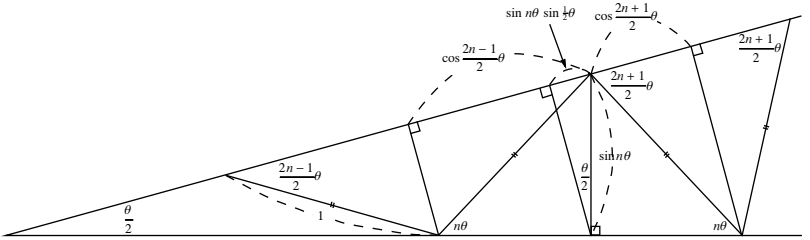
109.36 Proof without words: Lagrange's sine identity

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\dfrac{\theta}{2} - \cos\dfrac{(2n+1)\theta}{2}}{2\sin\dfrac{\theta}{2}}$$

Lemma: $\sin n\theta \sin\frac{1}{2}\theta = \frac{1}{2}\big(\cos\frac{1}{2}((2n-1)\theta) - \cos\frac{1}{2}((2n+1)\theta)\big)$. That is,

$$\sin n\theta = \frac{\cos\frac{1}{2}((2n-1)\theta) - \cos\frac{1}{2}((2n+1)\theta)}{2\sin\frac{1}{2}\theta}.$$



Theorem (Lagrange's sine identity):

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos\frac{1}{2}\theta - \cos\frac{1}{2}((2n+1)\theta)}{2\sin\frac{1}{2}\theta}.$$

10.1017/mag.2025.10137 © The Authors, 2025
Published by Cambridge University Press
on behalf of The Mathematical Association

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109.37 A visual proof of a theorem on divergent series

In mathematics, divergent series have been widely studied in the relevant literature and are always of interest due to their applications in various sciences. For example, in probability theory, the well-known Borel-Cantelli lemma (second version) is expressed with assumptions involving divergent series; it states that if the series of probabilities of an infinite sequence of independent events is divergent, then the probability that each event occurs infinitely often is equal to 1 (see, e.g., [1]).

Among the divergent series, the series that diverge slowly are more interesting. Some of the most popular examples of slowly diverging series are $\sum_{k=1}^{\infty}\frac{1}{k}$, $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ and $\sum_{k=3}^{\infty}\frac{1}{k\ln k\ln\ln k}$. Usually, to show the divergence of such series, the integral test or Abel-Dini theorem is used (see, e.g., [2] and [3]). In a recent paper [4], the author established the following general theorem, which is useful for showing the divergence of slowly diverging series.

https://doi.org/10.1017/mag.2025.10137 Published online by Cambridge University Press