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An inclination lemma for normally hyperbolic manifolds with an application to diffusion

LARA SABBAGH

Mathematics Institute, University of Warwick, UK
(e-mail: l.el-sabbagh@warwick.ac.uk)

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Abstract. Let (M, Ω) be a smooth symplectic manifold and $f : M \to M$ be a symplectic diffeomorphism of class C^l $(l \geq 3)$. Let N be a compact submanifold of M which is boundaryless and normally hyperbolic for f. We suppose that N is controllable and that its stable and unstable bundles are trivial. We consider a C^1-submanifold Δ of M whose dimension is equal to the dimension of a fiber of the unstable bundle of $T_N M$. We suppose that Δ transversely intersects the stable manifold of N. Then, we prove that for all $\varepsilon > 0$, and for $n \in \mathbb{N}$ large enough, there exists $x_n \in N$ such that $f^n(\Delta)$ is ε-close, in the C^1 topology, to the strongly unstable manifold of x_n. As an application of this λ-lemma, we prove the existence of shadowing orbits for a finite family of invariant minimal sets (for which we do not assume any regularity) contained in a normally hyperbolic manifold and having heteroclinic connections. As a particular case, we recover classical results on the existence of diffusion orbits (Arnold's example).

1. *Introduction*

In his famous note [**Arn64**], Arnold gave the first example of a three-degree-of-freedom system where diffusion orbits shadowing whiskered tori were constructed. More precisely, the system admits orbits for which the action undergoes a drift of length independent of the size of the perturbation. Arnold's example was chosen so that the Lagrangian invariant tori in the unperturbed system break down under the perturbation and give rise to partially hyperbolic tori in the perturbed system.

The diffusion mechanism is then based on the existence of a transition chain, that is, a family of invariant minimal tori with heteroclinic connections. One obtains the orbits shadowing the extremal tori of this chain by an 'obstruction argument' satisfied by each torus of the chain. This obstruction argument was first proved in [**Mar96**] as a corollary of a partially hyperbolic λ-lemma. The proof was then improved in [**FM00**] (see also [**Cre00**, **CW05**]).

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In the present paper, we prove a λ-lemma (also called an inclination lemma) for normally hyperbolic invariant manifolds, which turns out to be a new tool for proving the obstruction argument as well as several generalizations. This λ-lemma deals with normally hyperbolic manifolds instead of partially hyperbolic tori. This is not a genuine restriction since one can in general embed partially hyperbolic tori into their central manifolds which, as a rule, are normally hyperbolic. In that respect, this paper generalizes the results of [**Mar96**], [**Cre00**] and [**FM00**], and enables us to significatively simplify the previous proofs. Moreover, our λ-lemma can be applied to more general systems than that of Arnold (see, for example, [**DdlLS06**, **DH11**, **GR07**, **GR09**]) and can relate to the variational methods (which is another approach to diffusion problems) where significant contributions were given by Bernard, Bessi, Cheng, Kaloshin and many others.

We first state and prove a λ-lemma for normally hyperbolic invariant manifolds. Given a normally hyperbolic invariant manifold N for a diffeomorphism f, we consider a submanifold that transversely intersects the stable manifold of N and whose dimension is equal to the dimension of a fiber of the unstable bundle. We prove that under iteration by f, this submanifold is as close as desired (in the C^1 topology) to a suitable unstable leaf. The λ-lemma will enable us to prove the existence of drifting orbits along a chain of invariant minimal sets contained in a normally hyperbolic manifold, without any assumption on the nature of the invariant sets (in particular, they do not need to be submanifolds). As an easy particular case, we recover Arnold's example. In addition, the λ-lemma applies immediately to the examples of Delshams, De La Llave and Seara (see [**DdlLS06**] and the references therein) and yields the diffusion orbits.

In this paper, we will limit ourselves to the symplectic case and we will assume that our normally hyperbolic manifold has trivial stable and unstable bundles (this will, in particular, give us easy regularity conditions for the lamination of the invariant manifolds). This will be no restriction to us since all of the applications that we have in mind will fall into this category (diffusion orbits, Easton's windows, etc.). Moreover, we will adopt a very basic point of view and depict the geometry of the iterates of our transverse manifolds instead of using a more synthetic method (fixed-point theorem, for instance). In particular, this will enable us to directly use our various computations for the construction of windows and for estimating the transition times in a subsequent work. As a counterpart, we will have to use the existence of 'controlled' straightening neighborhoods for our manifold, which requires the previous (maybe unnecessary) assumptions.

The normally hyperbolic invariant manifolds we consider will be compact for technical simplicity but the non-compactness could easily be replaced with uniform lower bounds for the first and second derivatives of our diffeomorphisms and the constants of hyperbolicity (see equation (1) below). Finally, let us point out that even though we prove the λ-lemma for discrete systems, as usual analogous results hold for the continuous time Hamiltonian systems.

2. *A reminder on normally hyperbolic invariant manifolds and convention*

We begin with a reminder on normally hyperbolic manifolds in a general context and then specialize to the symplectic case where we can use a 'controlled' straightening neighborhood in which it is easy to depict the geometry of the invariant foliations induced by normal hyperbolicity.

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2.1. *General definitions.* Let M be a smooth n-dimensional manifold ($n \geq 3$) and let $f : M \to M$ be a C^l-diffeomorphism ($l \geq 1$) which leaves a smooth boundaryless compact submanifold N of M invariant. Given a Riemannian metric $\|\cdot\|$ on M and a subbundle E of $T_N M$ invariant under Df, we set

$$\mathrm{norm}(Df_{|E}) = \sup\{\|Df(a)_{|E_a}\| \mid a \in N\}, \quad \mathrm{conorm}(Df_{|E}) = (\mathrm{norm}(Df^{-1}_{|E}))^{-1}.$$

Definition 2.1. Let $q \leq l$ ($q \in \mathbb{N}^*$). The manifold N is q-normally hyperbolic for f if the tangent bundle of M restricted to N splits into three continuous subbundles $T_N M = TN \oplus E^s \oplus E^u$ invariant under Df, such that

$$\mathrm{norm}(Df_{|E^s}) < (\mathrm{conorm}(Df_{|TN}))^q \leq 1 \leq (\mathrm{norm}(Df_{|TN}))^q < \mathrm{conorm}(Df_{|E^u}). \tag{1}$$

This says that the behavior of f normal to N dominates the tangent behavior of f^q and is hyperbolic.

Now we state the local stable/unstable manifolds theorem. We do not mean to give the most general possible results, we rather limit ourselves to those which are strictly necessary for our purposes. For a more elaborate study on invariant manifolds, we refer the reader to [**HPS77**, **Cha04**, **BB13**].

THEOREM. [**HPS77**] *Let f, M and N be as above. We suppose that N is q-normally hyperbolic for f. Then if d is the distance associated with the Riemannian metric on M, the following properties hold true.*

(1) *Existence, characterization and smoothness. There exists a neighborhood $\mathcal{O}$ of N in M such that the sets*

$$W^s_{\mathrm{loc}}(N) = \{y \in \mathcal{O} \mid f^n(y) \in \mathcal{O}, \ \forall\, n \in \mathbb{N}\}$$

and

$$W^u_{\mathrm{loc}}(N) = \{y \in \mathcal{O} \mid f^{-n}(y) \in \mathcal{O}, \ \forall\, n \in \mathbb{N}\}$$

are C^q-manifolds that satisfy:

- *for all $y \in W^s_{\mathrm{loc}}(N)$, for all $\rho \in]\mathrm{norm}(Df_{|E^s}); \mathrm{conorm}(Df_{|TN})[$, we have $\lim_{n\to\infty} \rho^{-n} d(f^n(y), N) = 0$;*
- *for all $y \in W^u_{\mathrm{loc}}(N)$, for all $\rho \in]\mathrm{norm}(Df_{|TN}); \mathrm{conorm}(Df_{|E^u})[$, we have $\lim_{n\to\infty} \rho^{n} d(f^{-n}(y), N) = 0$.*

Moreover, $W^u_{\mathrm{loc}}(N)$ and $W^s_{\mathrm{loc}}(N)$ are tangent to $TN \oplus E^u$ and $TN \oplus E^s$, respectively, at each point of N.

(2) *Lamination. There exist two f-invariant laminations of $W^u_{\mathrm{loc}}(N)$ and $W^s_{\mathrm{loc}}(N)$, the leaves of which are unstable and stable leaves $W^{uu}_{\mathrm{loc}}(x)$ and $W^{ss}_{\mathrm{loc}}(x)$ associated with the points of N, defined as follows:*

$$W^{ss}_{\mathrm{loc}}(x) = \{y \in \mathcal{O} \mid \lim_{n\to\infty} d(f^n(y), f^n(x)) = 0\}$$

and

$$W^{uu}_{\mathrm{loc}}(x) = \{y \in \mathcal{O} \mid \lim_{n\to\infty} d(f^{-n}(y), f^{-n}(x)) = 0\}.$$

These leaves are C^q and tangent to the fibers E^u_x and E^s_x at each point x of N.

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Note that one gets the global stable (respectively unstable) manifolds by taking the union of the inverse (respectively direct) images of the local ones as follows:

$$W^s(N) = \bigcup_{n\in\mathbb{N}} f^{-n}(W^s_{\mathrm{loc}}(N)) \quad \text{and} \quad W^u(N) = \bigcup_{n\in\mathbb{N}} f^{n}(W^u_{\mathrm{loc}}(N)).$$

The same holds for the leaves:

$$W^{ss}(x) = \bigcup_{n\in\mathbb{N}} f^{-n}(W^{ss}_{\mathrm{loc}}(f^n(x))) \quad \text{and} \quad W^{uu}(x) = \bigcup_{n\in\mathbb{N}} f^{n}(W^{uu}_{\mathrm{loc}}(f^{-n}(x))).$$

These are immersed C^q-submanifolds of M. In the rest of the paper, we will drop the subscript loc from the notation. The local and the global invariant manifolds will be denoted by $W^{s,u}(N)$ since the context will always be clear. The same holds for the global and local leaves.

Definition 2.2. Let N be a q-normally hyperbolic manifold for f $(q \le l)$. We say that N is controllable if the following inequalities hold true:

$$\mathrm{norm}(Df_{|E^s}) \cdot \mathrm{norm}(Df_{|TN}) < 1 \quad \text{and} \quad \mathrm{conorm}(Df_{|TN}) \cdot \mathrm{conorm}(Df_{|E^u}) > 1. \tag{2}$$

We set $n_s := \dim(E^s)$, $n_u := \dim(E^u)$ and $n_0 := \dim(N)$, so that $n_0 + n_s + n_u = n$.

2.2. *Symplectic geometry and normal hyperbolicity.* Under symplecticity assumptions, the stable and unstable leaves are regular with respect to the points in N. More precisely, we have the following proposition which will enable us in the next section to introduce a *straightening* coordinate system in the vicinity of normally hyperbolic manifolds.

PROPOSITION A. (Marco) *Let (M, Ω) be a smooth symplectic manifold and let f be a C^l symplectic diffeomorphism of M $(l \ge 2)$. We suppose that N is a controllable q-normally hyperbolic manifold for f $(q \le l)$. Then:*

(i) *N is symplectic;*
(ii) *$W^u(N)$ and $W^s(N)$ are coisotropic;*
(iii) *$n_s = n_u$;*
(iv) *for all $x \in N$, $W^{uu}(x)$ and $W^{ss}(x)$ are isotropic and they coincide with the leaves of the characteristic foliations of $W^u(N)$ and $W^s(N)$.*

The proof of this proposition can be found in [**GM**]. Since the leaves of the characteristic foliations coincide with the leaves $W^{uu}(x)$ and $W^{ss}(x)$, the latter are C^{q-1} with respect to x. We get then the regularity we need for Proposition B below.

2.3. *Straightening neighborhood and convention.* Under the assumptions of Proposition A, one can find in the vicinity of a normally hyperbolic manifold a neighborhood in which the invariant manifolds and the leaves are straightened, making it easier to depict the behavior of f (see Figure 1). More precisely, we have the following proposition.

PROPOSITION B. (Tubular neighborhood and straightening) *Let M, N and f be as in Proposition A with $l \ge 3$. Let $p := n_s = n_u$. We suppose that N is 3-normally hyperbolic for f and that its stable and unstable bundles are trivial. Then, there exist a neighborhood U of N in M and a C^2-diffeomorphism $\varphi : U \longrightarrow V := N \times B^p \times B^p$, where B^p is an open ball centered at 0 in $\mathbb{R}^p$, such that for all $x \in N$:*

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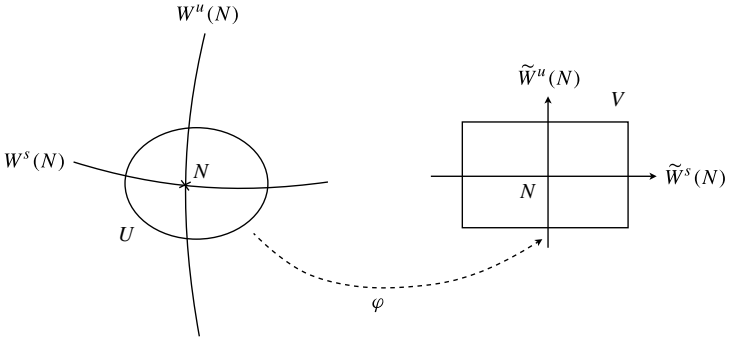


FIGURE 1. The straightening neighborhood.

(1) $\varphi(x) = (x, 0, 0)$;
(2) $\widetilde{W}^s(N) := \varphi(W^s(N) \cap U) = \{(x, s, u) \in V \mid u = 0\}$;
(3) $\widetilde{W}^u(N) := \varphi(W^u(N) \cap U) = \{(x, s, u) \in V \mid s = 0\}$;
(4) $\widetilde{W}^{ss}(x) := \varphi(W^{ss}(x) \cap U) = \{(x, s, 0) \mid s \in B^p\}$;
(5) $\widetilde{W}^{uu}(x) := \varphi(W^{uu}(x) \cap U) = \{(x, 0, u) \mid u \in B^p\}$.

The proof is straightforward once Proposition A is known. We will not prove Proposition B, we will content ourselves with the following few remarks. Near N, one can always find a tubular neighborhood. The straightening of the invariant manifolds is an immediate consequence of the graph property. We refer to [**LMS03**] and [**HPS77**] for details. When f is symplectic, the strongly stable/unstable leaves are straightened the same way.

Convention. The first B^p and the second B^p in $N \times B^p \times B^p$ do not play the same role since the first is the stable direction while the second is the unstable direction. In order to distinguish them from one another when we want to use them separately, we will add u and s in the notation as follows

$$N \times B_s^p \times B_u^p. \tag{3}$$

We use the same convention for $N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$.

In the rest of the paper, we will identify N with $\varphi(N) = N \times \{0\} \times \{0\}$ for notational simplicity. This will not lead to confusion since the context will always be clear enough.

Let d be the distance associated with the Riemannian metric on M. We will equip the neighborhood V defined in Proposition B with the distance given by the sup of $d_{|N}$ and the Euclidean distance on $\mathbb{R}^{2p}$. It is equivalent to the image under φ of d since V is relatively compact.

We will use the usual operator norms for the linear applications defined on Banach spaces that we will deal with throughout the paper. We will equip the product spaces with the sup norm and the subsets with the induced norm. For notational simplicity, we will denote all of our norms by the same symbol $\|\cdot\|$; the context will always be clear enough to avoid ambiguities.

To prove our results, we will use compositions of linear applications defined on the tangent spaces of some suitable manifolds. They will be normed algebras for the induced norm.

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3. *A λ-lemma for normally hyperbolic manifolds*

In this section, we prove a λ-lemma for normally hyperbolic manifolds. From now on, we suppose that f, M, N and Δ are as follows:

- (M, Ω) is a smooth symplectic Riemannian manifold;
- $f : M \longrightarrow M$ is a symplectic diffeomorphism of class C^l $(l \geq 3)$;
- N is a smooth submanifold of M, compact and boundaryless;
- N is a controllable 3-normally hyperbolic manifold for f;
- $n_s = n_u = p$;
- N has trivial stable and unstable bundles;
- Δ is a C^1-submanifold of M of dimension p which transversely intersects $W^s(N)$ at some point a.

We will state two versions of the λ-lemma. In §3.1, we will use the straightening neighborhood given in §2.3 to set out a simplified version of the λ-lemma (Theorem 1) and to properly define the notion of C^1-convergence. Then, in §3.2, we state the λ-lemma in a more general context (Theorem 2). We devote §§4 and 5 to the proofs of these theorems.

3.1. *Theorem 1: in the straightening neighborhood.*

In this section, we state the λ-lemma in the straightening neighborhood. Let us start with fixing the notation. We keep the notation of Proposition B. We will restrict our diffeomorphism φ to the open set $\mathcal{U} := U \cap f^{-1}(U)$, so that $F = \varphi \circ f \circ \varphi^{-1}$ is well defined on $\mathcal{V} := \varphi(\mathcal{U}) \subset V$ with values in V. A point in V will be written as a triple (x, s, u) and F as (F_x, F_s, F_u) according to the splitting $V = N \times B_s^p \times B_u^p$. Up to iterating Δ if necessary (and resetting the counters), we can suppose that $a \in \mathcal{U}$ without loss of generality, since we are interested in the behavior of Δ after a large number of iterations.

We introduce the projection $\Pi_N : \widetilde{W}^s(N) \longrightarrow N$ that sends each $(x, s, 0)$ to $(x, 0, 0)$. Let $P := \varphi(a) = (x, s, 0)$ be the intersection point of $\varphi(\Delta \cap \mathcal{U})$ and $\widetilde{W}^s(N)$. We set $P_0 := \Pi_N(P)$. For $n \geq 1$, we denote by $P^n = F^n(P)$, and $P_0^n := \Pi_N(P^n) = F^n_{|N}(P_0)$ which is the point in N such that $P^n \in \widetilde{W}^{ss}(P_0^n)$ (see Figure 2). We denote by $\widetilde{\Delta}$ the connected component of $\varphi(\Delta \cap \mathcal{U})$ in $\mathcal{V}$ containing P. For all $n \in \mathbb{N}$, we denote by $\widetilde{\Delta}^{n+1}$ the connected component of $F(\widetilde{\Delta}^n) \cap \mathcal{V}$ containing P^n (where $\widetilde{\Delta}^0 = \widetilde{\Delta}$).

Definition 3.1. (The graph property) Let Λ be a C^1-submanifold of $N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$. Let B be an open ball in $\mathbb{R}_u^p$. We say that Λ has the *graph property over* B, or equivalently that Λ is a *graph over* B, if there exists a C^1-map $\varpi : B \to N \times \mathbb{R}_s^p$ such that $\Lambda = \{(\varpi(u), u);\ u \in B\}$.

For δ small enough, we set

$$B_\delta := \{u \in B_u^p \mid \|u\| < \delta\} \quad \text{and} \quad D_\delta := \{(x, s, u) \in \mathcal{V} \mid u \in B_\delta\}.$$

For $n \in \mathbb{N}$, we introduce the constant map

$$\begin{array}{rccc} \ell_n : & B_\delta & \longrightarrow & N \times B_s^p \\ & u & \longmapsto & (P_0^n, 0) \end{array}$$

so that clearly $\widetilde{W}^{uu}(P_0^n) \cap D_\delta$ is the graph of ℓ_n, for all $n \in \mathbb{N}$.

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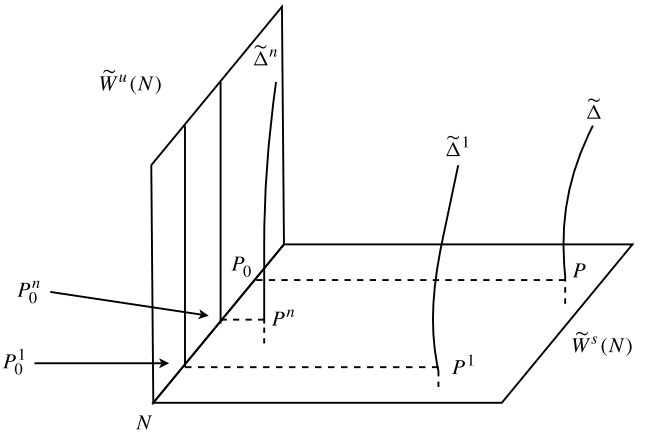


FIGURE 2. Straightening of $\widetilde{\Delta}$.

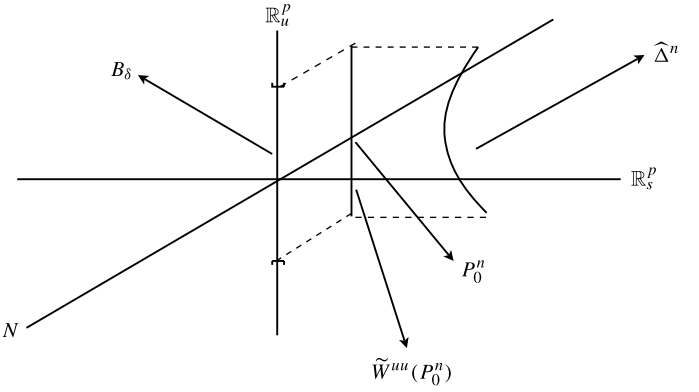


FIGURE 3. Graphs.

The λ-lemma in $\mathcal{V}$ takes the following form (see Figure 3 for illustration).

THEOREM 1. *For all $n \in \mathbb{N}$, let $\widetilde{\Delta}^n$ and ℓ_n be as above. Then, there exists $\delta > 0$ such that for all $n \in \mathbb{N}$, there exists a C^1-map $\xi_n : B_\delta \to N \times B_s^p$ such that $\widehat{\Delta}^n := \widetilde{\Delta}^n \cap D_\delta$ is the graph of ξ_n. Moreover,*

$$\lim_{n\to\infty} d_{C^1}(\xi_n, \ell_n) = 0,$$

where $d_{C^1}(\xi_n, \ell_n) = \sup_{u\in B_\delta}(d(\xi_n(u), \ell_n(u)) + \|\xi_n'(u) - \ell_n'(u)\|)$.

We will need four steps to prove Theorem 1 in §4. We will first show how, under iteration, arbitrary tangent vectors in $T_{P_0}\widetilde{\Delta}$ are straightened. We will then use the transversality of $\widetilde{\Delta}$ to $\widetilde{W}^s(N)$ to prove that some suitable part of $\widetilde{\Delta}$ (close to P) is a graph over a ball in $\mathbb{R}_u^p$. In the third step, we will show how this graph property is preserved under iteration *over the same domain in* $\mathbb{R}_u^p$. We will finally prove that tangent vectors along these graphs are straightened and a simple application of the mean value theorem ends the proof of Theorem 1.

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We end this section with the definition of a notion of 'closeness' for graphs which will be useful in what follows.

Definition 3.2. We keep the notation of Theorem 1. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We say that $\widehat{\Delta}^n$ and $\widetilde{W}^{uu}(P_0^n) \cap D_\delta$ are C^1 ε-close if $d_{C^1}(\xi_n, \ell_n) < \varepsilon$.

3.2. *Theorem 2: in an arbitrary compact subset of M.* In this section, we introduce a new notion of graphs and convergence in the C^1 compact open topology (in a fixed relatively compact set in M). See Figure 4 for illustration.

Definition 3.3. Let $\mathcal{U}$ and $\overline{U}$ be two neighborhoods of N in M such that $\overline{U} \subset \mathcal{U}$. We suppose that there exists a C^2-diffeomorphism $\varphi : \mathcal{U} \longrightarrow N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$. Let $m \in \mathbb{N}$ be fixed. We set $\psi_{(m,\overline{U})} := f^m \circ \varphi^{-1}|_{\varphi(\overline{U})}$. Let Q_1 be a C^1-submanifold of M contained in $W^u(N) \cap f^m(\overline{U})$ and Q_2 be a C^1-submanifold of M contained in $f^m(\overline{U})$. We say that Q_2 is a $(m, \overline{U})$-graph over Q_1 if $\psi^{-1}_{(m,\overline{U})}(Q_2)$ is a graph over $\Pi_3(\psi^{-1}_{(m,\overline{U})}(Q_1))$ in the sense of Definition 3.1, where Π_3 denotes the projection over the third variable.

If $\psi^{-1}_{(m,\overline{U})}(Q_2) = \operatorname{graph} \xi = \operatorname{graph}(X, S) = \{(X(u), S(u), u) \mid u \in \Pi_3(\psi_1^{-1}(Q_1))\}$, we define the following distance:

$$\begin{aligned} &d_{(C^1,m,\overline{U})}(Q_1, Q_2) \\ &\quad := \sup_{u \in \Pi_3(\psi^{-1}_{(m,\overline{U})}(Q_1))} d(\psi_{(m,\overline{U})}(X(u), S(u), u), \psi_{(m,\overline{U})}(X(0), 0, u)) \\ &\qquad + \sup_{\substack{u \in \Pi_3(\psi^{-1}_{(m,\overline{U})}(Q_1)) \\ v_1 \in B_{\mathbb{R}_u^p}}} \|D\psi_{(m,\overline{U})}(X(u), S(u), u).(X'(u).v_1, S'(u).v_1, v_1) \\ &\qquad - D\psi_{(m,\overline{U})}(X(0), 0, u).(0, v_1)\| \end{aligned}$$

where $B_{\mathbb{R}_u^p}$ is the unit ball in $\mathbb{R}_u^p$.

We now state the global version of the λ-lemma.

THEOREM 2. (λ-lemma) *Let f, M and N be as above. Let Δ be a p-dimensional C^1-submanifold transversely intersecting $W^s(N)$ at some point a, and let $\Delta^k = f^k(\Delta)$, for $k \geq 1$. Let a_0 be the point in N such that $a \in W^{ss}(a_0)$ and set $a_0^k := f^k(a_0)$.*

Then, there exist two neighborhoods $\mathcal{U}$ and $\overline{U}$ of N in M satisfying $\overline{U} \subset \mathcal{U}$, and a C^2-diffeomorphism $\varphi : \mathcal{U} \longrightarrow N \times \mathbb{R}_s^p \times \mathbb{R}_u^p$ such that for all $m \in \mathbb{N}$, for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$; for all $k \geq k_0$, there exists a C^1-submanifold $\overline{\Delta}^k$ in $f^k(\Delta) \cap f^m(\overline{U})$ such that $\overline{\Delta}^k$ is a $(m, \overline{U})$-graph over $W^{uu}(a_0^k) \cap f^m(\overline{U})$. Moreover,

$$d_{(C^1,m,\overline{U})}(\overline{\Delta}^k, W^{uu}(a_0^k) \cap f^m(\overline{U})) < \varepsilon.$$

We devote §5 to the proof of Theorem 2. It will be a direct consequence of the proof of Theorem 1.

Comments. Theorem 2 actually states the straightening property in any relatively compact set K (with a non-empty interior) in M intersecting all of the unstable leaves of the submanifold N. More precisely, let K be such a set. The sequence $(f^m(\overline{U}) \cap W^u(N))_{m \in \mathbb{N}}$

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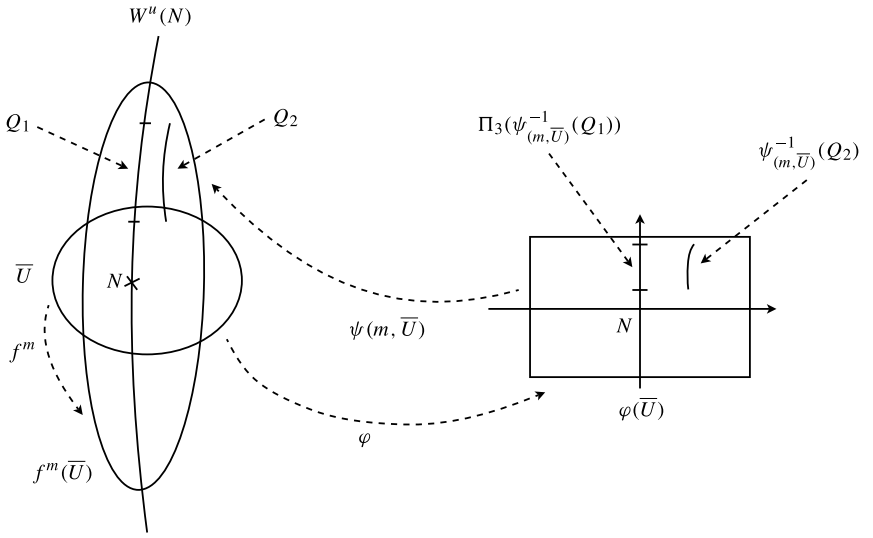


FIGURE 4. The $(m, \overline{U})$-graph property.

is clearly an exhaustion of $W^u(N)$ by relatively compact sets. By definition of the unstable manifold, there exists an integer m_0 such that $W^u(N) \cap K \subset f^{m_0}(\overline{U})$. Then, one can easily prove that for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, there exists a submanifold $\underline{\Delta}^k$ in $f^k(\Delta) \cap K$ such that $\underline{\Delta}^k$ is a $(m_0, \overline{U})$-graph over $W^{uu}(a_0^k) \cap K$. Moreover,

$$d_{(C^1, m_0, \overline{U})}(\underline{\Delta}^k, W^{uu}(a_0^k) \cap K) < \varepsilon.$$

Note that the convergence given by the basic λ-lemma is stronger than the Hausdorff one, for Δ and for its tangent space as well.

4. *Proof of Theorem 1*

In this section, we prove Theorem 1.

4.1. *General assumptions for Theorem 1.*

Here we keep the notation of Proposition B and of §3.1 and we limit ourselves to the behavior of F in $\mathcal{V}$. Recall that

$$\mathcal{V} \subset V = N \times B_s^p \times B_u^p,$$

where $B_{s,u}^p$ is an open ball centered at 0 in $\mathbb{R}^p$. Let $B_{s,u}^p$ be of radius ς.

Since $\widetilde{W}^{s,u}(N)$ are invariant under F, then

$$\text{for all } x \in N, \text{ for all } s \in B_s^p, \quad F_u(x, s, 0) = 0, \tag{4}$$

$$\text{for all } x \in N, \text{ for all } u \in B_u^p, \quad F_s(x, 0, u) = 0. \tag{5}$$

In addition, for all $x \in N$, $F(x, 0, 0) = (F_x(x, 0, 0), 0, 0)$. Since the stable and unstable *foliations* are invariant, then for all $(x, s, u) \in \mathcal{V}$,

$$F_x(x, 0, u) = F_x(x, s, 0) = F_x(x, 0, 0). \tag{6}$$

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Therefore, for $X = (x, 0, 0) \in N \times \{(0, 0)\}$, the derivative $DF(X)$ at X can be represented as a diagonal matrix:

$$DF(X) = \begin{pmatrix} \partial_x F_x(X) & 0 & 0 \\ 0 & \partial_s F_s(X) & 0 \\ 0 & 0 & \partial_u F_u(X) \end{pmatrix}. \tag{7}$$

The manifold $N \times \{(0, 0)\}$ being normally hyperbolic for F, one can find a real number $\lambda \in]0; 1[$ such that for all $x \in N$,

$$\|\partial_s F_s(x, 0, 0)\| < \lambda, \quad \|(\partial_u F_u)^{-1}(x, 0, 0)\| < \lambda,$$
$$\|\partial_s F_s(x, 0, 0)\| \cdot \|(\partial_x F_x(x, 0, 0))^{-1}\| < \lambda,$$
$$\|\partial_x F_x(x, 0, 0)\| \cdot \|(\partial_u F_u(x, 0, 0))^{-1}\| < \lambda.$$

Let $Y = (x, s, 0)$ be in $\widetilde{W}^s(N)$. Using equations (4) and (6), one easily sees that $DF(Y)$ takes the following form:

$$DF(Y) = \begin{pmatrix} \partial_x F_x(Y) & 0 & \partial_u F_x(Y) \\ \partial_x F_s(Y) & \partial_s F_s(Y) & \partial_u F_s(Y) \\ 0 & 0 & \partial_u F_u(Y) \end{pmatrix}. \tag{8}$$

One has an analogous property for the points of $\widetilde{W}^u(N)$.

We need to shrink $\mathcal{V}$ in order to have some estimates useful later on. Note first that $\mathcal{V}$ can be chosen so that $\partial_x F_x(Z)$ and $\partial_u F_u(Z)$ are invertible for all $Z \in \mathcal{V}$.

Let $\overline{\lambda}$ be in $]\lambda; 1[$. For simplicity, we choose $\overline{\lambda} = (1 + \lambda)/2$. However, all of the calculations in this proof can be adjusted so that they are compatible with any value of $\overline{\lambda} \in]\lambda; 1[$. Recall that $B^p_{s,u}$ is of radius ς.

PROPOSITION 4.1. *For ς small enough, there exist real positive constants C_1 and C_2 such that for all $Z = (x, s, u) \in \mathcal{V}$, the following inequalities hold true:*

(1) $\|s\| < (5 - 5\lambda)/2C_2(11 + \lambda)$;
(2) $\|DF(Z)\| \leq C_1$ *and* $\|D^2F(Z)\| \leq C_2$;
(3) $\|\partial_s F_s(Z)\| < \overline{\lambda}$ *and* $\|[\partial_u F_u(Z)]^{-1}\| < \overline{\lambda}$;
(4) $\|\partial_x F_x(Z)\| \cdot \|[\partial_u F_u(Z)]^{-1}\| < \overline{\lambda}$;
(5) $\max(\|\partial_s F_x(Z)\|, \|\partial_x F_s(Z)\|) < (5 - 5\lambda)/2(11 + \lambda)$.

Proof. Since F is at least C^2 and $\mathcal{V}$ is relatively compact, one can easily find the constants C_1 and C_2 that satisfy item 2. Items 3 and 4 are immediate, up to shrinking ς if necessary, for the same reasons. Note that the last item is immediate thanks to the form of DF in equation (7). As for the first item, once C_2 is chosen, and since λ is fixed above by normal hyperbolicity, it only remains to shrink ς if necessary, so that item 1 holds for each point of the (possibly shrinked) $\mathcal{V}$. □

4.2. *Linear straightening of $T_{P^m}\widetilde{\Delta}^m$.* The following proposition states the straightening of the tangent space of $\widetilde{\Delta}$ at its base point, under iteration by F.

PROPOSITION 4.2. *For all $m \in \mathbb{N}$, the tangent space $T_{P^m}\widetilde{\Delta}^m$ is the graph of a linear map $L_m = (B_m, C_m) : \mathbb{R}^p_u \longrightarrow T_{P^m_0}N \times \mathbb{R}^p_s$, whose norm satisfies*

$$\lim_{m \to \infty} \|L_m\| = 0.$$

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Proof. We start with a quick study of the dynamics in $\widetilde{W}^s(N)$. Recall that $P = (x, s, 0)$ is the intersection point of $\widetilde{\Delta}$ and $\widetilde{W}^s(N)$. Note first that by Proposition 4.1, $\|\partial_s F_s(P^i)\| < \overline{\lambda}$, for all $i \geq 0$. For $i \geq 1$, we set $s_i := F_s(P^{i-1})$. Then by the mean value theorem, one gets $\|s_i\| \leq \overline{\lambda}\|s_{i-1}\|$, and thus under iteration $\|s_i\| \leq \overline{\lambda}^i\|s\|$, that tends to zero with an exponential speed.

• We will see now where the graph property appears. By transversality of $\widetilde{\Delta}$ and $\widetilde{W}^s(N)$, and since $\dim \widetilde{\Delta} = p$, $T_P\widetilde{\Delta}$ is the graph of a linear map defined on $\mathbb{R}^p_u$, with values in $T_{P_0}N \times \mathbb{R}^p_s$. More precisely, recall that $\Pi_3 : N \times B^p_s \times B^p_u \longrightarrow B^p_u$ is the projection over the third variable. By transversality, $D\Pi_{3|_{\widetilde{\Delta}}}(P)$ is an isomorphism between $T_P\widetilde{\Delta}$ and $\mathbb{R}^p_u$. Therefore, there exist two linear maps B and C on $\mathbb{R}^p_u$, such that $T_P\widetilde{\Delta}$ is the image of the map

$$(B, C, I) : \mathbb{R}^p_u \longrightarrow T_{P_0}N \times \mathbb{R}^p_s \times \mathbb{R}^p_u,$$

where $I : \mathbb{R}^p_u \to \mathbb{R}^p_u$ is the identity map.

• Let us now see how the property of $T_P\widetilde{\Delta}$ being a graph of a linear map persists under iteration. We will proceed by induction. However, since the calculations are similar for all of the iterates, we will content ourselves with detailing the proof for the first iteration.

The image of $T_P\widetilde{\Delta}$ under $DF(P)$ is $T_{F(P)}F(\widetilde{\Delta})$. For notational convenience, we will identify our linear maps with the matrices below (in the suitable algebras of linear applications) and the partial derivatives with the blocks in the matrices. For instance, $T_{F(P)}F(\widetilde{\Delta})$ is identified with the image of the linear map

$$DF(P) \cdot \begin{pmatrix} B \\ C \\ I \end{pmatrix} : \mathbb{R}^p_u \longrightarrow T_{P^1_0}N \times \mathbb{R}^p_s \times \mathbb{R}^p_u.$$

Since P lies in $\widetilde{W}^s(N)$, this is nothing but the image of the following map

$$\begin{pmatrix} \partial_x F_x(P) & 0 & \partial_u F_x(P) \\ \partial_x F_s(P) & \partial_s F_s(P) & \partial_u F_s(P) \\ 0 & 0 & \partial_u F_u(P) \end{pmatrix} \begin{pmatrix} B \\ C \\ I \end{pmatrix} = \begin{pmatrix} \partial_x F_x(P) \cdot B + \partial_u F_x(P) \\ \partial_x F_s(P) \cdot B + \partial_s F_s(P) \cdot C + \partial_u F_s(P) \\ \partial_u F_u(P) \end{pmatrix}.$$

Since $\partial_u F_u(P) : \mathbb{R}^p_u \longrightarrow \mathbb{R}^p_u$ is invertible, $T_{F(P)}F(\widetilde{\Delta})$, that is, $T_{P^1}\widetilde{\Delta}^1$ coincides with the image of

$$\begin{pmatrix} \partial_x F_x(P) \cdot B + \partial_u F_x(P) \\ \partial_x F_s(P) \cdot B + \partial_s F_s(P) \cdot C + \partial_u F_s(P) \\ \partial_u F_u(P) \end{pmatrix} \cdot (\partial_u F_u(P))^{-1}$$
$$= \begin{pmatrix} \partial_x F_x(P) \cdot B \cdot (\partial_u F_u(P))^{-1} + \partial_u F_x(P) \cdot (\partial_u F_u(P))^{-1} \\ \partial_x F_s(P) \cdot B \cdot (\partial_u F_u(P))^{-1} + \partial_s F_s(P) \cdot C \cdot (\partial_u F_u(P))^{-1} + \partial_u F_s(P) \cdot (\partial_u F_u(P))^{-1} \\ I \end{pmatrix}.$$

This shows that $T_{P^1}\widetilde{\Delta}^1$ is also a graph. It is the image of the linear map

$$(B_1, C_1, I) : \mathbb{R}^p_u \longrightarrow T_{P^1_0}N \times \mathbb{R}^p_s \times \mathbb{R}^p_u,$$

where we have set

$$B_1 = \partial_x F_x(P) \cdot B \cdot (\partial_u F_u(P))^{-1} + \partial_u F_x(P) \cdot (\partial_u F_u(P))^{-1},$$

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and

$$C_1 = \partial_x F_s(P) \cdot B \cdot (\partial_u F_u(P))^{-1} + \partial_s F_s(P) \cdot C \cdot (\partial_u F_u(P))^{-1} + \partial_u F_s(P) \cdot (\partial_u F_u(P))^{-1}.$$

Pursuing the induction, one gets B_i and C_i ($i > 1$), by applying $DF(P^{i-1})$ to $T_{P^{i-1}}\widetilde{\Delta}^{i-1}$ (which is the image of (B_{i-1}, C_{i-1}, I)), and then normalizing by $(\partial_u F_u(P^{i-1}))^{-1}$. We set $b_i = \|B_i\|$ and $c_i = \|C_i\|$, for $i \in \mathbb{N}$, where $B_0 = B$ and $C_0 = C$.

• To end the proof, it is enough now to prove that (b_i) and (c_i) converge to zero. We begin with (b_i). We fix an arbitrary $\varepsilon > 0$. Proposition 4.1 yields, for all $i \in \mathbb{N}$,

$$\|\partial_x F_x(P^i)\| \cdot \|(\partial_u F_u(P^i))^{-1}\| < \overline{\lambda},$$

so that, since $\|(\partial_u F_u(P^i))^{-1}\| < 1$,

$$b_{i+1} \le \|\partial_x F_x(P^i)\| \cdot b_i \cdot \|(\partial_u F_u(P^i))^{-1}\| + \|\partial_u F_x(P^i))\| \cdot \|(\partial_u F_u(P^i))^{-1}\| \le \overline{\lambda} b_i + \beta_i,$$

where we have set $\beta_i := \|\partial_u F_x(P^i))\|$. Therefore, for $n \in \mathbb{N}^*$,

$$b_n \le \overline{\lambda}^n b_0 + \sum_{i=0}^{n-1} \overline{\lambda}^i \beta_{n-1-i}.$$

Note that we are not interested in giving the optimal expression for the convergence. Since $\overline{\lambda} < 1$, then for n large enough, $\overline{\lambda}^n b_0 \le \varepsilon/2$. On the other hand, by the mean value theorem, β_i satisfies

$$\beta_i \le C_2 \overline{\lambda}^i \|s\|$$

since $\|\partial_u F_x(P_0^i))\| = 0$. As a consequence of Proposition 4.1, it is easy to see that $C_2\|s\| \le 1$. Therefore

$$\sum_{i=0}^{n-1} \overline{\lambda}^i \beta_{n-1-i} \le \sum_{i=0}^{n-1} \overline{\lambda}^i \overline{\lambda}^{n-1-i} \le \sum_{i=0}^{n-1} \overline{\lambda}^{n-1} = n \cdot \overline{\lambda}^{n-1}.$$

Since $\overline{\lambda} < 1$, then for n large enough, $n \cdot \overline{\lambda}^{n-1} \le \varepsilon/2$. Then, for n large enough, $b_n \le \varepsilon$.

Note that one can also prove that the series $\sum b_i$ is convergent. This will be needed for the convergence of (c_i).

Let us now study the convergence of the sequence (c_i). For $i \ge 0$,

$$\begin{aligned} c_{i+1} \le{}& \|\partial_x F_s(P^i)\| \cdot b_i \cdot \|(\partial_u F_u(P^i))^{-1}\| + \|\partial_s F_s(P^i)\| \cdot c_i \cdot \|(\partial_u F_u(P^i))^{-1}\| \\ &+ \|\partial_u F_s(P^i)\| \cdot \|(\partial_u F_u(P^i))^{-1}\|. \end{aligned}$$

It is easy to see, using the mean value theorem, that $\|\partial_x F_s(P^i)\| < C_2\|s_i\| < 1$, for all i. As we did for (b_i), we define $\gamma_i := \|\partial_u F_s(P^i)\|$ and get $\gamma_i \le \overline{\lambda}^i$, following the same steps as for β_i. Therefore,

$$c_{i+1} \le b_i + \overline{\lambda} c_i + \overline{\lambda}^i,$$

and, for $n \ge 1$,

$$c_n \le \sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} b_i + \overline{\lambda}^n c_0 + \sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} \overline{\lambda}^i.$$

Since $\overline{\lambda} < 1$, for n large enough, one gets $\overline{\lambda}^n c_0 \le \varepsilon/3$. On the other hand, for n large enough, $\sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} \cdot \overline{\lambda}^i = n\overline{\lambda}^{n-1} \le \varepsilon/3$. Finally, let $s_{n-1} := \sum_{i=0}^{n-1} \overline{\lambda}^{(n-1-i)} b_i$.

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Observe that s_n is the general term of the Cauchy product of the series of general terms b_i and $\overline{\lambda}^i$, respectively. Since these series are both convergent, so is their Cauchy product. Then (s_n) converges to zero. More precisely, for n large enough, one has $s_{n-1} \leq \varepsilon/3$. This ends the proof of Proposition 4.2. □

4.3. *The graph property for* $\widetilde{\Delta}$. We have seen above that, because of the transversality, $D\Pi_3(P)$ restricted to $T_P\widetilde{\Delta}$ is an isomorphism between $T_P\widetilde{\Delta}$ and $\mathbb{R}^p_u$. Then, by the inverse function theorem, there exist a neighborhood $\mathcal{O}_1$ of P in $\widetilde{\Delta}$ and a neighborhood $\mathcal{O}_2$ of 0 in $\mathbb{R}^p_u$ such that $\Pi_{3|_{\widetilde{\Delta}}}$ is a diffeomorphism from $\mathcal{O}_1$ onto $\mathcal{O}_2$. More precisely, there exists a real number $\widetilde{\delta} > 0$ such that, if we set

$$B_{\widetilde{\delta}} := \{u \in B^p_u;\ \|u\| < \widetilde{\delta}\} \quad \text{and} \quad D_{\widetilde{\delta}} := \{(x, s, u) \in \mathcal{V} \mid u \in B_{\widetilde{\delta}}\},$$

then there exists a C^1-map $\xi : B_{\widetilde{\delta}} \to N \times B^p_s$, such that $\widetilde{\Delta} \cap D_{\widetilde{\delta}}$ is the graph of ξ (in the sense of Definition 3.1). We set $\xi = (X, S)$.

4.4. *The graph property for the iterates* $\widetilde{\Delta}^n$ *over a fixed strip.* We set

$$\widetilde{\nu} := \|\xi'\| = \max(\|X'\|, \|S'\|) = \sup_{u \in B_\delta}(\|\xi'(u)\|) \quad \text{and} \quad \nu := \max(1, \widetilde{\nu}).$$

We will see later on why we choose ν (and not just $\widetilde{\nu}$) to bound the norm of all of the derivatives of the graph maps. Let us set

$$\varepsilon_\nu = \frac{1-\lambda}{12\nu(1+\lambda)} = \frac{1-\overline{\lambda}}{12\nu\overline{\lambda}}. \tag{9}$$

The reason behind this choice will be clarified later on. By uniform continuity, and due to the form of DF on $\widetilde{W}^s(N)$ (equation (8)), there exists $\eta > 0$, such that for all $(x, s, u) \in \mathcal{V}$, if $\|u\| < \eta$, then

$$\|\partial_x F_u(x, s, u)\| < \varepsilon_\nu \quad \text{and} \quad \|\partial_s F_u(x, s, u)\| < \varepsilon_\nu. \tag{10}$$

We then set

$$\delta := \min\left(1, \widetilde{\delta}, \eta, \frac{1-\overline{\lambda}}{3C_2(2\nu+1)^2}\right). \tag{11}$$

PROPOSITION 4.3. *Let δ and ν be as above. Then, for all $n \in \mathbb{N}$, there exists a C^1-map $\xi_n : B_\delta \to N \times B^p_s$ such that $\widehat{\Delta}^n := \widetilde{\Delta}^n \cap D_\delta$ is the graph of ξ_n. Moreover, if for all $n \in \mathbb{N}$, $\xi_n = (X_n, S_n)$, then $\|\xi'_n\| := \max(\|X'_n\|, \|S'_n\|) = \sup_{u \in B_\delta}(\|\xi'_n(u)\|)$ satisfies $\|\xi'_n\| \leq \nu$.*

The rest of this subsection is devoted to the proof of Proposition 4.3. We will proceed by induction. We first prove these statements for the first iteration, by using intermediate lemmas which will be very useful for the estimates later on. All of the computations will be independent of n, which will easily yield the proof of the inductive step.

Note that when $n = 0$, the statement follows from §4.3 and the definition of ν. Therefore, we have to prove that if for $n \in \mathbb{N}$, $n \in \mathbb{N}$,

$$\widehat{\Delta}^n = \text{graph}\ \xi_n = \{(X_n(u), S_n(u), u) \mid u \in B_\delta\} \quad \text{with } \|\xi'_n\| \leq \nu,$$

then $F(\widehat{\Delta}^n)$ is also the graph of a map ξ_{n+1} over an open set in $\mathbb{R}^p_u$ strictly containing B_δ. We then set

$$\widehat{\Delta}^{n+1} = F(\widehat{\Delta}^n) \cap D_\delta = \widetilde{\Delta}^{n+1} \cap D_\delta = \text{graph}\ \xi_{n+1} = \{(X_{n+1}(u), S_{n+1}(u), u) \mid u \in B_\delta\}.$$

Note that we will keep the same notation for ξ_{n+1} and its restriction to B_δ. We also have to prove that $\|\xi'_{n+1}\| < \nu$.

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To simplify this step and to keep our formulas legible, we will actually prove that Proposition 4.3 holds true when $n = 1$. Since all of the computations will be independent of n, one can easily see that the statements are valid for an arbitrary n.

By applying F to $\widehat{\Delta} = \{(X(u), S(u), u) \mid u \in B_\delta\}$, one gets

$$F(\widehat{\Delta}) = \{(F_x(X(u), S(u), u), F_s(X(u), S(u), u), F_u(X(u), S(u), u)) \mid u \in B_\delta\}.$$

Let $G(u) := F_u(X(u), S(u), u) = h$. We will prove that G is a homeomorphism onto its image B'_δ and that the latter strictly contains B_δ. Then, it is easy to see that $F(\widehat{\Delta})$ restricted to $D'_\delta := \{(x, s, u) \in \mathcal{V} \mid u \in B'_\delta\}$ is the graph of (X_1, S_1), where

$$X_1(h) = F_x(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)),$$

and

$$S_1(h) = F_s(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)),$$

for $h \in B'_\delta$. We will need the following lemmas.

LEMMA 4.4. *For all $u \in B_\delta$, $G'(u)$ is an isomorphism on $\mathbb{R}^p_u$. Moreover,*

$$[G'(u)]^{-1} = \left(\sum_{m\geq 0}(-H(u))^m\right) \cdot [\partial_u F_u(X(u), S(u), u)]^{-1},$$

where

$$\begin{aligned} H(u) := {} & [\partial_u F_u(X(u), S(u), u)]^{-1} \cdot [\partial_x F_u(X(u), S(u), u) \cdot X'(u) \\ & + \partial_s F_u(X(u), S(u), u) \cdot S'(u)]. \end{aligned}$$

Proof. The equality $G(u) = F_u(X(u), S(u), u)$ gives by derivation

$$\begin{aligned} G'(u) = {} & \partial_x F_u(X(u), S(u), u) \cdot X'(u) + \partial_s F_u(X(u), S(u), u) \cdot S'(u) \\ & + \partial_u F_u(X(u), S(u), u). \end{aligned}$$

Recall that the linear map $\partial_u F_u(x, s, u)$ is invertible for all $(x, s, u) \in \mathcal{V}$, and satisfies

$$\|[\partial_u F_u(x, s, u)]^{-1}\| < \overline{\lambda} < 1.$$

Then one can write

$$G'(u) = [\partial_u F_u(X(u), S(u), u)] \cdot [I + H(u)],$$

where

$$\begin{aligned} H(u) := {} & [\partial_u F_u(X(u), S(u), u)]^{-1} \cdot [\partial_x F_u(X(u), S(u), u) \cdot X'(u) \\ & + \partial_s F_u(X(u), S(u), u) \cdot S'(u)]. \end{aligned}$$

Since $\partial_u F_u(X(u), S(u), u)$ is invertible, it is enough to prove that $I + H(u)$ is invertible too. It is the case if $\|H(u)\| < 1$ because it is an endomorphism of $\mathbb{R}^p_u$. So now we will prove that $\|H(u)\| < 1$. It is easy to see that, by definition of ε_ν (equation (9)), for all $u \in B_\delta$,

$$\|H(u)\| < 2\overline{\lambda}\nu\varepsilon_\nu = \frac{1-\overline{\lambda}}{6} < 1.$$

Therefore, $I + H(u)$ is invertible on $\mathbb{R}^p_u$ and $[I + H(u)]^{-1} = \sum_{m\geq 0}(-H(u))^m$. This ends the proof of Lemma 4.4. □

LEMMA 4.5. *For all $u \in B_\delta$, $\|[G'(u)]^{-1}\| < 1$.*

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Proof. This easily follows from the previous lemma. In fact,

$$\begin{aligned}\|[G'(u)]^{-1}\| &\le \left\|\sum_{m\ge 0}(-H(u))^m\right\| \cdot \|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\ &\le \frac{1}{1-\|H(u)\|} \cdot \|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\ &< \frac{1}{1-2\overline{\lambda}\nu\varepsilon_\nu} \cdot \|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\ &< \frac{6}{5+\overline{\lambda}} \cdot \|[\partial_u F_u(X(u), S(u), u)]^{-1}\| < \frac{6\overline{\lambda}}{5+\overline{\lambda}} < 1. \qquad \square\end{aligned}$$

We will now prove that G is invertible.

PROPOSITION 4.6. *There exists an open set B'_δ in $\mathbb{R}^p_u$ strictly containing B_δ, such that G is a homeomorphism from B_δ onto B'_δ.*

Proof. Without loss of generality, we can assume that ξ is defined on $\overline{B}_\delta$. We introduce an auxiliary map defined on $\overline{B}_\delta$,

$$\chi(u) = [\partial_u F_u(X(0), S(0), 0)]^{-1} \cdot G(u) = [\partial_u F_u(P)]^{-1} \cdot G(u).$$

We will first study the invertibility of χ, from which that of G easily follows. Let y be in a subset of $\mathbb{R}^p_u$ to be specified later on. We are looking for the conditions under which there exists a unique $x \in B_\delta$, such that $y = \chi(x)$. We let $\psi(x) := x - \chi(x) + y$, so that the point y has a unique preimage under χ if and only if ψ has a unique fixed point. To prove this last property, we will need the next lemma.

LEMMA 4.7. *For all $u \in \overline{B}_\delta$, $\|I - \chi'(u)\| < (1-\overline{\lambda})/2$.*

Proof. By derivating $\chi(u) = [\partial_u F_u(P)]^{-1} \cdot G(u)$, one gets

$$\chi'(u) = [\partial_u F_u(P)]^{-1} \cdot [\partial_u F_u(X(u), S(u), u)] \cdot [I + H(u)].$$

We set $\mathcal{W} := [\partial_u F_u(P)]^{-1} \cdot [\partial_u F_u(X(u), S(u), u)]$ and $\mathcal{T} := \mathcal{W} - I$ so that $\mathcal{W} = \mathcal{T} + I$ and $\chi'(u) - I = \mathcal{W}.(I + H(u)) - I$. Recall that $\|H(u)\| < 2\overline{\lambda}\nu\varepsilon_\nu = (1-\overline{\lambda})/6$ (see the proof of Lemma 4.4). Therefore,

$$\begin{aligned}\|\mathcal{T}\| &= \|[\partial_u F_u(P)]^{-1} \cdot [\partial_u F_u(X(u), S(u), u)] - I\| \\ &= \|[\partial_u F_u(P)]^{-1} \cdot [\partial_u F_u(X(u), S(u), u) - \partial_u F_u(P)]\| \\ &\le \|[\partial_u F_u(P)]^{-1}\| \cdot \|[\partial_u F_u(X(u), S(u), u) - \partial_u F_u(X(0), S(0), 0)\| \\ &\le \overline{\lambda}(2C_2\nu + C_2)\|u\|,\end{aligned}$$

by the mean value theorem. Writing

$$\chi'(u) - I = (\mathcal{T} + I) \cdot (I + H(u)) - I = H(u) + \mathcal{T} \cdot (I + H(u))$$

gives

$$\begin{aligned}\|\chi'(u) - I\| &\le \|H(u)\| + \|\mathcal{T}\| \cdot (1 + \|H(u)\|) \\ &< 2\overline{\lambda}\nu\varepsilon_\nu + \overline{\lambda}(2C_2\nu + C_2)\|u\|(1 + 2\varepsilon_\nu\overline{\lambda}\nu) \\ &< \frac{1-\overline{\lambda}}{6} + C_2(2\nu+1)^2\|u\|,\end{aligned}$$

because $\overline{\lambda} < 1$ and $\overline{\lambda}\varepsilon_\nu < 1$ using equation (9).

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Recall that $\|u\| < (1-\overline{\lambda})/3C_2(2\nu+1)^2$ by equation (11), which yields

$$\begin{aligned}\|\chi'(u) - I\| &< \frac{1-\overline{\lambda}}{6} + \frac{1-\overline{\lambda}}{3} \\ &< \frac{1-\overline{\lambda}}{2}.\end{aligned}$$

This ends the proof of Lemma 4.7. □

• We now go back to proving the invertibility of χ. Let $\kappa := (1-\overline{\lambda})/2$. Clearly $\kappa < 1$. The last lemma shows that $\psi = I_{B_\delta} - \chi + y$ is a contracting map. In order for it to have a unique fixed point, one needs to have $\psi(\overline{B}_\delta) \subset \overline{B}_\delta$. And this condition is satisfied if $\|y\| \le \delta(1-\kappa)$. Therefore, $\chi : \overline{B}_\delta \longrightarrow \operatorname{Im}\chi$ is bijective and satisfies $\overline{B}_{\delta(1-\kappa)} \subset \operatorname{Im}\chi$.

• The invertibility of G easily follows from that of χ. Recall that $\chi = [\partial_u F_u(P)]^{-1} \cdot G$. Therefore, $G : \overline{B}_\delta \longrightarrow \operatorname{Im} G$ is an homeomorphism and satisfies $B'_\delta := \operatorname{Im} G \supset \overline{B}_{\delta(1-\kappa)/\overline{\lambda}}$. Recall that $\kappa = (1-\overline{\lambda})/2$ which gives $(1-\kappa) > \overline{\lambda}$ and thus B'_δ, which contains $B_{\delta(1-\kappa)/\overline{\lambda}}$, strictly contains B_δ. This ends the proof of the proposition. □

Therefore, the proof of the graph property in Proposition 4.3 for the case $n = 1$ is complete. Let $\widehat{\Delta}^1 = F(\widehat{\Delta}) \cap D_\delta = \operatorname{graph}\xi_1 = \{(X_1(u), S_1(u), u);\ u \in B_\delta\}$. The next proposition will not only end the proof of the case $n = 1$, but will also be a preliminary step to estimating $\lim_{n\to\infty}\|\xi'_n\|$ in §4.5.

PROPOSITION 4.8. *If we set* $\|\xi'_1\| := \sup_{u\in B_\delta}(\|\xi'_1(u)\|) = \max(\|X'_1\|, \|S'_1\|)$, *then* $\|\xi'_1\| < \nu$.

Proof. We recall that for $h \in B'_\delta$,

$$X_1(h) = F_x(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)),$$

and

$$S_1(h) = F_s(X(G^{-1}(h)), S(G^{-1}(h)), G^{-1}(h)).$$

Since we are only interested in uniform norms over B_δ, we consider h to belong to B_δ from now on. We let $u := G^{-1}(h)$. Then $u \in G^{-1}(B_\delta) \subsetneq B_\delta$. We write

$$X_1(G(u)) = F_x(X(u), S(u), u),$$

and

$$S_1(G(u)) = F_s(X(u), S(u), u).$$

By derivating the two sides with respect to u and inverting $G'(u)$, one gets, for all $u \in B_\delta$,

$$\begin{aligned}X'_1(G(u)) = {} & \partial_x F_x(X(u), S(u), u) \cdot X'(u) \cdot [G'(u)]^{-1} \\ & + \partial_s F_x(X(u), S(u), u) \cdot S'(u) \cdot [G'(u)]^{-1} \\ & + \partial_u F_x(X(u), S(u), u) \cdot [G'(u)]^{-1},\end{aligned}$$

and

$$\begin{aligned}S'_1(G(u)) = {} & \partial_x F_s(X(u), S(u), u) \cdot X'(u) \cdot [G'(u)]^{-1} \\ & + \partial_s F_s(X(u), S(u), u) \cdot S'(u) \cdot [G'(u)]^{-1} \\ & + \partial_u F_s(X(u), S(u), u) \cdot [G'(u)]^{-1}.\end{aligned}$$

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Let us begin by studying $T := \|\partial_x F_x(X(u), S(u), u)\| \cdot \|[G'(u)]^{-1}\|$. Using the estimates in Lemma 4.5, one gets

$$\begin{aligned} T &\leq \|\partial_x F_x(X(u), S(u), u)\| \cdot \|[I + H(u)]^{-1}\| \cdot \|[\partial_u F_u(X(u), S(u), u)]^{-1}\| \\ &< \overline{\lambda} \cdot \|[I + H(u)]^{-1}\| < \frac{6\overline{\lambda}}{5 + \overline{\lambda}} := \widetilde{\alpha}, \end{aligned}$$

where we can easily see that $0 < \overline{\lambda} < \widetilde{\alpha} < 1$. Recall that by Proposition 4.1, for all $u \in B_\delta$,

$$\max(\|\partial_s F_x(X(u), S(u), u)\|, \|\partial_x F_s(X(u), S(u), u)\|) < \frac{5 - 5\lambda}{2(11 + \lambda)},$$

which yields

$$\begin{aligned} \|\xi_1'\| < &\left(\widetilde{\alpha} + \frac{5 - 5\lambda}{2(11 + \lambda)}\right)\|\xi'\| \\ &+ \sup_{u \in B_\delta} \max(\|\partial_u F_x(X(u), S(u), u)\|, \|\partial_u F_s(X(u), S(u), u)\|). \end{aligned}$$

On the one hand, $\widetilde{\alpha} = 6\overline{\lambda}/(5 + \overline{\lambda}) = (6 + 6\lambda)/(11 + \lambda)$, and, thus,

$$\widetilde{\alpha} + (5 - 5\lambda)/2(11 + \lambda) = (1 + \widetilde{\alpha})/2 := \beta \quad \text{with } 0 < \widetilde{\alpha} < \beta < 1.$$

On the other hand, using the particular form of F on the unstable manifold (equations (5) and (6)), for $X \in \widetilde{W}^u(N)$, the derivative $DF(X)$ at X has the following form

$$DF(X) = \begin{pmatrix} \partial_x F_x(X) & \partial_s F_x(X) & 0 \\ 0 & \partial_s F_s(X) & 0 \\ \partial_x F_u(X) & \partial_s F_u(X) & \partial_u F_u(X) \end{pmatrix}. \tag{12}$$

Therefore, using this particular form and the mean value theorem, one gets

$$\sup_{u \in B_\delta} \max(\|\partial_u F_x(X(u), S(u), u)\|, \|\partial_u F_s(X(u), S(u), u)\|) \leq C_2 \sup_{u \in B_\delta} \|S(u)\|,$$

and so

$$\|\xi_1'\| < \beta\|\xi'\| + C_2 \sup_{u \in B_\delta} \|S(u)\|. \tag{13}$$

Using item 1 of Proposition 4.1, and the fact that $\nu \geq 1$, one gets

$$\|\xi_1'\| < \left(\beta + \frac{5 - 5\lambda}{2(11 + \lambda)}\right)\nu = \nu.$$

This ends the proof of the lemma. □

Observe that the fact that ν is larger than one is crucial to show that $\|\xi_1'\| < \nu$ which explains our initial choice in the beginning of §4.4. Since all of the computations in the previous lemmas are independent of n, the proof of the inductive step easily follows.

We then set $\widehat{\Delta}^n = \widetilde{\Delta}^n \cap D_\delta = \text{graph } \xi_n = \{(X_n(u), S_n(u), u) \mid u \in B_\delta\}$ for all $n \in \mathbb{N}$. This ends the proof of Proposition 4.3.

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4.5. *Linear straightening along the graphs.* We will now see how tangent vectors along the graphs are straightened. We will use the estimates of the previous section to prove the following proposition.

Recall that $\widehat{\Delta}^n = \operatorname{graph} \xi_n = \operatorname{graph}(X_n, S_n) = \{(X_n(u), S_n(u), u) \mid u \in B_\delta\}$ for all $n \in \mathbb{N}$.

PROPOSITION 4.9. *For all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $\|\xi_n'\| < \varepsilon$.*

Proof. Generalizing inequality (13) to all of the iterates, since the estimates are uniform with respect to the order of the iteration, one gets

$$\|\xi_{n+1}'\| < \beta\|\xi_n'\| + C_2 \sup_{u \in B_\delta} \|S_n(u)\|. \tag{14}$$

By the mean value theorem, one can prove by induction that

$$\sup_{u \in B_\delta} \|S_n(u)\| \leq \overline{\lambda}^n \sup_{u \in B_\delta} \|S(u)\|.$$

More precisely, for all $u \in B_\delta$ and for all $n \in \mathbb{N}^*$, there exists $Z = (Z_1, Z_2, Z_3) \in \widehat{\Delta}^{n-1}$ such that $S_n(u) = F_s(Z) = F_s(Z) - F_s(Z_1, 0, Z_3)$. Therefore,

$$\|S_n(u)\| \leq \overline{\lambda}\|Z_2\| \leq \overline{\lambda} \sup_{u \in B_\delta} \|S_{n-1}(u)\|,$$

since $\widehat{\Delta}^{n-1} = \operatorname{graph}(X_{n-1}, S_{n-1})$, which proves our claim. Since $C_2 \sup_{u \in B_\delta} \|S(u)\| < 1$, then by inequality (14),

$$\|\xi_{n+1}'\| < \beta\|\xi_n'\| + \overline{\lambda}^n.$$

The proof of the convergence follows the same lines as that of (b_n) in §4.2, since $\beta < 1$. □

4.6. *Nonlinear straightening and the proof of Theorem 1.* We can now end the proof of Theorem 1 by a simple application of the mean value theorem. We get for $n \geq n_0$,

$$\begin{aligned} \sup_{u \in B_\delta} d(\xi_n(u), (P_0^n, 0)) &\leq \sup_{u \in B_\delta} d(\xi_n(u), \xi_n(0)) + d(\xi_n(0), (P_0^n, 0)) \\ &< \varepsilon + \|S_n(0)\| \\ &< \varepsilon + \overline{\lambda}^n \|S(0)\|, \end{aligned}$$

where we have used that $\|u\| < 1$. The convergence easily follows. This completes the proof of Theorem 1.

5. *Proof of Theorem 2*

We will now prove Theorem 2 which will be a consequence of Theorem 1. Let φ be the diffeomorphism given by Proposition B and $\mathcal{U}$ be as in §3.1. Let δ be given by Theorem 1. We set $\overline{U} := \varphi^{-1}(D_\delta)$. Let $m \in \mathbb{N}$ be fixed, then $\psi_{(m,\overline{U})} = f^m \circ \varphi^{-1}{}_{|_{D_\delta}}$. Let $(\widehat{\Delta}^n)$ be as in Theorem 1. For all $k \geq m$, let $\overline{\Delta}^k := \psi_{(m,\overline{U})}(\widehat{\Delta}^{k-m})$. See Figure 5 for illustration. The $(m, \overline{U})$-graph property of $\overline{\Delta}^k$ is an immediate consequence of Theorem 1. As for the convergence, the C^0 part of the convergence is obvious by uniform continuity of $\psi_{(m,\overline{U})}$. It is now enough to prove the convergence of the second term of the $(C^1, m, \overline{U})$-distance.

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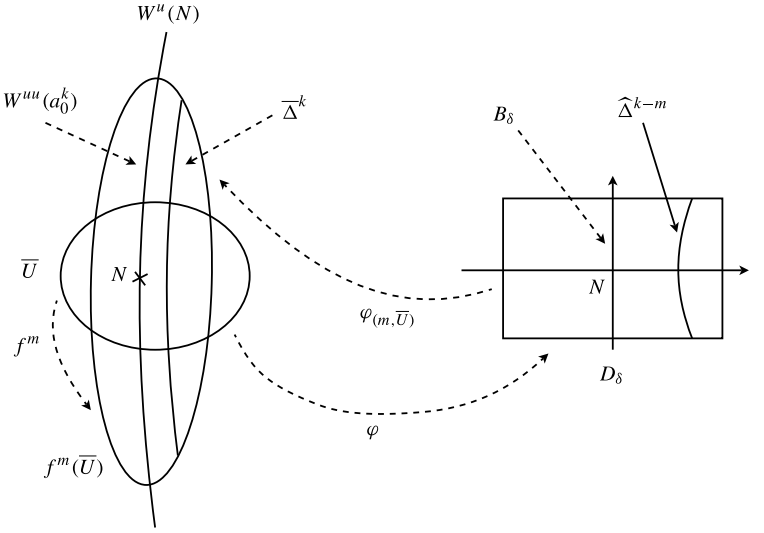


FIGURE 5. In $f^m(\overline{U})$.

There exist two positive real numbers $\overline{C}$ and $\overline{\overline{C}}$, such that for all $u \in B_\delta$, for all $v_1 \in B_{\mathbb{R}^p_u}$, for all $n \in \mathbb{N}$, if we set

$$T := \|D\psi_{(m,\overline{U})}(\xi_n(u), u) \cdot (\xi_n{}'(u) \cdot v_1, v_1) - D\psi_{(m,\overline{U})}(X_n(0), 0, u) \cdot (0, v_1)\|,$$

then

$$\begin{aligned} T &\leq \|D\psi_{(m,\overline{U})}(\xi_n(u), u) \cdot (\xi_n{}'(u) \cdot v_1, v_1) - D\psi_{(m,\overline{U})}(\xi_n(u), u) \cdot (0, v_1)\| \\ &\quad + \|D\psi_{(m,\overline{U})}(\xi_n(u), u) \cdot (0, v_1) - D\psi_{(m,\overline{U})}(X_n(0), 0, u) \cdot (0, v_1)\| \\ &\leq \|D\psi_{(m,\overline{U})}(\xi_n(u), u)\| \cdot \|\xi_n{}'(u)\| + \|D\psi_{(m,\overline{U})}(\xi_n(u), u) - D\psi_{(m,\overline{U})}(X_n(0), 0, u)\| \\ &\leq \overline{C} \cdot \|\xi_n{}'(u)\| + \overline{\overline{C}} d((\xi_n(u), u), (X_n(0), 0, u)) \end{aligned}$$

by the mean value theorem. The convergence follows from Theorem 1. By setting $n := k - m$, the proof of Theorem 2 is now complete.

6. *Application to diffusion*

We will now use the λ-lemma to prove a diffusion result. We will prove the existence of a shadowing orbit for a finite family of invariant dynamically minimal sets contained in a normally hyperbolic manifold and having successive heteroclinic connections. We will see that the existence of Arnold's diffusion orbit easily follows from this application.

6.1. *Shadowing orbits for a finite family of invariant minimal sets.*

In this section, we prove a corollary of the λ-lemma that gives the existence of a shadowing orbit for a transition chain. Let f, M and N be as in §3. If A is an invariant *dynamically minimal* set contained in N, that is, a set in which the orbit of each point is dense, we set

$$W^u(A) := \bigcup_{a \in A} W^{uu}(a).$$

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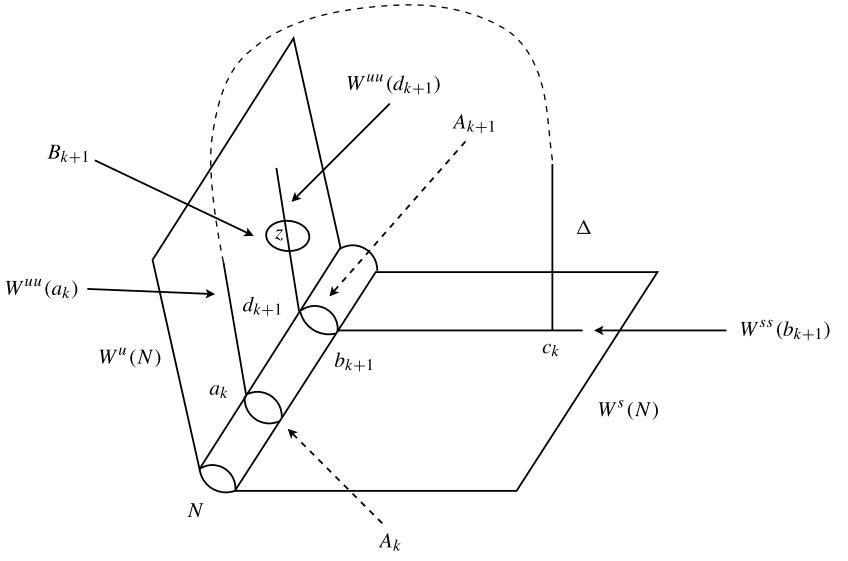


FIGURE 6. Heteroclinic connections.

Definition 6.1. (Transition chain) Let $n \in \mathbb{N}$, $(n > 1)$. Let $(A_k)_{1\le k\le n}$ be a finite family of invariant dynamically minimal sets contained in N. We say that (A_k) is a transition chain if, for all $k = 1, \ldots, n-1$, $W^u(A_k) \cap W^s(A_{k+1}) \neq \emptyset$.

Note that we do not require any regularity for the sets. In the Hamiltonian nearly integrable case, they can be general Aubry–Mather sets for instance.

We will only need the convergence in the C^0 topology stated in the λ-lemma to prove the following result. In Figure 6, we illustrate the assumptions of Corollary 6.2, in the particular case $n_0 = 2$ and $p = 1$. Of course, since the invariant manifolds are three-dimensional, this is only a rough representation of the situation.

COROLLARY 6.2. *Let f, M and N be as in §3. Let $(A_k)_{1\le k\le n}$ be a transition chain in N such that, for all $k = 1, \ldots, n-1$, there exist $a_k \in A_k$, $b_{k+1} \in A_{k+1}$ and $c_k \in W^{uu}(a_k) \cap W^{ss}(b_{k+1})$ such that $W^{uu}(a_k)$ and $W^s(N)$ transversely intersect at c_k. Then, for any $\varrho > 0$, there exists an orbit Γ such that, for all $k = 1, \ldots, n$, Γ intersects the ϱ-neighborhood of A_k.*

Proof. We fix $\varrho > 0$ and we denote by $\mathcal{V}_\varrho(A_k)$ the ϱ-neighborhood of A_k for all $k = 1, \ldots, n$. Without any loss of generality, we can suppose that, for all $k = 1, \ldots, n$, $\mathcal{V}_\varrho(A_k) \subset \mathcal{U}$ (defined in §3.1). Therefore, we can restrict the problem to the straightening neighborhood $\mathcal{V} \subset V$ (see Propositions B and 4.1).

Fix $k = 1, \ldots, n-1$ and fix a ball B_{k+1} centered at some point z of $W^u(A_{k+1})$ and contained in $\mathcal{V}_\varrho(A_{k+1})$. Therefore, since z is in $W^u(A_{k+1})$, there exists a unique $d_{k+1} \in A_{k+1}$ such that $z \in W^{uu}(d_{k+1})$. Let us set $\Delta := W^{uu}(a_k)$, then Δ is an immersed p-dimensional C^3-submanifold of M, transversely intersecting $W^s(N)$. For $m \in \mathbb{N}^*$,

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we let $b_{k+1}^m := F^m(b_{k+1})$ and Δ^m be the connected component of $F(\Delta^{m-1}) \cap \mathcal{V}$ containing $c_k^m := F^m(c_k)$.

• By the λ-lemma, for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $m \geq N_1$, Δ^m is ε-close to $W^{uu}(b_{k+1}^m)$ (in the sense of Definition 3.2).

• Since A_{k+1} is invariant, the sequence $(b_{k+1}^m)_{m\in\mathbb{N}}$ lies in A_{k+1}. Since this set is also minimal, we can extract a subsequence $(b_{k+1}^{m_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} b_{k+1}^{m_j} = d_{k+1}$. More precisely,

$$\text{for all } \varepsilon > 0, \text{ there exists } N_2 \in \mathbb{N};\ j \geq N_2 \Rightarrow d(b_{k+1}^{m_j}, d_{k+1}) < \varepsilon.$$

• The foliations being straightened, for j large enough, $W^{uu}(b_{k+1}^{m_j})$ is ε-close to $W^{uu}(d_{k+1})$.

• Therefore, for j large enough, Δ^{m_j} intersects B_{k+1}.

Let y be in $\Delta^{m_j} \cap B_{k+1}$ which is in $W^u(A_k)$. Then, for q large enough, we have $F^{-q}(y) \in \mathcal{V}_\varrho(A_k)$. Therefore there exists a ball B_k centered at $F^{-q}(y)$ and contained in $\mathcal{V}_\varrho(A_k)$ such that

$$F^q(B_k) \subset B_{k+1}.$$

We have proved then the existence of a ball B_k centered on $W^u(A_k)$ in $\mathcal{V}_\varrho(A_k)$ and a positive q such that $F^q(B_k) \subset B_{k+1}$.

Therefore, given a ball B_n centered on $W^u(A_n)$ and contained in $\mathcal{V}_\varrho(A_n)$, an immediate induction proves the existence of an integer q^* and a ball B_1 centered on $W^u(A_1)$ and contained in $\mathcal{V}_\varrho(A_1)$, whose sequence of iterates intersects each $\mathcal{V}_\varrho(A_k)$ and which moreover satisfies $F^{q^*}(B_1) \subset B_n$. This proves our claim. □

6.2. *Particular case: Arnold's example.* We will see in this section that Arnold's system [**Arn64**] satisfies all of the assumptions of Corollary 6.2 and, thus, one easily deduces the existence of drifting orbits. In Arnold's example, the stable manifold of a torus transversely intersects the unstable manifold of the next torus. These manifolds are Lagrangian and the Lagrangian/Lagrangian intersections will easily yield the isotropic/coisotropic intersections needed in Corollary 6.2 ($W^{uu}(a_k)$ and $W^s(N)$ transversely intersecting at c_k). We start with a reminder on Arnold's example and define the objects (F, M, N, the transition chain, etc.) needed to set up the context of Corollary 6.2.

The *autonomous* version of the Hamiltonian used by Arnold is defined on $\mathbb{T}^3 \times \mathbb{R}^3$ and is given by

$$H_{\varepsilon,\mu}(\theta, r) = \tfrac{1}{2}(r_1^2 + r_2^2) + r_3 + \varepsilon(\cos\theta_1 - 1) + \varepsilon\mu(\cos\theta_1 - 1)(\cos\theta_2 + \sin\theta_3),$$

where $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{T}^3$, $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ and $0 < |\mu| \ll |\varepsilon| \ll 1$.

THEOREM 6.3. (Arnold) *Given $A < B$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0; \varepsilon_0[$ there exists μ_0 such that for all $\mu \in]0; \mu_0[$, the system $H_{\varepsilon,\mu}$ admits an orbit whose projection on the action space $\mathbb{R}^3$ intersects the open sets $r_2 < A$ and $r_2 > B$.*

The Hamiltonian $H_{\varepsilon,\mu}$ is a perturbation of $H_{0,0} = \frac{1}{2}(r_1^2 + r_2^2) + r_3$, and the parameters ε and μ play asymmetric roles: ε preserves the integrability and creates hyperbolicity,

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and μ breaks down the integrability and causes instability. More precisely, when $\varepsilon = 0$, $\mathbb{T}^3 \times \mathbb{R}^3$ is foliated by invariant lagrangian tori, and when $\varepsilon > 0$ and $\mu = 0$, the system is equivalent to the uncoupled product of a pendulum ($H_p(\theta_1, r_1) = \frac{1}{2}r_1^2 + \varepsilon(\cos\theta_1 - 1)$) with the completely integrable system $H_r(\theta_2, \theta_3, r_2, r_3) = \frac{1}{2}r_2^2 + r_3$. The resonant surface given by the equation $r_1 = 0$, which is invariant and foliated by invariant tori when $\varepsilon = 0$, is destroyed. It gives rise to a one-parameter family of two-dimensional invariant tori which are partially hyperbolic, whose union is the normally hyperbolic invariant manifold $N' := \{0, 0\} \times \mathbb{T}^2 \times \mathbb{R}^2$. The invariant manifolds of N' are the product of those of the hyperbolic point ($\theta_1 = 0, r_1 = 0$) with the annulus $\mathbb{T}^2 \times \mathbb{R}^2$. When $|\mu| > 0$, we lose the integrability and the invariant manifolds of the tori do not coincide anymore. The Poincaré–Melnikov integrals show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0; \varepsilon_0[$ there exists μ_0 such that for all $\mu \in]0; \mu_0[$, the invariant manifolds transversely intersect along a homoclinic orbit. Note that μ has to be taken exponentially small with respect to ε. The diffusion question in the case where ε and μ are independent is still an open problem. Note also that Arnold chose the last term of the perturbation in such a way that it vanishes on the invariant tori (because $\theta_1 = 0$), and thus the previous partially hyperbolic tori are preserved when $\mu > 0$, as well as the normally hyperbolic manifold.

It is possible to choose a section $\mathcal{S}$ (see [**Mar96**]) contained in an energy level $\mathfrak{H}$ and transverse (in $\mathfrak{H}$) to the Hamiltonian flow. The Poincaré map associated to $\mathcal{S}$ and defined in a neighborhood of $N := N' \cap \mathcal{S}$ (which is also normally hyperbolic) in $\mathcal{S}$ will play the role of F (this of course is immediate with the non-autonomous form of the system). Note that $\mathcal{S}$ can be chosen so that the invariant manifolds of N are the intersections of those of N' with $\mathcal{S}$.

Let ω be irrational and let T_ω be the torus in N given by the equation $r_2 = \omega$. It is invariant and minimal (because ω is irrational). Arnold proved the existence of a finite family $(T_{\omega_i})_{i\in I}$ of those tori that have in addition Lagrangian invariant manifolds with transverse heteroclinic connections: $W^u(T_{\omega_i}) \pitchfork W^s(T_{\omega_{i+1}})$.

To get Arnold's orbits, it suffices now to apply Corollary 6.2 to the family (T_{ω_i}), since this family is contained in a normally hyperbolic manifold. The Lagrangian/Lagrangian intersection implies the isotropic/coisotropic intersection needed in the corollary. More precisely, for all $i \in I$, let $c_i \in W^u(T_{\omega_i}) \pitchfork W^s(T_{\omega_{i+1}})$. We set a_i the point in T_{ω_i} such that $c_i \in W^{uu}(a_i)$. It is easy to see that $W^{uu}(a_i)$ transversely intersects $W^s(N)$ at c_i. One gets then a transition chain as in Corollary 6.2.

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