Proc. Camb. Phil. Soc. (1968), **64**, 113
PCPS 64–18
Printed in Great Britain

Integral equations and relations for Lamé functions and ellipsoidal wave functions

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(*Received* 8 *December* 1966)

Abstract. Non-linear integral equations and relations, whose nuclei in all cases is the 'potential' Green's function, satisfied by Lamé polynomials and Lamé functions of the second kind are discussed. For these functions certain techniques of analysis are described and these find their natural generalization in ellipsoidal wave-function theory. Here similar integral equations are constructed for ellipsoidal wave functions of the first and third kinds, the nucleus in each case now being the 'free space' Green's function. The presence of ellipsoidal wave functions of the second kind is noted for the first time. Certain possible generalizations of the techniques and ideas involved in this paper are also discussed.

Introduction. In the problems of scalar scattering by ellipsoids and elliptic discs at low frequencies, ((10), (11)) one requires the use of non-linear integral equations satisfied by Lamé polynomials and Lamé functions of the second kind. The required integral equations have been discussed as long ago as 1846 when they were first introduced by Liouville (8), who demonstrated that Lamé polynomials satisfied certain non-linear integral equations. However, he failed to specify for which of the eight types of Lamé polynomials these equations were valid. Quite recently Arscott (2) has considered similar but slightly more specialized equations. He was able to show within this restricted framework that Liouville's integral equations could be extended to form integral relations satisfied by Lamé functions of the second kind and conjectured that Liouville's results were only valid for functions of type $uE_n^m(\gamma)$, $uF_n^m(\gamma)$.

In this paper, section 2, we return to Liouville's original equations and show, using the results obtained by Arscott, that integral equations and relations may be constructed which are satisfied by all eight types of Lamé polynomial and Lamé function of the second kind. The nucleus used for all these equations is the 'potential' Green's function.

In section 3 we generalize the ideas of section 2 in order to construct integral equations and relations satisfied by ellipsoidal wave functions of the first and third kinds, the common nucleus now being the 'free space' Green's function. As a special case of these relations appears the integral equations of Malurkar (9), later considered by Arscott (1), where we obtain a new class of nuclei for these equations. From the results obtained here, we show the presence of ellipsoidal wave functions of the second kind which do not appear to have been discussed before.

Finally, in section 4, some further generalizations are discussed.

https://doi.org/10.1017/S0305004100042638 Published online by Cambridge University Press

Section 1. *Preliminaries.* Lamé's equation is taken as

$$\frac{d^2\omega}{dz^2} + \{h - n(n+1)\,k^2 \operatorname{sn}^2 z\}\,\omega = 0, \tag{1·1}$$

where k is the modulus of the elliptic function $\operatorname{sn}(z, k)$. The parameter n is a non-negative integer and h has one of $2n+1$ characteristic values for which (1·1) has a Lamé polynomial as a first solution. Such polynomials are of the form

$$\operatorname{sn}^\rho z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z\, U_P(\operatorname{sn}^2 z), \tag{1·2}$$

where $\rho, \sigma, \tau = 0$ or 1, U_P denotes a polynomial of degree P in $\operatorname{sn}^2 z$ and $2P + \rho + \sigma + \tau = n$, so that the total degree of the polynomial is n. The notation is that of (3) and the Lamé polynomials will be normalized by specifying the highest term in U_P to be unity.

The second solution is defined as that solution which is of the form

$$\operatorname{sn}^\rho z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z\, V(\operatorname{sn} z), \tag{1·3}$$

where

$$V(\operatorname{sn} z) = \sum_{r=0}^{\infty} a_r (\operatorname{sn} z)^{-n-1-\rho-\sigma-\tau-2r} \tag{1·4}$$

and normalized by specifying $a_0 = 1$.

The second solution corresponding to a particular Lamé polynomial $E_n^m(z)$ is denoted by $F_n^m(z)$; in particular, if the first solution is of a specified type such as $uE_{2n}^m(z)$, the second solution is written as $uF_{2n}^m(z)$, etc. From (1·2), (1·3) it is easily shown that the Wronksian is given by

$$E_n^m(z)\,F_n'^m(z) - E_n'^m(z)\,F_n^m(z) = (-1)^{\sigma+\tau}(2n+1)\,k^{2\tau+1}. \tag{1·5}$$

By an ellipsoidal wave function, or more precisely an ellipsoidal wave function of the first kind, is meant a solution of the differential equation

$$\frac{d^2\omega}{dz^2} - (a + bk^2 \operatorname{sn}^2 z + qk^4 \operatorname{sn}^4 z)\,\omega = 0, \tag{1·6}$$

possessing the property of being uniform and doubly-periodic in z. Such functions exist for all values of q and suitably chosen values of a and b (depending on q), and are in one-to-one correspondence with the Lamé polynomials to which they reduce when $q = 0$. Such functions, denoted by $el_n^m(z)$, (cf. (3)), have the general form

$$el_n^m(z) = \operatorname{sn}^\rho z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z\, F(\operatorname{sn}^2 z), \tag{1·7}$$

where $\rho, \sigma, \tau = 0$ or 1 and F denotes an integral function of $\operatorname{sn}^2 z$. Let $el_n^m(z)$ and $E_n^m(z)$ (the Lamé polynomial to which $el_n^m(z)$ reduces when $q = 0$), be written respectively as

$$\operatorname{sn}^\rho z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z \sum_{P=0}^{\infty} A_P(q) \operatorname{sn}^{2P} z, \quad \operatorname{sn}^\rho z \operatorname{cn}^\sigma z \operatorname{dn}^\tau z \sum_{P=0}^{N} A_P(0) \operatorname{sn}^{2P} z; \tag{1·8}$$

then as we normalize $E_n^m(z)$ by specifying $A_N(0) = 1$, so we normalize $el_n^m(z)$ by the condition $A_N(q) = 1$ for all q. Denote by $wl_n^m(z)$ a solution of (1·6) which is independent of $el_n^m(z)$ and reduces to $F_n^m(z)$ when $q = 0$. Such a solution will be so normalized in order that the following Wronksian relation is satisfied

$$el_n^m(z)\,wl_n'^m(z) - el_n'^m(z)\,wl_n^m(z) = (-1)^{\sigma+\tau}(2n+1)\,k^{2\tau+1}. \tag{1·9}$$

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Obviously the above statement is insufficient to characterize $wl_n^m(z)$ completely. This point will be discussed in greater detail later.

The problems discussed in this paper, find their basis, in the following general theorem due to Arscott (4).

Theorem. Let (i) $\omega(z)$ be a solution of the differential equation

$$\frac{d^2\omega}{dz^2}+\{\lambda+\mu f(z)+g(z)\}\omega = 0. \tag{1·10}$$

(ii) $H(\alpha,\beta,\gamma)$ be a solution of the partial differential equation

$$\{f(\beta)-f(\gamma)\}\frac{\partial^2 H}{\partial\alpha^2}+\{f(\gamma)-f(\alpha)\}\frac{\partial^2 H}{\partial\beta^2}+\{f(\alpha)-f(\beta)\}\frac{\partial^2 H}{\partial\gamma^2}$$
$$= -[g(\alpha)\{f(\beta)-f(\gamma)\}+g(\beta)\{f(\gamma)-f(\alpha)\}+g(\gamma)\{f(\alpha)-f(\beta)\}]H, \tag{1·11}$$

such that $H(\alpha,\beta,\gamma)$ is analytic when α, β, γ lie in regions R_α, R_β, R_γ of the complex α-, β-, γ-planes respectively.

(iii) C_α, C_β be paths in R_α, R_β such that

$$\left[\frac{\partial H}{\partial\alpha}\omega(\alpha)-H(\alpha,\beta,\gamma)\frac{d\omega}{d\alpha}\right]_{C_\alpha} = 0, \tag{1·12a}$$

$$\left[\frac{\partial H}{\partial\beta}\omega(\beta)-H(\alpha,\beta,\gamma)\frac{d\omega}{d\beta}\right]_{C_\beta} = 0. \tag{1·12b}$$

(iv)
$$W(\gamma) = \int_{C_\alpha}\int_{C_\beta}\{f(\alpha)-f(\beta)\}H(\alpha,\beta,\gamma)\,\omega(\alpha)\,\omega(\beta)\,d\alpha\,d\beta \tag{1·13}$$

exist and if singular converge uniformly with respect to γ when α, β, γ lie in R_α, R_β, R_γ. Then $W(z)$ is a solution of (1·10).

Finally, we introduce a transformation which will be needed later and may be conveniently given here.

Ellipsoidal co-ordinates (α,β,γ) are related to Cartesian co-ordinates (x,y,z) by the relations

$$\left.\begin{aligned} x &= k^2 l\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma,\\ y &= -k^2 l k'^{-1}\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma,\\ z &= ilk'^{-1}\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma, \end{aligned}\right\} \tag{1·14}$$

where the elliptic functions are constructed with modulus k, and l is an arbitrary constant. It is not difficult to show that in order to cover all values of x, y and z we need only to have α, β, γ varying in the limited ranges

$$\left.\begin{aligned} &\alpha \text{ from } -2K \text{ to } 2K,\\ &\beta \text{ from } K \text{ to } K+2iK',\\ &\gamma \text{ from } iK' \text{ to } K+iK' \end{aligned}\right\} \tag{1·15}$$

and one set of values (x,y,z) then corresponds to one and only one set of values (α,β,γ). It will be also useful to mention that under the co-ordinate system (1·14), the surfaces β = constant, α = constant are hyperboloids of one and two sheets respectively while the surfaces γ = constant are ellipsoids.

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Section 2. Integral equations and relations for Lamé functions. From the differential equation (1·10) we see that Lamé's equation is obtained if

$$\lambda = h, \quad \mu = -n(n+1)k^2, \quad f(z) = \mathrm{sn}^2 z, \quad g(z) = 0 \tag{2·1}$$

and the partial differential equation satisfied by $H(\alpha, \beta, \gamma)$ in the theorem of section 1 becomes

$$(\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)\frac{\partial^2 H}{\partial\alpha^2} + (\mathrm{sn}^2\gamma - \mathrm{sn}^2\alpha)\frac{\partial^2 H}{\partial\beta^2} + (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\frac{\partial^2 H}{\partial\gamma^2} = 0. \tag{2·2}$$

In this paper we shall be particularly interested in the integral relation

$$W(\gamma) = \int_{C_\alpha}\int_{C_\beta} (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, H(\alpha, \beta, \gamma)\, E_n^m(\alpha)\, E_n^m(\beta)\, d\alpha\, d\beta. \tag{2·3}$$

If we relate α, β and γ in (2·2) via the co-ordinate system (1·14) we find that (2·2) reduces to Laplace's equation

$$\nabla^2 H = 0. \tag{2·4}$$

Let $R(P, P_0)$ be the distance, in ellipsoidal co-ordinates, of $P(\alpha, \beta, \gamma)$ from $P_0(\alpha_0, \beta_0, \gamma_0)$. Then clearly $1/R(P, P_0)$, the potential Green's function, is a solution of (2·2) provided $P \neq P_0$. If C_α is the line segment $(-2K, 2K)$, C_β is the line segment $(K - 2iK',\ K + 2iK')$ then it is easy to verify that equations (1·12a, b) are satisfied and hence

$$W(\gamma) = \iint_{\mathscr{S}} \frac{1}{R(P, P_0)} (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, E_n^m(\alpha)\, E_n^m(\beta)\, d\alpha\, d\beta, \tag{2·5}$$

where $\iint_{\mathscr{S}} d\alpha\, d\beta$ denotes $\int_{-2K}^{2K}\int_{K-2iK'}^{K+2iK'} d\alpha, d\beta$

is a solution of Lamé's equation.

If $1/R(P, P_0)$ is written in terms of spherical polar co-ordinates $(r, \theta, \phi), (r_0, \theta_0, \phi_0)$ then

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}},$$

which by straightforward calculation from (1·14) is found to have the value

$$r = l(k^2\,\mathrm{sn}^2\alpha - k^2\,\mathrm{cn}^2\beta - \mathrm{dn}^2\gamma)^{\frac{1}{2}}. \tag{2·6}$$

To express θ, ϕ in terms of α, β, γ is also straightforward, but one has a measure of choice in that any of the axes Ox, Oy, Oz can be taken as the polar axis, and then the remaining two axes can be interchanged if desired, giving altogether six permutations. One of these is the traditional orientation

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \end{aligned}\right\} \tag{2·7}$$

and then we obtain

$$\cos\theta = \frac{\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma}{k'(\mathrm{dn}^2\gamma - k^2\,\mathrm{sn}^2\alpha + k^2\,\mathrm{cn}^2\beta)^{\frac{1}{2}}}, \quad \tan\phi = -\frac{\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma}{k'\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma}. \tag{2·8}$$

Similar expressions are obtained for other arrangements of the axes.

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On using the coordinates (2·7), $1/R$ may be written as

$$\frac{1}{R} = \frac{1}{\sqrt{(r^2 + r_0^2 - 2rr_0\cos\omega)}}, \tag{2·9}$$

where $$\cos\omega = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0).$$

In terms of surface harmonics we have

$$\left.\begin{aligned} \frac{1}{R} = \frac{1}{r_>}\sum_{N=0}^{\infty}\left(\frac{r_<}{r_>}\right)^N P_N(\cos\omega), \quad r_> &= r \quad \text{if} \quad r > r_0, \\ &= r_0 \quad \text{if} \quad r < r_0, \\ r_< &= r \quad \text{if} \quad r < r_0, \\ &= r_0 \quad \text{if} \quad r > r_0. \end{aligned}\right\} \tag{2·10}$$

That is $$\frac{1}{R} = \frac{1}{r_0}\sum_{N=0}^{\infty}\left(\frac{r}{r_0}\right)^N P_N(\cos\omega) \quad \text{if} \quad P \text{ lies inside the sphere } r = r_0$$

$$= \frac{1}{r}\sum_{N=0}^{\infty}\left(\frac{r_0}{r}\right)^N P_N(\cos\omega) \quad \text{if} \quad P \text{ lies outside the sphere } r = r_0. \tag{2·11a,b}$$

If we now substitute for r, θ, ϕ, using equations (2·6), (2·8) then equations (2·11) may be restated as

$$\frac{1}{R} = \frac{1}{r_0}\sum_{N=0}^{\infty}\left(\frac{r}{r_0}\right)^N P_N(\cos\omega) \quad \text{if} \quad P \text{ lies inside the ellipsoid } \gamma = \gamma_0$$

$$\text{that is if} \quad \gamma \in (K + iK', \gamma_0);$$

$$\frac{1}{R} = \frac{1}{r}\sum_{N=0}^{\infty}\left(\frac{r_0}{r}\right)^N P_N(\cos\omega) \quad \text{if} \quad P \text{ lies outside the ellipsoid } \gamma = \gamma_0,$$

$$\text{that is if} \quad \gamma \in (\gamma_0, iK'). \tag{2·12a,b}$$

Consider (2·12a); expanding using the addition theorem for Legendre polynomials, we obtain

$$\frac{1}{R} = \frac{1}{r_0}\sum_{N=0}^{\infty}\left(\frac{r}{r_0}\right)^N P_N(\cos\theta)\,P_N(\cos\theta_0) + \frac{2}{r}\sum_{N=0}^{\infty}\sum_{s=1}^{N}\left(\frac{r}{r_0}\right)^N \frac{(N-s)!}{(N+s)!} P_N^s(\cos\theta)\,P_N^s(\cos\theta_0)\cos s(\phi - \phi_0). \tag{2·13}$$

It is important to point out here that the terms in (2·13) such as

$$r^N P_N^s(\cos\theta)\begin{Bmatrix}\cos s\phi, \\ \sin s\phi,\end{Bmatrix} \tag{2·14}$$

are precisely the nuclei considered by Arscott (2), and it is this fact that we use in order to determine the type of solution one obtains from equation (2·5).

Substituting (2·13) into (2·5) and taking

$$E_n^m(\alpha)\,E_n^m(\beta) = uE_{2n}^m(\alpha)\,uE_{2n}^m(\beta) = uEp_{2n}^m(\alpha, \beta)$$

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we have

$$W(\gamma) = \sum_{N=0}^{\infty} \frac{P_N(\cos\theta_0)}{r_0^{N+1}} \iint_{\mathscr{S}} r^N P_N(\cos\theta)\, uEp_{2n}^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta$$
$$+ 2\sum_{N=0}^{\infty}\sum_{s=1}^{N} \frac{1}{r_0^{N+1}} \frac{(N-s)!}{(N+s)!} P_N^s(\cos\theta_0)\cos s\phi_0$$
$$\times \iint_{\mathscr{S}} r^N P_N^s(\cos\theta)\cos s\phi\, uEp_{2n}^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta$$
$$- 2\sum_{N=0}^{\infty}\sum_{s=1}^{N} \frac{1}{r_0^{N+1}} \frac{(N-s)!}{(N+s)!} P_N^s(\cos\theta_0)\sin s\phi_0$$
$$\times \iint_{\mathscr{S}} r^N P_N^s(\cos\theta)\sin s\phi\, uEp_{2n}^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta. \quad (2{\cdot}15)$$

It is shown by Arscott (2) that the above integrals vanish unless the nuclei in each of them when considered as functions of γ, have the same parity and periodicity as the Lamé polynomial $uE_{2n}^m(\gamma)$, and also if $P_N^s(\cos\theta)$ is a polynomial in γ of the same degree as $uE_{2n}^m(\gamma)$. That is

$$\iint_{\mathscr{S}} r^N P_N^s(\cos\theta)\sin s\phi\, uEp_{2n}^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta \equiv 0$$

and

$$\iint_{\mathscr{S}} r^N P_N^s(\cos\theta)\cos s\phi\, uEp_{2n}^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta = 0 \quad (s \text{ any integer}). \quad (2.16a,b)$$

unless $N = 2n$ and s is even, in which case (2·16b) is a constant multiple (possibly zero) of $uE_{2n}^m(\gamma)$. Thus from equations (2·5), (2·15) and (2·16a, b) we obtain

$$uE_{2n}^m(\gamma) = \lambda_1 \iint_{\mathscr{S}} \frac{1}{R(P,P_0)}\, uEp_{2n}^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta, \quad \gamma \in (K+iK', \gamma_0) \quad (2{\cdot}17)$$

If we now replace $Ep_n^m(\alpha,\beta)$ in equation (2·5) by one of the remaining seven types of Lamé polynomial, then it is clear that in general

$$E_n^m(\gamma) = \lambda_1 \iint_{\mathscr{S}} \frac{1}{R(P,P_0)}\, Ep_n^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta, \quad \gamma \in (K+iK', \gamma_0) \quad (2{\cdot}18)$$

On the other hand, the same arguments may be applied if γ lies outside the ellipsoid $\gamma = \gamma_0$. In this case we must use equation (2·12b) and by noting that the functions

$$r^{-N-1} P_N^S(\cos\theta)\begin{Bmatrix}\cos s\phi, \\ \sin s\phi, \quad s \text{ any integer},\end{Bmatrix} \quad (2{\cdot}19)$$

are suitable nuclei for constructing integral relations for Lamé functions of the second kind, cf. Arscott (2), we prove the result

$$F_n^m(\gamma) = \lambda_2 \iint_{\mathscr{S}} \frac{1}{R(P,P_0)}\, Ep_n^m(\alpha,\beta)\,(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,d\alpha\,d\beta, \quad \gamma \in (\gamma_0, iK'). \quad (2{\cdot}20)$$

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In equation (2·20) it is a simple matter to show that the result is also true if P lies on the ellipsoid, that is if $\gamma = \gamma_0$, thus we have

$$F_n^m(\gamma) = \lambda_2 \iint_{\mathscr{S}} \frac{1}{R(P, P_0)} E p_n^m(\alpha, \beta) (\mathrm{sn}^2 \alpha - \mathrm{sn}^2 \beta) \, d\alpha \, d\beta, \quad \gamma \in [\gamma_0, iK']. \tag{2·21}$$

Obviously particular integral equations and relations may be obtained if P_0 is specified explicitly. For instance, suppose P_0 lies at the origin. This corresponds to taking

$$\alpha_0 = 0, \quad \beta_0 = K, \quad \gamma_0 = K + iK'. \tag{2·22}$$

Hence equation (2·6) evaluated at the origin gives $r_0 = 0$, and thus the expansion (2·12a) is undefined while (2·12b) vanishes apart from the first term. Therefore, in this case, we only obtain relations of the form (2·21) and further we see that (2·21) is only valid for functions of the type $uF_{2n}^m(\gamma)$. That is we have

$$uF_{2n}^m(\gamma) = \lambda_2 \iint_{\mathscr{S}} \frac{1}{r} uE p_{2n}^m(\alpha, \beta) (\mathrm{sn}^2 \alpha - \mathrm{sn}^2 \beta) \, d\alpha \, d\beta. \tag{2·23}$$

This integral relation has been obtained by Arscott (2). In order to obtain integral relations for the remaining seven types Arscott has shown, for this particular value of P_0, that the nucleus must be replaced by functions of the type

$$x/r^3, \quad y/r^3, \quad z/r^3, \quad \text{etc.} \tag{2·24}$$

An interesting question which has been raised, is that since the nucleus in equation (2·23) and those mentioned in (2·24) are independent of n (the degree of the corresponding Lamé polynomial), can integral equations of the type (2·18) be constructed with nuclei independent of n when P_0 lies at the origin. For the nuclei considered above we have taken the 'potential' Green's function $1/R$ and for this particular nucleus the answer appears to be no. However, more will be said on this point later.

It is fitting to remark here that equations (2·18), (2·21) are not entirely new. The former have been considered by Arscott (2) for the case when P_0 lies at the origin, while the latter is due to Liouville (8) and Halphen (7). The essential achievement in the above analysis is that we have been able to show that the potential Green's function is a suitable nucleus for integral equations and relations for all eight types of Lamé polynomials and Lamé functions of the second kind.

In equations (2·18), (2·21) the λ's, which depend on m and n, remain to be determined. This can be done by an elegant procedure also due to Liouville, and again repeated by Arscott. We repeat the method here, first because Liouville's notation is completely different and, secondly, because there appears to be an error in Arscott's development. The procedure is also important in that it suggests a generalization to similar problems associated with the ellipsoidal wave equation, to be discussed later.

Consider equation (2·21), then on substituting into the Wronksian given by (1·5), we have, on writing

$$E_n^m(\alpha) \, E_n^m(\beta) \, E_n^m(\gamma) = W(\alpha, \beta, \gamma), \tag{2·25}$$

$$(-1)^{\sigma+\tau}(2n+1) \, k^{2\tau+1} = \lambda_2 \iint_{\mathscr{S}} \left\{ W \frac{\partial}{\partial \gamma} \left(\frac{1}{R} \right) - \frac{1}{R} \frac{\partial W}{\partial \gamma} \right\} (\mathrm{sn}^2 \alpha - \mathrm{sn}^2 \beta) \, d\alpha \, d\beta. \tag{2·26}$$

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Now W and $1/R$ are both solutions of Laplace's equation and

$$\iint_{\mathscr{S}} \Phi(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, d\alpha\, d\beta,$$

represents twice the integral of Φ over the surface of an ellipsoid containing P_0. The procedure now is to employ Green's theorem. In order to do this we need to relate $\partial/\partial\gamma$ to the normal derivative $\partial/\partial n$ to the surface of the ellipsoid containing P_0, and also relate $(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, d\alpha\, d\beta$ to the surface element ds of the ellipsoid. Simple analysis shows, using equations (1·14)

$$\frac{\partial}{\partial n} = \frac{1}{k^2 l}(\mathrm{sn}^2\alpha - \mathrm{sn}^2\gamma)^{-\frac{1}{2}}(\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)^{-\frac{1}{2}}\frac{\partial}{\partial\gamma},$$

$$ds = ik^4 l^2(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)(\mathrm{sn}^2\alpha - \mathrm{sn}^2\gamma)^{\frac{1}{2}}(\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)^{\frac{1}{2}}\, d\alpha\, d\beta. \qquad (2\cdot27\,a, b)$$

Substituting from (2·27 a, b) into (2·26) we have

$$(-1)^{\sigma+\tau}(2n+1)\, k^{2\tau+1} = \frac{\lambda^2}{ik^2 l}\iint_{\mathscr{S}}\left\{W\frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial W}{\partial n}\right\} ds$$

$$= \frac{2\lambda_2}{ik^2 l}\int_V W\nabla^2\left(\frac{1}{R}\right) dV, \qquad (2\cdot28)$$

the integral being taken over the interior of the ellipsoid (the factor 2 appears because with the given range of α, β, γ the ellipsoid is described twice). The integrand vanishes except at P_0; so from (2·28) we have

$$(-1)^{\sigma+\tau}(2n+1)\, k^{2\tau+1} = -\frac{8\pi\lambda_2}{ik^2 l} E_n^m(\alpha_0)\, E_n^m(\beta_0)\, E_n^m(\gamma_0),$$

that is

$$\lambda_2 = \frac{il(-1)^{\sigma+\tau+1}(2n+1)\, k^{2\tau+3}}{8\pi E_n^m(\alpha_0)\, E_n^m(\beta_0)\, E_n^m(\gamma_0)}, \qquad (2\cdot29)$$

provided α_0, β_0 or γ_0 is not a zero of $E_n^m(z)$. In a similar manner, we find that for equation (2·18)

$$\lambda_1 = \frac{il(-1)^{\sigma+\tau+1}(2n+1)\, k^{2\tau+3}}{8\pi E_n^m(\alpha_0)\, E_n^m(\beta_0)\, F_n^m(\gamma_0)}. \qquad (2\cdot30)$$

Having discussed integral equations and relations satisfied by solutions of Lamé's equation we now consider its natural generalization, namely the ellipsoidal wave equation.

Section 3. *Integral equations and relations for ellipsoidal wave functions.* If from equation (1·10) we put

$$\lambda = -a, \quad \mu = -bk^2, \quad f(z) = -\mathrm{sn}^2 z, \quad g(z) = -qk^4\,\mathrm{sn}^4 z \qquad (3\cdot1)$$

then we obtain the ellipsoidal wave equation, namely equation (1·6). Hence in the theorem of section 1, the partial differential equation satisfied by the nucleus $H(\alpha, \beta, \gamma)$ now takes the form

$$(\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)\frac{\partial^2 H}{\partial\alpha^2} + (\mathrm{sn}^2\gamma - \mathrm{sn}^2\alpha)\frac{\partial^2 H}{\partial\beta^2} + (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\frac{\partial^2 H}{\partial\gamma^2}$$

$$= -qk^4(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)(\mathrm{sn}^2\gamma - \mathrm{sn}^2\alpha)(\mathrm{sn}^2\beta - \mathrm{sn}^2\gamma)H. \qquad (3\cdot2)$$

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If we transform this equation, using the co-ordinate system (1·14), we obtain, on putting $-q = \chi^2 l^2$

$$\nabla^2 H + \chi^2 H = 0, \tag{3·3}$$

which is the reduced Helmholtz equation (χ being the wave number). In equation (3·3) there are essentially two main cases of interest, that is when χ is purely real or purely imaginary. These cases correspond to q pure imaginary or purely real in (1·6) if l is assumed real. For the present we consider χ to be purely real.

On following the argument adopted in section 2 for the case of Lamé's equation, one would expect

$$\frac{\exp(i\chi R)}{R} \tag{3·4}$$

to be a suitable nucleus for integral equations and relations satisfied by ellipsoidal wave functions. It is a simple matter to show that this is indeed the case if C_α is the line segment $(-2K, 2K)$, C_β is the line segment $(K-2iK', K+2iK')$. The conditions of the theorem of section 1 are then satisfied if $\omega(z) = el_n^m(z)$ and hence we obtain

$$W(\gamma) = \iint_{\mathscr{S}} \frac{\exp(i\chi R)}{R} (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, el_n^m(\alpha)\, el_n^m(\beta)\, d\alpha\, d\beta. \tag{3·5}$$

It now remains to determine the form of $W(\gamma)$ in (3·5). As in section 2 we expand (3·4) in the spherical co-ordinates (2·7) and on making use of (2·6), (2·8) and (2·9) we have

$$\frac{\exp(i\chi R)}{R} = i\chi \sum_{N=0}^{\infty} \sum_{s=0}^{N} (2N+1)\epsilon_s \frac{(N-s)!}{(N+s)!} \cos s(\phi-\phi_0)\, P_N^s(\cos\theta_0)\, P_N^s(\cos\theta) \times \begin{cases} \psi_N^{(1)}(\chi r_0)\, \psi_N^{(3)}(\chi r), & \gamma \in (\gamma_0, iK'), \\ \psi_N^{(1)}(\chi r)\, \psi_N^{(3)}(\chi r_0), & \gamma \in (K+iK', \gamma_0), \end{cases} \tag{3·6}$$

where

$$\epsilon_s = 1 \quad (s = 0)$$
$$= 2 \quad (s \neq 0).$$

In equation (3·6) the $\psi_N^{(i)}(z)$ $(i = 1, 3)$ are the Sommerfeld functions, given by

$$\left.\begin{aligned} \psi_N^{(1)}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} J_{N+\frac{1}{2}}(z), \\ \psi_N^{(3)}(z) &= \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} H_{N+\frac{1}{2}}^{(1)}(z). \end{aligned}\right\} \tag{3·7a,b}$$

It will be useful to note that

$$\psi_N^{(3)}(z) = \psi_N^{(1)}(z) + i\psi_N^{(2)}(z), \tag{3·8a}$$

where

$$\psi_N^{(2)}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} Y_N(z). \tag{3·8b}$$

If we now wish to proceed in the same manner as for the Lamé functions we need to investigate the possibility of new nuclei for integral equations satisfied by ellipsoidal wave functions, generated by the various terms in equation (3·6). We shall therefore consider the functions

$$P_N^s(\cos\theta)\, \psi_N^{(1)}(\chi r) \begin{Bmatrix} \cos s\phi, \\ \sin s\phi, \end{Bmatrix} \tag{3·9}$$

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$$P_N^s(\cos\theta)\,\psi_N^{(3)}(\chi r)\begin{Bmatrix}\cos s\phi,\\ \sin s\phi,\end{Bmatrix} \tag{3·10}$$

and from (3·8a) we have the further possibility

$$P_N^s(\cos\theta)\,\psi_N^{(2)}(\chi r)\begin{Bmatrix}\cos s\phi,\\ \sin s\phi.\end{Bmatrix} \tag{3·11}$$

Consider the nuclei (3·9); they are solutions of the wave equation (3·3) and with C_α, C_β chosen as above, clearly satisfy the conditions of the theorem in section 1 provided the ellipsoidal wave functions are allotted correctly. That is, if we consider in particular the integral

$$W(\gamma) = \iint_{\mathscr{S}} P_N^s(\cos\theta)\,\psi_N^{(1)}(\chi r)\begin{Bmatrix}\cos s\phi\\ \sin s\phi\end{Bmatrix} uel_{2n}^m(\alpha)\,uel_{2n}^m(\beta)\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)\,d\alpha\,d\beta, \tag{3·12}$$

and wish to construct an integral equation satisfied by $uel_{2n}^m(\gamma)$, then the nucleus, considered as a function of γ, must have the same parity and periodicity as the ellipsoidal wave function $uel_{2n}^m(\gamma)$. Thus the nucleus must be of the form

$$P_N^{2s}(\cos\theta)\,\psi_N^{(1)}(\chi r)\cos 2s\phi, \quad s \text{ any integer.} \tag{3·13}$$

However in (3·13) the number N still remains to be determined and this we do by choosing N in such a way that as $\chi\to 0$ the integral equation (3·12) reduces to the corresponding equation satisfied by $uE_{2n}^m(\gamma)$.

Since (3·13) is a solution of the wave equation then clearly any constant multiple is also a solution of the same equation. If we take this constant to be χ^N, then if $\chi\to 0$ the resulting nucleus reduces to

$$P_N^{2s}(\cos\theta)\,r^N\cos 2s\phi. \tag{3·14}$$

When (3·14) is used as a nucleus for the integral equation satisfied by $uE_{2n}^m(\gamma)$ then $N = 2n$. Thus with this property we have the integral equation

$$uel_{2n}^m(\gamma) = \lambda\iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\,\psi_{2n}^{(1)}(\chi r)\cos 2s\phi\,uel_{2n}^m(\alpha)\,uel_{2n}^m(\beta)\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)\,d\alpha\,d\beta, \tag{3·15}$$

where, to be specific, $uel_{2n}^m(z)$ is an ellipsoidal wave function of the first kind. If we use the nucleus

$$P_{2n}^{2s}(\cos\theta)\,\psi_{2n}^{(3)}(\chi r)\cos 2s\phi, \tag{3·16}$$

then clearly the conditions of the theorem are satisfied and we obtain

$$W(\gamma) = \lambda\iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\,\psi_{2n}^{(1)}(\chi r)\cos 2s\phi\,uel_{2n}^m(\alpha)\,uel_{2n}^m(\beta)\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)\,d\alpha\,d\beta. \tag{3·17}$$

As $\gamma\to iK'$ then it is easy to see that $W(\gamma)$ behaves as

$$\frac{\exp(i\chi k\,\mathrm{sn}\,\gamma)}{k\,\mathrm{sn}\,\gamma},$$

and hence $W(\gamma)$ is, in the terminology of Arscott (3), an ellipsoidal wave function of the third kind and in this case is denoted by $uhl_{2n}^{(1)m}(\gamma)$.

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Thus we have

$$uhl_{2n}^{(1)m}(\gamma) = \lambda \iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(3)}(\chi r) \cos 2s\phi\, uel_{2n}^{m}(\alpha)\, uel_{2n}^{m}(\beta)\, (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, d\alpha, d\beta. \tag{3·18}$$

Finally, we note that the nucleus (3·11) also gives certain solutions of the ellipsoidal wave equation, and these we call ellipsoidal wave functions of the second kind, and denote them by $unl_{2n}^{m}(\gamma)$. Thus for the nucleus

$$P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(2)}(\chi r) \cos 2s\phi, \tag{3·19}$$

we have the integral representation

$$unl_{2n}^{m}(\gamma) = \lambda \iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(2)}(\chi r) \cos 2s\phi\, uel_{2n}^{m}(\alpha)\, uel_{2n}^{m}(\beta)\, (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\, d\alpha\, d\beta. \tag{3·20}$$

By following arguments similar to those which led to equation (3·15) for $uel_{2n}^{m}(\gamma)$ it may be shown that $uhl_{2n}^{(1)m}(\gamma)$, given by the relation (3·18), tends to $uF_{2n}^{m}(\gamma)$ as $\chi \to 0$. However, $unl_{2n}^{m}(\gamma)$ does not reduce to $uF_{2n}^{m}(\gamma)$ as $\chi \to 0$, but instead reduces to a linear combination of the functions $uE_{2n}^{m}(\gamma), uF_{2n}^{m}(\gamma)$. This was only to be expected, for as is shown below (equation (3·29)), an ellipsoidal wave function of the second kind is a linear combination of ellipsoidal wave functions of the first and third kinds. Hence we conclude that the functions $hl^{(1)}n^{m}(z)$ may be identified with the function $wl_{n}^{m}(z)$ of section 1 and so may be normalized via equation (1·9). Obviously the ellipsoidal wave functions of the second kind cannot be so normalized and some other convenient normalization must be adopted. A complete study of ellipsoidal wave function of the second kind will be made in a subsequent paper.

The following table gives the suitable nucleus for each type of ellipsoidal wave function of the first kind.

Nucleus	Ellipsoidal wave function
$P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(1)}(\chi r)\ \cos 2s\phi$	$uel_{2n}^{m}(\gamma)$
$P_{2n+1}^{2s}(\cos\theta)\, \psi_{2n+1}^{(1)}(\chi r) \cos 2s\phi$	$sel_{2n+1}^{m}(\gamma)$
$P_{2n+2}^{2s+1}(\cos\theta)\, \psi_{2n+2}^{(1)}(\chi r) \cos(2s+1)\phi$	$sdel_{2n+2}^{m}(\gamma)$
$P_{2n+1}^{2s+2}(\cos\theta)\, \psi_{2n+1}^{(1)}(\chi r) \cos(2s+1)\phi$	$del_{2n+1}^{m}(\gamma)$
$P_{2n+2}^{2s}(\cos\theta)\, \psi_{2n+2}^{(1)}(\chi r) \sin 2s\phi$	$cdel_{2n+2}^{m}(\gamma)$
$P_{2n+3}^{2s}(\cos\theta)\, \psi_{2n+3}^{(1)}(\chi r) \sin 2s\phi$	$scdel_{2n+3}^{m}(\gamma)$
$P_{2n+2}^{2s+1}(\cos\theta)\, \psi_{2n+2}^{(1)}(\chi r) \sin(2s+1)\phi$	$scel_{2n+2}^{m}(\gamma)$
$P_{2n+1}^{2s+1}(\cos\theta)\, \psi_{2n+1}^{(1)}(\chi r) \sin(2s+1)\phi$	$cel_{2n+1}^{m}(\gamma)$

(3·21)

Corresponding tables for ellipsoidal wave functions of the second and third kinds may be simply obtained from (3·21) by replacing $\psi_{n}^{(1)}(\chi r)$ by either $\psi_{n}^{(2)}(\chi r)$ or $\psi_{n}^{(3)}(\chi r)$. Also it is a simple matter to obtain the corresponding relations when χ is pure imaginary. Here the functions $\psi_{n}^{(i)}(\chi r)$ $(i = 1, 2, 3,)$ then take their modified form.

The integral equations discussed above are not new and have been discussed by Malurkar (9) and Arscott (1). However, both these writers only considered integral

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equations satisfied by ellipsoidal wave functions of the first kind, the representations for functions of the second and third kinds appear to be new. The above writers also only considered simple nuclei of exponential or hyperbolic type and we refer to their papers for details. The nuclei considered in this section appear to be new and although Arscott (3, 5) has discussed ellipsoidal wave functions of the third kind, the introduction of ellipsoidal wave functions of the second kind also appears to be new.

If in the integral representations for ellipsoidal wave functions of the third kind we formally expand the nucleus using the addition formulae for spherical Bessel functions (6) and integrate, then we obtain a series expansion for these functions in terms of Bessel functions. In particular, we find that $uhl_{2n}^{(1)m}$ may be formally expanded in the series

$$uhl_{2n}^{(1)m}(\gamma) = \sum_{r=0}^{\infty} a_r \psi_{2r}^{(3)}(kh \operatorname{sn}\gamma), \quad h = \sqrt{q}. \tag{3·22}$$

Series of the form (3·22) have been discussed by Arscott (5) and we refer to his paper for details. In a similar manner it is also clear that ellipsoidal wave functions of the second kind may also be expanded in series of Bessel functions. Thus for $unl_{2n}^{m}(\gamma)$ we have

$$unl_{2n}^{m}(\gamma) = \sum_{r=0}^{\infty} br\psi_{2r}^{(2)}(kh \operatorname{sn}\gamma). \tag{3·23}$$

We conjecture that a discussion of the expansions of the form (3·23) will follow along much the same lines as that given by Arscott (5) and so we shall not discuss this expansion further.

We now return to the integral relation (3·5) and by following analogous arguments to those adopted in section 2, it is clear that we may write

$$el_n^m(\gamma) = \lambda_1 \iint_{\mathscr{S}} \frac{\exp(i\chi R)}{R} el_n^m(\alpha)\, el_n^m(\beta)(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha\, d\beta, \quad \gamma \in (K + iK', \gamma_0), \tag{3·24}$$

$$hl_n^{(1)m}(\gamma) = \lambda_2 \iint_{\mathscr{S}} \frac{\exp(i\chi R)}{R} el_n^m(\alpha)\, el_n^m(\beta)(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha\, d\beta, \quad \gamma \in [\gamma_0, iK'). \tag{3·25}$$

It appears that an integral relation for the functions $nl_n^m(\gamma)$ similar to equations (3·24), (3·25) is not valid. This can be shown, by restricting the argument to the first functions of each kind, as follows.

We have from (3·15), (3·20), (3·18)

$$uel_{2n}^m(\gamma) = C_1 \iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(1)}(\chi r) \cos 2s\phi\, uel_{2n}^m(\alpha)\, uel_{2n}^m(\beta)(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha\, d\beta, \tag{3·26}$$

$$unl_{2n}^{(1)m}(\gamma) = C_2 \iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(2)}(\chi r) \cos 2s\phi\, uel_{2n}^m(\alpha)\, uel_{2n}^m(\beta)(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha\, d\beta, \tag{3·27}$$

$$uhl_{2n}^{(1)m}(\gamma) = C_3 \iint_{\mathscr{S}} P_{2n}^{2s}(\cos\theta)\, \psi_{2n}^{(3)}(\chi r) \cos 2\phi\, uel_{2n}^m(\alpha)\, uel_{2n}^m(\beta)(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha\, d\beta. \tag{3·28}$$

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Therefore, from (3·26), (3·27), (3·8a) we have

$$\frac{uel_{2n}^m(\gamma)}{C_1}+\frac{iunl_{2n}^m(\gamma)}{C_2}=\iint_{\mathscr{S}}P_{2n}^{2s}(\cos\theta)\,\psi_{2n}^{(3)}(\chi r)\cos 2s\phi\,uel_{2n}^m(\alpha)\,uel_{2n}^m(\beta)(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)\,d\alpha\,d\beta$$

$$=\frac{uhl_{2n}^{(1)m}(\gamma)}{C_3}.$$

That is

$$unl_{2n}^m(\gamma)=-iC_2\left\{\frac{uhl_{2n}^{(1)m}(\gamma)}{C_3}-\frac{uel_{2n}^m(\gamma)}{C_1}\right\}. \tag{3·29}$$

However, we cannot substitute for the right-hand side of (3·29) in terms of equations (3·24), (3·25), since there is no common domain of convergence.

Again we remark that particular integral equations and relations may be obtained if P_0 is specified explicitly. If P_0 lies at the origin, then on following arguments similar to those of section 2, we obtain only relations of the form (3·25) and further that in this case (3·25) gives rise to an integral relation for $uhl_{2n}^{(1)m}(\gamma)$ only. That is we obtain

$$uhl_{2n}^{(1)m}(\gamma)=\lambda_2\iint_{\mathscr{S}}\frac{\exp(i\chi r)}{r}uel_{2n}^m(\alpha)\,uel_{2n}^m(\beta)\,(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)\,d\alpha\,d\beta. \tag{3·30}$$

Finally we determine the constants λ_1, λ_2 in the relations (3·24), (3·25). The method followed is identical with that adopted in section 2 of this paper and so we shall not give the details and simply quote the results, namely

$$\lambda_1=\frac{il(-1)^{\sigma+\tau+1}(2n+1)\,k^{2\tau+3}}{8\pi el_n^m(\alpha_0)\,el_n^m(\beta_0)\,hl_n^{(1)m}(\gamma_0)}, \tag{3·31a}$$

$$\lambda_2=\frac{il(-1)^{\sigma+\tau+1}(2n+1)\,k^{2\tau+3}}{8\pi el_n^m(\alpha_0)\,el_n^m(\beta_0)\,el_n^m(\gamma_0)}. \tag{3·31b}$$

Section 4. *Generalizations.* As to generalizations, there are two obviously possible ones. The first is to consider the same problems but with more general Green's functions than those considered in this paper. If we then expand this nucleus either in terms of spherical harmonics or even other types of harmonics we may obtain a new class of suitable nuclei for integral equations and relations satisfied by Lamé functions and ellipsoidal wave functions. This idea may go some way to providing a positive answer to the question raised in section 2 of this paper; namely, can integral equations of the type (2·18) be constructed with nuclei independent of n (the degree of the Lamé polynomial) when P_0 lies at the origin.

The second possible generalization is to consider the class of non-linear integral equations which arise from the separation of the wave equation, or Laplace's equation, in general orthogonal curvilinear coordinates. These and other questions will be discussed in a subsequent paper.

The author is grateful to the referee for pointing out an oversight and also for useful suggestions which have much improved the presentation of this paper.

https://doi.org/10.1017/S0305004100042638 Published online by Cambridge University Press

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