P^-$ are horizontal vector fields then

$$\omega^-([X, Y])_{Fu} = -\Omega^-(X, Y)_{F(u)} = -\phi_- \Omega(\bar{X}, \bar{Y})_u = -(\Omega(\bar{X}, \bar{Y})_u)^-$$

where $\bar{X}, \bar{Y}$ are horizontal vector fields on P such that $dF(\bar{X}) = X \circ F$. Hence $A_X Y$ is a vertical vector field such that $\omega^-(A_X Y) = -\frac{1}{2}\Omega^-(X, Y)$. If V is a vertical vector field and $dF(\bar{V}) = V \circ F$ then

$$\begin{aligned} g_c(A_X V, Y)_{F(u)} &= -g_c(V, A_X Y)_{F(u)} \\ &= -\tfrac{1}{2}c^2 \operatorname{tr}(\omega^-(V)_{F(u)} \circ \omega^-(A_X Y)_{F(u)}) \\ &= \tfrac{1}{4}c^2 \operatorname{tr}(\omega^-(V)_{F(u)} \circ \Omega^-(X, Y)_{F(u)}) \\ &= \tfrac{1}{4}c^2 \operatorname{tr}(\phi_- \omega(\bar{V})_u \circ \phi_- \Omega(\bar{X}, \bar{Y})_u). \end{aligned}$$

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We can assume that X, Y are horizontal lifts of vector fields $X_*, Y_* \in \mathscr{X}(M)$. Consequently we obtain

$$g_c(A_X V, Y)_{F(u)} = \tfrac{1}{2}c^2 R(X_* \wedge Y_*, \alpha_u(\omega^-(V))) = \tfrac{1}{2}c^2 g(X_* \wedge Y_*, R^-(\alpha_u \omega^-(V))).$$

Note that $R_- = \alpha \,\mathrm{id}_{\wedge^-}$ where $\alpha = \tau_*/12$ and τ_* is the scalar curvature of the space (M, g). Let $\{V_1, V_2, V_3\}$ be an orthonormal basis of $\mathfrak{so}(4)_-$ and let $\{v_1, v_2, v_3\}$ be Killing vector fields corresponding (under a homomorphism F) to V_i. Then $g_c(v_i, v_i) = c^2$ and $g(\alpha_u(V_i), \alpha_u(V_i)) = 1$. Hence

$$c^2 \|A_X Y\|^2 = \sum_{i=1}^{3} g_c(A_X Y, v_i)^2 = \tfrac{1}{4}c^4 \sum_{i=1}^{3} g(X_* \wedge Y_*, R_-(\alpha_u(V_i)))^2. \tag{3.15}$$

From (3.15) it follows that if $\|X\| = \|Y\| = 1$ and $X \perp Y$ then

$$\|A_X Y\|^2 = \tfrac{1}{4}c^2\alpha^2 \|(X_* \wedge Y_*)_-\|^2 = \tfrac{1}{8}c^2\alpha^2$$

if $\|X_* \wedge Y_*\| = 1$. Let $\{Y_{1*}, Y_{2*}, Y_{3*}, Y_{4*}\}$ be a local orthonormal frame on M and $\{Y_1, Y_2, Y_3, Y_4\}$ be its horizontal lift. We can assume that $X = Y_1$. Note that if X is horizontal and $\|X\| = 1$ then for a vertical vector field V we have

$$\begin{aligned}
\|A_X V\|^2 &= \sum_{i=1}^{4} g_c(A_X V, Y_i)^2 \\
&= \tfrac{1}{4}c^4 \sum_{i=1}^{4} g((X_* \wedge Y_{i*})_-, R_-(\alpha_u(\omega^-(V))))^2 \\
&= \tfrac{1}{8}c^4 \|R_-(\alpha_u(\omega^-(V)))\|^2 \\
&= \tfrac{1}{8}c^4\alpha^2 \|\alpha_u(\omega^-(V))\|^2 \\
&= \tfrac{1}{8}c^2\alpha^2 \|V\|^2
\end{aligned}$$

in view of the relation $c^2 \|\alpha_u(\omega(V))\|^2 = g_c(V, V)$. Hence we have (we assume that $\|X\| = \|Y\| = 1$ and $X \perp Y$)

$$K(P_{v_i v_j}) = \frac{1}{2c^2} \quad \text{if } i \neq j$$

$$K(P_{X v_i}) = \tfrac{1}{8}c^2\alpha^2$$

$$K(P_{XY}) = K(P_{X_* Y_*}) - \tfrac{3}{8}c^2\alpha^2.$$

Consequently if V is a vertical vector field such that $|V\| = 1$ then

$$\rho(V, V) = c^{-2} + \tfrac{1}{2}c^2\alpha^2 \tag{3.16}$$

and if X is horizontal and $\|X\| = 1$ then

$$\rho(X, X) = \tfrac{1}{4}\tau_* - 3\tfrac{3}{8}c^2\alpha^2 + \tfrac{3}{8}c^2\alpha^2 = 3\alpha - \tfrac{3}{4}c^2\alpha^2. \tag{3.17}$$

Let $p(u) = x_0$ and let $\{E_1, E_2, E_3, E_4\}$ be an orthonormal basis of $T_{x_0} M$ and $X \in T_{F(u)} P^-$ be a horizontal vector. Let us write $X_* = dp(X)$. Then $X_* \in T_{x_0} M$. Let γ_i be the geodesics satisfying initial conditions $\gamma_i(0) = x_0$, $\gamma_i'(0) = E_i$. By X_* we shall denote the vector field along γ_i obtained by parallel translation of $X_* \in T_{x_0} M$ along γ_i and by X its horizontal lift to P^-. Let $\bar{\gamma}_i^h, \gamma_i^h$ be horizontal lifts of γ_i to P^- and P respectively such that $\bar{\gamma}_i^h(0) = F(u)$ and $\gamma_i^h(0) = u$. Then $F \circ \gamma_i^h = \bar{\gamma}_i^h$. Let us denote

$$B_i = \frac{d}{dt}\gamma_i^h, \quad \bar{B}_i = \frac{d}{dt}\bar{\gamma}_i^h.$$

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We can assume that $dF(B) = X \circ F$ where B is a horizontal vector field on P along the geodesics γ_i^h. Note that $\bar{B}_i \circ F = dF(B_i)$ and B_i are horizontal vector fields on P along γ_i^h. Consequently we derive from (3.8) and (1.2)

$$\overline{\nabla}_{\bar{B}_i} \bar{B}_i = 0, \quad \nabla_{B_i} B_i = 0, \quad \overline{\nabla}_{\bar{B}_i} X \in V, \quad \nabla_{B_i} B \in V \tag{3.18}$$

where by $\overline{\nabla}$ we denote the Levi–Civita connection of the metric g_c on P^-. Let $\xi = A^+$, $A \in \mathfrak{so}(3)$ be a vertical vector field and X be horizontal vector field on P_c^-. We shall show that $\rho(\xi, X) = 0$. Analogously as in the proof of Proposition 6 we have

$$\rho(\xi, X) = \sum_{i=1}^{4} R(\bar{B}_i, \xi, X, \bar{B}_i)$$

and

$$\begin{aligned} R(\bar{B}_i, \xi, X, \bar{B}_i) = {} & \bar{B}_i g_c(A(X, \bar{B}_i), \xi) \\ & - g_c(A(\overline{\nabla}_{\bar{B}_i} X, \bar{B}_i), \xi) - g_c(A(X, \overline{\nabla}_{\bar{B}_i} \bar{B}_i), \xi) - g_c(A(X, \bar{B}_i), \overline{\nabla}_{\bar{B}_i} \xi). \end{aligned}$$

Note that $\overline{\nabla}_{\bar{B}_i} \xi = \overline{\nabla}_{\xi} \bar{B}_i + [\bar{B}_i, \xi]$. We can assume here that $\bar{B}_i$ is a basic vector field. Since $\xi = A^+, A \in \mathfrak{so}(3)$ the field $[\bar{B}_i, \xi]$ is horizontal ($\omega^-([\bar{B}_i, \xi]) = -d\omega^-(\bar{B}_i, \xi) = -\Omega^-(\bar{B}_i, \xi) = 0$). In fact one can easily prove that $[\bar{B}_i, \xi] = 0$. Thus

$$\begin{aligned} g_c(A(X, \bar{B}_i), \overline{\nabla}_{\bar{B}_i} \xi) &= g_c(A(X, \bar{B}_i), \mathscr{V}(\overline{\nabla}_{\bar{B}_i} \xi)) \\ &= g_c(A(X, \bar{B}_i), \mathscr{V}(\overline{\nabla}_{\xi} \bar{B}_i)) \\ &= g_c(A(X, \bar{B}_i), T(\xi, \bar{B}_i)) = 0. \end{aligned}$$

Consequently we get

$$\begin{aligned} R(\bar{B}_i, \xi, X, \bar{B}_i) &= -\tfrac{1}{2} c^2 \bar{B}_i \langle \Omega^-(\bar{B}_i, X), \omega^-(\xi) \rangle \\ &= -\tfrac{1}{2} c^2 \bar{B}_i \langle \Omega^-(\bar{B}_i, F_* B), \omega^-(\xi) \rangle \\ &= -\tfrac{1}{2} c^2 \langle B_i \phi^- \Omega(B_i, B), \omega^-(\xi) \rangle. \end{aligned}$$

Since ϕ_- is a linear map and Ω is a horizontal form we obtain, in view of (3.18),

$$R(\bar{B}_i, \xi, X, \bar{B}_i) = -\tfrac{1}{2} c^2 \langle \phi_- B_i \Omega(B_i, B), \omega^-(\xi) \rangle = -\tfrac{1}{2} c^2 \langle \phi_- \nabla_{B_i} \Omega(B_i, B), \omega^-(\xi) \rangle.$$

Since (M, g) is an Einstein space we have $\delta \bar{R} = 0$ where $\bar{R}$ is the curvature tensor of (M, g). Thus (see (3.11))

$$\sum_{i=1}^{4} \nabla_{B_i} \Omega(B_i, B)_u = 0$$

and consequently we get

$$\rho(\xi, X) = 0$$

for any vertical vector field ξ and any horizontal vector field X. It follows that the Ricci endomorphism S of (P_c^-, g_c) preserves the horizontal and vertical distributions. From (3.16) and (3.17) we obtain that S has exactly two constant eigenvalues $\lambda = c^{-2} + \frac{1}{2} c^2 \alpha^2$ and $\mu = 3\alpha - \frac{3}{4} c^2 \alpha^2$. Let us note that $\lambda = \mu$ if and only if $c^2 = 2/\alpha$ or $c^2 = 2/5\alpha$ and for those values of c the manifold P_c^- is an Einstein space. For $c^2 \neq 2/\alpha$ and $c^2 \neq 2/5\alpha$ from Propositions 3, 4 and 5 it follows that (P^-, g_c) is a proper $\mathscr{A}$-manifold (that is, $\overline{\nabla} S \neq 0$).

Let us recall that A. Derdziński has constructed a family of Hermitian Einstein self-dual structures on $\mathbb{C}^2$ which are locally non-homogeneous (see [**4**]). Hence we get the following corollary.

COROLLARY 2. *Let (M, g) be a self-dual locally non-homogeneous Einstein manifold with nonzero scalar curvature τ_*. Then the principal bundle P^- of oriented orthogonal*

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normed bases of $\wedge^- M$ *admits a family* $P_c^- = (P^-, g_c)$, $c > 0$ *of* $\mathscr{A}$*-structures. For* $c^2 = 2/\alpha$ *or* $c^2 = 2/5\alpha$ *where* $\alpha = \tau_*/12$ *manifold* P_c^- *is an Einstein space and for* $c^2 \notin \{2/\alpha, 2/5\alpha\}$ *the manifold* (P_c^-, g_c) *is a proper locally non-homogeneous* $\mathscr{A}$*-manifold whose Ricci tensor has exactly two constant eigenvalues. If* $c^2 < 4/\alpha$ *then* (P_c^-, g_c) *has positive Ricci curvature.*

REMARK. Note that if (M, g) are Derdziński's examples then the manifolds P_c^- are not complete and consequently they are not compact. In view of the results of N. Hitchin [**6**] every complete self-dual Einstein manifold with positive scalar curvature is symmetric. It is also clear that an anti-self-dual manifold is self-dual with respect to the opposite orientation, and hence Proposition 8 holds also for an Einstein anti-self-dual manifold.

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