Math. Proc. Camb. Phil. Soc. (2011), **150**, 459 © *Eugenia Cheng* [2011] *Published by Cambridge Philosophical Society*
doi:10.1017/S0305004110000599

Iterated distributive laws

BY EUGENIA CHENG

Laboratoire J. A. Dieudonné, Université de Nice-Sophia Antipolis, Parc Valrose, 06198 Nice, France
and
*Department of Pure Mathematics, University of Sheffield, Hounsfield Road, Sheffield S*1 2*EH, UK*
e-mail: e.cheng@sheffield.ac.uk

(*Received* 19 *June* 2008; *revised* 4 *August* 2010)

Abstract

We give a framework for combining n monads on the same category via distributive laws satisfying Yang–Baxter equations, extending the classical result of Beck which combines two monads via one distributive law. We show that this corresponds to iterating n-times the process of taking the 2-category of monads in a 2-category, extending the result of Street characterising distributive laws. We show that this framework can be used to construct the free strict n-category monad on n-dimensional globular sets; we first construct for each i a monad for composition along bounding i-cells, and then we show that the interchange laws define distributive laws between these monads, satisfying the necessary Yang–Baxter equations.

Introduction

Monads give us a way of describing algebraic structures such as monoids, groups, rings and categories. Distributive laws between monads give us a way of combining two such algebraic structures in a coherent way. For example, a ring is a monoid under multiplication and an abelian group under multiplication, where the mulplication and addition must interact coherently. We can thus construct the monad for rings using the monad for monoids and the monad for abelian groups, via a distributive law. In this work we describe how to extend this to combine three or more algebraic structures in a coherent way. Our motivating example is the construction of strict n-categories; other examples are rings, constructed using three monads instead of the usual two, and rigs (semi-rings), constructed using four. The theory can also be used for other versions of n-category with strict interchange, as we will discuss later.

The starting point for distributive laws is that we have two monads S and T, say, on a category $\mathcal{C}$. We might then want to ask: can we compose them to get a monad TS? If we think of the action of a monad as building in algebraic structure freely then TS would first build in S-like structure freely, and then T-like structure "on top". For this to be a monad we can ask that the S-like structure "distribute" over the T-like structure, via a natural transformation

$$ST \overset{\lambda}{\Longrightarrow} TS$$

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

which we can think of as "moving the S structure across the T-structure". We can then apply the multiplication for S and T to get a putative multiplication for TS

$$TSTS \xrightarrow{T\lambda S} T^2S^2 \xrightarrow{\mu^T\mu^S} TS;$$

of course, we then need to check that this satisfies the associativity axiom for a monad.

A *distributive law* of S over T is defined in [**2**] to be a natural transformation λ as above, satisfying axioms ensuring coherent interaction with the monad structures for S and T. One consequence is that the putative multiplication we defined above really does make TS into a monad (with unit $\eta^T\eta^S : 1 \Rightarrow TS$); note that distributive laws are directional, and ST does not become a monad.

We can now examine the situation involving three monads A, B and C, say, on the same category, with distributive laws

$$\lambda : BA \Longrightarrow AB$$
$$\sigma : CA \Longrightarrow AC$$
$$\tau : CB \Longrightarrow BC.$$

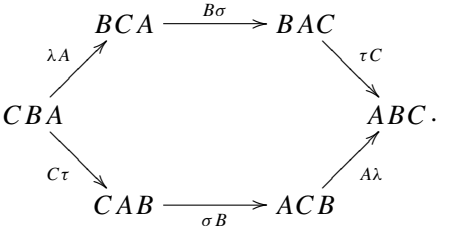
By the above distributive law result, we have canonical monads AB, AC and BC; we can further ask if we have a monad ABC. This would certainly follow from the theory of distributive laws if we had a distributive law

$$(BC)A \Longrightarrow A(BC)$$

or indeed

$$C(AB) \Longrightarrow (AB)C$$

each of which would give a canonical monad ABC. It turns out that although we can easily construct such maps, they will not automatically be distributive laws. However we can ensure that they are distributive laws by imposing the following axiom:



If we examine how A, B, and C are "switched" past each other in this diagram, we see that it is none other than the Yang–Baxter equation. This turns out to be all we need to make sure we induce the two new distributive laws above. Moreover, the two resulting monad structures on ABC coincide. The BD-laws of [**9**] are an example of this sort of structure, with $A = B = C$ and $\lambda = \sigma = \tau$.

More generally we can consider a series of n monads $T_1, \ldots, T_n$ on a category, with pairwise distributive laws going in one "direction" only, that is, for all $i > j$ a distributive law

$$T_iT_j \Longrightarrow T_jT_i$$

where every three monads satisfies the Yang–Baxter equation; we state this precisely in the main theorem (Theorem 1·6) and call it a "distributive series of monads". This turns out to

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

be enough to ensure that the monads are as coherent as we want – they can be composed (in fixed order) to produce one combined monad. There are no further conditions that need to be imposed other than the three-way Yang–Baxter conditions. Note that this idea is essentially the same as the fact that, to express the braid group by generators and relations, a single crossing and the Yang–Baxter equation suffice.

There are (at least) two ways to prove this theorem:

(1) direct proof by a diagram chase and induction – this is easy but unilluminating;
(2) proof by iterating the $\mathbf{Mnd}(\mathcal{B})$ of construction of [**14**] and using the lax Gray tensor product.

The second, more abstract, proof is the subject of Section 2, but we include the proof by induction in the Appendix. The idea is that we can define monads inside any 2-category $\mathcal{B}$ – "ordinary" monads happen to be inside **Cat**, the 2-category of categories, functors and natural transformations. Moreover, in [**14**] Street also defines monad functors and monad transformations, so that given any 2-category $\mathcal{B}$ there is a 2-category $\mathbf{Mnd}(\mathcal{B})$ of monads in $\mathcal{B}$. So we can iterate this and take monads in $\mathbf{Mnd}(\mathcal{B})$; at the end of [**14**] Street shows that monads in $\mathbf{Mnd}(\mathcal{B})$ are precisely distributive laws. That is, a 0-cell in $\mathbf{Mnd}(\mathbf{Mnd}(\mathcal{B}))$ is given by a 0-cell in $\mathcal{B}$, two monads on it, and a distributive law of one over the other. We now iterate this n times and get the 2-category $\mathbf{Mnd}^n(\mathcal{B})$, and prove that an object in $\mathbf{Mnd}^n(\mathcal{B})$ is precisely a distributive series of n monads as in the main theorem described above. Again a direct proof by induction is not difficult, but we prefer a more abstract approach using the characterisation of monads as 2-functors from the "free-living monad" 2-category Δ to $\mathcal{B}$.

Using the closed structure of **2-Cat** with respect to the lax Gray tensor product (see [**5**, **6**]), we see that

$$\mathbf{Mnd}^n(\mathcal{B}) \cong \mathbf{Str}(\Delta^{\otimes n}, \mathcal{B}) \cong \mathbf{LaxCubical}(\Delta^n, \mathcal{B})$$

where **Str** denotes the 2-category of strict 2-functors, strict transformations and modifications, and **LaxCubical** denotes the 2-category of *lax cubical functors in n variables*, lax transformations and modifications. Cubical functors may be thought of as a kind of "semi-strict" functor in which only some of the constraint data is required to be strict.

Examining the elementary characterisation of cubical functors given by Gray [**5**], we see exactly how the series of n monads, the distributive laws, and the Yang–Baxter equations arise in Theorem 1·6, as the data for a cubical functor consists of functors on each variable, "interchange" constraints involving pairs of variables, and Yang–Baxter-like conditions. This gives our most abstract and conceptual proof of the main theorem.

In Section 3 we present our motivating example, the construction of the free strict n-category monad (on n-dimensional globular sets) by building up the monad from n separate monads for composition. That is, we isolate composition along i-cell boundaries for each $0 \leqslant i \leqslant n-1$, and construct for each i a monad T_i that gives this composition alone. These monads are simply a generalisation of the usual free category monad on graphs. It is then an interesting fact that the required distributive laws come from the interchange laws for a strict n-category – for all $i > j$ there is a strict interchange law for i-composition over j-composition generalising the usual interchange law in 2-categories, and this does indeed define a distributive law of the monad T_i over the monad T_j. Moreover, these distributive laws satisfy the Yang–Baxter equation for all $i > j > k$. Thus, using Theorem 1·6 we can construct the free strict n-category monad as the composite of the monads T_i, in order.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

This is reminiscent of the definition of strict n-category as an n-globular set in which every sub-2-globular set is a 2-category, that is, where interchange holds for every pair of dimensions (see [**15**]).

It is worth commenting briefly on the notion of strict interchange. Interchange in n-categories is sometimes thought of as "the only part of weak n-categories that cannot be strictified". More precisely, we know that not every tricategory is triequivalent to a completely strict 3-category; the well-known coherence result is that every tricategory is triequivalent to a Gray-category [**4**, **7**]. This has led to a generally accepted conjecture that every weak n-category should be equivalent to a semi-strict one, where "semi-strict" means that every thing is strict *apart* from interchange – interchange thus being the only part that cannot be strictified. However, recent work of Joyal and Kock [**8**] has shown that a different type of "semi-strict 3-category" is also fruitful: one in which everything is strict apart from units. Joyal and Kock have shown that such 3-categories do model homotopy 3-types, and do give rise to braided monoidal categories in the suitably degenerate case, two crucial things that fully strict 3-categories cannot do. Indeed Simpson [**12**, **13**] conjectures that n-categories with weak units (and everything else strict) are enough to model n-types for all n.

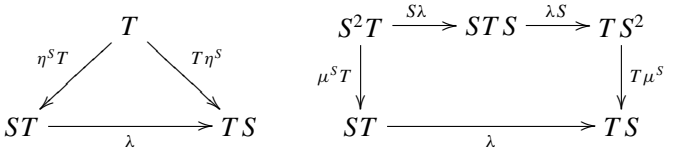
In the light of these recent results we consider the study of n-categories with strict interchange to be important, and the results of the present work will contribute towards that study. In [**3**] we proceed to use the theory of iterated distributive laws to study Trimble's notion of n-category [**16**], which has strict interchange but weak units. The results of the present work enable us to construct a comparison functor from Trimble n-categories to those defined by Batanin [**1**].

1. *Basic definitions and examples*

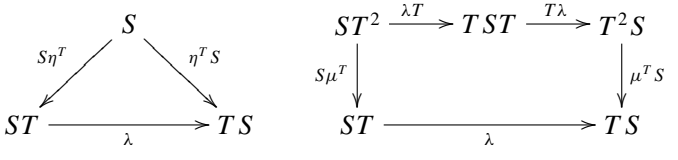
1·1. *Distributive laws*

We first recall the classical theory of distributive laws.

Definition 1·1. (Beck [**2**]). Let S and T be monads on a category $\mathcal{C}$. A **distributive law of S over T** consists of a natural transformation $\lambda\colon ST \Rightarrow TS$ such that the following diagrams commute.



(1·1)



(1·2)

Note that the first pair of axioms (1·1) says that "λ interacts coherently with the monad structure for S" and the second pair (1·2) that "λ interacts coherently with the monad structure for T".

The main theorem about distributive laws tells us about new monads that arise canonically as a result of the distributive law.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

THEOREM 1·2 ([**2**]). *The following are equivalent*:
(i) *a distributive law of S over T*;
(ii) *a lifting of the monad T to a monad T' on S-*Alg;
(iii) *an extension of the monad S to a monad S' on* Kl(T).
It follows that TS canonically acquires the structure of a monad, whose category of algebras coincides with that of the lifted monad T', and whose Kleisli category coincides with that of S'.

Remark. We will not be using the Kleisli part of this theorem in this work.

The idea is that T lifts to a monad on S-algebras via λ, with its action on an S-algebra θ given by

$$\begin{array}{ccc} SA & & STA \\ \downarrow{\scriptstyle\theta} & \longmapsto & \downarrow{\scriptstyle\lambda_A} \\ A & & TSA \\ & & \downarrow{\scriptstyle T\theta} \\ & & TA \end{array},$$

TS becomes a monad with multiplication

$$TSTS \xRightarrow{T\lambda S} T^2S^2 \xRightarrow{\mu^T\mu^S} TS;$$

and unit $\eta^T\eta^S\colon 1 \Rightarrow TS$. The axioms for λ precisely ensure that both of these satisfy the axioms for a monad.

Example 1·3 (**Rings**).
$\mathcal{C} =$ **Set**;
$S =$ free monoid monad;
$T =$ free abelian group monad;
$\lambda =$ the usual distributive law for multiplication and addition, e.g.,

$$(a+b)(c+d) \longmapsto ac + bc + ad + bd.$$

Then the composite monad TS is the free ring monad.

Example 1·4 (**Monoids**).
$\mathcal{C} =$ **Set**;
$S =$ monad for non-unital associative multiplication;
$T =$ monad for pointed sets ie $TA = A \coprod \{*\}$;
λ ensures that $*$ acts as a unit for multiplication:

$$\begin{array}{ccc} S(A \coprod \{*\}) & \longrightarrow & SA \coprod \{*\} \\ a_1 \ldots . a_i . * . a_{i+1} \ldots . a_n & \longmapsto & a_1 \ldots . a_n . \end{array}$$

Then the composite monad TS is the free monoid monad.

Example 1·5 (**2-categories**).
$\mathcal{C} =$ **2-GSet**, the category of 2-globular sets, that is diagrams in **Set**

$$A(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1) \underset{t}{\overset{s}{\rightrightarrows}} A(0)$$

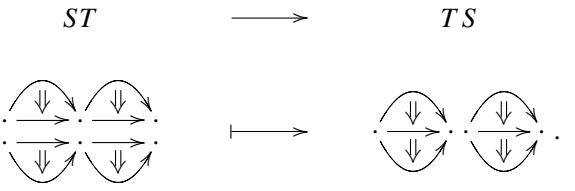
https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

with $ss = ts, ts = tt$;

$S =$ monad for vertical composition of 2-cells ($A(1)$ and $A(0)$ are unchanged);

$T =$ monad for horizontal composition of 2-cells and 1-cells ($A(0)$ is unchanged);

λ is given by the interchange law, e.g.;



Note that this distributive law can only go in this direction; we will discuss this in more detail in Section 3. The fact that this is a distributive law can be proved by direct calculation; alternatively it follows from abstract results that we give in Theorem 3·8.

1·2. *Iterated distributive laws*

The main theorem of this paper generalises the notion of distributive law to the case when we have more than two monads interacting with each other. Here we state the theorem with some basic examples; the proof is the subject of the next section.

1·3. *The main theorem*

THEOREM 1·6. *Fix $n \geqslant 3$. Let $T_1, \ldots, T_n$ be monads on a category $\mathcal{C}$, equipped with:*

(i) *for all $i > j$ a distributive law $\lambda_{ij} : T_i T_j \Rightarrow T_j T_i$, satisfying*:

(ii) *for all $i > j > k$ the "Yang–Baxter" equation given by the commutativity of the following diagram*:

$$\begin{array}{ccccc} & & T_j T_i T_k & \xrightarrow{T_j \lambda_{ik}} & T_j T_k T_i & & \\ & \overset{\lambda_{ij} T_k}{\nearrow} & & & & \overset{\lambda_{jk} T_i}{\searrow} & \\ T_i T_j T_k & & & & & & T_k T_j T_i \\ & \underset{T_i \lambda_{jk}}{\searrow} & & & & \underset{T_k \lambda_{ij}}{\nearrow} & \\ & & T_i T_k T_j & \xrightarrow[\lambda_{ik} T_j]{} & T_k T_i T_j & & \end{array} \tag{1·3}$$

Then for all $1 \leqslant i < n$ we have canonical monads

$$T_1 T_2 \cdots T_i \quad \text{and} \quad T_{i+1} T_{i+2} \cdots T_n$$

together with a distributive law of $T_{i+1} T_{i+2} \cdots T_n$ over $T_1 T_2 \cdots T_i$ i.e.

$$(T_{i+1} T_{i+2} \cdots T_n)(T_1 T_2 \cdots T_i) \Longrightarrow (T_1 T_2 \cdots T_i)(T_{i+1} T_{i+2} \cdots T_n)$$

given by the obvious composites of the λ_{ij}. Moreover, all the induced monad structures on $T_1 T_2 \cdots T_n$ are the same.

Definition 1·7. A **distributive series of n monads** is a system of monads and distributive laws as in Theorem 1·6.

We conclude this section with two brief examples of iterated distributive laws; our main example, that of interchange for n-categories, will be treated in Section 3.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

Example 1·8 (**Rings**). The usual example for distributive laws involves constructing the free ring monad from the free commutative monoid monad and the free abelian group monad (Example 1·3). However, we can also construct it from the following distributive series of 3 monads on **Set**, combining Example 1·3 with Example 1·4:

A = monad for associative non-unital binary multiplication $\times$;

B = monad for pointed sets i.e. $X \mapsto X \coprod \{1\}$;

C = free additive abelian group monad.

We have distributive laws:

(i) $AB \Rightarrow BA$ as in Example 1·4, ensuring that 1 acts as a unit for $\times$;

(ii) $AC \Rightarrow CA$ as in Example 1·3, the usual distributive law of $\times$ over $+$, but without units; and

(iii) $BC \Rightarrow CB$ is the obvious embedding;

and it is easy to check the Yang–Baxter equation. The composite monad CBA is then the free ring monad.

Example 1·9 (**Rigs**). If we have no negatives we can further decompose the situation with the following four monads:

A = monad for associative non-unital binary multiplication $\times$;

B = monad for pointed sets i.e. $X \mapsto X \coprod \{1\}$;

C = monad for associative non-unital non-invertible $+$;

$D = X \mapsto X \coprod \{0\}$;

and we have distributive laws as follows:

(i) $AB \Rightarrow BA$ as above;

(ii) $AC \Rightarrow CA$ as above but without 0;

(iii) $AD \Rightarrow DA$ ensures that $x \times 0 = 0$;

(iv) $BC \Rightarrow CB$ as above but without 0;

(v) $BD \Rightarrow DB$ is given by the obvious map $X \coprod \{0\} \coprod \{1\} \longrightarrow X \coprod \{1\} \coprod \{0\}$;

(vi) $CD \Rightarrow DC$ ensures that $x + 0 = 0$.

Then we can check all the required Yang–Baxter equations, and the resulting composite monad $DCBA$ is the free rig monad.

2. *Iterated distributive laws via the formal theory of monads*

In his classic paper *The formal theory of monads* [**14**] Street defines for any 2-category $\mathcal{B}$ a 2-category **Mnd**($\mathcal{B}$) of monads in $\mathcal{B}$. Then distributive laws arise as monads in **Mnd**($\mathcal{B}$). In this section we show that iterating this process gives the distributive series of monads defined in the previous section. There we only considered the case $\mathcal{B} =$ **Cat**, but we now treat the case for a general 2-category $\mathcal{B}$.

2·1. *The* 2*-category of monads in* $\mathcal{B}$

First we recall the basic definitions as given in [**14**]; none of the material in this section is new.

Definition 2·1. Let $\mathcal{B}$ be a 2-category. A monad in $\mathcal{B}$ is given by:

(i) a 0-cell X;

(ii) a 1-cell $S : X \to X$; and

(iii) 2-cells $1 \overset{\eta}{\Rightarrow} S \overset{\mu}{\Leftarrow} S^2$; satisfying

(iv) the usual monad axioms

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

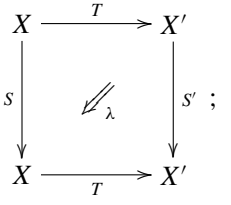


(2·1)

A monad morphism $(X, S) \to (X', S')$ is given by:

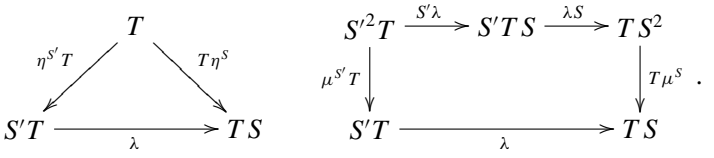
(i) a 1-cell $T : X \to X'$; and

(ii) a 2-cell $\lambda : S'T \Rightarrow TS$; i.e.



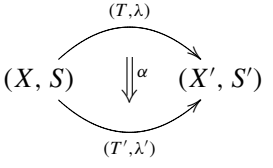
such that:

(iii) the following diagrams commute:



(2·2)

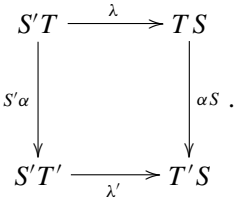
A monad transformation



is given by:

(i) a 2-cell $\alpha : T \Rightarrow T'$; such that

(ii) the following diagram commutes



(2·3)

Furthermore, these data organise themselves into a 2-category **Mnd**($\mathcal{B}$) as follows:

(i) 1-cell composition

$$(X, S) \xrightarrow{(T,\lambda)} (X', S') \xrightarrow{(T',\lambda')} (X'', S'') \;=\; (X, S) \xrightarrow{(T'T,\; T'\lambda \circ \lambda' T)} (X'', S'');$$

(ii) 1-cell identity (1, 1);

(iii) 2-cell composition is inherited from $\mathcal{B}$.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

Remark 2·2. In fact there is a functor

$$\begin{aligned} \mathbf{Mnd}\colon \mathbf{2\text{-}Cat} &\longrightarrow \mathbf{2\text{-}Cat} \\ \mathcal{B} &\longmapsto \mathbf{Mnd}(\mathcal{B}), \end{aligned}$$

and moreover it is a monad [**14**].

Given the notion of monads in any 2-category, we of course have the notion of distributive law between monads in any 2-category. The following theorem of Street characterises distributive laws abstractly.

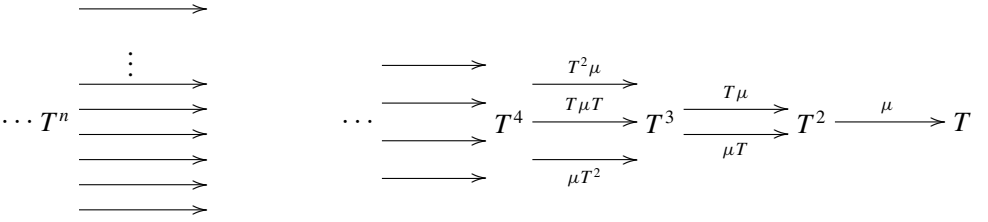
THEOREM 2·3 (Street [**14**]). *A monad in* $\mathbf{Mnd}(\mathcal{B})$ *is a* 0*-cell* X*, a pair of monads* S *and* T *and a distributive law* $\lambda : ST \Rightarrow TS$.

Proof (*sketch*). A monad in $\mathbf{Mnd}(\mathcal{B})$ consists of:
(i) a 0-cell (X, S);
(ii) a 1-cell $(X, S) \xrightarrow{(T,\lambda)} (X, S)$; and
(iii) 2-cells $(1, 1) \overset{\eta}{\Rightarrow} (T, \lambda) \overset{\mu}{\Leftarrow} (T, \lambda)^2 = (T^2, T\lambda \circ \lambda T)$;
satisfying axioms. Hence *a priori* we have a monad S, an endomorphism T, and a 2-cell $\lambda : ST \Rightarrow TS$. Now η and μ are 2-cells of $\mathbf{Mnd}(\mathcal{B})$ so are given by 2-cells of $\mathcal{B}$; these together with the axioms making (T, λ) a monad in $\mathbf{Mnd}(\mathcal{B})$ make T into a monad in $\mathcal{B}$. The axioms for (T, λ) being a 1-cell of $\mathbf{Mnd}(\mathcal{B})$ give the first two axioms (1·1) for a distributive law (interaction with the monad structure of S) and the axioms for η and μ to be 2-cells of $\mathbf{Mnd}(\mathcal{B})$ give the second two axioms (1·2) for a distributive law (interaction with the monad structure of T).

Our aim now is to iterate the $\mathbf{Mnd}(\mathcal{B})$ construction. We will write $\mathbf{Mnd}^2(\mathcal{B})$ for $\mathbf{Mnd}(\mathbf{Mnd}(\mathcal{B}))$. Theorem 2·3 tells us what the 0-cells of $\mathbf{Mnd}^2(\mathcal{B})$ are; we will prove that a 0-cell of $\mathbf{Mnd}^n(\mathcal{B})$ is precisely a distributive series of n monads, as defined in the previous section. A direct proof by induction is not difficult, but a more elegant approach uses the *lax Gray tensor product* and simplicial resolution of monads. This is the subject of the rest of this section.

2·2. *Simplicial resolution of a monad*

Recall that given any monad T we can construct its simplicial resolution:



with various commuting conditions ensuring, among other things, that the diagram yields a unique morphism from T^n to T. Note that we have only drawn the multiplications (face maps) in this diagram; there are also degeneracies corresponding to applications of the unit for the monad.

We can use the simplicial resolution of monads to express monads in $\mathcal{B}$ as 2-functors from a certain 2-category Δ to $\mathcal{B}$.

Let Δ be the "free-living monad" 2-category of (possibly empty) finite ordinals (also known as "algebraic Δ"). Δ is more commonly thought of as a *category* whose objects are

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

the finite ordinals, but it has a monoidal structure given by addition; thus it can be considered as a bicategory with only one 0-cell, 1-cells given by the natural numbers, and 2-cells $n \to m$ given by order-preserving maps

$$\{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}.$$

Then Δ is the universal 2-category containing a monad, thus a monad in a 2-category $\mathcal{B}$ can be expressed as a (strict) functor $\Delta \to \mathcal{B}$. The single 0-cell of Δ picks out an underlying 0-cell X of $\mathcal{B}$ and the rest of Δ picks out a monad on X by specifying its entire simplicial resolution. Furthermore, a monad map is a lax transformation between functors, and a monad transformation is a modification.

Definition 2·4. Given 2-categories A and B, write $\mathbf{Lax}(A, B)$ for the 2-category whose 0-cells are strict functors $A \to B$, 1-cells are lax transformations and 2-cells are modifications.

The following is a standard result; the fact that the monoidal category Δ classifies monoids in a monoidal category is given in [**11**].

PROPOSITION 2·5. *There is an isomorphism of* 2*-categories*

$$\mathbf{Mnd}(\mathcal{B}) \cong \mathbf{Lax}(\Delta, \mathcal{B}).$$

In fact $\mathbf{Lax}(A, B)$ gives us a monoidal closed structure with respect to the *lax Gray tensor product*, and this greatly facilitates the iteration of the $\mathbf{Mnd}(\mathcal{B})$ construction; this is the subject of the next section.

2·3. *The lax Gray tensor product and lax cubical functors*

In this section we recall some basic results about the lax Gray tensor product. The idea is that since this tensor product gives a closed structure on the category **2-Cat** i.e. a natural isomorphism

$$\mathbf{2\text{-}Cat}(A \otimes B, C) \cong \mathbf{2\text{-}Cat}(A, \mathbf{Lax}(B, C)),$$

we can then write

$$\mathbf{Mnd}^2(\mathcal{B}) = \mathbf{Lax}(\Delta, \mathbf{Lax}(\Delta, \mathcal{B})) \cong \mathbf{Lax}(\Delta \otimes \Delta, \mathcal{B})$$

and indeed

$$\mathbf{Mnd}^n(\mathcal{B}) \cong \mathbf{Lax}(\Delta^{\otimes n}, \mathcal{B}).$$

We begin with a gentle introduction to the lax Gray tensor product which experts may wish to skip. This tensor product was introduced in [**5**]; we will also be using material from [**4**].

We first consider the closed structure in the strict case. Writing $\mathbf{Str}(B, C)$ for the 2-category of strict functors, strict transformations and modifications, we get a straightforward cartesian closed structure on **2-Cat** with

$$\mathbf{2\text{-}Cat}(A \times B, C) \cong \mathbf{2\text{-}Cat}(A, \mathbf{Str}(B, C)).$$

In order to understand the lax case giving rise to the lax Gray tensor product, it is instructive to consider the "free-living 1-cell" 2-category $\mathbb{2}$, with two 0-cells x, y and one 1-cell $f : x \to y$. A functor $\mathbb{2} \to C$ just picks out a 1-cell in C, and a functor

$$\mathbb{2} \longrightarrow \mathbf{Str}(\mathbb{2}, C)$$

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

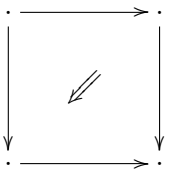
picks out a commuting square in C. We can then check the left-hand-side of the above isomorphism and we see that $\mathbb{2} \times \mathbb{2}$ is indeed the "free-living commuting square"

$$\begin{array}{ccc} (0,0) & \xrightarrow{(1,f)} & (0,1) \\ {\scriptstyle (f,1)}\downarrow & & \downarrow{\scriptstyle (f,1)} \\ (1,0) & \xrightarrow[(1,f)]{} & (1,1) \end{array}.$$

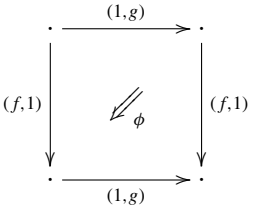
If we now consider the case with **Str**(B,C) replaced with the lax version **Lax**(B, C), we see that a functor

$$\mathbb{2} \longrightarrow \mathbf{Lax}(\mathbb{2}, C)$$

now picks out a square in C filled in by a non-invertible 2-cell:



Thus we need the lax Gray tensor product $\mathbb{2} \otimes \mathbb{2}$ to be the free-living such diagram. This gives us all the ideas we need to define $A \otimes B$ – essentially, whenever we have 1-cells $f \in A$, $g \in B$, we must have a 2-cell



The full definition by generators and relations is a little laborious as we have:

(i) 0-cells given by the 0-cells of $A \times B$;

(ii) 1-cells generated by cells of the form $(f, 1)$ and $(1, g)$;

(iii) 2-cells generated by cells of the form $(\alpha, 1)$ and $(1, \beta)$ similarly, together with 2-cells of the form ϕ as above; and

(iv) relations ensuring all this interacts properly with the existing structure of A and B.

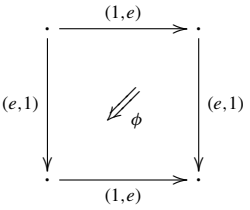
We write **LaxGray** for the resulting monoidal category of 2-categories, strict 2-functors and the lax Gray tensor product.

Example 2·6. Let us write the 1-cells of Δ as $1, e, e^2, e^3, \ldots$. Then $\Delta \otimes \Delta$ has 1-cells generated by $(1, e)$ and $(e, 1)$. Now whereas in $\Delta \times \Delta$ we have the relation

$$(e, 1) \circ (1, e) = (1, e) \circ (e, 1)$$

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

in $\Delta \otimes \Delta$ we instead have a 2-cell generator $(e, 1) \circ (1, e) \Rightarrow (1, e) \circ (e, 1)$



We now describe how a functor

$$\alpha : \Delta \otimes \Delta \longrightarrow \mathcal{B}$$

corresponds to an object of $\mathbf{Mnd}^2(\mathcal{B})$, that is, monads S and T and a distributive law $\lambda : ST \Rightarrow TS$.

First recall that a functor $\theta : \Delta \to \mathcal{B}$ gives us a monad in $\mathcal{B}$ by giving us the entire simplicial resolution of a monad, thus the image of e gives the functor part of the monad. So for the case above we can set $S = \alpha(e, 1)$ and $T = \alpha(1, e)$ and these are automatically monads. For the distributive law recall that in $\Delta \otimes \Delta$ we have the 2-cell ϕ above. Now α applied to the source of ϕ gives ST, and applied to the target it gives TS. So we have

$$\alpha(\phi) : ST \Longrightarrow TS$$

and this can be shown to be a distributive law. Furthermore, to find the composite monad TS we use the diagonal functor, which is the strict functor generated by

$$\begin{array}{ccc} \Delta & \xrightarrow{\delta} & \Delta \otimes \Delta \\ e & \longmapsto & (1, e) \circ (e, 1). \end{array}$$

Then given any $\alpha : \Delta \otimes \Delta \to \mathcal{B}$ corresponding to (S, T, λ) we get the functor

$$\Delta \xrightarrow{\delta} \Delta \otimes \Delta \xrightarrow{\alpha} \mathcal{B}$$

corresponding to the monad TS.

A more elegant (and for our purposes efficacious) way to deal with this tensor product is via its universal property. Just as the tensor product $A \otimes B$ of vector spaces is universal with respect to bilinear maps out of $A \times B$, so the lax Gray tensor product is universal with respect to "cubical functors" out of $A \times B$. Informally, strict functors $A \otimes B \to C$ correspond to cubical functors $A \times B \to C$. This is relevant for us as we are interested in studying strict functors $\Delta^{\otimes n} \to \mathcal{B}$; we have reduced the problem to studying cubical functors $\Delta^n \to \mathcal{B}$.

The following definition is introduced in [**5**] under the name "quasi-functor of n variables". The idea is that a lax cubical functor is a "semi-strict" lax 2-functor – some of the functoriality constraint cells are required to be the identity.

Definition 2·7. Let $A_1, \ldots, A_n, C$ be 2-categories. A **lax cubical functor of n variables**

$$F : A_1 \times \cdots \times A_n \longrightarrow C$$

is a lax 2-functor such that, given composable morphisms

$$(a_1, \ldots, a_n) \xrightarrow{(f_1, \ldots, f_n)} (a'_1, \ldots, a'_n) \xrightarrow{(f'_1, \ldots, f'_n)} (a''_1, \ldots, a''_n) \in A_1 \times \cdots \times A_n$$

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

satisfying the condition

$$i < j \implies \text{either } f_i = \mathbf{id} \text{ or } g_i = \mathbf{id}$$

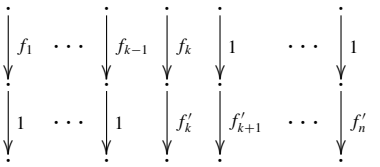
then the constraint cell

$$F(f'_1, \ldots, f'_n) \circ F(f_1, \ldots, f_n) \longrightarrow F((f'_1, \ldots, f'_n) \circ (f_1, \ldots, f_n))$$

is the identity.

LaxCubical$(A \times B, C)$ is the 2-category whose 0-cells are the lax cubical functors $A \times B \to C$, 1-cells are lax transformations between them, and 2-cells are modifications.

Remark 2·8. Note that this condition for lax cubical functors means that the constraint cell is the identity for composites of the form



Note also that this condition ensures in particular that if we fix all but the ith variable, we get a *strict* 2-functor $A_i \to C$.

THEOREM 2·9 (Gray [**5**, I·4·2). *There is natural isomorphism of* 2*-categories*

$$\mathbf{LaxCubical}(A \times B, C) \cong \mathbf{Lax}(A, \mathbf{Lax}(B, C));$$

note that these are the components of a natural isomorphism between functors, not 2*-functors.*

Remark 2·10. It follows that $\mathbf{LaxCubical}(A \times B, C) \cong \mathbf{Str}(A \otimes B, C)$.

COROLLARY 2·11. *There is an isomorphism of* 2*-categories.*

$$\mathbf{Mnd}(\mathbf{Mnd}(\mathcal{B})) \cong \mathbf{LaxCubical}(\Delta \times \Delta, \mathcal{B}).$$

Remark 2·12. Thus a cubical functor $F\colon \Delta \times \Delta \to \mathcal{B}$ specifies a pair of monads in $\mathcal{B}$ equipped with a distributive law between them. It will help us later to make this correspondence more explicit.

The 2-category Δ has only one 0-cell $*$, say, and we write $F(*) = X$ which will be the underlying 0-cell of $\mathcal{B}$ on which our two monads act.

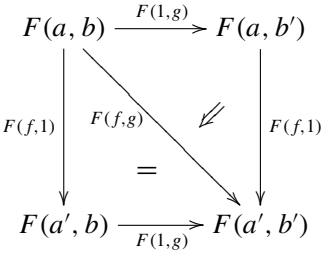
As before, we write e for the generating 1-cell of Δ. Then fixing each variable of the cubical functor in turn gives us a strict functor $\Delta \to \mathcal{B}$, thus a monad on X. We write $F(e, 1) = S$ and $F(1, e) = T$, the 1-cell parts of the monads in question.

For the distributive law $ST \Rightarrow TS$ we note that for a cubical functor

$$A \times B \longrightarrow C$$

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

and any morphism $(f, g) : (a, b) \to (a', b') \in A \times B$ we have a 2-cell given by the following composite of constraint cells:



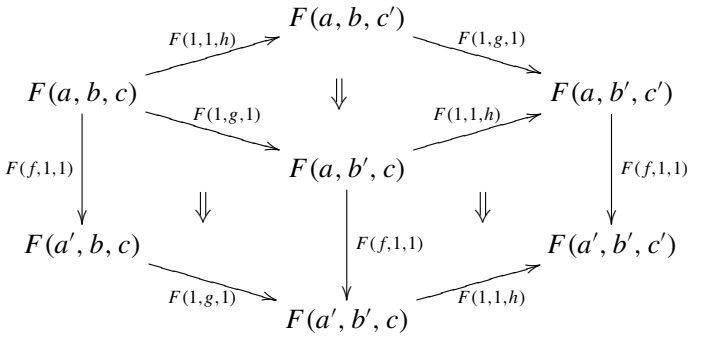
where, by the condition for cubical functors, the bottom left triangle is filled in by a constraint cell which is the identity. We call this 2-cell $F_{f,g}$. We can then check that according to the correspondence in Corollary 2·11 the 2-cell $F_{e,e} : ST \to TS$ gives us the distributive law in question.

The following gives an equivalent characterisation of cubical functors of 3 or more variables. In fact this is the definition originally given by Gray in [**5**]; the first definition we have given above is given in [**4**], with Gray's original then given as an equivalent characterisation.

PROPOSITION 2·13. *A cubical functor of* 3 *variables* $F: A \times B \times C \to D$ *consists of cubical functors of* 2 *variables*

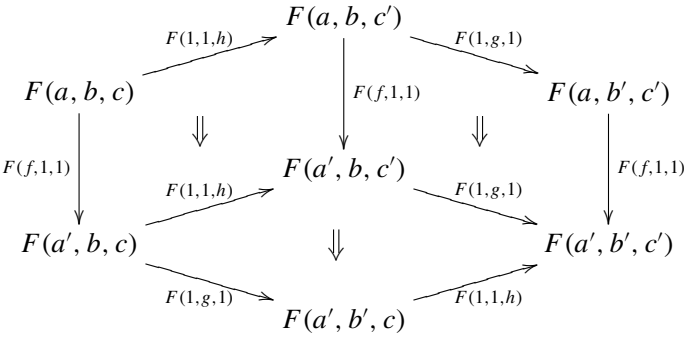
$$F(a, -, -) : B \times C \longrightarrow D$$
$$F(-, b, -) : A \times C \longrightarrow D$$
$$F(-, -, c) : A \times B \longrightarrow D$$

such that for all 1*-cells* $(f, g, h) : (a, b, c) \to (a', b', c') \in A \times B \times C$ *the following equality of* 2*-cells holds*:



=

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press



where all the 2*-cells in the diagram are of the form described in Remark* 2·12 *as a composite of two constraint cells, one being the identity.*

For $n > 3$ *a cubical functor* $F\colon A_1 \times \cdots \times A_n \to D$ *consists of, for all* $i > j > k$ *and* $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ *a cubical functor of* 3 *variables*

$$F(a_1, \ldots, \hat{a}_k, \ldots, \hat{a}_j, \ldots, \hat{a}_i, \ldots, a_n) : A_i \times A_j \times A_k \longrightarrow D$$

where as usual $\hat{a}_i$ *indicates omission of the variable* a_i.

Remark 2·14. In the above commuting 2-cell condition, the 2-cells fit together into a cube shape; we will refer to this as the "cubical condition". It should be clear from examining the different 1-cell composites in this diagram that this condition is essentially the Yang–Baxter equation.

2·4. *Iterated distributive laws*

The characterisation of $\mathbf{Mnd}^n(\mathcal{B})$ now follows immediately from the above characterisation of lax cubical functors of n variables.

PROPOSITION 2·15. $\mathbf{Mnd}^n(\mathcal{B}) \cong \mathbf{LaxCubical}(\Delta^n, \mathcal{B})$.

COROLLARY 2·16. *An object of* $\mathbf{Mnd}^n(\mathcal{B})$ *is a distributive series of n monads, that is, a tuple*

$$\left(X, \{T_i\}_{1 \leqslant i \leqslant n}, \{\lambda_{ij}\}_{1 \leqslant j > i \leqslant n}\right)$$

where:

(i) X *is a 0-cell of* $\mathcal{B}$;
(ii) *each* T_i *is a monad on* X;
(iii) *each* λ_{ij} *is a distributive law* $T_i T_j \to T_j T_i$*; and*
(iv) *for all* $i > j > k$ *the monads* T_i, T_j, T_k *satisfy the Yang–Baxter equation.*

Proof. We know that $\mathbf{Mnd}^2(\mathcal{B}) \cong \mathbf{LaxCubical}(\Delta^2, \mathcal{B})$ has objects (X, S, T, λ) where:
(i) X is a 0-cell of $\mathcal{B}$;
(ii) S and T are monads on X; and
(iii) $\lambda : TS \Rightarrow ST$ is a distributive law.

Now $\mathbf{Mnd}^n(\mathcal{B}) \cong \mathbf{LaxCubical}(\Delta^n, \mathcal{B})$ so by Proposition 2·13 a 0-cell of this 2-category is given by, for every $i > j$ a cubical functor of 2 variables $\Delta^2 \to \mathcal{B}$, such that for every $i > j > k$ the cubical condition holds. But such cubical functors are just distributive laws $\lambda_{ij}\colon T_i T_j \to T_j T_i$ as required. Examining the cubical condition for $i > j > k$ and the case $f = g = h = e$ we see that the 2-cells involved are the distributive laws $\lambda_{ij}\colon T_i T_j \to T_j T_i$ giving the Yang–Baxter equations as required; all other cases of f, g, h follow since:

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

(1) e generates the 1-cells of Δ; and
(2) a lax cubical functor is strict in each variable.

Remark 2·17. Note that we could use $\mathbf{Mnd}^n(\mathcal{B}) \cong \mathbf{Lax}(\Delta^{\otimes n}, \mathcal{B})$ and directly "read off" the result from the data for the n-fold Gray tensor product. However, while it is clear what this data should be, writing out the whole presentation of the n-fold tensor product by generators and relations (and proving that it is correct) is laborious.

We now use Corollary 2·16 to prove Theorem 1·6, which we state here for monads in a 2-category $\mathcal{B}$.

THEOREM 1·6. Let $T_1, \ldots, T_n$ be a distributive series of monads in a 2-category $\mathcal{B}$ with distributive laws $\lambda_{ij} : T_i T_j \Rightarrow T_j T_i$ for all $i > j$. Then for all $1 \leqslant i < n$ we have canonical monads

$$T_1 T_2 \cdots T_i \quad \text{and} \quad T_{i+1} T_{i+2} \cdots T_n$$

together with a distributive law of $T_{i+1} T_{i+2} \cdots T_n$ over $T_1 T_2 \cdots T_i$ i.e.

$$(T_{i+1} T_{i+2} \cdots T_n)(T_1 T_2 \cdots T_i) \Longrightarrow (T_1 T_2 \cdots T_i)(T_{i+1} T_{i+2} \cdots T_n)$$

given by the obvious composites of the λ_{ij}. Moreover, all the induced monad structures on $T_1 T_2 \cdots T_n$ are the same.

Proof. A distributive series of n monads is given by a strict functor

$$\chi : \Delta^{\otimes n} \longrightarrow \mathcal{B}.$$

First note that Δ is a comonoid in **LaxGray**, the monoidal category of 2-categories, strict 2-functors and the lax Gray tensor product. Comultiplication for the comonoid Δ is given by the diagonal functor, the strict functor generated by

$$\begin{aligned} \delta \colon \Delta &\longrightarrow \Delta \otimes \Delta \\ e &\longmapsto (1, e) \circ (e, 1) \end{aligned}$$

and the counit is the terminal map $\Delta \to \mathbb{1}$. Now for all $i \in \mathbb{N}$ we get a unique (up to coherence cells) map

$$\delta_i : \Delta \longrightarrow \Delta^{\otimes i}$$

built from the comultiplication, and thus for each $1 \leqslant i \leqslant n-1$ we have a strict functor

$$\Delta \otimes \Delta \xrightarrow{\delta_i \otimes \delta_{n-i}} \Delta^{\otimes i} \otimes \Delta^{\otimes n-i} \cong \Delta^{\otimes n} \xrightarrow{\chi} \mathcal{B}$$

giving the distributive law

$$(T_{i+1} T_{i+2} \cdots T_n)(T_1 T_2 \cdots T_i) \Longrightarrow (T_1 T_2 \cdots T_i)(T_{i+1} T_{i+2} \cdots T_n).$$

For each i the composite monad resulting from this distributive law is given by the functor

$$\Delta \xrightarrow{\delta} \Delta \otimes \Delta \xrightarrow{\delta_i \otimes \delta_{n-i}} \Delta^{\otimes i} \otimes \Delta^{\otimes n-i} \cong \Delta^{\otimes n} \xrightarrow{\chi} \mathcal{B}.$$

But for each i this is just the composite

$$\Delta \xrightarrow{\delta_n} \Delta^{\otimes n} \xrightarrow{\chi} \mathcal{B}$$

by coherence for comonoids, showing that in each case the induced composite monad is the same.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

Remark 2·18. The fact that the functor $\mathbf{Mnd} = \mathbf{Lax}(\Delta, _)$ is a monad follows immediately from the fact that Δ is a comonoid in **LaxGray**. We can then use Corollary 2·16 to relate the simplicial resolution of the monad **Mnd** to the above result: the unique morphism from cat^n to **Mnd** in the simplicial resolution gives us the unique composite monad $T_1T_2\cdots T_n$; the maps to $\mathbf{Mnd}^2$ give us the distributive laws $T_{i+1}T_{i+2}\cdots T_n$ over $T_1T_2\cdots T_i$.

3. *Interchange for n-categories*

In this section we discuss our motivating example, the free strict n-category monad on n-dimensional globular sets. In an n-category, interchange laws govern the interaction between different types of composition. These different types of composition can be expressed using monads, and the main result of this section is that the interchange laws define distributive laws between those monads, giving a distributive series of monads. Using the theory of iterated distributive laws, the resulting composite monad is the standard "free strict n-category" monad induced by the adjunction:

$$n\textbf{-GSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \textbf{Str-}n\textbf{-Cat}.$$

Throughout this section we will omit the word "strict" and understand all our n-categories to be strict. In fact the key for us is that *interchange* is strict; this theory could in principle be used for notions of n-category that are weaker, as long as interchange is still strict. An example of this is Trimble's definition [**3**, **16**].

3·1. *Composition in n-categories*

The underlying data for an n-category is an n-globular set, that is, a diagram of sets and functions

$$A(n) \underset{t}{\overset{s}{\rightrightarrows}} A(n-1) \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} A(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1) \underset{t}{\overset{s}{\rightrightarrows}} A(0)$$

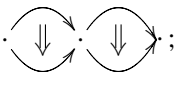
such that $ss = st, ts = tt$. The elements of each $A(m)$ are generally referred to as "m-cells", and the functions s and t give the "source" and "target" m-cells of an $(m+1)$-cell, also generally known as the boundary or bounding cells. Then n-globular sets form a category n**-GSet** with the obvious morphisms; note that n**-GSet** can be expressed as the category of presheaves on the category

$$n \underset{t}{\overset{s}{\leftrightarrows}} n-1 \underset{t}{\overset{s}{\leftrightarrows}} \cdots \underset{t}{\overset{s}{\leftrightarrows}} 2 \underset{t}{\overset{s}{\leftrightarrows}} 1 \underset{t}{\overset{s}{\leftrightarrows}} 0$$

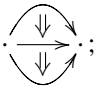
with the relations $ss = ts, st = tt$; this fact helps many of the proofs in this section.

An n-category should be an n-globular set with, for all $0 \leqslant m \leqslant n-1$, composition along bounding m-cells, which we will call m-composition and denote by $\circ_m$. For example 2-categories have:

(i) 0-composition = horizontal composition, usually denoted $*$, or in diagrams



(ii) 1-composition = vertical composition, usually denoted $\circ$, or in diagrams

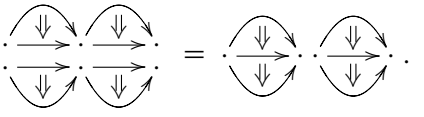


https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

and each is strictly unital and associative. In a 2-category we also have the interchange law

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

or in diagrams



In effect this, together with associativity and unit laws, ensures that any given diagram of composable cells has a unique composite. For m-cells in an n-category there are m different kinds of composition, along bounding i-cells for all $0 \leqslant i \leqslant m-1$, and an interchange law for all pairs i, j with $0 \leqslant i < j \leqslant m-1$

$$(a \circ_j b) \circ_i (c \circ_j d) = (a \circ_i c) \circ_j (b \circ_i d)$$

ensuring that any diagram of composable cells (perhaps including more than two types of composition) has a unique composite.

3·2. *Monads for i-composition*

We construct, for each $0 \leqslant i < n$ a monad T_i on n-**GSet** which forms i-composites freely (leaving k-cells alone for $k \leqslant i$). Each of these monads is a completely straightforward generalisation of the ordinary free category monad on graphs; full details are given in Appendix B.

First we recall the free enriched category monad as described in [**10**], which acts on the category of $\mathcal{V}$-graphs.

Definition 3·1. *Given a category $\mathcal{V}$, a $\mathcal{V}$-**graph** A is given by*:
(i) *a set A_0 of objects; and*
(ii) *for every pair of objects a, a', an object $A(a, a') \in \mathcal{V}$.*
A morphism $F : A \to B$ of $\mathcal{V}$-graphs is given by;
(i) *a function $F : A_0 \to B_0$; and*
(ii) *for every pair of objects a, a', a morphism $A(a, a') \to B(Fa, Fa') \in \mathcal{V}$.*
*$\mathcal{V}$-graphs and their morphisms form a category $\mathcal{V}$-**Gph**.*

Note that

$$\begin{aligned} \textbf{Set-Gph} &\equiv \textbf{1-GSet} \\ \textbf{Gph-Gph} &\equiv \textbf{2-GSet} \\ &\vdots \\ \textbf{(}n\textbf{-GSet)-Gph} &\equiv (n+1)\textbf{-GSet}. \end{aligned}$$

We will also write n-**Gph** for n-**GSet**, so (n-**Gph**)-**Gph** = $(n+1)$-**Gph**.

If $\mathcal{V}$ is monoidal we can construct categories enriched in $\mathcal{V}$, but to make the free $\mathcal{V}$-category construction we need $\mathcal{V}$ to be suitably well-behaved. The following theorem is given in [**10**] for presheaf categories $\mathcal{V}$; in fact the result holds for any "infinitely distributive category", that is, a category with finite products and small coproducts in which the former distribute over the latter.

THEOREM 3·2 (Leinster [**10**]). *If $\mathcal{V}$ is an infinitely distributive category then the forgetful*

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

functor

$$\mathcal{V}\textbf{-Cat} \longrightarrow \mathcal{V}\textbf{-Gph}$$

is monadic. The induced monad is the "free $\mathcal{V}$-category monad" $\mathbf{fc}_{\mathcal{V}}$. *Its action is given by*

$$(\mathbf{fc}_{\mathcal{V}}A)(a,a') = \coprod_{a=a_0,a_1,\ldots,a_{k-1},a_k=a'} A(a_{k-1},a_k)\times\cdots\times A(a_0,a_1).$$

PROPOSITION 3·3. *Let $n \geqslant 1$. Then for all $0 \leqslant i \leqslant n-1$ we have a monadic adjunction*

$$\underbrace{\textbf{Gph-Gph-}\cdots\textbf{-Gph}}_{(n-i-1)\ \text{times}}\textbf{-Cat-}\underbrace{\textbf{Gph-}\cdots\textbf{-Gph}}_{i\ \text{times}} \rightleftarrows n\textbf{-Gph}$$

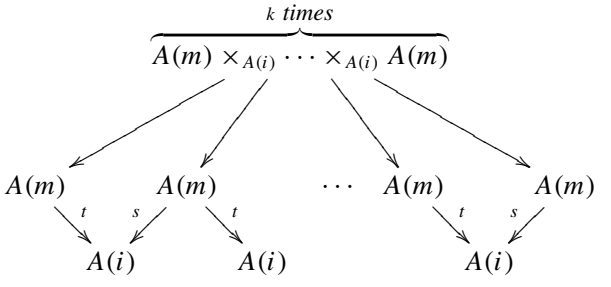
which we could also write as:

$$\big[(n-i-1)\textbf{-Gph}\big]\textbf{-Cat-}\big[i\textbf{-Gph}\big] \rightleftarrows n\textbf{-Gph}.$$

We write the induced monad as $T_i^{(n)}$, and its action is given by

$$T_iA(m) = \begin{cases} A(m) & m \leqslant i \\ \displaystyle\coprod_{k\geqslant 0} \underbrace{A(m)\times_{A(i)}\cdots\times_{A(i)}A(m)}_{k\ \text{times}} & m > i. \end{cases}$$

For the proof, see Appendix B. Note that the above formula produces k-length strings of i-composable m-cells – we are taking k-fold wide pullbacks



where now s and t denote the composites along the top and bottom of

$$A(m) \underset{t}{\overset{s}{\rightrightarrows}} A(m-1) \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} A(i).$$

We now show how to construct the distributive laws we require. We will use the following proposition of Leinster; in fact this is only part of [**10**, proposition F·1·1].

Definition 3·4. For any functor $F : \mathcal{V} \to \mathcal{W}$ we get a functor $F_* : \mathcal{V}\textbf{-Gph} \to \mathcal{W}\textbf{-Gph}$ as follows. Given a $\mathcal{V}$-graph A, the graph F_*A is defined by:

(i) $(F_*A)_0 = A_0$

(ii) $(F_*A)(a,a') = F(A(a,a'))$

and we extend this to morphisms in the obvious way.

Similarly given functors $F, G : \mathcal{V} \to \mathcal{W}$ and a natural transformation $\alpha : F \Rightarrow G$, we get a natural transformation α_* with components $F_*A \to G_*A$ given by:

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

(i) on objects the identity, which makes sense since $(F_*A)_0 = A_0 = (G_*A)_0$;
(ii) on hom-objects

$$\alpha_{A(a,a')} : F(A(a,a')) = (F_*A)(a,a') \longrightarrow (G_*A)(a,a') = G(A(a,a')).$$

This makes $()_*$ into a 2-functor **Cat** $\longrightarrow$ **Cat**; in particular it preserves distributive laws.

The notation in the following Proposition may seem austere, but we will immediately give a motivating example below.

PROPOSITION 3·5 (Leinster [**10**]). *Let $\mathcal{V}$ be an infinitely distributive category and T a monad on $\mathcal{V}$ that preserves coproducts. Write $\mathcal{V}^T$ for the category of algebras of T. Then we have monads on $\mathcal{V}$-**Gph** given by T_* and $\mathbf{fc}_\mathcal{V}$, and a distributive law*

$$\lambda : T_* \circ \mathbf{fc}_\mathcal{V} \Longrightarrow \mathbf{fc}_\mathcal{V} \circ T_*$$

*whose resulting composite monad $\mathbf{fc}_\mathcal{V} \circ T_*$ is the free $\mathcal{V}^T$-**Cat** monad, that is*

$$(\mathcal{V}\text{-}\mathbf{Gph})^{\mathbf{fc}_\mathcal{V} \circ T_*} \cong \mathcal{V}^T\text{-}\mathbf{Cat}.$$

Proof. Since T preserves coproducts we have

$$(T_* \circ \mathbf{fc}_\mathcal{V})(A) = \coprod_{\substack{k \geqslant 0 \\ a_0,\ldots,a_k \in A(0)}} T\big(A(a_{k-1},a_k) \times \cdots \times A(a_0,a_1)\big)$$

and we also have

$$(\mathbf{fc}_\mathcal{V} \circ T_*)(A) = \coprod_{\substack{k \geqslant 0 \\ a_0,\ldots,a_k \in A(0)}} T\big(A(a_{k-1},a_k)\big) \times \cdots \times T\big(A(a_0,a_1)\big).$$

Now the universal property of the product

$$T\big(A(a_{k-1},a_k)\big) \times \cdots \times T\big(A(a_0,a_1)\big)$$

induces a canonical morphism from

$$T\big(A(a_{k-1},a_k) \times \cdots \times A(a_0,a_1)\big)$$

and this gives us the components of a natural transformation λ as required. It is straightforward to check that λ is a distributive law.

The following example is a "prototype" for the construction of the strict n-category monad for general n.

Example 3·6. **2-categories revisited.**
We put $\mathcal{V} =$ **Gph** and $T =$ free category monad, which does indeed preserve coproducts. Then we have:
(i) $\mathcal{V}^T =$ **Cat**;
(ii) $\mathcal{V}$-**Gph** = **2-Gph**;
(iii) T_* is the monad on **2-Gph** forming 1-composites of 2-cells freely;
(iv) $\mathbf{fc}_\mathcal{V}$ is the monad on **2-Gph** making free 0-composites; and
(v) λ is given by the usual interchange law for 2-categories.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

The composite monad $\mathbf{fc}_{\mathcal{V}} \circ T_*$ resulting from this distributive law is the free 2-category monad on **2-Gph**. By the theory of distributive laws (Theorem 1·2) we also get a lift of the monad $\mathbf{fc}_{\mathcal{V}}$ to T_*-Alg = **Cat-Gph**, whose algebras are precisely 2-categories. This expresses 2-categories as graphs enriched in categories, with certain extra composition structure, which in effect gives us the usual definition of a 2-category as a category enriched in categories.

Example 3·7. ***n*-categories as constructed by Leinster.**
In this example we recall Leinster's construction of the monad for strict n-categories, which is given as part of [**10**, theorem F·2·1]. The construction proceeds by induction. We construct for each $n \geqslant 1$ a monad S_n on n-**Gph**, whose algebras are precisely strict n-categories. We begin by taking S_1 to be the usual free category monad on **Gph**. Then for all $n \geqslant 2$ we apply Proposition 3·5 with:

(i) $\mathcal{V} = (n-1)$-**Gph**, and

(ii) $T = S_{n-1}$, the free $(n-1)$-category monad that we have constructed by induction.

Then we have

(i) $\mathcal{V}^T = (n-1)$-**Cat**, and

(ii) $\mathcal{V}$-**Gph** = n-**Gph**,

and the composite monad $\mathbf{fc}_{\mathcal{V}} \circ T_*$ resulting from the distributive law given by the Proposition has as its category of algebras

$$\mathcal{V}^T\text{-}\mathbf{Cat} = ((n-1)\text{-}\mathbf{Cat})\text{-}\mathbf{Cat} = n\text{-}\mathbf{Cat}$$

which is to say that we have indeed constructed the free strict n-category monad. As in the 2-category example above, we have essentially expressed n-categories as graphs enriched in $(n-1)$-categories, together with certain extra composition structure, which in effect gives us the usual definition of an n-category as a category enriched in $(n-1)$-categories.

We now have everything we need to form all the distributive laws for interchange and thereby construct the monad for strict n-categories – we simply start with a special case of Proposition 3·5 and then apply the $(-)_*$ construction repeatedly. This is the content of Theorem 3·8 and its proof.

THEOREM 3·8. *The monads $T_0^{(n)}, \ldots, T_{n-1}^{(n)}$ on n-***GSet** *defined in Proposition* 3·3 *form a distributive series of monads as in Theorem* 1·6. *For all $n > i > j \geqslant 0$ the distributive law*

$$\lambda_{ij}^{(n)} : T_i^{(n)} T_j^{(n)} \Longrightarrow T_j^{(n)} T_i^{(n)}$$

*is given by interchange. The resulting composite monad $T_0^{(n)} T_1^{(n)} \cdots T_{n-1}^{(n)}$ is the free strict n-category monad on n-***GSet**.

Proof. First observe that each of the monads $T_0^{(n)}, \ldots, T_{n-1}^{(n)}$ preserves coproducts; this follows from results of [**10**] or by direct calculation.

We now construct the distributive laws. Note that this, and indeed this whole proof, can be done directly by writing down and examining the formulae. However we will take the more abstract approach.

We begin by examining the case $j = 0$, so we seek a distributive law

$$T_i^{(n)} \circ T_0^{(n)} \Longrightarrow T_0^{(n)} \circ T_i^{(n)}$$

for each $n > i > 0$. But we can check from the formula that

$$T_i^{(n)} = (T_{i-1}^{(n-1)})_*$$

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

(see also Appendix B) and

$$T_0^{(n)} = \mathbf{fc}_{(n-1)}\textbf{-Gph}$$

so this is just a special case of Proposition 3·5 above.

For $j > 0$ we use the distributive law

$$T_{i-j}^{(n-j)} \circ T_0^{(n-j)} \Longrightarrow T_0^{(n-j)} \circ T_{i-j}^{(n-j)}$$

and apply the $(-)_*$ construction j times; since this is a 2-functor, the result of applying it to a distributive law must be a distributive law.

Finally it is straightforward to check that these distributive laws obey all the necessary Yang–Baxter equations, since all the maps involved are induced by universal properties of products. (In fact it is straightforward to check that all the distributive laws are inclusions, thus both legs of the Yang–Baxter equation are the same inclusion.)

To show that the composite monad $T_0^{(n)} T_1^{(n)} \cdots T_{n-2}^{(n)} T_{n-1}^{(n)}$ is the free strict n-category monad we also proceed by induction. The result is clearly true for $n = 1$. For $n > 1$ we know by Theorem 1·6 that the this composite monad arises from various different composite distributive laws; in particular it arises from the distributive law

$$\left(T_1^{(n)} T_2^{(n)} \cdots T_{n-1}^{(n)}\right) \circ T_0^{(n)} \Longrightarrow T_0^{(n)} \circ \left(T_1^{(n)} T_2^{(n)} \cdots T_{n-1}^{(n)}\right).$$

Now by definition we have

$$\begin{aligned} T_1^{(n)} T_2^{(n)} \cdots T_{n-1}^{(n)} &= (T_0^{(n-1)})_* (T_1^{(n-1)})_* \cdots (T_{n-2}^{(n-1)})_* \\ &= \left(T_0^{(n-1)} T_1^{(n-1)} \cdots T_{n-2}^{(n-1)}\right)_* \end{aligned}$$

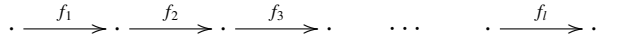
but by induction

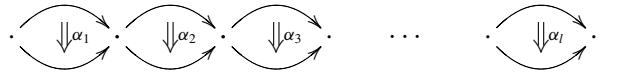
$$T_0^{(n-1)} T_1^{(n-1)} \cdots T_{n-2}^{(n-1)}$$

is the free strict $(n-1)$-category monad. So this distributive law is exactly the one that Leinster uses to construct the free strict n-category monad.

We will now illustrate this construction for the case $n = 2$, $j = 0$, $i = 1$, which should be the usual interchange law between horizontal and vertical composition.

(i) A cell of $T_0^{(2)}(A)$ is an l-length string of 0-composable cells, for example

1-cell 

2-cell 

(ii) A cell of $T_1^{(2)}(A)$ is an h-height "column" of 1-composable cells, for example a 2-cell

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

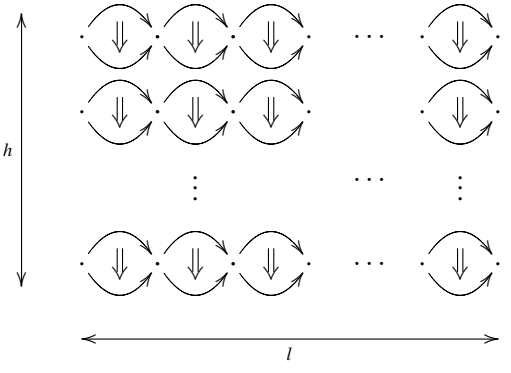
has the form



(iii) A 2-cell of $T_1^{(2)}T_0^{(2)}(A)$ is thus an

"h-height column of 1-composable { strings of 0-composable 2-cells }".

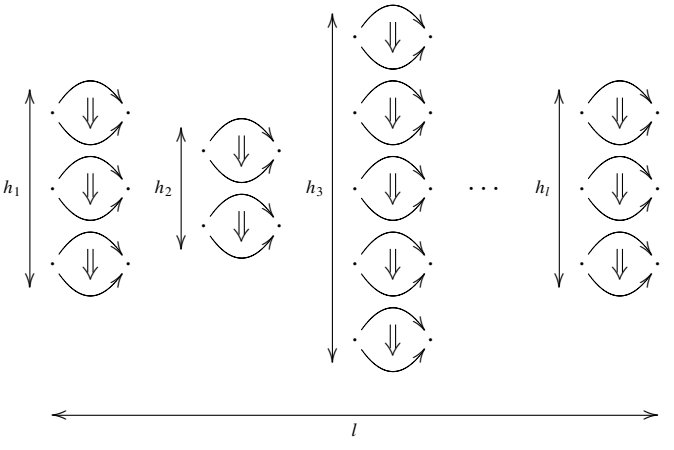
Note that for these strings of 0-composable cells to be 1-composable, they must all have the same length l. So we have an $l \times h$ composable "grid" of 2-cells



(iv) On the other hand a 2-cell of $T_0^{(2)}T_1^{(2)}(A)$ is an

"l-length string of 0-composable {columns of 1-composable 2-cells}".

Note that for the columns to be 0-composable they do not have to be the same height, so we have a configuration of 2-cells such as:



https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

There is evidently a natural map from an $l \times h$ grid to one with columns of height $h_1, \ldots, h_l$ as above – we simply put $h_1 = h_2 = \cdots = h_l = h$. This is the canonical map

$$T\big(A(a_{l-1}, a_l) \times \cdots \times A(a_0, a_1)\big) \longrightarrow T\big(A(a_{l-1}, a_l)\big) \times \cdots \times T\big(A(a_0, a_1)\big)$$

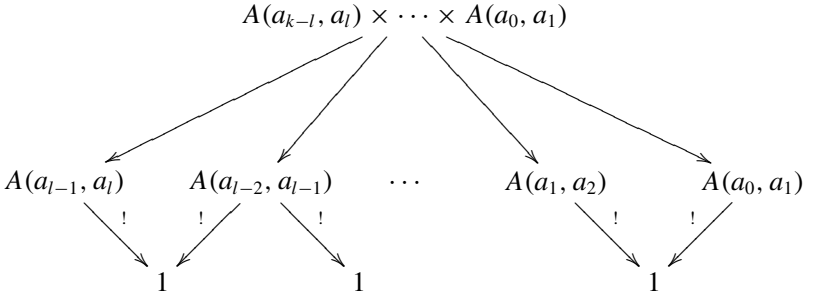
as described in the proof of Proposition 3·5. In our case we are using $T_* = T_1^{(2)}$ so $T = T_0^{(1)}$ which is just the free category monad. Note that the left-hand side

$$T\big(A(a_{l-1}, a_l) \times \cdots \times A(a_0, a_1)\big)$$

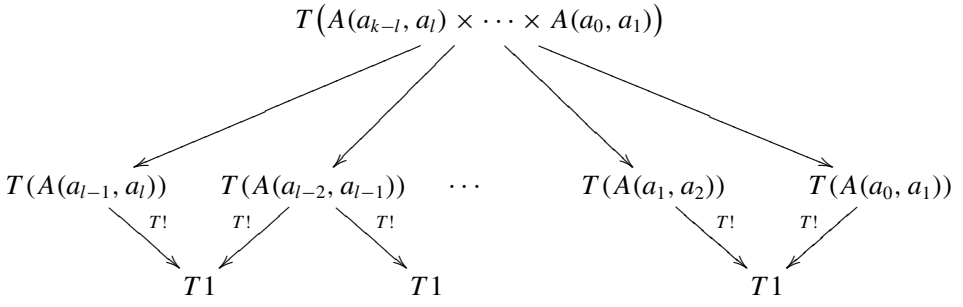
is T applied to the product

$$A(a_{l-1}, a_l) \times \cdots \times A(a_0, a_1).$$

We can express this product as a (trivial) wide pullback



and since T is cartesian we know that applying it to this product gives the wide pullback



The commuting condition over $T1$ is what ensures that all the "columns" must now have equal height.

On the other hand there is not a distributive law going in the opposite direction, since given a grid with columns of possibly varying height, there is no canonical way to map it to a grid with columns of equal height. We might attempt to insert 2-cell identities to "extend" the shorter columns, but the possible choice of positions for the inserted identity cells means that this will not satisfy the axioms for a distributive law.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

Appendix A. Direct proof of Theorem 1·6 *by induction*

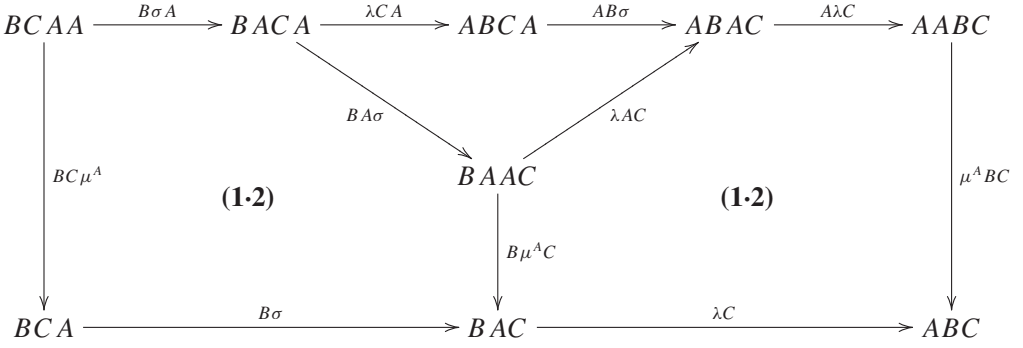
The case $n = 3$ is a simple diagram chase as follows. For ease of notation we write the monads T_1, T_2, T_3 as A, B, C with distributive laws

$$\lambda : BA \Rightarrow AB$$
$$\sigma : CA \Rightarrow AC$$
$$\tau : CB \Rightarrow BC.$$

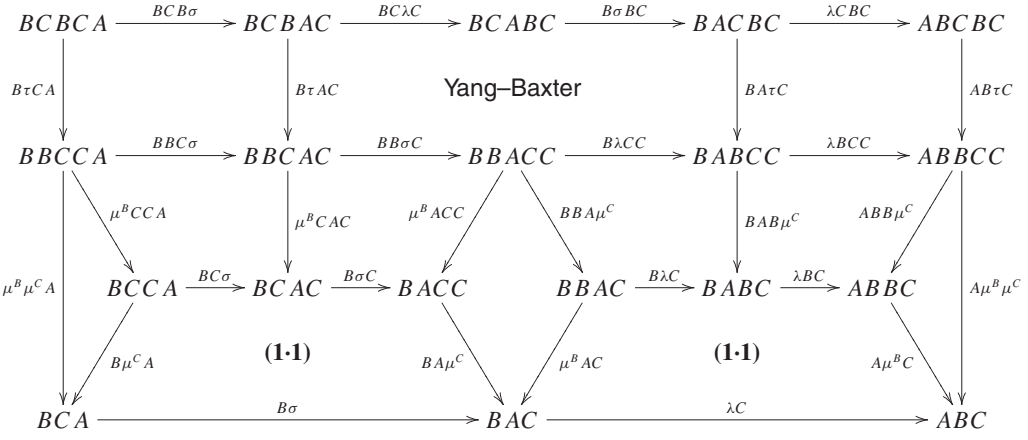
and we will check that the following is a distributive law:

$$BCA \overset{B\sigma}{\Longrightarrow} BAC \overset{\tau C}{\Longrightarrow} ABC\,.$$

Now, the triangular axioms for a distributive law follow immediately without any need for the Yang–Baxter condition, as does the pentagon axiom (1·2):



where the unmarked region commutes by naturality. The pentagon axiom (1·1) is seen to commute as follows:



where all the unmarked parts commute by naturality, and the Yang–Baxter equation marked is that for C, B, A with an extra B applied on the left throughout, and a C on the right. The result for the other distributive law follows similarly.

Now consider $n > 3$. Given i with $1 \leqslant i < n$, the series of monads $T_1, \ldots, T_i$ and the series $T_{i+1}, \ldots, T_n$ is each a series of monads satisfying the induction hypothesis; each series has fewer than n monads so by induction we have the monads $T_1T_2 \cdots T_i$ and $T_{i+1}T_{i+2} \cdots T_n$ as required. Now define monads $S_1, \ldots, S_{n-1}$ by

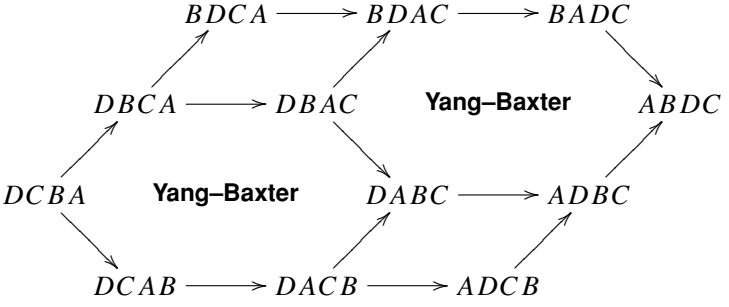
https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

(i) $S_i = T_i$ for all $1 \leqslant i < n-1$, and
(ii) $S_{n-1} = T_{n-1}T_n$.

We now check that the monads $S_1, \ldots, S_{n-1}$ satisfy the hypothesis of the theorem:

(1) We need for all $i > j$ a distributive law of S_i over S_j. For the cases $i < n-1$ this is just the distributive law of T_i over T_j. For the case $i = n-1$ we need for all $j < n-1$ a distributive law of $T_{n-1}T_n$ over T_i; this follows from the result for $n = 3$ applied to the monads T_i, T_{n-1}, T_n.

(2) We need for all $i > j > k$ the Yang–Baxter equation for S_i, S_j, S_k. Again, for the cases $i < n-1$ this is just the Yang–Baxter eqation for T_i, T_j, T_k. For the case $i = n-1$ we need for all $k < j < n-1$ the Yang–Baxter equation for the monads S_{n-1}, T_j, T_k, that is, the monads $(T_{n-1}T_n), T_j, T_k$. This follows from the Yang–Baxter equations for T_{n-1}, T_j, T_k and T_n, T_j, T_k. This is seen by the following diagram, where for clarity we have labelled the four monads in question A, B, C, D:



Here the lower left hexagon is the Yang–Baxter equation for B, C, A (with D applied on the left), and the upper right hexagon is the Yang–Baxter equation for D, B, A (with C applied on the right).

So by the result for $n-1$ we have distributive laws for all $1 \leqslant i < n-1$

$$(S_{i+1}S_{i+2}\cdots S_{n-1})(S_1S_2\cdots S_i) \Longrightarrow (S_1S_2\cdots S_i)(S_{i+1}S_{i+2}\cdots S_{n-1})$$

that is,

$$(T_{i+1}T_{i+2}\cdots T_{n-1}T_n)(T_1T_2\cdots T_i) \Longrightarrow (T_1T_2\cdots T_i)(T_{i+1}T_{i+2}\cdots T_{n-1}T_n)$$

all inducing the same monad. We are still missing the distributive law for $i = n-1$; for this we just repeat the above proof using monads $(T_1T_2), T_3, \ldots, T_n$, and the full result follows.

Appendix B. Proof of Proposition 3·3

We proceed in the following steps:

(1) Use the enriched version of the free category monad to construct a monad for "free 0-composition" on $(n-i)$-**GSet**.

(2) Shift the dimensions up i times by inserting lower dimensions trivially, which turns this into the monad for "free i-composition" on n-**GSet**.

We begin by using Theorem 3·2 to construct the monad on n-**Gph** for "free 0-composition". We use $\mathcal{V} = (n-1)$-**Gph**.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

COROLLARY B·1. *For all $n \geqslant 1$ we have a monadic adjunction*

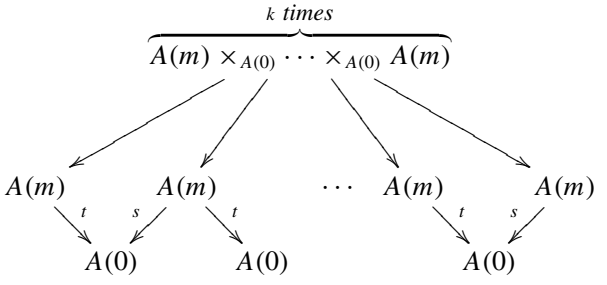
$$((n-1)\textbf{-Gph})\textbf{-Cat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} ((n-1)\textbf{-Gph})\textbf{-Gph} = n\textbf{-Gph}.$$

The induced monad T constructs 0-composites freely:

$$TA(m) = \coprod_{\substack{k \geqslant 0 \\ a_0, \ldots, a_k \in A(0)}} A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1).$$

Proof. Put $\mathcal{V} = (n-1)$**-Gph** in Theorem 3·2. Then $T = \mathbf{fc}_{(n-1)\textbf{-Gph}}$ and the formula is exactly the formula given in [**10**].

Note that this formula produces k-length strings of 0-composable cells. It is a coproduct over k of k-fold wide pullbacks as below:



where s and t denote the composites along the top and bottom of

$$A(m) \underset{t}{\overset{s}{\rightrightarrows}} A(m-1) \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} A(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1) \underset{t}{\overset{s}{\rightrightarrows}} A(0)$$

Now in order to make free i-composites and not just free 0-composites we just need to "shift" the monad up i dimensions. We do this using the $()_*$ construction described in Definition 3·4.

PROPOSITION B·2 (Leinster [**10**]). *A monadic adjunction*

$$\mathcal{V} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{W}$$

induces a monadic adjunction

$$\mathcal{V}\textbf{-Gph} \underset{U_*}{\overset{F_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{W}\textbf{-Gph}.$$

Writing T for the original monad FU, the induced monad is given by $F_*U_* = T_*$.

Example B·3. **2-categories.**
We put:
(i) $\mathcal{V} =$ **Gph**;
(ii) $\mathcal{W} =$ **Cat**;
(iii) $F =$ the free category functor; and
(iv) $U =$ the usual forgetful functor.

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

Then Proposition B·2 gives us an adjunction

$$\mathbf{2\text{-}Gph} \underset{U_*}{\overset{F_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Cat\text{-}Gph}.$$

Here the functor F_* sends the graph

$$A(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1) \underset{t}{\overset{s}{\rightrightarrows}} A(0)$$

to the cat-graph with underlying 2-graph

$$FA(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1) \underset{t}{\overset{s}{\rightrightarrows}} A(0)$$

where by abuse of notation we have written

$$FA(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1)$$

to denote the graph of the free category on

$$A(2) \underset{t}{\overset{s}{\rightrightarrows}} A(1)$$

so in effect we are forming 1-composites of 2-cells freely. This naturally has the structure of a cat-graph. The monad T_* induced by this adjunction is the free 2-category monad, and T_*-Alg $= \mathbf{2\text{-}Cat}$.

We can now prove Proposition 3·3 by induction.

Proof. By induction over n and i. Put $T_0^{(n)} = \mathbf{fc}_{(n-1)\text{-}\mathbf{Gph}}$, and for $i > 0$ put $T_{i-1}^{(n)} = (T_i^{(n-1)})_*$.

Acknowledgements. I would like to thank Michael Shulman for useful conversations relating to this work.

REFERENCES

[1] M. A. BATANIN. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Adv. Math.* **136**(1) (1998), 39–103.
[2] J. BECK. Distributive laws. *Lecture Notes in Math.* **80** (1969), 119–140.
[3] E. CHENG. Comparing operadic theories of n-category, 2008. E-print `0809.2070`.
[4] R. GORDON, A. J. POWER and R. STREET. Coherence for tricategories. *Mem. Amer. Math. Soc.* **117**(558), (1995).
[5] J. W. GRAY. *Formal Category Theory: Adjointness for* 2*-Categories*. Lecture Notes in Math. vol. 391 (Springer-Verlag, 1974).
[6] J. W. GRAY. Coherence for the tensor product of 2-categories, and braid groups. In *Algebra, Topology and Category Theory, a collection in honor of Samuel Eilenberg*, LMS, pages 63–76 (Academic Press, 1976).
[7] N. GURSKI. *An algebraic theory of tricategories*. PhD thesis. University of Chicago (2006). Available via `http://www.math.yale.edu/~mg622/tricats.pdf`.
[8] A. JOYAL and J. KOCK. Weak units and homotopy 3-types. In Batanin, Davydov, Johnson, Lack, and Neeman, editors, *Categories in Algebra, Geometry and Mathematical Physics, proceedings of Streetfest*. *Contemp. Math.* vol. 431, pages 257–276 (AMS, 2007).
[9] S. KASANGIAN, S. LACK and E. VITALE. Coalgebras, braidings, and distributive laws. *Theory and Applications of Categories*. **13** (2004), 129–146.
[10] T. LEINSTER. *Higher operads, higher categories*. London Math. Soc. Lecture Note Series, no. 298 (Cambridge University Press, 2004).

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press

[**11**] S. MAC LANE. *Categories for the working mathematician*. volume 5 *Graduate Texts in Mathematics* vol. 5 (Springer-Verlag, second edition, 1998).
[**12**] C. SIMPSON. A closed model structure for n-categories, internal Hom, n-stacks and generalized Seifert-Van Kampen, 1997. E-print `alg-geom/9704006`.
[**13**] C. SIMPSON. Limits in n-categories, 1997. E-print `alg-geom/9708010`.
[**14**] R. STREET. The formal theory of monads. *J. Pure Appl. Alg.* **2** (1972), 149–168.
[**15**] R. STREET. The algebra of oriented simplexes. *J. Pure Appl. Alg.* **49**(3) (1987), 283–335.
[**16**] T. TRIMBLE. What are 'fundamental n-groupoids'? Seminar at DPMMS (Cambridge, 24 August 1999).

https://doi.org/10.1017/S0305004110000599 Published online by Cambridge University Press