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CLOUD STORAGE FACILITY AS A FLUID QUEUE CONTROLLED BY MARKOVIAN QUEUE

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Cloud storage faces many problems in the storage process which badly affect the system's efficiency. One of the most problems is insufficient buffer space in cloud storage. This means that the packets of data wait to have storage service which may lead to weakness in performance evaluation of the system. The storage process is considered a stochastic process in which we can determine the probability distribution of the buffer occupancy and the buffer content and predict the performance behavior of the system at any time. This paper modulates a cloud storage facility as a fluid queue controlled by Markovian queue. This queue has infinite buffer capacity which determined by the $M/M/1/N$ queue with constant arrival and service rates. We obtain the analytical solution of the distribution of the buffer occupancy. Moreover, several performance measures and numerical results are given which illustrate the effectiveness of the proposed model.

Keywords: buffer content distribution, cloud deployments, fluid queue, markovian queue, performance measures

1. INTRODUCTION

Nowadays, cloud computing is considered one of the most important technologies which has several useful applications in our life. Cloud computing may be defined as shared of computing resources over the internet; see Figure 1. Cloud may be a public, private, community or some combination of them. One of the important services introduced by cloud computing is a storage service; see Ghazouani and Slimani [6] and Marinescu [9].

In cloud storage, the data are stashed in a remote physical location on hardware. Clients send files to the cloud via the internet where the files have discrete behavior then they are served by one server in the simplest case or more than one. After that, they are sent as a stream of data packets to storage facility in the cloud by networks. A Dropbox is a good example for cloud storage which lets clients store and share their files.

Arrival of files to storage facility is considered a stochastic process, and the files may wait for storage service. The waiting may cause a queue in the system. So, to avoid or at

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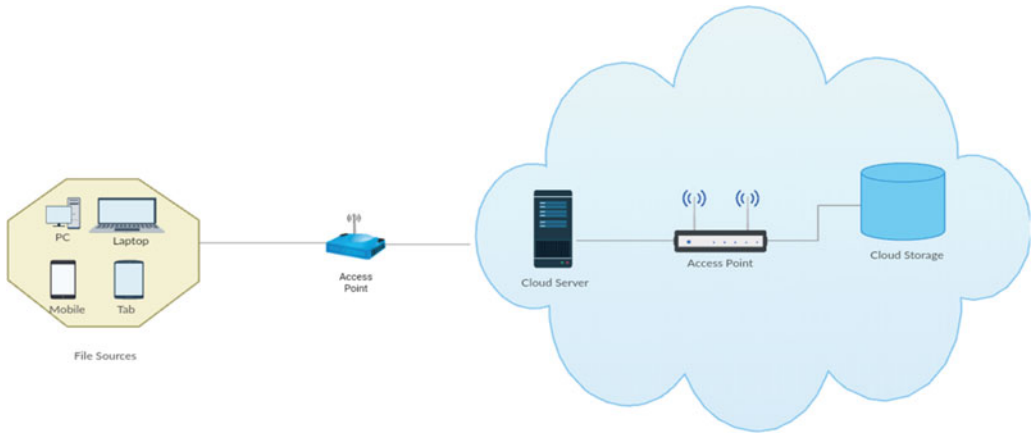


FIGURE **1.** Cloud storage model.

least reduce congestion, the distribution of the buffer occupancy and the buffer content at any time should be studied.

Fluid flow can be studied as stochastic models which have a lot of applications in different disciplines such as networks, telecommunications and manufacturing, for more details, see Refs. [2,4,5,10,11,13,14]. The bursts of data in such models are transmitted with high-speed networks in cells or packets [1,12,16].

Many researchers have applied queueing models in cloud computing and they have obtained various performance measures [15]. To the best of our knowledge, this study is the first which applies fluid queue models in cloud storage facilities. In such models, the files or data are transmitted to the cloud in packets as a stream flow. The process starting with the arrival of files to the cloud server and then they are served, stored and released from the cloud storage facility.

In this study, a private cloud model is considered where the cloud infrastructure is operated solely for a specific organization and is managed by the organization or a third party. In this model, users send files or tasks which they arrive to cloud server in discrete manner then after manipulate, they will transfer in packets or cells into cloud storage facility by using high-speed networks where the behavior of the packets will be converted from the discrete behavior to continuous behavior, that is, stream of the packets [3]. These packets inflow to the storage facility and after they are served, they will release with the same continuous manner.

We modulate the previous scenario in three models as a fluid queue controlled by the $M/M/1/N$ queue with different efficient variable input rates. Also, we obtain the analytical solutions of the distribution of the buffer occupancy and the buffer content in the steady state for each model. Furthermore, we obtain various performance measures such as expected buffer content, server utilization, average throughput and mean delay to ensure that the models work efficiently. Finally, we present numerical computations to illustrate the efficiency of the proposed models.

The outline of the paper is as follow: In Section 2, general model formulation is provided. Analytical solutions for the proposed models are obtained in Section 3. In Section 4, the performance measures for these models are given. In Section 5, the numerical computations for the proposed models are discussed. Finally, the conclusion is summarized in Section 6.

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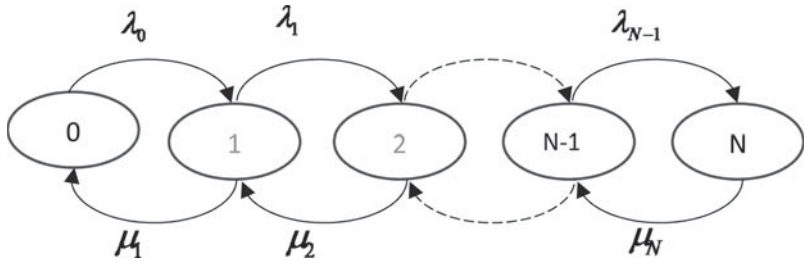


FIGURE **2.** State transition diagram of the CTMC $Y(t)$.

2. GENERAL MODEL FORMULATION

In the proposed model, the data are transmitted in packets from a finite number of users to the cloud server according to the exponential distribution with the rate λ and served according to the exponential distribution with the rate μ. This process is considered a stochastic process of continuous time Markov chain (CTMC) which has the form $\{Y(t),\ t \geq 0\}$ with constant parameters λ and μ which denote the arrival and service rates, respectively, and $Y(t)$ is a random variable which refers to the number of files need to serve by the cloud server at time t. From the other hand, this process is considered a finite birth–death process which has a transition probability diagram shown in Figure 2.

The state space S of this process is finite where $S = \{0, 1, 2, \ldots, N\}$. After data departure from the cloud server, they are transmitted to the cloud storage facility as a stream of packets. This process is considered a stochastic process of CTMC which has the form $\{Q(t), t \geq 0\}$ with input flow rate q_j and output flow rate c_j and with the efficient input rate r_j such that $r_j = q_j - c_j$, where $Q(t)$ refers to the buffer content at a time t and $j \in S$. Let u denotes the amount of fluid in the buffer. Furthermore, the efficient input rate is determined by the $M/M/1/N$ queue with constant parameters λ and μ where $\rho = (\lambda/\mu)$. The previous scenario can be modulated as fluid queue driven by Markovian queue which has the form $\{(Y(t), Q(t)),\ t \geq 0\}$. The difference between the two stochastic processes is that in the first one, it is easy to distinguish between the files in spite of each file is considered as continuous stream of packets, but in the second one, all the files are transmitted as a fluid flow to the storage facility where the size of each file is counted as one unit.

The previous scenario can be modulated as three models as follow:

In the first model: $r_0 = (r\sqrt{\rho}/(\sqrt{\rho} - 1))$, $r_j = r,\ j = 1, 2, 3, \ldots, N$.
In the second model: $r_0 = (r\sqrt{\rho}/(\sqrt{\rho} - 1))$, $r_j = r,\ j = 1, 2, 3, \ldots, N-1$ and $r_N = (r/(1 - \sqrt{\rho}))$.
In the third model: $r_0 = (r\sqrt{\rho}/(\sqrt{\rho} - 1))$, $r_j = r$, $j = 1, 2, 3, \ldots, N-1$ and $r_N = (r/(1 + \sqrt{\rho}))$.

We study the three models under the following considerations:

$$r_0 < 0,\ r_j > 0,\ j = 1, 2, 3, \ldots, N,\ r_j = q_j - c_j \text{ also } \mu > \lambda \text{ and } u \geq 0.$$

In the fluid queue, whenever the reservoir is empty the buffer content cannot decrease. So,

$$\frac{dQ(t)}{dt} = \begin{cases} 0, & \text{if } Q(t) = 0, Y(t) = 0, \\ r_0, & \text{if } Q(t) > 0, Y(t) = 0, \\ r_j,\ j = 1, 2, 3, \ldots, N & \text{elsewhere.} \end{cases}$$

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The stability condition in the stationary state can be written as follows:

$$r_0 p_0 + \sum_{j=1}^{N} p_j r_j < 0,$$

where $p_j,\ 1 \le j \le N$ are the probabilities of the background $M/M/1/N$ queue in the steady state.

Let

$$F_j(t,u) = P\{Y(t) = j,\, Q(t) \le u\},\ j \in S,\ u \ge 0,$$

such that $F_j(t,u)$ is the distribution of the buffer occupancy and P denotes the probability.

In the steady state case, we have

$$F_j(u) = P\{Y = j,\, Q \le u\},\ j \in S,\ u \ge 0,$$

with boundary conditions

$$F_0(0) = \frac{p_0}{r_0} \sum_{j=0}^{N} r_j \rho^j \text{ and } F_j(0) = 0, \quad \text{where } p_0 = \frac{1-\rho}{1-\rho^{N+1}} \text{ and } \rho = \frac{\lambda}{\mu}.$$

The system of partial differential equations for the models is given by

$$\frac{\partial F_j(t,u)}{\partial t} = \lambda_{j-1} F_{j-1}(t,u) + \mu_{j+1} F_{j+1}(t,u) - (q_j - c_j)\frac{\partial F_j(t,u)}{\partial u} - (\lambda_j + \mu_j) F_j(t,u),\ u > 0,\ 0 \le j \le N.$$

As a special case: when $j = 0$ and $j = N$, we get, respectively,

$$\frac{\partial F_0(t,u)}{\partial t} + (q_0 - c_0)\frac{\partial F_0(t,u)}{\partial u} = \mu_1 F_1(t,u) - \lambda_0 F_0(t,u),$$

$$\frac{\partial F_N(t,u)}{\partial t} + (q_N - c_N)\frac{\partial F_N(t,u)}{\partial u} = \lambda_{N-1} F_{N-1}(t,u) - (\lambda_N + \mu_N) F_N(t,u) + \mu_{N+1} F_{N+1}(t,u).$$

For the stationary case, we have $(\partial F_j(t,u)/\partial t) = 0$; hence, the general system of differential equations becomes

$$\begin{aligned}
\dot{F}_0(u) &= \frac{-\lambda}{r_0} F_0(u) + \frac{\mu}{r_0} F_1(u), && j = 0,\\
\dot{F}_j(u) &= \frac{\lambda}{r} F_{j-1}(u) - \frac{(\lambda+\mu)}{r} F_j(u) + \frac{\mu}{r} F_{j+1}(u), && 1 \le j \le N-1,\\
\dot{F}_N(u) &= \frac{\lambda}{r_N} F_{N-1}(u) - \frac{\mu}{r_N} F_N(u), && j = N,
\end{aligned}$$

such that $\dot{F}_j(u) = (dF_j(u)/du)$.

By taking Laplace transform, we get

$$\begin{aligned}
&\left[\theta + \frac{\lambda}{r_0}\right] \psi_0(\theta) - \frac{\mu}{r_0}\psi_1(\theta) = F_0(0), && j = 0,\\
&\frac{-\lambda}{r}\psi_{j-1}(\theta) + \left[\theta + \frac{\lambda+\mu}{r}\right] \psi_j(\theta) - \frac{\mu}{r}\psi_{j+1}(\theta) = 0, && 1 \le j \le N-1,\\
&\frac{-\lambda}{r_N}\psi_{N-1}(\theta) + \left[\theta + \frac{\mu}{r_N}\right] \psi_N(\theta) = 0, && j = N,
\end{aligned}$$

where $\psi_j(\theta) = L[F_j(u)]$,

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Reformulate the previous system in the matrix form, we obtain

$$A \cdot \psi(\theta) = B\,,$$

where $\psi(\theta) = [\psi_j(\theta)]^T_{0\le j\le N}$, $B = [F_0(0), 0, \ldots, 0]^T_{(N+1)\times 1}$,

and

$$A = \begin{pmatrix} \theta + \frac{\lambda}{r_0} & \frac{-\mu}{r_0} & & & & \\ \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & \frac{-\mu}{r} & \\ & & & \frac{-\lambda}{r_N} & \theta + \frac{\mu}{r_N} & \end{pmatrix}_{(N+1)\times(N+1)}$$

The above three models come from the fact that at the above-mentioned values of the input rate, the matrix A has exactly zero eigenvalue and N nonzero eigenvalues, see Mitra [10] and Stern and Elwalid [14].

3. MODEL SOLUTION

To obtain the solution for the above system, $\psi(\theta)$ can be written as

$$\psi(\theta) = [A_{0j}]^T{}_{(N+1)\times 1} \cdot [\det(A)]^{-1} \cdot (F_0(0)), \quad 0 \le j \le N,$$

where A_{0j} is the minor element in the 0th row and the jth column of the matrix A and $\det(A)$ is the determinant of matrix A.

Let

$$D_N = \det \begin{pmatrix} \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & & & & \\ \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & \frac{-\mu}{r} & \\ & & & \frac{-\lambda}{r_N} & \theta + \frac{\mu}{r_N} & \end{pmatrix}_{(N)\times(N)},$$

THEOREM 1: *For any nonnegative integer N, the value of D_N of the matrix A is given as follows:*

$$D_N = (\xi_1 - \xi_2)^{-1} \left[{\xi_1}^{N+1} - {\xi_2}^{N+1} + \left(-\left(\frac{\lambda+\mu}{r}\right) + \left(\frac{\mu}{r_N}\right) \right) ({\xi_1}^N - {\xi_2}^N) + \left(\frac{\lambda\mu}{r^2} - \frac{\lambda\mu}{r.r_N} \right) ({\xi_1}^{N-1} - {\xi_2}^{N-1}) \right],$$

where ξ_1 and ξ_2 are given as

$$\xi_1 = \frac{\sqrt{\lambda\mu}}{r} e^{J\varphi},\ \xi_2 = \frac{\sqrt{\lambda\mu}}{r} e^{-J\varphi},\ J = \sqrt{-1} \text{ and } \varphi = \cos^{-1}\left(\frac{\theta + (\lambda+\mu)/r}{2\sqrt{(\lambda\mu/r^2)}} \right).$$

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PROOF: The determinant of D_N is written as follows:

$$D_N = (\xi_1 - \xi_2)^{-1} \left[({\xi_1}^{N-1} - {\xi_2}^{N-1}) \times \left(\left(\theta + \frac{(\lambda+\mu)}{r} \right) \left(\theta + \frac{\mu}{r_N} \right) - \frac{\lambda\mu}{r.r_N} \right) + \left((\xi_1 {\xi_2}^{N-1} - \xi_2 {\xi_1}^{N-1}) \left(\theta + \frac{\mu}{r_N} \right) \right) \right], \quad \textbf{(1)}$$

where

$$\xi_1, \xi_2 = \frac{\theta + ((\lambda+\mu)/r) \pm \sqrt{(\theta + ((\lambda+\mu)/r))^2 - 4(\lambda\mu/r^2)}}{2},$$

by using

$$\xi_1 + \xi_2 = \theta + \frac{(\lambda+\mu)}{r} \text{ and } \theta + \frac{\mu}{r_N} = \xi_1 + \xi_2 - \left(\frac{(\lambda+\mu)}{r} \right) + \frac{\mu}{r_N}$$

in (1), we obtain

$$D_N = (\xi_1 - \xi_2)^{-1} \left[{\xi_1}^{N+1} - {\xi_2}^{N+1} + \left(- \left(\frac{\lambda+\mu}{r} \right) + \left(\frac{\mu}{r_N} \right) \right) ({\xi_1}^N - {\xi_2}^N) + \left(\frac{\lambda\mu}{r^2} - \frac{\lambda\mu}{r.r_N} \right) ({\xi_1}^{N-1} - {\xi_2}^{N-1}) \right].$$

This completes the proof. ■

LEMMA 1: *For any nonnegative integer state j, A_{0j} has the following form:*

$$A_{0j} = \left(\frac{\lambda}{r} \right)^j D_{N-j}, \quad j \in S.$$

PROOF: Since A_{0j} is the minor element in the 0th row and the jth column of the matrix A. So, A_{0j} is written as

$$A_{0j} = \left(\begin{array}{c|c} U & V \\ \hline O & W \end{array} \right)_{(N)\times(N)},$$

where

$$U = \begin{pmatrix} \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & 0 & \cdots & \vdots \\ & \ddots & & \ddots & & & 0 \\ & & & & & & \frac{-\mu}{r} \\ \vdots & & & & & \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} \\ 0 & & \cdots & & & 0 & \frac{-\lambda}{r} \end{pmatrix}_{(j)\times(j)},$$

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$$V = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & \vdots \\ \frac{-\mu}{r} & & \\ \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} \quad 0 \quad \cdots & 0 \end{pmatrix}_{(j)\times(N-j)},$$

$$W = \begin{pmatrix} \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & 0 & 0 & \cdots & 0 \\ \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{-\mu}{r} & 0 & \cdots & 0 \\ 0 & & & & & \vdots \\ & \ddots & \ddots & \ddots & & 0 \\ \vdots & & \frac{-\lambda}{r} & \theta + \frac{(\lambda+\mu)}{r} & & \frac{-\mu}{r} \\ 0 & \cdots & 0 & \frac{-\lambda}{r_N} & & \theta + \frac{\mu}{r_N} \end{pmatrix}_{(N-j)\times(N-j)},$$

and O is $(N-j) \times j$ zero matrix. Hence,

$$\begin{aligned} A_{0j} &= \det U \cdot \det W \\ &= \left(\frac{-\lambda}{r}\right)^j D_{N-j}. \end{aligned}$$

■

THEOREM 2: *The determinant of the matrix A is given by*

$$\det(A) = \theta(\xi_1 - \xi_2)^{-1} d \left(\frac{r}{r_N}\right) ({\xi_1}^{N+1} - {\xi_2}^{N+1}),$$

where

$$d = \left[\frac{r_N}{r} + \left(\frac{\mu}{r}\right)\left(1 - \frac{r_N}{r} - \frac{r_N}{r}\sqrt{\rho}\right)\left(\frac{{\xi_1}^N - {\xi_2}^N}{{\xi_1}^{N+1} - {\xi_2}^{N+1}}\right) + \left(\frac{\mu}{r}\right)\left(\frac{r_N}{r} - 1\right)\left(\frac{{\xi_1}^{N-1} - {\xi_2}^{N-1}}{{\xi_1}^{N+1} - {\xi_2}^{N+1}}\right)\right].$$

PROOF: The determinant of the matrix A is given by

$$\det(A) = \left(\theta + \frac{\lambda}{r_0}\right) D_N - \left(\frac{\lambda\mu}{r \cdot r_0}\right) D_{N-1}.$$

It can be written as

$$\det(A) = \det \begin{pmatrix} \theta + \frac{\lambda}{r_0} & \frac{\lambda}{r_0} & & & \\ \frac{\mu}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{\lambda}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\lambda}{r} \\ & & & \frac{\mu}{r_N} & \theta + \frac{\mu}{r_N} \end{pmatrix}_{(N+1)\times(N+1)}$$

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As in Losonczi [8], the previous matrix becomes

$$\det(A) = \theta \cdot \det(A'), \tag{2}$$

where

$$\det(A') = \det \begin{pmatrix} \theta + \frac{\lambda}{r_0} + \frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & & \\ \frac{\sqrt{\lambda\mu}}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{\sqrt{\lambda\mu}}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\sqrt{\lambda\mu}}{r} \\ & & & \frac{\sqrt{\lambda\mu}}{r} & \theta + \frac{\lambda}{r} + \frac{\mu}{r_N} \end{pmatrix}_{N\times N}$$

$\det(A')$ is written as follows:

$$\det(A') = \left(\theta + \frac{\lambda}{r_0} + \frac{\mu}{r}\right) D_{N-1} - \left(\frac{\lambda\mu}{r^2}\right) D_{N-2}, \tag{3}$$

$$D_i = \det \begin{pmatrix} \theta + \frac{(\lambda+\mu)}{r} & \frac{\sqrt{\lambda\mu}}{r} & & & \\ \frac{\sqrt{\lambda\mu}}{r} & \theta + \frac{(\lambda+\mu)}{r} & \frac{\sqrt{\lambda\mu}}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\sqrt{\lambda\mu}}{r} \\ & & & \frac{\sqrt{\lambda\mu}}{r} & \theta + \frac{\lambda}{r} + \frac{\mu}{r_N} \end{pmatrix}_{(i)\times(i)}$$

and $i = N-1$ and $N-2$.

D_i is written as

$$D_i = (\xi_1 - \xi_2)^{-1} \left[{\xi_1}^{i+1} - {\xi_2}^{i+1} + \left(-\left(\frac{\mu}{r}\right) + \left(\frac{\mu}{r_N}\right) \right) ({\xi_1}^{i} - {\xi_2}^{i}) \right].$$

Substituting by the previous relation in (3), we have

$$\begin{aligned} \det(A') = (\xi_1 - \xi_2)^{-1} \Bigg[& {\xi_1}^{N+1} - {\xi_2}^{N+1} + \left(\frac{\lambda}{r_0} - \frac{\lambda+\mu}{r} + \frac{\mu}{r_N} \right) ({\xi_1}^{N} - {\xi_2}^{N}) \\ & + \left(\frac{\lambda}{r_0} - \frac{\lambda}{r} \right) \left(\frac{\mu}{r_N} - \frac{\mu}{r} \right) ({\xi_1}^{N-1} - {\xi_2}^{N-1}) \Bigg] \end{aligned}$$

Substituting by $r_0 = (\lambda r/(\lambda - \sqrt{\lambda\mu}))$ in (2), we obtain

$$\begin{aligned} \det(A) = \theta(\xi_1 - \xi_2)^{-1} \Bigg[& {\xi_1}^{N+1} - {\xi_2}^{N+1} + \left(\frac{\mu}{r_N} - \frac{\mu}{r} - \frac{\sqrt{\lambda\mu}}{r} \right) ({\xi_1}^{N} - {\xi_2}^{N}) \\ & + \left(\frac{\mu}{r} - \frac{\mu}{r_N} \right) \left(\frac{\sqrt{\lambda\mu}}{r} \right) ({\xi_1}^{N-1} - {\xi_2}^{N-1}) \Bigg] \end{aligned}$$

It is complete the proof. ■

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LEMMA 2: *The general solution, $\psi_j(\theta)$, is given by*

$$\psi_j(\theta) = \frac{c_0}{\theta} + \sum_{k=1}^{N} \frac{c_k}{(\xi_1 + \xi_1 - 2\beta_{N,k}\sqrt{\xi_1\xi_2})},$$

where $c_0, c_1, c_2, \ldots, c_N$ are unknown coefficients and $\beta_{N,k} = \cos(\upsilon)$, $\upsilon = (((2k-1)\pi)/(2N+1))$.

PROOF: We have

$$\psi_j(\theta) = \frac{[(\lambda/r)^j D_{N-j}](F_0(0))}{\det(A)},$$

and

$$({\xi_1}^{N+1} - {\xi_2}^{N+1}) = (\xi_1 - \xi_2)^{-1} \sum_{k=0}^{N} {\xi_1}^k {\xi_2}^{N-k},$$

which can be written as

$$({\xi_1}^{N+1} - {\xi_2}^{N+1}) = (\xi_1 - \xi_2) \prod_{k=1}^{N} (\xi_1 + \xi_2 - 2\beta_{N,k}\sqrt{\xi_1\xi_2}).$$

So,

$$\det(A) = \theta d \left(\frac{r}{r_N}\right) \prod_{k=1}^{N} (\xi_1 + \xi_2 - 2\beta_{N,k}\sqrt{\xi_1\xi_2}),$$

hence

$$\psi_j(\theta) = \frac{[(\lambda/r)^j D_{N-j}](F_0(0)/d)(r_N/r)}{\theta \prod_{k=1}^{N} (\xi_1 + \xi_2 - 2\beta_{N,k}\sqrt{\xi_1\xi_2})}.$$

By using the partial fraction method, the proof is completed. ■

THEOREM 3: *For any nonnegative integer $j \in S$, the buffer occupancy distribution for the first model is given by*

$$F_j(u) = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})} + \frac{4\rho^{((j+1)/2)} \cdot F_0(0)}{(2N+1)} \times \sum_{k=1}^{N} \frac{(-1)^k \cos(\upsilon/2)[\sin(N-j+1)\upsilon - \sqrt{\rho}\sin(N-j)\upsilon]}{(1+\rho-2\sqrt{\rho}\cos\upsilon)} e^{[((-(\lambda+\mu))/r)+2(\sqrt{\lambda\mu}/r)\cos\upsilon]u}.$$

where

$$F_0(0) = 1 - \frac{(1-\rho^N)\sqrt{\rho}}{(1-\rho^{N+1})},\ \upsilon = \frac{(2k-1)\pi}{2N+1}\ \textit{and}\ \rho = \frac{\lambda}{\mu}.$$

PROOF: Now, we will compute the unknown coefficients in Lemma 2.

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To compute c_0, multiply $\psi_j(\theta)$ by θ and take the limit when θ convergence to zero, we obtain

$$c_0 = \lim_{\theta \to 0}[\theta\ \psi_j(\theta)]. \tag{4}$$

Substituting in the roots in Theorem 1 by $\theta = 0$, we get

$$\xi_1 = \frac{\lambda}{r} \text{ and } \xi_2 = \frac{\mu}{r}.$$

Substituting in Theorem 2 by $r_N = r$, we obtain

$$d = 1 - \frac{(1-\rho^N)\sqrt{\rho}}{(1-\rho^{N+1})}.$$

Substituting in Lemma 1 by $r_N = r$, we get

$$D_{N-j} = (\xi_1 - \xi_2)^{-1}\left[({\xi_1}^{N-j+1} - {\xi_2}^{N-j+1}) - \frac{\lambda}{r}({\xi_1}^{N-j} - {\xi_2}^{N-j})\right],$$

from the Appendix in case $(r_N = r)$, we get

$$F_0(0) = 1 - \frac{(1-\rho^N)\sqrt{\rho}}{1-\rho^{N+1}}.$$

By virtue of Lemma 2, we obtain

$$c_0 = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}}. \tag{5}$$

Similarly, we can obtain c_k as follows:

Multiply $\psi_j(\theta)$ in Lemma 2 by $[\theta(r) + \lambda + \mu - 2\beta_{N,k}\sqrt{\lambda\mu}]$ and take the limit when θ convergence to θ_k

where

$$\theta_k = \frac{-\lambda - \mu + 2\beta_{N,k}\sqrt{\lambda\mu}}{r}, \quad \beta_{N,k} = \cos\left(\frac{(2k-1)\pi}{2N+1}\right),$$

$$c_k = \frac{1}{r}\{\lim_{\theta \to \theta_k}([\theta(r) + \lambda + \mu - 2\beta_{N,k}\sqrt{\lambda\mu}]\psi_j(\theta))\}. \tag{6}$$

The value of the determinant of matrix A can be written in different form when $r_N = r$ as

$$\det(A) = \theta\prod_{i=1}^{N}\left(\theta + \frac{(\lambda+\mu)}{r} - 2\frac{\sqrt{\lambda\mu}}{r}\cos\left(\frac{(2k-1)\pi}{2N+1}\right)\right).$$

Hence, Eq. (6) can be written as

$$c_k = \left(\frac{(\lambda/r)^j}{(1/r)^N}\right)F_0(0)\left[\frac{\lim_{\theta \to \theta_k}[(\xi_1 - \xi_2)D_{N-j}]}{\lim_{\theta \to \theta_k}\left[\theta(\xi_1 - \xi_2)\prod_{\substack{i=1 \\ i \neq k}}^{N}(\theta(r) + \lambda + \mu - 2\beta_{N,i}\sqrt{\lambda\mu})\right]}\right],$$

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we get

$$c_k = \left(\frac{4\rho^{((j+1)/2)} \cdot F_0(0)}{(2N+1)}\right) \sum_{k=1}^{N} \frac{(-1)^k \cos(\upsilon/2)[\sin(N-j+1)\upsilon - \sqrt{\rho}\sin(N-j)\upsilon]}{(1+\rho-2\sqrt{\rho}\cos\upsilon)}, \quad \textbf{(7)}$$

substituting from (5) and (7) in Lemma 2 and after taking Laplace transform the proof is completed.

Special cases: For $N = 1$.

The stationary distribution of the buffer occupancy is given by

$$F_0(u) = \frac{1}{1+\rho} - \frac{\sqrt{\rho}(1-\sqrt{\rho})}{1+\rho} e^{[((-(\lambda+\mu))/r)+(\sqrt{\lambda\mu}/r)]u} \text{ and}$$

$$F_1(u) = \frac{\rho}{1+\rho} - \frac{\rho}{1+\rho} e^{[((-(\lambda+\mu))/r)+(\sqrt{\lambda\mu}/r)]u}.$$

So, the marginal distribution of the buffer occupancy is given by

$$P(C \le u) = F_0(u) + F_1(u) = 1 - \frac{\sqrt{\rho}}{1+\rho} e^{[((-(\lambda+\mu))/r)+(\sqrt{\lambda\mu}/r)]u}.$$

Hence, the probability density function of the buffer content is given by

$$b(u) = \frac{d}{du}[P(C \le u)] = \frac{\mu}{r}\left(\frac{\sqrt{\rho}(1+\rho-\sqrt{\rho})}{1+\rho}\right) e^{[((-(\lambda+\mu))/r)+(\sqrt{\lambda\mu}/r)]u}.$$

Which agree with the results mentioned in Lenin and Parthasarathy [7]. ■

THEOREM 4: *For any nonnegative integer $j \in S$, the buffer occupancy distribution for the second model is given by*

$$F_j(u) = \frac{(1-\rho)\rho^j}{(1-\rho^{N+1})} - (\rho^{\frac{j+1}{2}} \cdot F_0(0)) \sum_{k=1}^{N} \times \frac{\sin\upsilon_k(\cos\upsilon_k+1)[\sin(N-j+1)\upsilon_k - (1+\sqrt{\rho})\sin(N-j)\upsilon_k + \sqrt{\rho}\sin(N-j-1)\upsilon_k]}{\sin\upsilon_k(N\cos N\upsilon_k \cdot \sin\upsilon_k + \sin N\upsilon_k)(1+\rho-2\sqrt{\rho}\cos\upsilon_k)} e^{[\frac{-(\lambda+\mu)}{r}+2\frac{\sqrt{\lambda\mu}}{r}\cos\upsilon_k]u}.$$

where

$$F_0(0) = \frac{(1-\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}},\ \upsilon = \frac{(k-1)\pi}{N} \text{ and } \rho = \frac{\lambda}{\mu}.$$

Proof:

The proof is similar to the argument of Theorem 1 and is omitted.

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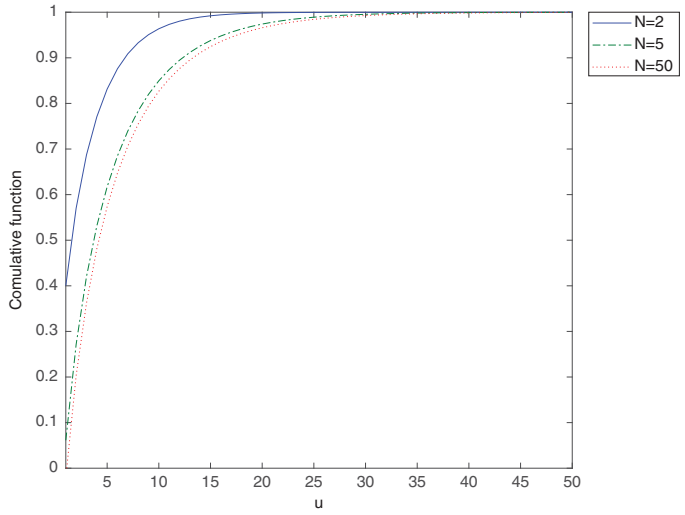


FIGURE **3.** Relation between the cumulative distribution function and buffer content for different values of files number at $c = 0.1$ and $r = 4$.

THEOREM 5: *For any nonnegative integerj* $\in S$, *the buffer occupancy distribution for the third model is given by*

$$F_j(u) = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}} - \left(\frac{\rho^{((j+1)/2)} \cdot F_0(0)}{N}\right) \times \sum_{k=1}^{N} \frac{[\sin(N-j+1)\upsilon - (1-\sqrt{\rho})\sin(N-j)\upsilon - \sqrt{\rho}\sin(N-j-1)\upsilon]}{(1+\rho-2\sqrt{\rho}\cos\upsilon)(\sin N\upsilon)} \times e^{[((-(\lambda+\mu))/r)+2(\sqrt{\lambda\mu}/r)\cos\upsilon]u}.$$

where

$$F_0(0) = \frac{(1+\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}},\ \upsilon = \frac{(2k-1)\pi}{2N} \text{ and } \rho = \frac{\lambda}{\mu}.$$

Proof:

The proof is similar to the argument of Theorem 1 and is omitted.

4. NUMERICAL ILLUSTRATION

Numerical calculations for the explicit solutions which have obtained in Theorems 3–5 are introduced to show the effectiveness of the obtained results.

Figure 3 illustrates the relation between cumulative distribution of the buffer occupancy and the buffer content u for different values of the number of files. It is clear that by increasing the number of files leads to a considerable increasing in the cumulative distribution of the buffer occupancy. Also, when the number of files tends to infinity, the curve of cumulative function tends to 1.

Figure 4 gives the relation between the cumulative distribution function and buffer content u for different values of efficient input rate r, the figure shows that the increasing in efficient input rate leads to the increasing in cumulative distribution function. Also, when efficient input rate r tends to infinity, the curve of cumulative function tends to 1.

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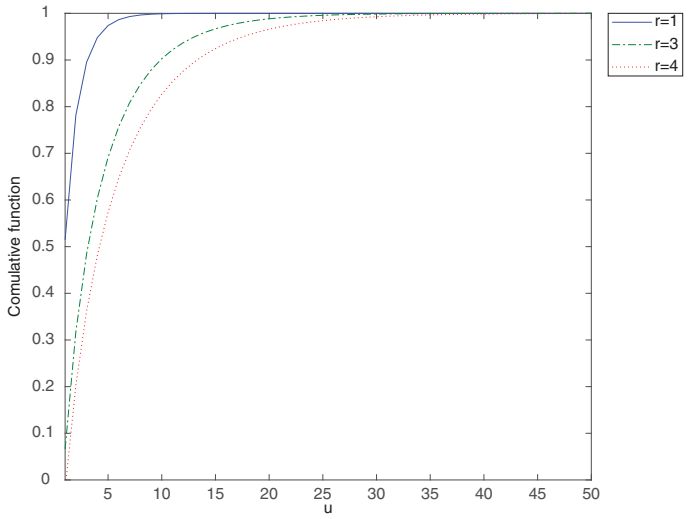


FIGURE **4.** Relation between the cumulative distribution function and buffer content for different values of efficient input rate at $c = 0.1$ and $N = 50$.

5. PERFORMANCE MEASURES

A well-known and widely used performance measures of the proposed system as average throughput, server utilization, expected buffer content and mean delay are introduced [3].

1. Average throughput is the number of data packets outflow from the buffer per unit time. It can be written in other terms as: Throughput equals service rate times probability that the system is non-empty.

The probability that the system is non-empty $= 1 - F_0(0) = 1 - \lim_{t\to\infty} p\{Y(t) = 0,\ Q(t) \le 0\}$,
hence

$$\text{throughput} = c\left(\frac{(1-\rho^N)\sqrt{\rho}}{1-\rho^{N+1}}\right).$$

Figure 5 shows that the throughput increases with an increasing in the arrival rate λ and a decreasing the service rate μ.

2. Server utilization means that the buffer is busy, that is,

utilization = probability that the buffer does not empty

$$= P(Q > 0) = 1 - P(Q \le 0) = 1 - F(0)$$

$$= 1 - \sum_{j=0}^{N} F_j(0).$$

Figure 6 shows that the utilization increases with an increasing in the arrival rate which is expected because more arrivals mean more loads on the server.

3. Expected buffer content means the expectation of the buffer content after long

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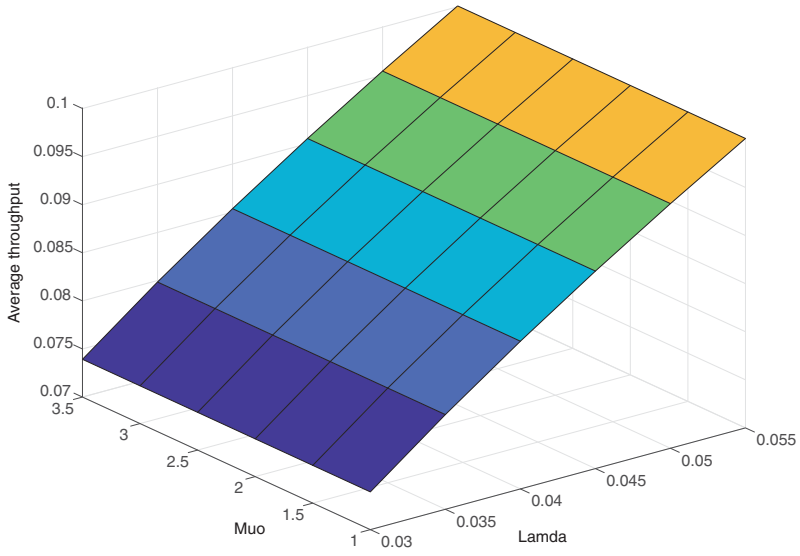


FIGURE **5.** Relation between average throughput versus different values of arrival and service rate at $c = 0.1$ and $N = 50$.

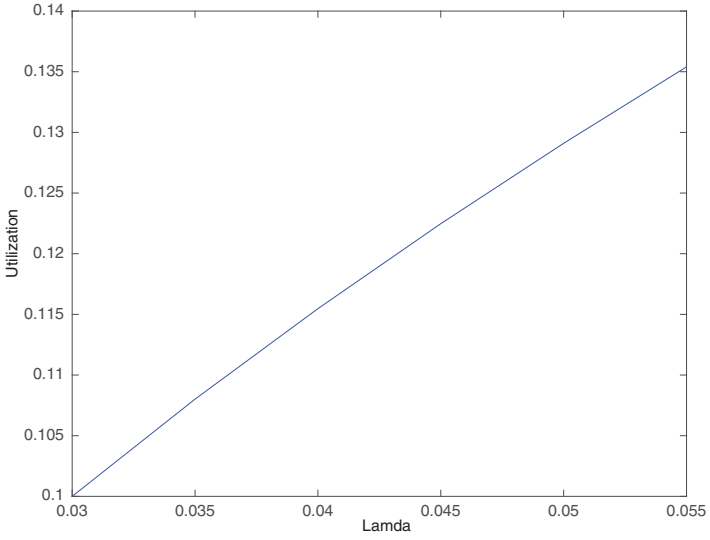


FIGURE **6.** Relation between server utilization and arrival rate λ at $\mu = 3$ and $N = 50$.

work period, which can be written as

$$\text{Expected buffer content} = E(u) = \int_0^{\infty} \left[1 - \sum_{j=0}^{N} F_j(u)\right] du.$$

Figure 7 shows that the expected buffer content increases with an increasing in the arrival rate which is predictable because when more arrivals occur, the buffer content will increase immediately.

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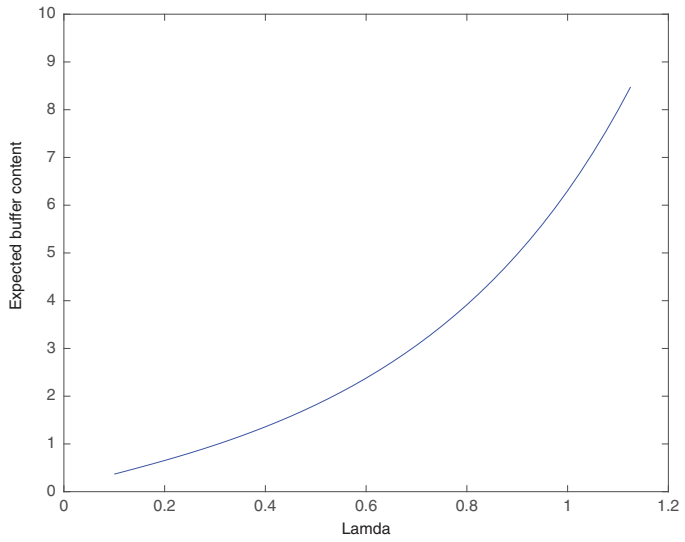


FIGURE **7.** Relation between expected buffer content versus arrival rate λ at $c = 0.1$, $r = 4$ and $N = 50$.

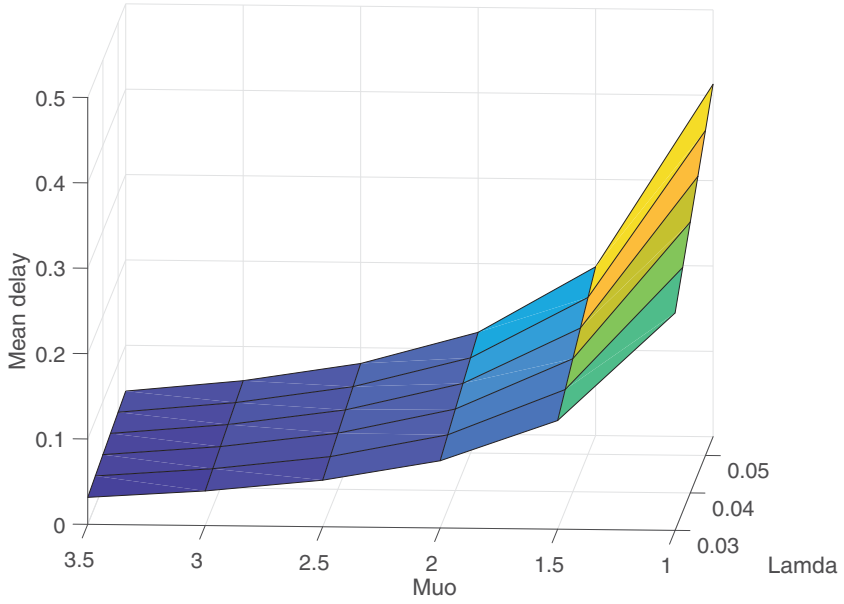


FIGURE **8.** Relation between mean delay versus different values of arrival and service rate at $c = 0.1$, $r = 4$ and $N = 50$.

4. Mean delay means the average time will be taken until the files depart from the storage server; in other words, it can be defined as expected buffer content divided by the mean inflow rate.

Hence,

$$\text{Mean delay} = \frac{\int_0^\infty \left[1 - \sum_{j=0}^{N} F_j(u)\right] du}{q}.$$

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Figure 8 shows that the mean delay increases with an increasing the arrival rate λ and decreasing the service rate μ because when arrivals of files increases this cause the server will be busy; hence, it will increase the delay.

From the obtained performance measures, we can conclude that there is a good agreement of the obtained results with those related to the practical models. So, the proposed models in this study are communicated well with the practical situations.

6. CONCLUSION

The system of storage data in the cloud is modulated in this study. We obtain explicit solutions for the stationary distributions of the buffer occupancy in the storage cloud facility and the density function of the buffer content. Also, various performance measures of the proposed system are obtained. The obtained numerical results and performance measures ensure that the proposed models describe the actual system efficiently. As a future work, we may get the distribution of the buffer occupancy in the transient state and we may study the catastrophe on the behavior of the system.

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APPENDIX

In this appendix, we obtain the value of $F_0(0)$ as follows:

We have

$$r_0 \cdot F_0^{\cdot}(u) + r \sum_{j=1}^{N-1} F_j^{\cdot}(u) + r_N \cdot F_N^{\cdot}(u) = 0.$$

Take the integration from zero to infinity, we get

$$r \sum_{j=1}^{N-1} [F_j^{\cdot}(\infty) - F_j^{\cdot}(0)] = -r_0.[F_0^{\cdot}(\infty) - F_0^{\cdot}(0)] - r_N \cdot [F_N^{\cdot}(\infty) - F_N^{\cdot}(u)].$$

Here

$$F_0^{\cdot}(\infty) = p_0 = \frac{1-\rho}{1-\rho^{N+1}},\ F_j^{\cdot}(\infty) = p_j = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}},\ F_j^{\cdot}(0) = 0, \quad 1 \le j \le N.$$

After simple calculations, we get

$$F_0(0) = \left(\frac{p_0}{r_0}\right) \sum_{j=0}^{N} r_j \rho^j.$$

Case 1: $r_N = r$.

$$F_0(0) = \left(\frac{p_0}{r_0}\right) \left[r_0 + r \sum_{j=1}^{N} \rho^j\right] = p_0 + \left(\frac{r \cdot p_0}{r_0}\right) [\rho + \rho^2 + \cdots + \rho^N],$$

$$= 1 - \frac{(1-\rho^N)\sqrt{\rho}}{1-\rho^{N+1}}.$$

Case 2: $r_N = (\mu r/(\mu - \sqrt{\lambda\mu}))$.

$$F_0(0) = \left(\frac{p_0}{r_0}\right) \left[r_0 + r \sum_{j=1}^{N-1} \rho^j + r_N \rho^N\right] = p_0 + \left(\frac{r \cdot p_0}{r_0}\right) \sum_{j=1}^{N-1} \rho^j + \left(\frac{r_N \cdot p_0}{r_0}\right) \rho^N,$$

$$= \frac{(1-\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}}.$$

Case 3: $r_N = (\mu r/(\mu + \sqrt{\lambda\mu}))$.

$$F_0(0) = \left(\frac{p_0}{r_0}\right) \left[r_0 + r \sum_{j=1}^{N-1} \rho^j + r_N \rho^N\right] = p_0 + \left(\frac{r \cdot p_0}{r_0}\right) \sum_{j=1}^{N-1} \rho^j + \left(\frac{r_N \cdot p_0}{r_0}\right) \rho^N,$$

$$= \frac{(1+\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}}.$$

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