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# DIMENSIONS OF TRIANGLE SETS

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*Abstract.* In this paper we discuss some dimension results for triangle sets of compact sets in $\mathbb{R}^2$. In particular we prove that for any compact set $F$ in $\mathbb{R}^2$, the triangle set $\Delta(F)$ satisfies

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \tfrac{3}{2} \dim_{\mathrm{A}} F.$$

If $\dim_{\mathrm{A}} F > 1$, then we have

$$\dim_{\mathrm{A}} \Delta(F) \geqslant 1 + \dim_{\mathrm{A}} F.$$

If $\dim_{\mathrm{A}} F > 4/3$, then we have the following better bound:

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \min\{\tfrac{5}{2} \dim_{\mathrm{A}} F - 1, 3\}.$$

Moreover, if $F$ satisfies a mild separation condition, then the above result holds also for the box dimensions, namely,

$$\underline{\dim}_{\mathrm{B}} F \geqslant \tfrac{3}{2} \underline{\dim}_{\mathrm{B}} \Delta(F) \quad \text{and} \quad \overline{\dim}_{\mathrm{B}} F \geqslant \tfrac{3}{2} \overline{\dim}_{\mathrm{B}} \Delta(F).$$

§1. *Introduction.* The main goal of this paper is to count triangles in a given subset of $\mathbb{R}^2$. To be precise, for any set $F \subset \mathbb{R}^2$ we consider the following triangle set:

$$\Delta(F) = \{r_1, r_2, r_2 \in \mathbb{R}^3_+ : \exists x, y, z \in F, r_1 = |x-y|, r_2 = |z-y|, r_3 = |x-z|\}.$$

When $F$ is a finite set, $\#\Delta(F)$ is roughly the number of different triangles spanned by $F$, where we used $\#B$ for the cardinality of a set $B$. Here we consider two triangles to be the same if they are isometric with respect to the Euclidean metric. In fact, $\#\Delta(F)$ counts a particular triangle at least once and at most six times depending on whether the triangle is equilateral, isosceles or scalene.

On one hand, estimating the size of the triangle set $\Delta(F)$ in terms of the size of $F$ can be naturally considered as a "higher order simplex" version of the distance set problem, as we can regard line segments as 1-simplices and triangles as 2-simplices. Some Hausdorff dimension results for triangle sets and more generally $n$-simplex sets can be found in [6–8]. On the other hand, the study of triangle sets belongs to the more general class of problems which can be best described as "finding patterns". For example, consider $F \subset \mathbb{R}^2$. How large must

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$F$ be to ensure that $F$ contains three vertices of an equilateral triangle? In [13] is was shown that a set of full Hausdorff dimension can avoid containing three vertices of equilateral triangles. However, in [1, Corollary 1.7] it was shown that if $F \subset \mathbb{R}^2$ satisfies some regularity condition concerning Fourier decay, then $F$ contains three vertices of an equilateral triangle. For more recent results see [17] and the references therein.

Now we introduce a bilinear separation condition which is motivated by [11, Bilinear Distance Conjecture 1.5]. In what follows we will encounter the notions of dimensions, covering numbers and approximation symbols. They are discussed in detail in §2.

*Definition* 1.1. For a set $F \subset \mathbb{R}^2$, we say that $F$ is weak-bilinearly separated if the following hold.

(1) There exist three disjoint sets $F_1, F_2, F_3$ such that

$$F_1 \cup F_2 \cup F_3 \subset F$$

and such that

$$\min\{\underline{\dim}_{\mathrm{B}} F_1, \underline{\dim}_{\mathrm{B}} F_2, \underline{\dim}_{\mathrm{B}} F_3\} = \underline{\dim}_{\mathrm{B}} F.$$

(2) There is a positive number $r > 0$ such that any triple $(x, y, z) \in F_1 \times F_2 \times F_3$ forms a triangle such that

$$|\langle x - y, y - z\rangle| = |x - y||y - z||\sin\theta(x - y, y - z)| > r.$$

Here $\langle \, . \, , \, . \, \rangle$ denotes the Euclidean outer product and $\theta(x - y, y - z)$ is the angle spanned by $x - y, y - z$.

We say that $F$ is strong-bilinearly separated if the condition (1) above is replaced by the following stronger condition.

(1′) There are three disjoint sets $F_1, F_2, F_3$ such that

$$F_1 \cup F_2 \cup F_3 \subset F$$

and such that, for all $\delta \in (0, 1)$,

$$\min\{N_\delta(F_1), N_\delta(F_2), N_\delta(F_3)\} \gtrsim N_\delta(F).$$

The intuitive idea behind the above bilinear separations is that $F$ can be essentially decomposed into three large parts. If we take one point from each part, the three points form a triangle which is far away from being degenerate. Following this idea, it is natural to guess that if $F$ is not small, then we can find many triangles in $F$. The following theorem formulates this idea more precisely.

THEOREM 1.2. *Let $F \subset \mathbb{R}^2$ be a compact and weak-bilinearly separated set. Then we have the following inequality:*

$$\overline{\dim}_{\mathrm{B}} \Delta(F) \geqslant \tfrac{3}{2} \underline{\dim}_{\mathrm{B}} F.$$

*If $F$ is strong-bilinearly separated, then we have*

$$\underline{\dim}_{\mathrm{B}} \Delta(F) \geqslant \tfrac{3}{2} \underline{\dim}_{\mathrm{B}} F \quad \textit{and} \quad \overline{\dim}_{\mathrm{B}} \Delta(F) \geqslant \tfrac{3}{2} \overline{\dim}_{\mathrm{B}} F.$$

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THEOREM 1.3. *Let $F \subset \mathbb{R}^2$ be a compact set. Then the following result holds:*

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \tfrac{3}{2} \dim_{\mathrm{A}} F.$$

*If $\dim_{\mathrm{A}} F > 1$, then we have the following better bound:*

$$\dim_{\mathrm{A}} \Delta(F) \geqslant 1 + \dim_{\mathrm{A}} F.$$

*If $\dim_{\mathrm{A}} F > 4/3$, then we have the following even better bound:*

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \min\{\tfrac{5}{2} \dim_{\mathrm{A}} F - 1, 3\}.$$

Note that for the Assouad dimension result we do not need any bilinear separation condition on $F$. In this paper, we will use three different methods to study the triangle sets. For the first part of Theorem 1.2 we will use a direct and straightforward counting method, see Lemma 4.3, and for Theorem 1.3 we will make use of the fact that there is a close relationship between the triangle sets and the distance sets. For the second part of Theorem 1.3 we shall use a harmonic analysis method introduced in [**8**]. All these three methods have their advantages. The distance set method is more powerful in $\mathbb{R}^2$ and the direct counting method, as well as the harmonic analysis method, has a potential to be generalized for higher-dimensional situations, see §8.

§2. *Notation.* We briefly introduce the notions of dimensions in this section. For more details on the Hausdorff and box dimensions, see [**2**, Chs 2, 3] and [**14**, Chs 4, 5]. For the Assouad dimension see [**3**] for more details.

We shall use $N_r(F)$ for the minimal covering number of a set $F$ in $\mathbb{R}^n$ by cubes of side length $r > 0$.

2.1. *Hausdorff dimension.* For any $s \in \mathbb{R}^+$ and $\delta > 0$ define the following quantity:

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} (\mathrm{diam}(U_i))^s : \bigcup_i U_i \supset F, \mathrm{diam}(U_i) < \delta\right\}.$$

The $s$-Hausdorff measure of $F$ is

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

The Hausdorff dimension of $F$ is

$$\dim_{\mathrm{H}} F = \inf\{s \geqslant 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geqslant 0 : \mathcal{H}^s(F) = \infty\}.$$

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2.2. *Box dimensions.* The upper box dimension of $F$ is

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{r\to 0}\left(-\frac{\log N_r(F)}{\log r}\right).$$

The lower box dimension of $F$ is

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{r\to 0}\left(-\frac{\log N_r(F)}{\log r}\right).$$

If the upper and lower box dimensions of $F$ are equal, then we call this value the box dimension of $F$ and we denote it as $\dim_{\mathrm{B}} F$.

2.3. *The Assouad dimension and weak tangents.* The *Assouad dimension* of $F$ is

$$\dim_{\mathrm{A}} F = \inf\Big\{s \geqslant 0\colon (\exists C > 0)(\forall R > 0)(\forall r \in (0, R))(\forall x \in F)$$
$$N_r(B(x, R) \cap F) \leqslant C\left(\frac{R}{r}\right)^s\Big\},$$

where $B(x, R)$ denotes the closed ball of centre $x$ and radius $R$.

We now introduce the notion of *weak tangent* which is very useful in studying the Assouad dimension. We start by defining the Hausdorff distance between two compact sets $A, B \subset \mathbb{R}^n$ as follows:

$$d_{\mathcal{H}}(A, B) = \inf\{\delta > 0 : A \subset B_\delta, B \subset A_\delta\},$$

where $A_\delta$ is the closed $\delta$-neighbourhood of $A$. Let $X = [0, 1]^n$ for some $n$. Then $(\mathcal{K}(X), d_{\mathcal{H}})$, the space of compact subsets of $X$, is a compact metric space.

*Definition* 2.1. We call $D \in \mathcal{K}(X)$ a *miniset* of $F \in \mathcal{K}(R^n)$ if $D = (cF + t) \cap X$ for some constants $c \geqslant 1$ and $t \in \mathbb{R}^n$. A set $E \in \mathcal{K}(X)$ is called a *weak tangent* if it is the limit of a sequence of minisets (under the Hausdorff metric).

Due to [4, 12] we know that for compact subsets $F \subset \mathbb{R}^n$

$$\dim_{\mathrm{A}} F \geqslant \sup\{\dim_{\mathrm{A}} E\colon E \text{ is a weak tangent of } F\}.$$

The following result can be found in [10, Proposition 5.7]; the positivity of Hausdorff measure can be found in [4, Theorem 1.3].

LEMMA 2.2 (Käenmäki–Ojala–Rossi). *Let $F$ be a compact set with $\dim_{\mathrm{A}} F = s$. Then there exist a weak tangent $E$ of $F$ such that*

$$\dim_{\mathrm{H}} E = s.$$

*Moreover, we have*

$$\mathcal{H}^s(E) > 0.$$

In particular, we see that, for compact subsets $F \subset \mathbb{R}^n$,

$$\dim_{\mathrm{A}} F = \max\{\dim_{\mathrm{H}} E\colon E \text{ is a weak tangent of } F\}.$$

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2.4. *Approximation symbols for box counting.* When counting covering numbers, it is convenient to introduce notions $\approx, \lesssim, \gtrsim$ for approximately equal, approximately smaller and approximately larger. As our box counting procedure always involves scales, later we use $1 > \delta > 0$ to denote a particular scale. Then for two quantities $f(\delta), g(\delta)$ we define the following:

$$f \lesssim g \iff \text{there exists } M > 0, \text{ for all } \delta > 0,\ f(\delta) \leqslant Mg(\delta),$$
$$f \gtrsim g \iff g \lesssim f,$$
$$f \approx g \iff f \lesssim g \text{ and } g \lesssim f.$$

Thus $f \approx g$ in our box counting procedure can be intuitively read as "$f$ and $g$ give the same box dimension". Later in context, $f, g$ can be either box covering number of scale $\delta$ or Lebesgue measure of a $\delta$ neighbourhood of a set.

§3. *Some recent results.* In this section we review some results on triangle sets $\Delta(F)$ for finite $F \subset \mathbb{R}^2$. A result from [9] shows that there exists a constant $C > 0$ such that

$$\#D(F) \geqslant C\frac{\#F}{\log \#F},$$

where we write $D(F)$ for the distance set of $F$ defined as follows:

$$D(F) = \{|x - y| : (x, y) \in F \times F\}.$$

From here (see [7, §6] and the references therein) it is possible to show that

$$\#\Delta(F) \geqslant C\frac{\#F^2}{\log \#F}.$$

For Borel sets, by a result due to [8, Theorem 1.5] it is possible to deduce the following result.

THEOREM 3.1 (Greenleaf–Iosevich–Liu–Palsson). *Let $F \subset \mathbb{R}^2$ be a Borel set. If $\dim_{\mathrm{H}} F > \frac{8}{5}$, then $\Delta(F)$ has positive Lebesgue measure.*

We note here a recent result on the distance sets that will be used later in this paper. Notice that the theorem we stated below is weaker than its original version [16, Theorem 1.1]. In what follows for $x \in \mathbb{R}^2$ we denote the pinned distance set $D_x(F)$ to be the following set:

$$D_x(F) = \{|x - y| : y \in F\}.$$

THEOREM 3.2. *Let $F \subset \mathbb{R}^2$ be a Borel set with $\dim_{\mathrm{H}} F = \overline{\dim}_{\mathrm{B}} F > 1$. Then we have the following result:*

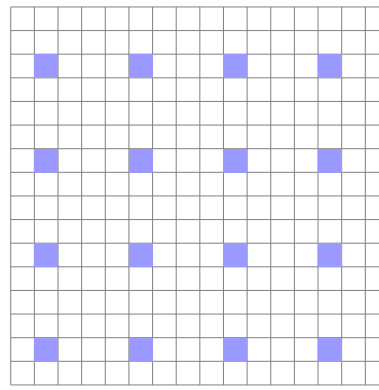
$$\dim_{\mathrm{H}}\{x \in \mathbb{R}^2 : \dim_{\mathrm{H}} D_x(F) < 1\} \leqslant 1.$$

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§4. *Some combinatorics.* To prove Theorem 1.2 we need some combinatorial results. In this section, we shall prove Lemma 4.3. This result is based on a private communication with Héra. In order to improve readability and to make our paper self-contained, we provided detailed proofs here. We begin with two standard combinatorial tools.

LEMMA 4.1 (Chessboard). *Let $n \geqslant 1$ be an integer. Let $\delta > 0$ and consider a disjoint covering of $[0,1]^n$ with $\delta$-cubes. For any collection of $N \geqslant 100^n$ such cubes, we can find a sub-collection with at least $N/100^n$ cubes such that each two different cubes in this sub-collection are separated by at least $100\delta$.*

*Proof.* The idea can be presented in the following picture in $\mathbb{R}^2$.



We can assume $\delta = 1/K$ for some integer $K$. Then our $\delta$-cubes can be identified with a subset of $\mathbb{N}^n$. Namely

$$A = \{(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n : \forall i \in [1,n], a_i \in [1,K]\}.$$

Let $i \in [1,n]$, $r_i \in [0,99]$ be $n$ integers. Consider the set

$$A_{r_1,\ldots,r_n} = \{(a_1, a_2, \ldots, a_n) \in A : i \in [1,n], a_i \equiv r_i \bmod 100\}.$$

There are $100^n$ many such sets, they are pairwise disjoint, and they form a decomposition of $A$. Therefore for any subset of $A$ with $N$ elements, we can find integers $r_i, i \in [1,n]$, such that $A_{r_1,\ldots,r_n}$ contains at least $N/100^n$ many elements. By construction, two different points in $A_{r_1,\ldots,r_n}$ are at least 100 separated. The proof concludes by scaling the whole configuration back by multiplying $\delta = K^{-1}$. □

LEMMA 4.2 (Dyadic pigeonhole). *Let $N$ be an integer, and suppose there are $N$ positive integers $m_i, i \in \{1,\ldots,N\}$ such that $\sum_i m_i = M$. Then we can find a positive integer $k$ such that $\sum_{i:m_i \in [2^{k-1},2^k)} m_i \geqslant M/\log M$.*

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*Proof.* For each $k \in \mathbb{N}$ consider

$$D_k = \{i \in \{1, \dots, N\} : m_i \in [2^{k-1}, 2^k)\}.$$

It is clear that for each $i$ we have $1 \leqslant m_i \leqslant M$ and therefore we have at most $\log M$ many non-empty sets $D_k$. Therefore there is at least one $k$ such that

$$\sum_{i \in D_k} m_i \geqslant \frac{M}{\log M}. \qquad \square$$

We will now prove the following combinatorial result.

LEMMA 4.3. *Let $0 < r_1 \leqslant r_2$ be positive real numbers. Let $\gamma > 0$ be another positive number such that $0 < 10\gamma < r_2 + r_1$. Let $\delta > 0$ be a positive real number which can be chosen arbitrarily small. Let $\alpha, \beta$ be two non-negative numbers such that we can find a set $C \subset \mathbb{R}^2$ such that the following hold.*

(1) *The set $C$ is $100\delta$ separated and $\#C = \delta^{-\alpha}$.*
(2) *For any $c \in C$, two circles $c + r_1 S^1$ and $c + r_2 S^1$ both contain $\delta^{-\beta}$ many $100\delta$ separated points. We denote the two sets of points as $C_1(c)$ and $C_2(c)$ respectively.*
(3) *For any $c \in C$, we have the following condition:*

$$\begin{aligned}&\#\{(x, y) \in C_1(c) \times C_2(c) : r_2 - r_1 + 10\gamma < |x - y| < r_2 + r_1 - 10\gamma\} \\ &\quad \geqslant \tfrac{1}{2}\#C_1(c)\#C_2(c).\end{aligned}$$

*Then $\bigcup_{c \in C}(C_1(c) \cup C_2(c))$ contains $\gtrsim \delta^{-0.5\alpha - \beta}$ many $\delta$-separated points. The implicit constant in $\gtrsim$ depends on $r_1, r_2, \gamma$ but not on $\alpha, \beta$.*

*Proof.* We can create a $\delta$-grid and count how many disjoint $\delta$-squares we need to cover $\bigcup_{c \in C} C_1(c) \cup C_2(c)$. We have in total $2\delta^{-\alpha-\beta}$ many possible points, but they might not be all $\delta$-separated. We denote $\mathcal{F}$ the set of $\delta$-squares we need to cover all points. Consider the following incidence set:

$$\begin{aligned}\mathcal{I} = \{&(x_1, x_2, c) \in F \times F \times C : \\ &x_1 \cap c + r_1 S^1 \neq \emptyset, x_2 \cap c + r_2 S^1 \neq \emptyset, \\ &|x_1 - x_2| \subset (r_2 - r_1 + \gamma, r_2 + r_1 - \gamma)\}.\end{aligned}$$

We now count $\#\mathcal{I}$ in two different ways. First, there are at most $\#\mathcal{F}^2$ many elements in $\mathcal{F} \times \mathcal{F}$. For any pair $(x_1, x_2) \in \mathcal{F} \times \mathcal{F}$ such that $|x_1 - x_2| \subset (r_2 - r_1 + \gamma, r_2 + r_1 - \gamma)$, it is not hard to see that there are not many $c \in C$ such that

$$x_1 \cap c + r_1 S^1 \neq \emptyset, \qquad x_2 \cap c + r_2 S^1 \neq \emptyset.$$

To be precise, consider $T^{3\delta}_{r_1}(x_1)$, $T^{3\delta}_{r_2}(x_2)$ to be the annulus cocentred with $x_1, x_2$ respectively. The inner and outer radii of $T^{3\delta}_{r_i}(x_i)$ are $r_i - 1.5\delta, r_i + 1.5\delta$. Then we see that

$$\{c \in C : x_1 \cap c + r_1 S^1 \neq \emptyset, x_2 \cap c + r_2 S^1 \neq \emptyset\} \subset T^{3\delta}_{r_1}(x_1) \cap T^{3\delta}_{r_2}(x_2).$$

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Now observe that there is a constant $A$ which depends on $r_1, r_2, \gamma$ such that

$$T^{3\delta}_{r_1}(x_1) \cap T^{3\delta}_{r_2}(x_2)$$

can be covered by two squares side length $A\delta$. Because $C$ is $\delta$-separated we see that

$$\{c \in C : x_1 \cap c + r_1 S^1 \neq \emptyset, x_2 \cap c + r_2 S^1 \neq \emptyset\}$$

contains at most $1000A^2$ many elements. Therefore we see that

$$\#\mathcal{I} \leqslant 1000A^2 \#\mathcal{F}^2. \tag{1}$$

To obtain a lower bound, we consider each individual $c \in C$. By assumption for each $c \in C$ we can find at least $0.5\#C_1(c)\#C_2(c)$ many pairs in $\mathcal{F} \times \mathcal{F}$ such that the distance set between each pair of cubes are contained in $(r_2 - r_1 + \gamma, r_2 - r_1 - \gamma)$. Therefore we see that

$$\#\mathcal{I} \geqslant \tfrac{1}{2}\delta^{-\alpha-2\beta}. \tag{2}$$

The final result follows from (1), (2) by absorbing constants into $\gtrsim$ symbol. □

§5. *Proof of Theorem 1.2.* In the proof we will be counting covering numbers with a fixed scale $\delta \in (0, 1)$. All the quantities might depend on $\delta$. We will also use $\gtrsim, \lesssim, \approx$ symbols. In this proof, we can see that there is a fixed number $M > 0$ such that all the implicit constants in the approximation symbols can be chosen as $M$. For example

$$f(\delta) \lesssim g(\delta) \text{ can be written as } f(\delta) \leqslant Mg(\delta).$$

Let $F_1, F_2, F_3$ be stated as in the statement of Theorem 1.2. We first show the result with weak-bilinear separation condition. For all $\delta \in (0, 1)$ and $\epsilon > 0$ we have the following result:

$$\min\{N_\delta(F_1), N_\delta(F_2), N_\delta(F_3)\} \gtrsim \delta^{-\underline{\dim}_{\mathrm{B}} F+\epsilon}. \tag{$*$}$$

Later we shall let $\epsilon \to 0$ but for now it is a fixed number. By applying the chessboard argument (Lemma 4.1) we can find $100\delta$-separated subsets $C_i \subset F_i$, $i \in \{1, 2, 3\}$ such that

$$\#C_i \gtrsim \delta^{-\underline{\dim}_{\mathrm{B}} F+\epsilon}.$$

We see that each $(c_1, c_2, c_3) \in C_1 \times C_2 \times C_3$ forms a triangle which is far away from being degenerate. (A precise description can be found in Definition 1.1.) Now we can use disjoint $\delta$ cubes to cover $\Delta(F)$. Denote this set of $\delta$-cubes to be $\mathcal{N}_\delta(\Delta(F))$. We see that the triple $(|c_1 - c_2|, |c_2 - c_3|, |c_3 - c_1|)$ belongs to one of the cubes in $\mathcal{N}_\delta(\Delta(F))$. We denote $K(c_1, c_2, c_3)$ to be this cube. For each $K \in \mathcal{N}_\delta(\Delta(F))$ we define the following set:

$$S(K) = \{(c_1, c_2, c_3) \in C_1 \times C_2 \times C_3 : (|c_1 - c_2|, |c_2 - c_3|, |c_3 - c_1|) \in K\}.$$

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Clearly we have the following relation:

$$\sum_{K \in \mathcal{N}_\delta(\Delta(F))} \#S(K) = \#C_1 \#C_2 \#C_3.$$

Then by pigeonhole principle we see that there exists at least one $K$ such that

$$\#S(K) \geqslant \frac{\#C_1 \#C_2 \#C_3}{N_\delta(\Delta(F))} \gtrsim \frac{(\delta^{-\underline{\dim}_B F+\epsilon})^3}{N_\delta(\Delta(F))}.$$

Intuitively this means that we can find a triangle and we can find a lot of copies of that triangle in $F$. Now we fix this choice of cube $K$. For each $c_1 \in C_1$, we define the following set:

$$S(K, c_1) = \{(c_2, c_3) \in C_2 \times C_3 : (c_1, c_2, c_3) \in S(K)\}.$$

Then we see that

$$\sum_{c_1 \in C_1} \#S(K, c_1) = \#S(K).$$

By the dyadic pigeonhole principle (Lemma 4.2) we can find two integers $N_1$, $N_2$ such that

$$N_1 N_2 \geqslant \frac{\#S(K)}{2 \log \#S(K)}$$

and such that there are $N_1$ many $c_1 \in C_1$ with

$$\#S(K, c_1) \in [N_2, 2N_2].$$

Now we take a closer look at the set $S(K, c_1)$ for $c_1$ described as above. We can find many pairs $(c_1, c_2, c_3)$ in $C_2 \times C_3$ such that $c_1, c_2, c_3$ is $\delta$-close to a fixed non-degenerate triangle. Because of the bilinear separation condition, there exist positive constants $r_1, r_2, \gamma$ such that

$$|c_1 - c_2| \in [r_1 - 3\delta, r_1 + 3\delta], \qquad |c_1 - c_3| \in [r_2 - 3\delta, r_2 + 3\delta],$$
$$|c_2 - c_3| \in [|r_1 - r_2| + 10\gamma, r_1 + r_2 - 10\gamma].$$

If we further fix $c_2$, then there exist at most a bounded number (which does not depend on $\delta$) of $c_3$ such that

$$(c_2, c_3) \in S(K, c_1).$$

To summarize, around each $c_1$ we can find two (not necessary distinct) annuli of inner, outer radii in $[r_1 - 3\delta, r_1 + 3\delta]$, $[r_2 - 3\delta, r_2 + 3\delta]$. Those annuli contain $\gtrsim \#S(K, c_1)$ many points in $C_2, C_3$ respectively. Now we can apply Lemma 4.3 to deduce that

$$N_\delta(F) \gtrsim \sqrt{N_1 N_2^2}.$$

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We also have the following obvious bound:

$$N_\delta(F) \geqslant \max\{N_1, N_2\}.$$

So we see that

$$N_\delta(F) \gtrsim (N_1 N_2)^{2/3} \gtrsim \frac{-1}{\log \delta} \frac{(\delta^{-\underline{\dim}_B F+\epsilon})^2}{N_\delta(\Delta(F))^{2/3}}.$$

Therefore we see that for all $\delta > 0$

$$N_\delta(\Delta(F)) \gtrsim \left(\frac{-1}{\log \delta}\right)^{3/2} \frac{(\delta^{-\underline{\dim}_B F+\epsilon})^3}{(N_\delta(F))^{3/2}}.$$

We know that there exist arbitrarily small $\delta \in (0, 1)$ such that

$$N_\delta(F) \leqslant \delta^{-\underline{\dim}_B F-\epsilon}.$$

This implies that

$$\overline{\dim}_B \Delta(F) \geqslant \tfrac{3}{2}\underline{\dim}_B F - 4.5\epsilon.$$

By letting $\epsilon \to 0$ we see that

$$\overline{\dim}_B \Delta(F) \geqslant \tfrac{3}{2}\underline{\dim}_B F.$$

Now we shall show the result for strong-bilinear separation condition. In this case we can replace the inequality (∗) with the following:

$$\min\{N_\delta(F_1), N_\delta(F_2), N_\delta(F_3)\} \gtrsim N_\delta(F).$$

The rest of this proof is very similar to that of the weak-bilinear separation case. We omit the full details.

§6. *Proof of Theorem 1.3, part I.* Before we prove Theorem 1.3, let us examine an extreme case when the weak-bilinear separation condition does not hold. Suppose that $F$ is contained in a line segment and for simplicity we shall assume that $F \subset [0, 1]$. For any $\delta \in (0, 1)$ we can cover $F$ with disjoint $\delta$-boxes and we need $N_\delta(F)$ many of them. Then we simply find $x \in F$ such that

$$N_\delta(F \cap [0, x]) = N_\delta(F \cap [x, 1])$$

and therefore for any $y, z \in F$ with $y < x < z$ we see that $(x - y, z - x) \in \Delta(F)$. It is easy to see that this gives at least $0.25N_\delta(F)^2$ many contributions to $N_\delta(\Delta(F))$. As this holds for all $\delta \in (0, 1)$ we see that $\underline{\dim}_B \Delta(F) \geqslant 2\underline{\dim}_B F$. We will use this result later.

LEMMA 6.1. *If $F \subset \mathbb{R}^2$ is a compact subset, then $\Delta(F)$ is a compact subset of $\mathbb{R}^3$.*

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*Proof.* It is clear that $\Delta(F) \subset \mathbb{R}^3$ is bounded. We show that $\Delta(F)$ is also closed. Let $a_i \in \Delta(F), i \geqslant 1$ be a sequence of points converging to $a \in \mathbb{R}^3$. Then we can find points $(x_i, y_i, z_i) \in F \times F \times F$ such that

$$a_i = (|x_i - y_i|, |x_i - z_i|, |y_i - z_i|).$$

By taking a subsequence if necessary we assume that $x_i \to x, y_i \to y, z_i \to z$ for $(x, y, z) \in F \times F \times F$. Then we see that

$$|x_i - y_i| \to |x - y|, |x_i - z_i| \to |x - z|, |z_i - y_i| \to |z - y|.$$

Thus we see that $a \in \Delta(F)$. This concludes the proof. □

LEMMA 6.2. *Let $E$ be a weak tangent of $F$. Then $\Delta(E)$ is a subset of a weak tangent of $\Delta(F)$. In particular we see that* $\dim_{\mathrm{A}} \Delta(E) \leqslant \dim_{\mathrm{A}} \Delta(F)$.

*Proof.* By definition, $E$ is the limit in Hausdorff metric of sets $E_i = (r_i E + b_i) \cap [0, 1]^2$ where

$$b_i \in \mathbb{R}^2, \qquad r_i > 0, \qquad \lim_{i \to \infty} r_i = \infty.$$

Now we can take the sequence $\Delta(E_i)$ and it is easy to see that

$$\Delta(E_i) \subset r_i \Delta(F) \cap [0, 1]^3.$$

By taking a subsequence of the sets $E_i$ if necessary we can assume that

$$E_i, \Delta(E_i), r_i \Delta(F) \cap [0, 1]^3$$

all converge as $i \to \infty$ with respect to the Hausdorff metric. Now fix a positive number $\epsilon > 0$ and suppose that $(a, b, c) \in \Delta(E)$. Then for all large enough $i$ we can find three points $(a_i, b_i, c_i)$ in $E_i$ such that

$$\max\{|a_i - a|, |b_i - b|, |c_i - c|\} < \epsilon.$$

Then we see that

$$|a_i - b_i| \leqslant |a_i - a| + |a - b| + |b - b_i| < |a - b| + 2\epsilon,$$
$$|a - b| \leqslant |a - a_i| + |a_i - b_i| + |b_i - b| < |a_i - b_i| + 2\epsilon.$$

Similar relations hold for $|b - c|, |b_i - c_i|$ and $|a - c|, |a_i - c_i|$ as well. We see that

$$\Delta(E) \subset \lim_{i \to \infty} \Delta(E_i) \subset \lim_{i \to \infty} (r_i \Delta(F) \cap [0, 1]^3).$$

This is what we want. □

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Because of Lemmas 2.2, 6.1 and 6.2, by taking a weak tangent if necessary, we can assume that $F$ has equal Hausdorff dimension and Assouad dimension, say, $s > 0$ and $\mathcal{H}^s(F) > 0$. We shall try to find an integer $k > 0$ and three disjoint dyadic cubes

$$c_1, c_2, c_3 \in \mathcal{D}_k$$

such that for $i \in \{1, 2, 3\}$

$$\underline{\dim}_{\mathrm{B}}(c_i \cap F) = s.$$

In what follows, for each $i$ we write $F_i$ as $c_i \cap F$. Furthermore, any triple $(x, y, z) \in F_1 \times F_2 \times F_3$ form a triangle that is far away from being degenerate, namely, there is a constant $r > 0$ such that for all such triples

$$|\langle x - y, y - z\rangle| > r.$$

If we can find such dyadic cubes, then $F_1 \cup F_2 \cup F_3$ is weak-bilinearly separated and by Theorem 1.2 we see that

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \overline{\dim}_{\mathrm{B}}\Delta(F) \geqslant \overline{\dim}_{\mathrm{B}}\Delta(F_1 \cup F_2 \cup F_3) \geqslant \tfrac{3}{2}\underline{\dim}_{\mathrm{B}}(F_1 \cup F_2 \cup F_3) \geqslant \tfrac{3}{2}s.$$

Now we are going to find those dyadic cubes. We shall see that it is always possible to find such cubes unless $F$ is essentially contained in a line in a precise sense. First we want to find $k_1$ such that at least two non-adjacent cubes in $\mathcal{D}_{k_1}$ whose intersections with $F$ have positive $\mathcal{H}^s$ measures. If such $k_1$ does not exist, then we see that $\mathcal{H}^s$ has singleton support and this is not possible. This contradiction gives us the existence of $k_1$. Then we can find two non-adjacent cubes $A_1, A_2$ whose intersections with $F$ has positive $\mathcal{H}^s$ measures. Then for any $k > k_1$ we can find $A_1^k, A_2^k \in \mathcal{D}_k$ with $A_1^k \subset A_1$, $A_2^k \subset A_2$. Then we define the following "line" set:

$$L(A_1^k, A_2^k) = \bigcup_{l \text{ is a line}}^{l \cap A_1^k \neq \emptyset, l \cap A_2^k \neq \emptyset} l \cap F.$$

Since $F$ is bounded, we see that as $k \to \infty$, $L(A_1^k, A_2^k)$ converges to a line segment $L$ with respect to the Hausdorff metric. Now if for any $k > k_1$ we can find $A_3^k \in \mathcal{D}_k$ whose intersection with $F$ has positive $\mathcal{H}^s$ measure such that

$$2A_3^k \cap L(A_1^k, A_2^k) = \emptyset,$$

then we can choose $c_i = A_i^k$ for $i \in \{1, 2, 3\}$ and we are done. Otherwise we see that $\mathcal{H}^s$ is supported in a $3 \times 2^{-k}$ neighbourhood of $L$ for all $k > k_1$ and therefore $\mathcal{H}^s$ is supported in $L$. Therefore we can actually focus on $F \cap L$, and in this case we saw that $s \leqslant 1$ and

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \underline{\dim}_{\mathrm{B}}\Delta(F) \geqslant 2\,\underline{\dim}_{\mathrm{B}} F \geqslant 2\dim_{\mathrm{H}} F = 2\dim_{\mathrm{A}} F.$$

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In the right-most inequality we have used the assumption that $\dim_{\mathrm{H}} F = \dim_{\mathrm{A}} F$. This shows that

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \tfrac{3}{2} \dim_{\mathrm{A}} F. \tag{\%}$$

Now we assume that $\dim_{\mathrm{A}} F = s > 1$, and in this case, because of Lemmas 2.2, 6.1 and 6.2, as before we can assume that $\dim_{\mathrm{H}} F = \dim_{\mathrm{A}} F = s > 1$ and $\mathcal{H}^s(F) > 0$. Then we see that $F$ is weakly-bilinearly separated in this case because $s > 1$. As above, we can find subsets $F_1, F_2, F_3 \subset F$ with positive $\mathcal{H}^s$ measures. Since $s > 1$ and $\dim_{\mathrm{H}} F_1 \leqslant \dim_{\mathrm{A}} F = s$ we see that $\dim_{\mathrm{H}} F_1 = \dim_{\mathrm{A}} F_1$, and the same relation holds for $F_2, F_3$ as well. By Theorem 3.2 we see that there exists $x \in F_1$ such that $\dim_{\mathrm{H}} D_x(F_2) = 1$. Then we see that

$$\underline{\dim}_{\mathrm{B}} D_x(F_2) = 1.$$

For each $\epsilon > 0$, for all small enough numbers $\delta > 0$ we see that

$$N_\delta(D_x(F_2)) \geqslant \delta^{-1+\epsilon}. \tag{\#}$$

Now we choose $l \in D_x(F_2)$ and $y \in F_2$ such that $|x - y| = l$. Because of the construction of $F_1, F_2, F_3$ we can assume that $l \geqslant c$ for a constant $c > 0$ which does not depend on $\delta$. Therefore we see that there exists constant $M > 0$ such that for each pair of two points $z, z' \in F_3$ with $|z - z'| \geqslant M\delta$, the triangle spanned by $xyz$ and the triangle spanned by $xyz'$ separate each other by at least $\delta$ when regarded as points in $\mathbb{R}^3$. Since $\dim_{\mathrm{H}} F_3 = \dim_{\mathrm{A}} F_3 = s$, we see that $\underline{\dim}_{\mathrm{B}} F_3 = s$ and for all small enough $\delta$ we have $N_\delta(F_3) \geqslant \delta^{-s+\epsilon}$. Then together with (#) we see that, for all small enough $\delta > 0$,

$$N_\delta(\Delta(F_1 \cup F_2 \cup F_3)) \geqslant \delta^{-1-s+2\epsilon}.$$

This implies that $\underline{\dim}_{\mathrm{B}} \Delta(F) \geqslant 1 + s - 2\epsilon$. Since $\epsilon > 0$ can be chosen arbitrarily small we see that

$$\dim_{\mathrm{A}} \Delta(F) \geqslant \underline{\dim}_{\mathrm{B}} \Delta(F) \geqslant 1 + s.$$

The above result holds for $\dim_{\mathrm{A}} F = s > 1$ and together with (%) we see that the first two conclusions of Theorem 1.3 hold.

§7. *Proof of Theorem 1.3, part II.* In this section, we closely follow [**8**]. At this stage, we are not aiming at self-containing. The reader is strongly recommended to read [**8**] and convince himself/herself that the result we are going to prove "naturally follows". In fact, a fairly large part of the main proof in this section shares arguments with [**8**].

As we use some different notation from that used in [**8**], we reintroduce some definitions from [**8**].

*Definition* 7.1. Let $n \geqslant 2$ and $2 \leqslant k \leqslant n+1$ be integers. Given a set $F \subset \mathbb{R}^n$, define

$$\Delta_k(F) = \{(r_{ij}, 1 \leqslant i < j \leqslant k) \in \mathbb{R}^{k(k-1)/2} : x_1, \ldots, x_k \in F, \\ |x_i - x_j| = r_{ij}, 1 \leqslant i < j \leqslant k\}.$$

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Notice that $\Delta_3$ has the same meaning as $\Delta$ we have dealt with. Taking permutations of vertices into account, $\Delta_k(F)$ counts a particular simplex at least once and at most $c(k)$ times for an integer $c(k)$ depending only on $k$. For $k = 3$ we know that $c(3) = 6$.

*Definition* 7.2. Let $F \subset \mathbb{R}^n$ be a compact set and let $\mu$ be a probability measure supported on $F$. For $g \in \mathbb{O}(n)$, the orthogonal group on $\mathbb{R}^n$, we construct a measure $\nu_g$ as follows:

$$\int_{\mathbb{R}^n} f(z)\, d\nu_g(z) = \int_F \int_F f(u - gv)\, d\mu(u)\, d\mu(v), \quad f \in C_0(\mathbb{R}^n).$$

In other words, $\nu_g = \mu * g\mu$, where $g\mu$ is the pushed forward measure of $\mu$ under the map $g$. We also construct a measure $\nu$ on $\Delta_k(F) \subset \mathbb{R}^{k(k-1)/2}$ by

$$\int f(t)\, d\nu(t) = \int f(|x_1 - x_2|, \ldots, |x_i - x_j|, \ldots, |x_{k-1} - x_k|)\, d\mu(x_1) \ldots d\mu(x_k),\ f \in C_0(\mathbb{R}^{k(k-1)/2}),$$

where $t$ is a $k(k-1)/2$-vector with entries $|x_i - x_j|$ for $1 \leqslant i < j \leqslant k$.

For a given set $F \subset \mathbb{R}^n$ we can choose $\mu$ supported on $F$ with some regularities, for example, a Frostman measure. Then with these regularities we are able to obtain some results of $\nu$. Since $\nu$ supports on $\Delta_k(F)$ we can get some information for $\Delta_k(F)$. Thus the difference between $\Delta_k(F)$ in the above definition and $T_k(E)$ in [**8**, Definition 1.1] is that we count the same simplex multiple times due to permutations of its vertices. For example, a triangle $\Delta ABC$ would count differently from $\Delta BAC$ in our triangle set counting, but they are actually the same triangle. A bit of caution should be given here. We frequently use $B_\delta(x)$ for the (closed) $\delta$-ball around $x$ in a metric space. We will encounter the situation where we use $B_\delta(x)$, $B_\delta(z)$ in the same expression but $x, z$ are in different spaces. We hope no confusion will rise here as the closed balls should be in the same space as their centres.

THEOREM 7.3. *Let $F \subset \mathbb{R}^n$ be a compact set with $\dim_{\mathrm{H}} F = s > n/2$. Then we have*

$$\dim_{\mathrm{H}} \Delta(F) \geqslant \min\{2s + \gamma_s - 2n + 3, 3\},$$

*where $\gamma_s = (n+2s-2)/4$ if $s \in [n/2, (n+2)/2]$ and $\gamma_s = s-1$ if $s \geqslant (n+2)/2$.*

In particular, when $n = 2$ we see that

$$\dim_{\mathrm{H}} \Delta(F) \geqslant 2.5s - 1.$$

Then the third conclusion of Theorem 1.3 follows by the above Theorem and the weak tangent trick introduced in the previous section. When $s$ is large enough so that $2s + \gamma_s - 2n + 3 > 3$, then $\Delta(F)$ has positive measure; this was shown in [**8**].

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*Proof of Theorem 7.3.* We need to choose cutoff functions in various places. Unless otherwise mentioned, all cutoff functions are assumed to be real-valued and radial. Let $\phi(.)$ be a function taking arguments in $\mathbb{R}^l$ with $l \geqslant 1, l \in \mathbb{N}$. We say that $\phi$ is radial if $\phi(x) = \phi(y)$ whenever $|x| = |y|$. A particular reason for requiring the radial property is that Fourier transforms of real-valued radial functions are also real-valued and radial. Let $\mu$ be an $s$-Frostman measure supported on $F$; $s$ can be arbitrarily chosen as long as $s < \dim_{\mathrm{H}} F$. Then we construct measures $\nu_g, g \in \mathbb{O}(n), \nu$ as in Definition 7.2. To start with, we choose a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^{k(k-1)/2})$ bounded by 1.5 whose support is contained in $B_1(0)$ and equal to 1.5 on $B_{1/2}(0)$. We also require that $\|\phi\|_1 = 1$. For any positive number $\delta > 0$, let $\phi_\delta(.) = \delta^{-k(k-1)/2}\phi(./\delta)$. Let $\nu_\delta = \phi_\delta * \nu$. We know that $\nu_\delta \to \nu$ as $\delta \to 0$ in the weak-* sense. By the argument in [**8**, §2] we see that

$$\int \nu_\delta^2(z)\, dz \lesssim \delta^{-n(k-1)} \int \mu^{2k} \{(x_1, \ldots, x_k, y_1, \ldots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leqslant \delta, 1 \leqslant i < j \leqslant k\}\, dg,$$

where $dg$ is the Haar measure on $\mathbb{O}(n)$. Now we claim that for each $g \in \mathbb{O}(n)$,

$$\mu^{2k} \{(x_1, \ldots, x_k, y_1, \ldots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leqslant \delta, 1 \leqslant i < j \leqslant k\} \leqslant \int \nu_g^{k-1}(B_{2\delta}(z))\, d\nu_g(z).$$

To see this, let $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$ be given, and consider the following section (we omit the coordinates $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$ as they are fixed):

$$\{(x_k, y_k) : |(x_i - gy_i) - (x_j - gy_j)| \leqslant \delta, 1 \leqslant i < j \leqslant k\}.$$

It is easy to see that the above section is contained in

$$E = \{(x_k, y_k) : |(x_k - gy_k) - (x_1 - gy_1)| \leqslant \delta\}.$$

Then we see that the $\mu^{2k}$ measure is now bounded from above by

$$\mu^{2(k-1)} \{(x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}) \in \mathbb{R}^{2(k-1)n} : |(x_i - gy_i) - (x_j - gy_j)| \leqslant \delta, 1 \leqslant i < j \leqslant k-1\} \times \int 1_E(x_k, y_k)\, d\mu(x_k)\, d\mu(y_k).$$

We see that $1_E(x_k, y_k) = f(x_k - gy_k)$ for $f : z \in \mathbb{R}^n \to f(z) = 1_{\{a : |a - (x_1 - gy_1)| \leqslant \delta\}}(z)$. Then by the definition of $\nu_g$ we see that

$$\int 1_E(x_k, y_k)\, d\mu(x_k)\, d\mu(y_k) \leqslant \nu_g(B_{2\delta}(x_1 - gy_1)).$$

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In fact, if $\nu_g$ does not give positive measure on any spheres then $\nu_g(B_{2.5\delta}(.))$ would be continuous with compact support and we would get

$$\int 1_E(x_k, y_k)\, d\mu(x_k)\, d\mu(y_k) = \nu_g(B_\delta(x_1 - gy_1)).$$

However, we do not assume this continuity of $\nu_g$ and we only have an upper bound by choosing a real-valued function in $C_0(\mathbb{R}^n)$ which is bounded from above by one, equal to one on $B_\delta(x_1 - gy_1)$ and vanishes outside $B_{2\delta}(x_1 - gy_1)$. We can do the above step $k-1$ times and as a result we see that, for each fixed $x_1, y_1$, the section (we omit the coordinates $x_1, y_1$ as they are fixed)

$$\{(x_2, \ldots, x_k, y_2, \ldots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leqslant \delta, 1 \leqslant i < j \leqslant k\}$$

has $\mu^{2k-2}$ measure at most

$$\nu_g^{k-1}(B_{2\delta}(x_1 - gy_1)).$$

By Fubini's theorem, we see that

$$\begin{aligned}\mu^{2k}\, &\{(x_1, \ldots, x_k, y_1, \ldots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leqslant \delta,\\ &\quad 1 \leqslant i < j \leqslant k\}\\ &\leqslant \int \nu_g^{k-1}(B_{2\delta}(x_1 - gy_1))\, d\mu(x_1)\, d\mu(y_1) \leqslant \int \nu_g^{k-1}(B_{2.5\delta}(z))\, d\nu_g(z).\end{aligned}$$

If $\nu_g(B_{2\delta}(.))$ were continuous, then we would have

$$\int \nu_g^{k-1}(B_{2\delta}(x_1 - gy_1))\, d\mu(x_1)\, d\mu(y_1) = \int \nu_g^{k-1}(B_{2\delta}(z))\, d\nu_g(z).$$

In general, we choose a continuous function sandwiched by $\nu_g(B_{2\delta}(.))$ and $\nu_g(B_{2.5\delta}(.))$ (by taking a convolution of a suitable smooth cutoff function with $\nu_g$) then apply the definition of $\nu_g$ to arrive at the above inequality. Thus we have shown the following estimate:

$$\int \nu_\delta^2(z)\, dz \lesssim \delta^{-n(k-1)} \iint \nu_g^{k-1}(B_{2.5\delta}(z))\, d\nu_g(z)\, dg.$$

If $\nu_g$ were absolutely continuous with respect to the Lebesgue measure for almost all $g \in \mathbb{O}(n)$, then the right-hand side above could be replaced by

$$\iint \nu_g^k(z)\, dz\, dg.$$

Then we would arrive at the same situation as in [8]. In general, we do not have this continuity at hand. To deal with this issue, let $\phi^{DD}(.)$ be a radial Schwartz function such that $\hat{\phi}^{DD}$ is non-negative, vanishes outside the ball of radius $0.5c'' > 0$ around the origin and is equal to a positive number $c > 0$ on a

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ball of radius $c' > 0$ around the origin. Now we take the square $\phi^D = (\phi^{DD})^2$ and see that

$$\hat{\phi}^D = \hat{\phi}^{DD} * \hat{\phi}^{DD}.$$

We see that $\hat{\phi}^D$ is non-negative, and vanishes outside the ball of radius $c''$ around the origin. Unlike $\hat{\phi}^{DD}$, $\hat{\phi}^D$ is no longer a constant function on any ball centred at the origin. However there is a number $c''' > 0$ such that for each $\omega$ inside $B_{c'''}(0)$, $\hat{\phi}^D(\omega)$ is greater than $0.5\hat{\phi}^D(0) > 0$ and less than $2\hat{\phi}^D(0)$. By rescaling, we may assume that $\phi^D(x) \geqslant 1$ for $x \in B_{2.5}(0)$. This can be done because $\phi^D$ is real-valued and Schwartz and $\phi^D(0) > 0$. Since $\hat{\phi}^D$ is compactly supported, we can denote $c'''' = \|\hat{\phi}^D\|_\infty$. We write $h_{g,\delta} = \nu_g * \phi^D(\delta^{-1}.)$. We see that

$$\nu_g(B_{2.5\delta}(z)) = \int_{B_{2.5\delta}(z)} d\nu_g(x) \leqslant \int \phi^D((z-x)/\delta)\, d\nu_g(x) = h_{g,\delta}(z).$$

Now we write $f_{g,\delta}(.) = \delta^{-n} h_{g,\delta}(.)$. As a result we see that

$$\int \nu_\delta^2(z)\, dz \lesssim \iint f_{g,\delta}^{k-1}(z)\, d\nu_g(z)\, dg.$$

Let $\psi$ be a smooth real-valued cutoff function supported in $\{\omega \in \mathbb{R}^n : |\omega| \in [0.5, 4]\}$ and identically equal to 1 in $\{\omega \in \mathbb{R}^n : |\omega| \in [1, 2]\}$ and bounded from above by 1. Let $f_{g,\delta,j}, \nu_{g,j}$ be the $j$th Littlewood–Paley piece of $f_{g,\delta}$, $\nu_g$ respectively, namely, $\hat{f}_{g,\delta,j}(\omega) = \hat{f}_{g,\delta}(\omega)\psi(2^{-j}\omega)$, and similarly for $\nu_{g,j}$. The rest of the argument is essentially the same as in [**8**, §3]. We need to bound $\|f_{g,\delta,j}\|_\infty$ as well as $\|\nu_{g,j}\|_\infty$. The later can be bounded by

$$C2^{j(n-s)}$$

for any $s < \dim_{\mathrm{H}} F$ with a constant $C$ depending on the function $\psi$. This was shown in [**8**, p. 805]. For the former, we will be interested in estimating $\|f_{g,\delta,j}\|_\infty$ when $2^j$ is not as large as $\delta^{-1}$. In this case, recall that $f_{g,\delta} = \nu_g * \phi_\delta^D$, and in terms of Fourier transform we have

$$\hat{f}_{g,\delta,j} = \hat{\nu}_g \hat{\phi_\delta^D} \psi(2^{-j}).$$

Recall that $\phi_\delta^D(.) = \delta^{-n}\phi^D(./\delta)$; therefore we have $\hat{\phi_\delta^D}(.) = \hat{\phi^D}(\delta.)$. Then we see that

$$\begin{aligned}
\|f_{g,\delta,j}\|_\infty &\leqslant \|\hat{f}_{g,\delta,j}\|_1 \\
&\leqslant c'''' \int |\hat{\nu}_g(\omega)\psi(2^{-j}\omega)|\, d\omega \\
&\leqslant c'''' \int_{B_{2j+2}(0)} |\hat{\mu}(\omega)\hat{\mu}(g\omega)|\, d\omega \\
&\leqslant c'''' \sqrt{\int_{B_{2j+2}(0)} |\hat{\mu}(\omega)|^2\, d\omega \int_{B_{2j+2}(0)} |\hat{\mu}(g\omega)|^2\, d\omega}.
\end{aligned}$$

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By the discussion in [15, §3.8] we see that

$$\int_{B_{2^{j+1}}(0)} |\hat{\mu}(\omega)|^2 \, d\omega \lesssim 2^{(j+2)(n-s)}.$$

The same estimate holds for $\int_{B_{2^{j+2}}(0)} |\hat{\mu}(g\omega)|^2 \, d\omega$ as well. Therefore we see that

$$\| f_{g,\delta,j} \|_\infty \leqslant C' 2^{j(n-s)},$$

where $C' > 0$ is a constant which does not depend on $g$, $j$, $\delta$. Observe that if $2^{j-1} > c''\delta^{-1}$, then $f_{g,\delta,j} = 0$, and this is the reason for considering $2^j$ to be not much larger than $\delta^{-1}$. By [15, Formula (3.27)], we have

$$\int f_{g,\delta}^{k-1}(z) \, d\nu_g(z) = \int \hat{\nu}_g(\omega) \hat{f}_{g,\delta} * \cdots * \hat{f}_{g,\delta}(-\omega) \, d\omega. \qquad (*)$$

We can apply exactly the same argument as in [8, §3]. As a result we see that

$$\int f_{g,\delta}^{k-1}(z) \, d\nu_g(z) \lesssim \sum_j 2^{j(n-s)(k-2)} \int |f_{g,\delta,j}(x) \nu_{g,j}(x)| \, dx.$$

By the Cauchy–Schwartz inequality we see that

$$\int |f_{g,\delta,j}(x) \nu_{g,j}(x)| \, dx \leqslant \sqrt{\int |f_{g,\delta,j}(x)|^2 \, dx \int |\nu_{g,j}(x)|^2 \, dx}.$$

By Plancherel's formula we see that

$$\begin{aligned} \int |f_{g,\delta,j}(x)|^2 \, dx &= \int |\hat{f}_{g,\delta,j}(\omega)|^2 \, d\omega \\ &= \int |\hat{\nu}_{g,j}(\omega)|^2 |\hat{\phi}_\delta^D(\omega)|^2 \, d\omega \leqslant (c'''')^2 \int |\hat{\nu}_{g,j}(\omega)|^2 \, d\omega. \end{aligned}$$

However, if $2^{j-1} \geqslant c''\delta^{-1}$ we see that $\hat{\phi}_\delta^D(\omega) = 0$ whenever $|\omega| \in [2^{j-1}, 2^{j+2}]$. Thus in this case we see that

$$\int |f_{g,\delta,j}(x)|^2 \, dx = \int |\hat{\nu}_{g,j}(\omega)|^2 |\hat{\phi}_\delta^D(\omega)|^2 \, d\omega = 0.$$

Therefore we see that, for $2^{j-1} \leqslant c''\delta^{-1}$,

$$\int |f_{g,\delta,j}(x) \nu_{g,j}(x)| \, dx \lesssim \int |\nu_{g,j}(x)|^2 \, dx.$$

Then by integrating the above inequality against $dg$ we see that

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$$\int \nu_\delta^2(z)\,dz \lesssim \iint f_{g,\delta}^{k-1}(z)\,d\nu_g(z)\,dg \lesssim \sum_{j:2^{j-1}\leqslant c''\delta^{-1}} 2^{j(n-s)(k-2)} \iint |\nu_{g,j}(x)|^2\,dx\,dg.$$

Notice that $\nu_{g,j}$ is real-valued. By Plancherel's formula we have

$$\iint |\nu_{g,j}(x)|^2\,dx\,dg = \iint |\hat{\nu}_{g,j}(\omega)|^2\,d\omega\,dg = \iint |\hat{\nu}_g(\omega)|^2 |\psi(2^{-j}\omega)|^2\,d\omega\,dg.$$

Since $\nu_g = \mu * g\mu$ we see that

$$\iint |\hat{\nu}_g(\omega)|^2 |\psi(2^{-j}\omega)|^2\,d\omega\,dg = \iint |\hat{\mu}(\omega)|^2 |\hat{\mu}(g\omega)|^2 |\psi(2^{-j}\omega)|^2\,d\omega\,dg.$$

Since $\psi$ is radial, for $t \geqslant 0$ we write $\psi(t)$ for the value $\psi(x)$ with an arbitrary $x$ with norm $|x| = t$. This value is well defined. Then, up to a multiple constant, the above expression on the right-hand side is equal to

$$\int \left( \int_{S^{n-1}} |\hat{\mu}(t\tau)|^2\,d\tau \right)^2 |\psi(2^{-j}t)|^2 t^{n-1}\,dt, \tag{$**$}$$

where $d\tau$ is the Lebesgue measure on the unit sphere. We see that ($**$) is bounded from above by

$$\int_{2^{j-1}}^{2^{j+2}} \left( \int_{S^{n-1}} |\hat{\mu}(t\tau)|^2\,d\tau \right)^2 t^{n-1}\,dt.$$

Now we have the following estimate:

$$\int f_{g,\delta}^{k-1}(z)\,d\nu_g(z)\,dg \lesssim \sum_{j:2^j\leqslant 2c''\delta^{-1}} 2^{j(n-s)(k-2)} \int_{2^{j-1}}^{2^{j+2}} \left( \int_{S^{n-1}} |\hat{\mu}(t\tau)|^2\,d\tau \right)^2 t^{n-1}\,dt.$$

If $k = 3$, then we have (the sum of negative values of $j$ gives a constant)

$$\int f_{g,\delta}^{2}(z)\,d\nu_g(z)\,dg \lesssim \sum_{j:1\leqslant 2^j\leqslant 2c''\delta^{-1}} 2^{j(n-s)} \int_{2^{j-1}}^{2^{j+2}} \left( \int_{S^{n-1}} |\hat{\mu}(t\tau)|^2\,d\tau \right)^2 t^{n-1}\,dt.$$

Now we are going to use [**8**, Theorem 3.1].

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THEOREM 7.4. *Let $\mu$ be a compactly supported Borel measure on $\mathbb{R}^n$. Then for $s \in (n/2, \dim_{\mathrm{H}} \mu)$, $\epsilon > 0$,*

$$\int_{S^{n-1}} |\hat{\mu}(t\tau)|^2 \, d\tau \lesssim_{\epsilon,s} t^{\epsilon-\gamma_s},$$

*where $\gamma_s = (n+2s-2)/4$ if $s \in [n/2, (n+2)/2]$ and $\gamma_s = s-1$ if $s \geqslant (n+2)/2$.*

Here $\dim_{\mathrm{H}} \mu$ is defined to be the following value:

$$\sup\left\{s > 0 : \iint |x-y|^s \, d\mu(x)\, d\mu(y) < \infty\right\}.$$

For a compact set $F$ we have $\dim_{\mathrm{H}} F = \sup_{\mu \in \mathcal{P}(F)} \dim_{\mathrm{H}} \mu$. In our situation, we recall that $s < \dim_{\mathrm{H}} F$, which was mentioned in the beginning of this proof.

If $2^{j-1} \leqslant c''\delta^{-1}$ we see that

$$\int_{2^{j-1}}^{2^{j+2}} \left(\int_{S^{n-1}} |\hat{\mu}(t\tau)|^2 \, d\tau\right)^2 t^{n-1} \, dt \lesssim 2^{j(n-s-\gamma_s)}.$$

This is because (see the beginning of [**15**, §3.8]),

$$\int_{2^{j-1}}^{2^{j+2}} \left(\int_{S^{n-1}} |\hat{\mu}(t\tau)|^2 \, d\tau\right) t^{n-1} \, dt \lesssim \int_{B(0,2^{j+2})} |\hat{\mu}(\omega)|^2 \, d\omega \lesssim 2^{j(n-s)},$$

since $\mu$ is an $s$-Frostman measure. For the sum with small $j$, we assume that $2n - 2s - \gamma_s > 0$ for otherwise $\nu$ is absolutely continuous as shown in [**8**, §3]. We have

$$\sum_{j:1\leqslant 2^j \leqslant 2c''\delta^{-1}} 2^{j(2n-2s-\gamma_s)} \lesssim_{n,s} \delta^{-(2n-2s-\gamma_s)}.$$

In all we have obtained that, for $k = 3$ and $2n - 2s - \gamma_s > 0$,

$$\int f_{g,\delta}^2(z) \, dv_g(z) \, dg \lesssim \delta^{-(2n-2s-\gamma_s)}.$$

Therefore we see that

$$\int \nu_\delta^2(z) \, dz \lesssim \int f_{g,\delta}^2(z) \, dv_g(z) \, dg \lesssim \delta^{-(2n-2s-\gamma_s)}.$$

The above estimate holds for each $\delta$ with a suitable constant in the symbol $\lesssim$. We claim the following estimate holds:

$$\int_{B_{\delta^{-1}}(0)} |\hat{\nu}(\omega)|^2 \, d\omega \lesssim \delta^{-(2n-2s-\gamma_s)}.$$

To see this, observe that $\hat{\phi}$ is Schwartz, real-valued and $\hat{\phi}(0) > 0$. Then there is a positive number $r$ such that $\hat{\phi}(\omega) \geqslant 0.5\hat{\phi}(0)$ whenever $|\omega| \leqslant r$. Then we know that

$$\begin{aligned}\int_{B_{r\delta^{-1}}(0)} |\hat{\nu}(\omega)|^2 \, d\omega &\leqslant (0.5\hat{\phi}(0))^{-2} \int_{B_{r\delta^{-1}}(0)} |\hat{\nu}(\omega)\hat{\phi}_\delta(\omega)|^2 \, d\omega \\ &\leqslant (0.5\hat{\phi}(0))^{-2} \int |\hat{\nu}(\omega)\hat{\phi}_\delta(\omega)|^2 \, d\omega.\end{aligned}$$

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Since $\nu_\delta$ is real-valued we see that

$$\int |\hat{\nu}(\omega)\hat{\phi}_\delta(\omega)|^2\,d\omega = \int |\hat{\nu}_\delta(\omega)|^2\,d\omega = \int \nu_\delta^2(z)\,dz.$$

From here the claim follows. Therefore we see that

$$\begin{aligned}&\int |\hat{\nu}(\omega)|^2|\omega|^{t-3}\,d\omega\\ &\lesssim \int_{B_1(0)} |\omega|^{t-3}\,d\omega + \sum_{j\geqslant 0} 2^{j(t-3)} \int_{|\omega|\in[2^j,2^{j+1}]} |\hat{\nu}(\omega)|^2\,d\omega < \infty\end{aligned}$$

whenever $0 < t < 3 + 2s + \gamma_s - 2n$. This implies that $\nu$ has finite $t$-energy if $t < 3 + 2s + \gamma_s - 2n$. Therefore we see that

$$\dim_{\mathrm{H}} \nu \geqslant 2s + \gamma_s - 2n + 3.$$

This concludes the proof. □

§8. *Further comments and problems.* A crucial point for the proof of Lemma 4.3 is that if we fix a point in $\mathbb{R}^2$ and we want to put $n$ different triangles with the same shape and the corresponding vertex $x$, then the other two vertices trace out a subset of two concentric circles. On each of these circles, we have at least $n/2$ points. So we get $n^2/4$ many different pairs for incidence counting. This does not hold in $\mathbb{R}^3$. For example, take a 3-simplex and put $n$ rotated copies of this simplex around a fixed point in $\mathbb{R}^3$. Then the other three points trace out three (not necessarily different) concentric spheres. However, it can happen that one of the spheres contains only one point. So we cannot hope to get roughly $n^3$ many triples for the incidence. Instead, it is possible to show that the number of triples is at least roughly $n^2$. The situation is more complicated for higher dimensions. We think that the following general result holds.

CONJECTURE 8.1. *Let $F \subset \mathbb{R}^n$, $n \geqslant 1$ be a compact set. Then the following result holds:*

$$\dim_{\mathrm{A}} \Delta_n(F) \geqslant \frac{n}{2} \dim_{\mathrm{A}} F.$$

More generally we can ask the following question.

*Question* 8.2. Let $F \subset \mathbb{R}^n$ with $n \geqslant 1$. Then for an integer $k \in [2, n+1]$ what is the relation between $\dim_{\mathrm{A}} F$ and $\Delta_k(F)$?

In particular, when $k = 2$ we encounter the distance set problem and in this case we have the following result by [**5**, Theorem 2.9].

THEOREM. *Let $F \subset \mathbb{R}^n$ with $n \geqslant 1$. We have the following result:*

$$\dim_{\mathrm{A}} D(F) \geqslant \frac{1}{d} \dim_{\mathrm{A}} F.$$

*The inequality is strict unless* $\dim_{\mathrm{A}} D(F) = 1$.

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## *References*

**1.** V. Chan, I. Łaba and M. Pramanik, Finite configurations in sparse sets. *J. Anal. Math.* **128** (2016), 289–335.
**2.** K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley (2004).
**3.** J. Fraser, Assouad type dimensions and homogeneity of fractals. *Trans. Amer. Math. Soc.* **366** (2014), 6687–6733.
**4.** J. Fraser, Distance sets, orthogonal projections, and passing to weak tangents. *Israel J. Math.* **226** (2018), 851–875.
**5.** J. Fraser, D. Howroyd and H. Yu, Dimension growth for iterated sumsets. *Math. Z.* (to appear). *Preprint*, 2018, arXiv:1802.03324.
**6.** L. Grafakos, A. Greenleaf, A. Iosevich and E. Palsson, Multilinear generalized Radon transforms and point configurations. *Forum Math.* **27** (2015), 2323–2360.
**7.** A. Greenleaf and A. Iosevich, On triangles determined by subsets of the Euclidean plane, the associated bilinear operators and applications to discrete geometry. *Anal. PDE* **5**(2) (2012), 397–409.
**8.** A. Greenleaf, A. Iosevich, B. Liu and E. Palsson, A group-theoretic viewpoint on Erdős–Falconer problems and the Mattila integral. *Rev. Mat. Iberoam.* **31**(3) (2015), 799–810.
**9.** L. Guth and N. Katz, On the Erdős distances problem in the plane. *Ann. of Math. (2)* **181** (2015), 155–190.
**10.** A. Käenmäki, T. Ojala and E. Rossi, Rigidity of quasisymmetric mappings on self-affine carpets. *Int. Math. Res. Not. IMRN* **2018** (2018), 3769–3799.
**11.** N. Katz and T. Tao, Some connections between Falconer's Distance set conjecture and sets of Furstenburg type. *New York J. Math.* **7** (2001), 149–187.
**12.** J. Mackay and J. Tyson, *Conformal Dimension. Theory and Application* (*University Lecture Series* **54**), American Mathematical Society (Providence, RI, 2010).
**13.** P. Maga, Full dimensional sets without given patterns. *Real Anal. Exchange* **36**(1) (2010/11), 79–90.
**14.** P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability* (*Cambridge Studies in Advanced Mathematics*), Cambridge University Press (1999).
**15.** P. Mattila, *Fourier Analysis and Hausdorff Dimension* (*Cambridge Studies in Advanced Mathematics*), Cambridge University Press (2015).
**16.** P. Shmerkin, On the Hausdorff dimension of pinned distance sets. *Israel J. Math.* (to appear). *Preprint*, 2017, arXiv:1706.00131.
**17.** P. Shmerkin and V. Suomala, Patterns in random fractals. *Amer. J. Math.* (to appear). *Preprint*, 2017, arXiv:1703.09553.

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