Supergravity amplitudes 12

We have seen that on-shell methods are particularly powerful for theories with gauge redundancy. Gravity has in a sense even more redundancy than Yang–Mills theory because of the diffeomorphism invariance. So perhaps there are hidden structures waiting to be discovered in gravity amplitudes? In this chapter, we discuss what is currently known about the scattering amplitudes in perturbative supergravity theories, including their UV behavior, and we review the interesting connections between gauge theory amplitudes and gravity amplitudes, relations that are often phrased loosely as "gravity = (gauge theory)^2." These relations are explored further in Chapter 13.

12.1 Perturbative gravity

In a typical course on General Relativity you learn about Einstein's equation and its solutions, for example the Schwarzchild black hole and Friedmann–Robertson–Walker cosmology. (And if you have a hot course, you will also learn about black rings.) These are solutions to the classical equations of motion of gravity, just as the point-particle Coulomb potential, electromagnetic waves, or Dirac monopoles are solutions to the Maxwell equations in electromagnetism. Here we are interested in the scattering of perturbative states at weak coupling. From the point of view of perturbation theory, monopoles and black holes are considered non-perturbative states that are typically suppressed by powers e^{-1/g^2} in the weak-coupling $g \ll 1$ scattering processes.

As you know well from your QFT courses, scattering amplitudes are obtained after quantization of the field theory: start with the Lagrangian, extract the Feynman rules, and off we go to calculate scattering perturbatively. Of course, in the previous 200-something pages, we have tried to convince you that recursion relations and other on-shell methods offer much more insight and efficiency than the good old Feynman rules, but to understand what we mean by ***perturbative gravity***, let us start with the Lagrangian approach and Feynman rules. This will also give us a greater appreciation for the powers of the modern on-shell methods.

The Einstein equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, is the classical equation of motion that follows from the variational principle applied to the ***Einstein–Hilbert action***

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int d^D x\,\sqrt{-g}\,R \;+\; S_{\rm matter}\,, \tag{12.1}$$

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where R is the Ricci scalar and $2\kappa^2 = 16\pi G_N$. We have written the action in D spacetime dimensions with a D-dimensional Newton's constant G_N. The metric $g_{\mu\nu}(x)$ is a field in the field theory described by (12.1). The variation $\delta g_{\mu\nu}$ of $\sqrt{-g}\,R$ gives (after partial integration and a little work [175–177]) the Einstein tensor part, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, of Einstein's equation, while the metric variation of the "matter" action $S_{\rm matter}$ in (12.1) gives the stress-tensor part, $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}$. In the following, we use the term ***pure gravity*** to describe the field theory (12.1) without matter fields, $S_{\rm matter} = 0$.

Quantum field theory in curved spacetime is a highly non-trivial and interesting subject which has important consequences such as Hawking radiation of black holes. But this is not what we are going to discuss here. Our focus is the application of standard quantum field theory in flat spacetime to scattering of the particles associated with the quantization of the gravitational field $g_{\mu\nu}$. More precisely, we expand the gravitational field around flat space $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and regard the fluctuating field $h_{\mu\nu}$ as the ***graviton field***. To start with, let us just consider pure gravity without matter and expand the Einstein–Hilbert action in powers of $\kappa h_{\mu\nu}$. Since the Ricci-scalar R involves two derivatives, every term in the expansion has two derivatives. Suppressing the increasingly intricate index-structure, we write these terms schematically as $h^{n-1}\partial^2 h$ for $n = 2, 3, 4, \ldots$, so that the action becomes

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int d^D x\,\sqrt{-g}\,R = \int d^D x\left[h\partial^2 h + \kappa\, h^2\partial^2 h + \kappa^2\, h^3\partial^2 h + \kappa^3\, h^4\partial^2 h + \cdots\right]. \tag{12.2}$$

There are infinitely many terms. There are two reasons for this: (a) in R, the series expansion of the inverse metric generates an infinite series and (b) the expansion of the determinant $g = \det g_{\mu\nu}$ is finite, but the square root in $\sqrt{-g}$ generates an infinite series.[1] There are no mass terms in (12.2), so the particles associated with quantization of the gravitational field $h_{\mu\nu}$ are massless: they have spin-2 and are called ***gravitons***.

In order to extract Feynman rules from (12.2) we first have to gauge fix the action. A typical choice is the ***de Donder gauge***, $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h_\mu{}^\mu$, which brings the quadratic terms in the action to the form

$$h\partial^2 h \to -\frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} + \frac{1}{4}h_\mu{}^\mu\Box h_\nu{}^\nu\,. \tag{12.3}$$

The propagator resulting from these quadratic terms is

$$P_{\mu_1\nu_1,\mu_2\nu_2} = -\frac{i}{2}\Big(\eta_{\mu_1\mu_2}\,\eta_{\nu_1\nu_2} + \eta_{\mu_1\nu_2}\,\eta_{\nu_1\mu_2} - \frac{2}{D-2}\,\eta_{\mu_1\nu_1}\,\eta_{\mu_2\nu_2}\Big)\frac{1}{k^2}\,. \tag{12.4}$$

Each graviton leg is labeled by two Lorentz-indices. The external line rule is to dot in graviton polarization vectors. In 4d, the polarizations encode the two helicity $h = \pm 2$ physical graviton states. They can be constructed as products of the spin-1 polarization vectors (2.50):

$$e^{\mu\nu}_-(p_i) = \epsilon^\mu_-(p_i)\epsilon^\nu_-(p_i)\,, \qquad e^{\mu\nu}_+(p_i) = \epsilon^\mu_+(p_i)\epsilon^\nu_+(p_i)\,. \tag{12.5}$$

[1] By a field redefinition, we can use $g_{\mu\nu} = e^{-h_{\mu\nu}}$ instead; this brings the metric and its inverse on an equal footing and therefore offers a simpler expansion [178].

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Note that this ensures the correct little group scaling t^{-2h_i} of the on-shell graviton scattering amplitude.

The infinite set of 2-derivative interaction terms $h^{n-1}\partial^2 h$ yields Feynman rules for n-graviton vertices for *any* $n = 3, 4, 5, \ldots$ For example, the de Donder gauge 3-vertex takes the form

$$V_3(p_1, p_2, p_3) = p_1^{\mu_3} p_2^{\nu_3} \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} + (\text{many other terms with various index-structures})\,. \tag{12.6}$$

You can look up the full expression for the 3-vertex in [179].

The 3-term de Donder propagator (12.4) and the infinite set of complicated interaction terms should make it clear that calculation of even tree-level graviton scattering amplitudes from Feynman diagrams is not a business for babies. The 4-point graviton tree amplitude was calculated brute force with Feynman diagrams in [22] where each of the four contributing Feynman diagrams fill about a page or so of elaborate index-delight. Nonetheless, the final result can be brought to a very simple form: in 4d, it can be written in spinor helicity formalism as

$$M_4^{\text{tree}}(1^-2^-3^+4^+) = \frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2} = \frac{\langle 12\rangle^4[34]^4}{stu}\,. \tag{12.7}$$

We have already encountered this expression in Exercise 2.33. Note that we will be using M_n to denote (super)gravity amplitudes to distinguish them from (super) Yang–Mills amplitudes A_n.

Of course, you already know where we are headed: on-shell methods and recursion relations make the calculation of tree-level graviton scattering amplitudes much more fun and efficient – and they have the power to reveal structures in the amplitudes that were not visible at the level of the Lagrangian. The short version of the story is that little group scaling fixes the possible 3-graviton amplitudes and recursion then allows you to compute all other tree-level graviton processes. Loop-level amplitudes can be constructed with unitarity techniques (Chapter 6). Thus the infinite set of interaction terms in the Lagrangian is not needed from the point of view of the on-shell scattering amplitudes: the terms' role in life is to ensure off-shell diffeomorphism invariance of the gravitational action. It is an interesting aspect of on-shell recursion relations that they eliminate the need for infinitely many interaction terms.

Let us specialize to $D = 4$ and be more explicit about the graviton scattering amplitudes. Dimensional analysis and little group scaling fix the 3-point graviton amplitudes to be

$$\begin{aligned} M_3(1^-2^-3^+) &= \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2} = A_3[1^-2^-3^+]^2\,, \\ M_3(1^+2^+3^-) &= \frac{[12]^6}{[23]^2[31]^2} = A_3[1^+2^+3^-]^2\,. \end{aligned} \tag{12.8}$$

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At tree-level, the graviton amplitudes with all-plus or all-minus helicity arrangements vanish in pure gravity as do those with just one ±-helicity:

$$M_n^{\text{tree}}(1^+2^+\ldots n^+) = M_n^{\text{tree}}(1^-2^+\ldots n^+) = M_n^{\text{tree}}(1^+2^-\ldots n^-) \\ = M_n^{\text{tree}}(1^-2^-\ldots n^-) = 0\,. \tag{12.9}$$

This is most easily proven using the supersymmetry Ward identities, just as we did in (4.26), (4.27) for gluon amplitudes. The tree gravity amplitudes have to obey these same Ward identities as in a supergravity theory because the supersymmetric partners couple quadratically; hence it is only at loop-level the pure graviton amplitudes can distinguish themselves from the supergravity amplitudes. In particular, (12.9) has to hold at tree-level.

▶ **Exercise 12.1**

For simplicity, we dropped the explicit powers of the gravitational coupling κ in (12.7) and (12.8), and we continue to do so henceforth. What is the mass-dimension of κ in 4d? Show that the 4-graviton amplitude (12.7) has the correct mass-dimension (cf. (2.102)).

We categorize graviton amplitudes the same way as gluon amplitudes with designation N^KMHV. An important difference is that the graviton scattering amplitudes are not color-ordered. Using BCFW recursion relations,[2] relatively compact graviton amplitudes can be found for the MHV sector. One of the earliest formulas is BGK (Berends, Giele, and Kuijf) [180] written here in the form presented in [84] valid for $n > 4$:

$$M_n^{\text{tree}}(1^-2^-3^+\ldots n^+) = \sum_{P(3,4,\ldots,n-1)} \frac{\langle 12\rangle^8 \prod_{l=3}^{n-1}\langle n|2+3+\cdots+(l-1)|l]}{\big(\prod_{i=1}^{n-2}\langle i,i+1\rangle\big)\langle 1,n-1\rangle\langle 1n\rangle^2\langle 2n\rangle^2\big(\prod_{l=3}^{n-1}\langle ln\rangle\big)}\,. \tag{12.10}$$

The sum is over all permutations of the labels $(3, 4, \ldots, n-1)$.

Another form of the same MHV graviton tree amplitude expresses M_n^{tree} in terms of color-ordered gluon amplitudes A_n^{tree} of Yang–Mills theory:

$$M_n^{\text{tree}}(1^-2^-3^+\ldots n^+) = \sum_{P(i_3,i_4,\ldots,i_n)} s_{1i_n}\left(\prod_{k=4}^{n-1}\beta_k\right) A_n^{\text{tree}}\big[1^-2^-i_3^+\,i_4^+\ldots i_n^+\big]^2\,, \tag{12.11}$$

where $n \geq 4$ and

$$\beta_k = -\frac{\langle i_k\, i_{k+1}\rangle}{\langle 2 i_{k+1}\rangle}\langle 2|i_3+i_4+\cdots+i_{k-1}|i_k]\,. \tag{12.12}$$

The result (12.11) can be derived [181, 182] using a $[-,-\rangle$ BCFW shift.

[2] BCFW for tree graviton amplitudes is valid when based on shifts $[-,-\rangle$, $[-,+\rangle$, and $[+,+\rangle$.

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There are also other graviton MHV formulas available in the literature, for example the "soft-factor" formula [183]. You may find that these MHV expressions are terribly complicated compared with Parke–Taylor; however, they are remarkably simple when compared with the impenetrable mess a Feynman diagram calculation would produce.

Beyond the MHV level, one can readily use BCFW to calculate explicit results. You might be curious if there is also a CSW-like expansion for gravity amplitudes. The MHV vertex expansion [35] based on the Risager-shift (see discussion below Exercise 3.10) works for NMHV graviton amplitudes with $n < 12$ particles. It fails [36] for $n \geq 12$ because the large-z falloff of the n-point amplitude under Risager-shift is $1/z^{12-n}$ and the Cauchy contour deformation argument needed to derive the recursion relations therefore picks up a term at infinity for $n > 11$. The all-line shift discussed in Section 3.4 also fails (for interesting reasons [26]). For further discussion of CSW for gravity, see [17, 20, 26, 36].

The relation between gravity and gauge theory amplitudes is clearly visible in the 4d MHV expressions (12.8) and (12.11), but there are more general relations available. The first such example is the ***KLT relations***, derived in string theory by Kawai, Lewellen and Tye [21]: the KLT relations give the n-point tree-level closed string scattering amplitude as a sum over products of pairs of n-point open string partial amplitudes, with coefficients that depend on the kinematic variables as well as the string tension $1/(2\pi\alpha')$. This is natural, albeit non-trivial, since the closed string vertex operators are products of open string vertex operators. The non-triviality of the KLT relations is that the factorization into open string amplitudes survives the integrals over the insertion points of the vertex operators. In the limit of infinite tension, $\alpha' \to 0$, the closed string amplitudes with massless spin-2 string external states become the regular graviton scattering amplitudes M_n we have discussed above. And in this limit, the open-string partial amplitudes with external massless spin-1 states become the color-ordered gluon amplitudes A_n. Thus, in the limit $\alpha' \to 0$, KLT offers a relationship between tree-level M_n and A_n for each n. For $n = 4$, 5, and 6 the field theory KLT relations are

$$
\begin{aligned}
M_4^{\text{tree}}(1234) &= -s_{12}\, A_4^{\text{tree}}[1234]\, A_4^{\text{tree}}[1243]\,, \\
M_5^{\text{tree}}(12345) &= s_{23}s_{45}\, A_5^{\text{tree}}[12345]\, A_5^{\text{tree}}[13254] + (3 \leftrightarrow 4)\,, \\
M_6^{\text{tree}}(123456) &= -s_{12}s_{45} A_6^{\text{tree}}[123456]\Big(s_{35} A_6^{\text{tree}}[153462] + (s_{34}+s_{35}) A_6^{\text{tree}}[154362]\Big) \\
&\quad + \mathcal{P}(2,3,4)\,.
\end{aligned}
\tag{12.13}
$$

In the 6-point case, $\mathcal{P}(2,3,4)$ stands for the sum of all permutations of legs 2, 3, 4. At 7-point and higher, the KLT relations are more complicated; they can be found in Appendix A of [184]. The relation between gravity and gluon scattering is not at all visible in the Lagrangian (12.2), although field redefinitions and clever gauge choices can bring the first few terms in the gravitational action into a more KLT-like form; see [185–187] and the review [188].

Note that there is no specification of helicities of the external states in (12.13): this is because the KLT relations are valid in D-dimensions. In 4d, the KLT relations work for any helicity assignments of the gravitons on the LHS; if the ith graviton has helicity $h_i = +2$,

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then the gluons labeled i in the amplitudes on the RHS have helicity $h_i = +1$; similarly for negative helicity. This ensures that the little group scaling works out on both sides of the KLT relations. We may then say that KLT in 4d uses

$$\text{graviton}^{\pm 2}(p_i) = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\pm 1}(p_i)\,. \tag{12.14}$$

This is also encoded in the graviton polarizations (12.5).

Inspecting the relationship (12.14) between gravitons and gluons, we could also ask what happens if we combine gluons of *opposite* helicity in the KLT relations. The result is something that has the little group scaling of a scalar on the gravity side. In fact, what you get is the ***dilaton and axion***:

$$\left.\begin{matrix}\text{dilaton}\\ \text{axion}\end{matrix}\right\} = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\mp 1}(p_i)\,. \tag{12.15}$$

This is completely natural from the string theory point of view where the graviton state comes together with an antisymmetric tensor $B_{\mu\nu}$ and a scalar "trace" mode. The latter is the dilaton and the former has a 3-form field strength $H = dB$ which means that in 4d it is dual to a scalar with shift-symmetry, i.e. an axion. Therefore we can write the relation between the spectra

$$\text{4d axion-dilaton gravity} = (\text{YM theory}) \otimes (\text{YM theory})\,. \tag{12.16}$$

Given the attention we have poured into the study of amplitudes in (planar) $\mathcal{N} = 4$ SYM in 4d, you may also be curious about what we would get if we tensored the 2^4 states of $\mathcal{N} = 4$ SYM *à la* (12.16). The answer is a very good one: we get the 2^8 states of $\mathcal{N} = 8$ supergravity, which is the 4d supergravity theory with maximal supersymmetry. Supergravity amplitudes, especially those in $\mathcal{N} = 8$ supergravity, are the main focus in the following. We return to the study of "gravity =(gauge theory)^2" in Chapter 13, though you will see more of it in the following sections too.

12.2 Supergravity

Supergravity is the beautiful union of gravity and supersymmetry. It is the result of making the supersymmetry transformations local in the sense that the SUSY parameter ϵ is space-time dependent. If you have not previously studied supergravity, you should immediately read the essay [189] and then textbooks such as [37, 39, 190].

The supersymmetry partner of the graviton is called the ***gravitino***. It has spin-$\frac{3}{2}$ and (when supersymmetry is unbroken) it is massless. In 4d we characterize a gravitino by its two helicity states $h = \pm\frac{3}{2}$; its Feynman rule for the external line simply combines a $\pm$-helicity spin-1 polarization vector with the $\pm$-helicity spin-$\frac{1}{2}$ fermion wavefunction.

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In a 4d supergravity theory with $\mathcal{N}$ supercharges, Q^A and $\widetilde{Q}_A$, the graviton has $\mathcal{N}$ gravitino-partners. We can construct the spectrum by starting with the negative helicity graviton h^- as the highest-weight state and apply the supercharges $\widetilde{Q}_A$. At each step, $\widetilde{Q}_A$ raises the helicity by $\frac{1}{2}$, so when $\widetilde{Q}_A$ is applied once to h^- it produces a negative helicity gravitino ψ_A^-. If $\mathcal{N} = 1$, then the process terminates because of the Grassmann nature of the supercharge. So the $\mathcal{N} = 1$ ***pure supergravity*** multiplet consists of the two CPT conjugate pairs of graviton and gravitino:

$$\mathcal{N} = 1 \text{ supergravity:} \quad (h^-, \psi^-) \quad \text{and} \quad (\psi^+, h^+)\,. \tag{12.17}$$

When we say ***pure supergravity*** we mean that there are no other matter-supermultiplets included; we only have the states that are related to the graviton via supersymmetry.

Pure $\mathcal{N} = 2$ ***supergravity*** has 2×2^2 states

$$\mathcal{N} = 2 \text{ supergravity:} \quad (h^-, \psi_A^-, v^-) \quad \text{and} \quad (v^+, \psi^{A+}, h^+)\,, \tag{12.18}$$

where the two gravitinos ψ_A^- and ψ^{A+} are labeled by $A = 1, 2$ and $v^\pm$ denotes the two helicity states of the spin-1 ***gravi-photon***.

Fast-forward to pure $\mathcal{N} = 4$ ***supergravity***. Its 2×2^4 states can be characterized as

$$\mathcal{N} = 4 \text{ supergravity} = \big(\mathcal{N} = 4 \text{ SYM}\big) \otimes \big(\mathcal{N} = 0 \text{ SYM}\big)\,. \tag{12.19}$$

By $\mathcal{N} = 0$ SYM we just mean pure Yang–Mills theory. The spectrum (12.19) should be read as follows: the two graviton states are given in terms of the gluon states as in (12.14). Using the spectrum (4.30) of $\mathcal{N} = 4$ SYM, we find:

$$\begin{aligned}
&\text{gravitons:} && h^\pm = g^\pm \otimes g^\pm\,,\\
&\text{gravitinos:} && \psi^{A+} = \lambda^{A+} \otimes g^+ \ \text{ and } \ \psi_A^- = \bar{\lambda}_A^- \otimes g^-\,,\\
&\text{gravi-photons:} && v_{AB}^\pm = S_{AB} \otimes g^\pm\,,\\
&\text{gravi-photinos:} && \psi^{A-} = \lambda^{A+} \otimes g^- \ \text{ and } \ \psi_A^+ = \bar{\lambda}_A^- \otimes g^+\,,\\
&\text{scalars (dilaton–axion):} && g^\pm \otimes g^\mp\,,
\end{aligned} \tag{12.20}$$

where $g^\pm$ are gluons, $\bar{\lambda}_A^-$ and λ^{A+} are gluinos, and S_{AB} are the six scalars of $\mathcal{N} = 4$ SYM. Totaling the states, we get $2 \times 1 + 2 \times 4 + 2 \times 6 + 2 \times 4 + 2 = 32 = 2 \times 2^4$.

▶ **Exercise 12.2**

Identify the supermultiplets in the theory whose spectrum is $\big(\mathcal{N} = 2 \text{ SYM}\big) \otimes \big(\mathcal{N} = 0 \text{ YM}\big)$.

What is the difference between the two $\mathcal{N} = 4$ supergravity theories whose spectra are $(\mathcal{N} = 4 \text{ SYM}) \otimes (\mathcal{N} = 0 \text{ SYM})$ and $(\mathcal{N} = 2 \text{ SYM}) \otimes (\mathcal{N} = 2 \text{ SYM})$?

Applying the $\mathcal{N}$ supersymmetry generators $\widetilde{Q}_A$ to the graviton top state h^-, we cannot avoid states with spin greater than 2 if $\mathcal{N} > 8$. There are no consistent interactions in flat space for particles with spin greater than 2, so that tells us that maximal supersymmetry

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in 4d is $\mathcal{N} = 8$. The $\mathcal{N} = 8$ ***supergravity*** theory is unique: the ungauged theory, which is our focus here, was first written down in [191, 192].[3] Its spectrum of 2^8 states forms a CPT-self-conjugate supermultiplet (just as in $\mathcal{N} = 4$ SYM). As noted at the end of Section 12.1, the spectrum can be characterized as

$$\mathcal{N} = 8 \text{ supergravity} = \big(\mathcal{N} = 4 \text{ SYM}\big) \otimes \big(\mathcal{N} = 4 \text{ SYM}\big)\,. \tag{12.21}$$

In any supergravity theory, there are ***supersymmetry Ward identities*** that restrict the amplitudes, just as discussed for gauge theories in Sections 4.2 and 4.4. In particular, the *graviton amplitudes in supergravity* satisfy

$$M_n(1^+2^+\ldots n^+) = M_n(1^-2^+\ldots n^+) = M_n(1^+2^-\ldots n^-) = M_n(1^-2^-\ldots n^-) = 0 \tag{12.22}$$

at all orders in perturbation theory. There are also simple Ward identities among graviton and gravitino MHV amplitudes that give

$$\mathcal{N} \geq 1 \text{ supergravity:} \qquad M_n(1^-\psi^-\psi^+4^+\ldots n^+) = \frac{\langle 13\rangle}{\langle 12\rangle}\, M_n(1^-2^-3^+4^+\ldots n^+)\,, \tag{12.23}$$

just as for gluons and gluinos. In extended ($\mathcal{N} > 1$) supergravity there are further relations, as you will see shortly from the superamplitudes in $\mathcal{N} = 8$ supergravity. The example

$$\mathcal{N} = 8 \text{ supergravity:} \qquad M_n(1^+\ldots i^-\ldots j^-\ldots n^+) = \frac{\langle ij\rangle^8}{\langle 12\rangle^8}\, M_n(1^-2^-3^+\ldots n^+) \tag{12.24}$$

is the $\mathcal{N} = 8$ supergravity analogue of the supersymmetry Ward identity (4.35) in $\mathcal{N} = 4$ SYM.

12.3 Superamplitudes in $\mathcal{N} = 8$ supergravity

The spectrum (12.21) of $\mathcal{N} = 8$ supergravity consists of 128 bosons and 128 fermions. Organized by helicity $h = 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2$, we can write it out as

$$\begin{gathered} 1 \text{ graviton } h^+, \quad 8 \text{ gravitinos } \psi^A, \quad 28 \text{ gravi-photons } v^{AB}, \\ 56 \text{ gravi-photinos } \chi^{ABC}, \quad 70 \text{ scalars } S^{ABCD}, \quad 56 \text{ gravi-photinos } \chi^{ABCCDE}, \\ 28 \text{ gravi-photons } v^{ABCDEF}, \quad 8 \text{ gravitinos } \psi^{ABCDEFG}, \quad 1 \text{ graviton } h^- = h^{12345678}. \end{gathered} \tag{12.25}$$

[3] The gauged $\mathcal{N} = 8$ supergravity theory was presented in [193].

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Here $A, B, \ldots = 1, 2, \ldots, 8$ are $SU(8)$ R-symmetry indices and each state above is fully antisymmetric in these labels; this simply reflects that the helicity-h state transforms in the rank $r = 4 - 2h$ fully antisymmetric irreducible representation (irrep) of $SU(8)$ and the multiplicity given in (12.25) is the dimension of the irrep. The 70 scalars are self-dual and satisfy $\overline{S}_{ABCD} = \frac{1}{4!}\epsilon_{ABCDEFGH}S^{EFGH}$. Supersymmetry generators Q^A and $\widetilde{Q}_A$ act on the states in an obvious generalization of (4.31).

Just as in $\mathcal{N}=4$ SYM, it is highly convenient to combine the states into a superfield, or super-wavefunction, with the help of an on-shell superspace with Grassmann variables η_{iA} whose index $i = 1, \ldots, n$ is a particle label and $A = 1, 2, \ldots, 8$ is a fundamental $SU(8)$ R-symmetry index. The $\mathcal{N}=8$ superfield is then

$$\Phi_i = h^+ + \eta_{iA}\,\psi^A - \frac{1}{2}\eta_{iA}\eta_{iB}\,v^{AB} + \cdots + \eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4}\eta_{i5}\eta_{i6}\eta_{i7}\eta_{i8}\,h^- \,. \tag{12.26}$$

The $SU(8)$ R-symmetry requires that the superamplitudes are degree $8k$ polynomials in the Grassmann variables. This directly gives us the $\mathcal{N}=8$ supergravity version of the $\mathrm{N}^K\mathrm{MHV}$ classification: the Kth sector contains the superamplitudes of degree $8(K+2)$ polynomials in the η_{iA}s. It should be clear from (12.26) that the MHV sector ($K=0$) includes the graviton component amplitude $M_n(1^-2^-3^+\ldots n^+)$.

The super-Poincaré generators – momentum $P^{\dot{a}b}$, rotations/boosts, and the supercharges Q^A and $\widetilde{Q}_A$ are given in (5.1) and (4.46), with the only difference that now $A = 1, 2, \ldots, 8$. Momentum- and supermomentum-conservation require that the general superamplitudes in $\mathcal{N}=8$ supergravity are of the form

$$\mathcal{M}_n^{\mathrm{N}^K\mathrm{MHV}} = \delta^4\big(P\big)\,\delta^{(16)}\big(\widetilde{Q}\big)\,P_{8K}\,, \tag{12.27}$$

where P_{8K} is annihilated by Q^A which acts by differentiation: $Q^A P_{8K} = 0$.

At MHV level, we are already home safe. The Grassmann delta function (12.27) eats up all 16 fermionic variables, so P_0 is η-independent. It can be fixed by requiring that $\mathcal{M}_n$ projects out the correct pure graviton MHV amplitude $M_n(1^-2^-3^+\ldots n^+)$. This is easily accomplished:

$$\mathcal{M}_n^{\mathrm{MHV}} = \delta^4\big(P\big)\,\delta^{(16)}\big(\widetilde{Q}\big)\,\frac{M_n(1^-2^-3^+\ldots n^+)}{\langle 12\rangle^8}\,. \tag{12.28}$$

In particular, the supersymmetry Ward identity (12.24) for graviton amplitudes in $\mathcal{N}=8$ supergravity follows from (12.28).

Beyond the MHV level, one can solve the supersymmetric Ward identities $Q^A\mathcal{M}_n = \widetilde{Q}_A\mathcal{M}_n = 0$ (just as in the $\mathcal{N}=4$ SYM case) to find a basis of input-amplitudes that completely determine the full superamplitude. The basis can be labeled by the $K \times 8$ rectangular Young tableaux of $SU(n-4)$ irreps [42]. Another approach is to use the super-BCFW recursion relations; they are valid for super-shift of any two lines [19, 47].[4]

[4] A super-shift version of CSW was discussed in [44] and while it works for all tree superamplitudes in $\mathcal{N}=4$ SYM, it has limited validity in $\mathcal{N}=8$ supergravity.

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The set-up for the MHV superamplitude (12.28) is perhaps a bit "cheap" because the component amplitude $M_n(1^-2^-3^+\ldots n^+)$, as we have seen in Section 12.1, does not take a particularly compact form and it does not clearly reflect symmetries such as full permutation symmetry of identical external states. So there has been quite a lot of effort towards building an MHV superamplitude that more clearly encodes the symmetries. One representation [194] of the superamplitude builds on a super-BCFW shift in the $\mathcal{N}=7$ formulation of $\mathcal{N}=8$ supergravity.[5] Other MHV formulas use the Grassmannian representations [195, 196] or the twistor string [197, 198]. More recently, new compact formulas for both Yang–Mills and gravity amplitudes have been proposed to be valid in any spacetime dimensions [199, 200].

$\mathcal{N}=8$ supergravity has, as we have noted above, a ***global*** $SU(8)$ ***R-symmetry***. This symmetry is realized linearly, as you can see on the spectrum and on the amplitudes which vanish unless the external states form an $SU(8)$ singlet. However, the theory also has a "hidden" symmetry: the equations of motion of $\mathcal{N}=8$ supergravity have a ***continuous global*** $E_{7(7)}(\mathbb{R})$ ***symmetry***. The group $E_{7(7)}$ is a non-compact version of the exceptional group E_7; its maximal compact subgroup is $SU(8)$. $E_{7(7)}$ has rank 7 and is 133 dimensional. It is *not* a symmetry of the action of $\mathcal{N}=8$ supergravity. The best way to think of this is that the $E_{7(7)}$ is spontaneously broken to $SU(8)$. There are $133-63=70$ broken generators, giving 70 Goldstone bosons. Those are exactly the 70 scalars S^{ABCD} in the spectrum (12.25) and they "live" in the coset space $E_{7(7)}/SU(8)$.

As a spontaneously broken symmetry, $E_{7(7)}$ is not linearly realized on the on-shell scattering amplitudes, but instead it manifests itself via ***low-energy theorems***.[6] If the momentum of an external scalar S^{ABCD} is taken soft, then the amplitude must vanish because the Goldstone scalars are derivatively-coupled. Basically this says that the moduli space $E_{7(7)}/SU(8)$ is homogeneous: it does not matter what the vevs of the scalars are, all points on moduli space are equivalent. The soft scalar limit probes the neighborhood of a point in moduli space and since the moduli space is homogeneous, the soft scalar limit vanishes. There are also double-soft limits that involve the commutator of two coset generators and these therefore directly reveal, from the on-shell point of view, the coset structure $E_{7(7)}/SU(8)$.

▶ Exercise 12.3

Project the amplitude $M_4(S^{1234}S^{5678}h^-h^+)$ out from the MHV superamplitude (12.28). Show that

$$\lim_{p_1\to 0} M_4\big(S^{1234}S^{5678}h^-h^+\big)=0\,. \tag{12.29}$$

In contrast, the scalars in $\mathcal{N}=4$ SYM are not Goldstone bosons, so the soft-scalar limits do not have to vanish. For example, show

$$\lim_{p_1\to 0} A_4\big[S^{12}g^-S^{34}g^+\big]\neq 0\,. \tag{12.30}$$

[5] Just as $\mathcal{N}=3$ SYM is identical to $\mathcal{N}=4$ SYM, so is $\mathcal{N}=7$ supergravity identical to $\mathcal{N}=8$ supergravity. The validity of the super-BCFW shifts in $\mathcal{N}=7$ supergravity is proven in [87].

[6] Low-energy theorems were originally developed in pion-physics [201]. For a review, see [202].

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The soft limit explores the points of moduli space in the neighborhood of the origin: away from the origin, the $\mathcal{N} = 4$ SYM theory is on the ***Coulomb branch***, part of the gauge group is broken, and some of the $\mathcal{N} = 4$ supermultiplets become massive. The non-vanishing limit (12.30) has a nice interpretation. Set $p_1 = \epsilon\, q$ for some light-like $q = -|q\rangle[q|$ and take the soft limit as $\epsilon \to 0$. The limit (12.30) then depends on $|q\rangle$. The soft limit $p_1 \to 0$ leaves an object with momentum conservation on three particles: the result can be interpreted as the small-mass limit of the Coulomb branch amplitude $A_3[W^- S^{34}\, W^+]$, where $W^\pm$ are the longitudinal modes of the massive spin-1 W-bosons of a massive $\mathcal{N} = 4$ supermultiplet and S^{34} is a massless scalar. From this point of view, q is a reference vector that allows us to project the massive momenta of $W^\pm$ such that the corresponding angle spinors are well-defined. This is actually also needed to define the helicity basis because helicity is not a Lorentz-invariant concept for massive particles; but q breaks Lorentz-invariance and allows us to define a suitable q-helicity basis [26, 140].

The 3-point amplitude $A_3[W^- S^{34}\, W^+]$ violates the $SU(4)$ R-symmetry of $\mathcal{N} = 4$ SYM at the origin of moduli space. This is fine, because the Coulomb branch breaks the R-symmetry. Minimally, one has $SU(4) \sim SO(6) \to SO(5) \sim Sp(4)$.

The moral of the story is that single-scalar soft limits for $\mathcal{N} = 8$ supergravity amplitudes *vanish* because the 70 scalars are Goldstone bosons of $E_{7(7)} \to SU(8)$. And single-scalar soft limits for $\mathcal{N} = 4$ SYM are *non-vanishing* and reproduce the small-mass limit of the Coulomb branch amplitudes [140]. One can in fact re-sum the entire small-mass expansion from multiple-soft-scalar limits and recover the general-mass Coulomb branch amplitudes [140, 203].

The single-soft-scalar limits of tree amplitudes in $\mathcal{N} = 8$ supergravity were first studied in [36]. Single and double soft limits were discussed extensively and clarified in [47]. The soft-scalar limits play a key role for us in Section 12.5.

12.4 Loop amplitudes in supergravity

It is taught in all good kindergartens that a point-particle theory of gravity is badly UV divergent and non-renormalizable. This means that it is not a good quantum theory. So what is perturbative gravity all about?

Naive power-counting gives a clear indication that gravity with its 2-derivative interactions generically has worse UV behavior than for example Yang–Mills theory with its 1- and 0-derivative interactions. Consider for example a generic 1-loop m-gon diagram. In gravity, the numerator of the loop-integrand can have up to $2m$ powers of momenta, while in Yang–Mills theory it is at most m. Both have m propagators, so in gravity this gives

$$\text{gravity 1-loop } m\text{-gon diagram} \sim \int^{\Lambda} d^4\ell\, \frac{(\ell^2)^m}{(\ell^2)^m} \sim \Lambda^4\,. \tag{12.31}$$

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This is power-divergent as the UV cutoff Λ is taken to ∞ for all m. On the other hand, for Yang–Mills theory the m-gon integral has at most ℓ^m in the numerator, so it is manifestly UV finite for $m > 4$.

Now, the power-counting is too naive. There can be cancellations within each diagram. Moreover, we have learned that we should not take individual Feynman diagrams seriously if they are not gauge invariant. So cancellations of UV divergences can take place in the sum of diagrams, rendering the on-shell amplitude better behaved than naive power-counting indicates.

In fact, pure gravity in 4d is finite at 1-loop order [204]: all the 1-loop UV divergences cancel! This is difficult to see by direct Feynman diagram calculations, but it follows rather trivially from the absence of any valid *counterterms*. We review this approach in detail in Section 12.5.

At 2-loop order, it has been demonstrated by Feynman diagram calculations that pure gravity indeed has a UV divergence [205, 206]. In Yang–Mills theory we are not too scared of UV divergences because we know how to treat them with the procedure of renormalization. However, in gravity, it would take an infinite set of local counterterms to absorb the divergences and hence the result is unpredictable: pure gravity is a non-renormalizable theory.

So what is the theory described by the Einstein–Hilbert action? Because it is non-renormalizable, it is not by itself a well-defined *quantum theory* of gravity. Instead, we should regard the field theory defined by the Einstein–Hilbert action as an ***effective field theory***, valid at scales much smaller than the Planck scale $M_{\text{Planck}} \sim 10^{19}\,\text{GeV}$. To see this, recall that the 4d gravitational coupling κ has mass-dimension -1. So when we do perturbation theory, we should really use the dimensionless coupling $E\kappa$ where E is the characteristic energy of the process. At high enough energies, this dimensionless coupling is no longer small and we cannot trust perturbation theory. So we should not extrapolate to such high energies. In energy units, we have $\kappa^{-1} \sim G_N^{-1/2} = M_{\text{Planck}}$, so this tells us to use gravity, as described by the Einstein–Hilbert action, for energies $E \ll M_{\text{Planck}}$. As a classical effective field theory, General Relativity is enormously successful and captures classical gravitational phenomena stunningly as shown by experimental tests.

Regarding gravity as an effective theory, we can study the low-energy perturbative amplitudes: the tree amplitudes capture the classical physics and there are no divergences to worry about. At 1-loop level, we have mentioned that pure gravity is finite. Could we imagine adding matter to gravity in such a way that its higher-loop amplitudes were also finite? Gravity with generic matter is 1-loop divergent [204, 207], but we know from gauge theories that supersymmetry improves the UV behavior of loop amplitudes, even to such an extreme extent that the maximally supersymmetric Yang–Mills theory, $\mathcal{N} = 4$ SYM, is UV finite: the UV divergences cancel completely at each order in the loop expansion. Could something like that also happen in supergravity? If it did, it would eliminate the need for renormalization and the problems of non-renormalizability would be obsolete. There would still be important questions unresolved about non-perturbative aspects of supergravity; perturbative finiteness would not mean that the theory is UV complete. The question of perturbative UV finiteness of (maximal) supergravity in 4d has received increased attention in the past few years and the on-shell amplitude techniques have facilitated

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multiple explicit calculations of supergravity loop amplitudes. It should be emphasized that whether or not the perturbative calculations eventually encounter a divergence, one should appreciate that the study of loop amplitudes in supergravity has resulted in a number of new insights, of independent value, about gravity scattering amplitudes. An example is the connection between gravity and Yang–Mills amplitudes via the so-called BCJ dualities (see Chapter 13).

Pure supergravity in 4d is better behaved in the UV than pure gravity: ***all pure supergravity theories in 4d are finite at 1-loop*** [208] ***and 2-loop order*** [209–211], i.e. the first possible UV divergence can appear only at 3-loop order, improving on the 2-loop UV divergence of pure gravity [205, 206]. In the spirit of "the more supersymmetry, the better," it is natural to focus on maximal supersymmetry, i.e. $\mathcal{N} = 8$ supergravity in 4d. An explicit calculation, using the generalized unitarity method, demonstrated that the 3-loop 4-graviton amplitude is UV finite in $\mathcal{N} = 8$ supergravity in 4d [212–214]. This and related observations of unexpected cancellations motivated Bern, Dixon, and Roiban [212] to ask if $\mathcal{N} = 8$ supergravity in 4d is UV finite? They further proposed that the ***critical dimension*** D_c for the first UV divergence of maximal supergravity in D-dimensions follows the same pattern as for maximal super Yang–Mills theory [184, 215], namely

$$D_c(L) = \frac{6}{L} + 4 \quad \text{for} \quad L > 1\,. \tag{12.32}$$

It has been shown [216, 217] that the 4-loop 4-graviton amplitude is UV finite in $\mathcal{N} = 8$ supergravity in 4d and that it follows the pattern (12.32). How about 5-loops? A pure-spinor based argument [218] leads to (12.32) for $L = 2, 3, 4$, but implies that the critical dimension for $L = 5$ is $D = 24/5$ and not $26/5$ as (12.32) predicts. This question can be settled by direct computation: at the time of writing, the 5-loop calculation is still in progress, so you will have to watch the news feeds for a resolution.

For $D = 4$, the symmetries of $\mathcal{N} = 8$ supergravity can be used to establish that *all* amplitudes of the theory are UV finite for $L \leq 6$: this explains the finiteness of the 3- and 4-loop 4-graviton amplitudes and predicts that no UV divergence appears in any higher point amplitudes for $L \leq 6$ in 4d. For $L \geq 7$, the known symmetries do not suffice to rule out UV divergences in 4d. The following section reviews how these results are obtained using an on-shell amplitude-based approach [43, 219–221] to counterterms in $\mathcal{N} = 8$ supergravity. We then provide in Section 12.6 an overview of the current status of the UV behavior of supergravity as a function of dimensions D, supersymmetries $\mathcal{N}$, and loop order L.

12.5 $\mathcal{N} = 8$ supergravity: loops and counterterms

Suppose that a supergravity has its *first* UV divergence in an n-point amplitude at L-loop order. Then the effective action for the theory must have a ***local diffeomorphism invariant counterterm*** constructed from n fields (corresponding to the n external states)

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and $(2L+2)$ derivatives. The latter statement follows from dimensional analysis because the gravitational coupling κ has mass-dimension -1: for given n, the ratio of the L-loop supergravity amplitude to the tree amplitude has an overall factor of κ^{2L}, so the corresponding local counterterm has to make up the mass-dimension by having $2L$ more derivatives than the 2-derivative tree-level theory.[7]

▶ **Exercise 12.4**

Show that the n-graviton 1-loop amplitude has an overall factor of κ^2 compared with the n-graviton tree amplitude.

If we consider just pure gravity, the possible local diff-invariant counterterms must be Lorentz scalars formed from contractions of Riemann tensors and possibly covariant derivatives. Each Riemann tensor contributes 2-derivatives, so at 1-loop (4-derivatives), the possible candidates are $\sqrt{-g}R^2$, $\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$, and $\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. If we suppress the index contractions, we can write schematically

$$S_{\text{eff}} = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\Big(\underbrace{R}_{L=0} + \underbrace{\kappa^2 R^2}_{L=1} + \underbrace{\kappa^4 R^3}_{L=2} + \underbrace{\kappa^6 R^4}_{L=3} + \underbrace{\kappa^6\big(D^2R^4 + R^5\big)}_{L=4} + \underbrace{\kappa^8\big(D^4R^4 + D^2R^5 + R^6\big)}_{L=5} + \cdots\Big), \tag{12.33}$$

where R denote Riemann tensors and D are covariant derivatives. This should be viewed as a list of possible candidate counterterms; the operators in (12.33) are not necessarily generated in perturbation theory.

▶ **Exercise 12.5**

Trivia question: why are we not including operators $D^{2k}R$ in (12.33)?
How about operators $D^{2k}R^2$?

[Hint: use the Bianchi identity for the Riemann tensor.]
Show that operators of the form $D^{2k}R^3$ have vanishing 3-point matrix elements for $k \geq 1$.

Since we consider on-shell amplitudes, we can enforce the equations of motion on the candidate counterterms. In pure gravity, the Einstein equation gives $R_{\mu\nu} = 0$, so this leaves $\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ as the only possibility at 1-loop. However, we are free to add zero to convert $\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ to the Gauss–Bonnet term $\sqrt{-g}\big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\big)$ which equals a total derivative. Therefore there is ***no local counterterm for pure gravity at 1-loop order!*** And since there is no counterterm, ***pure gravity is not UV divergent at 1-loop***. A cleaner way to say this is that one can do a field redefinition that changes

[7] To be a little more precise, the above statement is true for amplitudes with purely bosonic fields. Since external fermions dress the amplitude with dimensionful wavefunctions, each pair of fermions counts one derivative for the purpose of dimensional analysis.

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$\sqrt{-g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ to the Gauss–Bonnet term, and since a field redefinition does not change the amplitude there cannot be a 1-loop divergence.

At 2-loop order, the candidate counterterm has to be composed of some index contractions of three Riemann tensors – let us denote it by R^3, and here and henceforth leave the $\sqrt{-g}$-factor implicit in candidate counterterm operators. The R^3 counterterm is present for pure gravity, which (as noted in Section 12.4) is 2-loop divergent.

Counterterms have to respect any non-anomalous symmetries of the theory. Supersymmetry is preserved at loop-level, so in supergravity theories counterterm candidates must be supersymmetrizable. We showed in Exercise 2.35 that a matrix element produced by R^3 is fixed by little group scaling to be

$$M_3(1^-2^-3^-)_{R^3} = \text{constant} \times \langle 12\rangle^2\langle 23\rangle^2\langle 13\rangle^2 \,. \tag{12.34}$$

But we also know from (12.22) that all-minus amplitudes violate the supersymmetry Ward identities. So this means that any operator that produces a non-vanishing matrix element $M_3(1^-2^-3^-)_{R^3}$ violates supersymmetry. Since R^3 produces a supersymmetry-violating amplitude, we conclude that R^3 cannot be supersymmetrized [209, 222]. Therefore R^3 is not a viable counterterm and hence ***any pure supergravity must be 2-loop finite!***

In the above argument, you may object that we may not have to care about the 3-point amplitude (12.34) since it vanishes in real kinematics. But it is easy to show (using for example an all-line shift [26]) that a non-vanishing all-minus 3-point amplitude implies that there is a non-vanishing 4-graviton amplitude $M_4(1^-2^-3^-4^-)$ and clearly this violates supersymmetry. Another objection could be: "what if the constant in (12.34) is zero, then there is no contradiction with supersymmetry?" That is true, but if the matrix element vanishes that means that the 3-field part of R^3 is a total derivative, and then we don't care about it anyway because there are no available gravity diff-invariant 4-field operators at 6-derivative order. So, either way, we conclude that pure supergravity is finite at 2-loops.

Let us now specialize to $\mathcal{N}=8$ supergravity in 4d. The candidate counterterms have to respect $\mathcal{N}=8$ supersymmetry and also be $SU(8)$ invariant, since the global R-symmetry is non-anomalous [89, 90]. Moreover, they should be compatible with the "hidden" $E_{7(7)}$ symmetry [223]; we will come back to $E_{7(7)}$ later – for now, we explore the constraints of supersymmetry and R-symmetry on the candidate counterterms in $\mathcal{N}=8$ supergravity.

It is in general difficult to analyze the candidate counterterm operators directly: a full field theory $\mathcal{N}=8$ supersymmetrization of the independent contractions of Rs and Ds is complicated in component form; for R^4 it has been done explicitly at the linearized level only [224]. A better approach is to use a superfield formalism. There is no off-shell superfield formalism for $\mathcal{N}=8$ supergravity, but harmonic superspace techniques have been used to constrain the possible counterterms in supergravity theories in various dimensions. We are going to highlight some of the results of the superspace approach in Section 12.6, but otherwise we do not discuss these methods here: this is a book about amplitudes and that is the path we take.

The supersymmetry and R-symmetry constraints on the candidate counterterm operators translate into Ward-identity constraints on the matrix elements produced by counterterms. Let us list the translation of constraints between an operator, whose lowest interaction-term

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is an n-vertex, and the corresponding n-point matrix element:

n-field operator		n-point matrix element
local with $2L+2$ derivatives	$\leftrightarrow$	polynomial in $\langle ij\rangle$ and $[kl]$ of degree $2L+2$
$\mathcal{N}=8$ SUSY	$\leftrightarrow$	$\mathcal{N}=8$ SUSY Ward identities
$SU(8)$ R-symmetry	$\leftrightarrow$	$SU(8)$ Ward identities

(12.35)

($E_{7(7)}$ constraints will be treated separately, starting on page 267.) In addition, the matrix elements have to respect Bose/Fermi symmetry under exchange of identical external states.

The condition that the matrix element is polynomial follows from the locality of the operator and the insistence that it corresponds to the *leading* (i.e. first) UV divergence in the theory; with other operators present, there could be pole terms. The matrix element we consider here is strictly the amplitude calculated from the n-point vertex of the given n-field operator, and therefore it cannot have any poles, i.e. it must be a polynomial in the kinematic variables $\langle ij\rangle$ and $[kl]$. The degree of the polynomial follows from dimensional analysis.

Let us be clear about what our approach is: for a given L, we ask if the *first* UV divergence could appear in an n-point amplitude. If this is so, then there must be a corresponding n-field $(2L+2)$-derivative counterterm. For example, for $L=3$ the lowest-n candidate counterterm would be an $SU(8)$-invariant $\mathcal{N}=8$ supersymmetrization of R^4. To analyze if such an operator exists, we write down all possible matrix elements satisfying the constraints (12.35). If there are no such matrix elements, we conclude there is no corresponding $\mathcal{N}=8$ SUSY and $SU(8)$-invariant operator and therefore the first divergence in the theory cannot be in the n-point L-loop amplitude. On the other hand, if one or more such matrix elements exist, then the corresponding operator respects *linearized* $\mathcal{N}=8$ supersymmetry and $SU(8)$ and we may consider it as a candidate counterterm. That does not mean that perturbation theory actually produces the corresponding UV divergence; that would have to be settled by other means, such as an explicit L-loop computation. Thus, the approach here is to use the matrix elements to *exclude* counterterm operators as well as *characterize* candidate counterterms as operators that respect $\mathcal{N}=8$ SUSY and $SU(8)$-symmetry at the linearized level.

To illustrate the idea, consider R^4.[8] Its 4-point matrix element $M_4(1^-2^-3^+4^+)_{R^4}$ has to be a degree 8 polynomial in angle and square brackets. Taking into account the little group scaling, there is only one option: the matrix element has to be $M_4(1^-2^-3^+4^+)_{R^4} = c_{R^4}\langle 12\rangle^4[34]^4$, where c_{R^4} is some undetermined constant. By the same arguments as above, dimensional analysis and little group scaling, we know that $M_4(1^-2^+3^-4^+)_{R^4} = c_{R^4}\langle 13\rangle^4[24]^4$. We can then check the MHV-level $\mathcal{N}=8$ SUSY Ward identity (12.24) which at n-point reads

$$\mathcal{N}=8 \text{ supergravity:} \quad M_n(1^+\dots i^-\dots j^-\dots n^+)_{\mathcal{O}} = \frac{\langle ij\rangle^8}{\langle 12\rangle^8} M_n(1^-2^-3^+\dots n^+)_{\mathcal{O}}\,. \tag{12.36}$$

[8] The relevant contraction of four Riemann tensors is the square of the Bel–Robinson tensor [225].

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For our 4-point R^4 matrix elements, we have

$$M_4(1^-2^+3^-4^+)_{R^4} = \frac{\langle 13\rangle^8}{\langle 12\rangle^8} M_4(1^-2^-3^+4^+)_{R^4} = \frac{\langle 13\rangle^8}{\langle 12\rangle^8} c_{R^4}\langle 12\rangle^4[34]^4 = c_{R^4}\langle 13\rangle^4[24]^4\,, \tag{12.37}$$

thanks to momentum conservation $\langle 13\rangle[34] = -\langle 12\rangle[24]$. It is not hard to see that the 4-point super-matrix-element

$$\mathcal{M}_4(1234)_{R^4} = \delta^4\big(P\big)\,\delta^{(16)}\big(\widetilde{Q}\big)\,\frac{[34]^4}{\langle 12\rangle^4} \tag{12.38}$$

fulfills all criteria (12.35).[9] This means that linear $\mathcal{N}=8$ supersymmetry and $SU(8)$ do *not* rule out R^4. But it does not mean that it will occur in perturbation theory: in fact, we know from the explicit 3-loop calculation [212] that the 4-graviton amplitude is finite, so R^4 does not occur. Why not? Well, read on, we will get to that later in this section.

Let us now see an example of how the on-shell matrix elements can be used to rule out a counterterm. At 4-loop order, we can write down two pure-gravity 10-derivative operators, D^2R^4 and R^5. The first one, D^2R^4, stands for the possible scalar contractions of two covariant derivatives acting (in some way) on four Riemann tensors. No matter how the indices are contracted, its matrix element turns out to be proportional to $(s+t+u)\mathcal{M}_4(1234)_{R^4}$ and therefore it vanishes. This means that the 4-point interaction in the operator D^2R^4 is a total derivative when evaluated on the equations of motion. So we can rule out the 4-loop 4-graviton amplitude as the first instance of a UV divergence in $\mathcal{N}=8$ supergravity. The second 10-derivative operator R^5 is a little more interesting.

The 5-point MHV matrix element of R^5 is fixed uniquely by locality, dimensional analysis, and little group scaling up to an overall constant:

$$M_5(1^-2^-3^+4^+5^+)_{R^5} = a_{R^5}\langle 12\rangle^4[34]^2[45]^2[53]^2\,. \tag{12.39}$$

Next, we check the supersymmetry Ward identity (12.36) in much the same way as for R^4. But now we find

$$\begin{aligned} M_5(1^-2^+3^-4^+5^+)_{R^5} &\overset{?}{=} \frac{\langle 13\rangle^8}{\langle 12\rangle^8} M_5(1^-2^-3^+4^+5^+)_{R^5} \\ \Longrightarrow \qquad a_{R^5}\langle 13\rangle^4[24]^2[45]^2[52]^2 &\overset{!}{=} a_{R^5}\frac{\langle 13\rangle^8[34]^2[45]^2[53]^2}{\langle 12\rangle^4}\,. \end{aligned} \tag{12.40}$$

This time momentum conservation does not save us. The LHS and RHS of (12.40) are not equal for generic momenta, in particular the LHS is local (i.e. does not have any poles) while the RHS has a pole $1/\langle 12\rangle^4$. This is a contradiction that can only be resolved when $a_{R^5}=0$. So that means that the operator R^5 does not have an $\mathcal{N}=8$ supersymmetrization. And that in turn rules out the 5-point 4-loop amplitude as the first UV divergence in $\mathcal{N}=8$ supergravity. One can further argue [219, 226] that there are no other possible 10-derivative operators compatible with $\mathcal{N}=8$ supersymmetry and $SU(8)$, so this means that $\mathcal{N}=8$ supergravity cannot have its first UV divergence at 4-loop order.

[9] For details of how the R-symmetry acts on the superamplitudes, see Section 2.2 of [42].

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Equation (12.40) illustrates a conflict between supersymmetry and locality, a conflict that can be exploited to rule out potential counterterms. We will describe the method for operators of the form $D^{2k}R^5$, then outline the general results. The strategy is to construct the most general matrix element $M_5(1^-2^-3^+4^+5^+)_{D^{2k}R^5}$ that respects the little group scaling, has mass-dimension $2k+10$, and is Bose symmetric in exchange of same-helicity gravitons. Then ask if there exists a linear combination that respects the SUSY Ward identities. Practically, this was done in [219] using Mathematica. (For more advanced cases, Gröbner basis techniques are very useful [221], and the results found can be reproduced and extended by an analysis based on the superconformal group $SU(2,2|8)$ [221].)

▷ *Example.* As an example of the procedure in [219], consider D^2R^5. First we find that there are 40 angle-square bracket monomials of degree 12 that have the correct little group scaling of $M_5(1^-2^-3^+4^+5^+)_{D^2R^5}$. This does not take into account redundancy under Schouten or momentum conservation. Now take linear combinations of the 40 monomials to enforce Bose symmetry: this leaves six polynomials as candidates for $M_5(1^-2^-3^+4^+5^+)_{D^2R^5}$. Then impose Schouten and momentum conservation and one finds that only one polynomial survives: this means that the MHV matrix element of D^2R^5 is unique; it takes the form

$$M_5(1^-2^-3^+4^+5^+)_{D^2R^5} = a_{D^2R^5}\, s_{12}\, \langle 12\rangle^4 [34]^2[45]^2[53]^2 \,. \tag{12.41}$$

This is actually just s_{12} times $M_5(1^-2^-3^+4^+5^+)_{R^5}$ in (12.39). Now repeat the SUSY test (12.40) for M_{5,D^2R^5} to find that the RHS has a pole $1/\langle 12\rangle^3$, contradicting the locality of the LHS. So the operator D^2R^5 is excluded as a counterterm for $\mathcal{N}=8$ supergravity, which means that the first divergence cannot be in the 6-loop 5-graviton amplitude.

Let us summarize the result of the process outlined above for a few more operators in the same class – after each step given in the first column, we list how many polynomials remain:

	R^5	D^2R^5	D^4R^5	D^6R^5	D^8R^5
Little grp	1	40	595	4983	29397
Bose symmetry	1	6	63	454	2562
Schouten, mom-cons	1	1	6	9	24
Weakest pole in SUSY Ward id	$\langle 12\rangle^{-4}$	$\langle 12\rangle^{-3}$	$\langle 12\rangle^{-2}$	$\langle 12\rangle^{-1}$	no pole

(12.42)

It follows from the last line that the $D^{2k}R^5$ operators are excluded as counterterm candidates for $k=0,1,2,3$, but not for $k=4$ where there is one unique matrix element that solves the supersymmetry Ward identities. Thus there is a unique operator D^8R^5 that passes the tests of linearized SUSY; if present, this would correspond to a first UV divergence at 8-loop order in the 5-graviton amplitude. ◁

▶ Exercise 12.6

Show that for the operator R^n with $n \geq 3$ there are no n-point MHV matrix elements compatible with the $\mathcal{N}=8$ supersymmetry Ward identities.

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For operators with $n > 5$ fields, one has to distinguish between the different N^{K}MHV sectors. Beyond MHV level, this can be done using the solutions to the SUSY Ward identities [42] in $\mathcal{N}=8$ supergravity. As an example of a non-MHV result, the lowest order permissible NMHV-level operator is D^4R^6 at 7-loops. Actually, there are two independent NMHV matrix elements, so this means that there are two independent linearly-supersymmetrizable operators D^4R^6.

A detailed analysis of possible counterterm operators was carried out in [219] (see also [226]) and it was found that

below 7-loop order, the only operators compatible with linearized $\mathcal{N}=8$ supersymmetry and $SU(8)$ R-symmetry are

$$\underbrace{R^4}_{\text{3-loop}}\,, \qquad \underbrace{D^4R^4}_{\text{5-loop}}\,, \qquad \underbrace{D^6R^4}_{\text{6-loop}}\,. \tag{12.43}$$

At 7-loop order, an infinite tower of linearized $\mathcal{N}=8$ supersymmetry and $SU(8)$ R-symmetry permissible operators was found

$$\text{7-loops:} \quad D^8R^4\,, \quad D^4R^6\,, \quad R^8\,, \quad \phi^2R^8\,, \quad \phi^4R^8\,, \quad \ldots \tag{12.44}$$

The operators $\phi^{2k}R^8$ should be viewed as representatives for the linearized $\mathcal{N}=8$ supersymmetrization of some contraction of eight Riemann tensors multiplied by an $SU(8)$-singlet combination of $2k$ scalars S^{ABCD}. These do not have purely gravitational $(8+2k)$-point matrix elements, so they cannot be in the MHV or anti-MHV sector. For example, it was shown [221] that ϕ^2R^8 only gives N^{3}MHV matrix elements (four distinct ones).

At 8-loop order and beyond, there are infinite towers of operators that respect linearized $\mathcal{N}=8$ supersymmetry and $SU(8)$.

You can find a detailed characterization of the counterterms in Table 1 of [221].

There is one symmetry we did not use to restrict the candidate counterterm operators in the above discussion, and that is the $E_{7(7)}$ ***"hidden" symmetry***.[10] As discussed at the end of Section 12.3, it manifests itself in the amplitudes of $\mathcal{N}=8$ supergravity through ***low-energy theorems***. These also have to apply to the matrix elements of any acceptable candidate counterterm operator $\mathcal{O}$. In particular, the single-soft-scalar limit must vanish, for example

$$\lim_{p_1\to 0} M_6\big(S^{1234}S^{5678}h^-h^-h^+h^+\big)_{\mathcal{O}} = 0\,. \tag{12.45}$$

If the matrix element of an operator does not pass the single-soft-scalar test, then it is not compatible with $E_{7(7)}$ symmetry. If it does pass the test, then we conclude nothing: it could be $E_{7(7)}$ at play or just a coincidence.

[10] See also [227, 228] for related aspects of $E_{7(7)}$ symmetry.

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It turns out that the single-soft-scalar test is only non-trivial starting at $n = 6$. So precisely (12.45) can be used to test the operators that survived the $\mathcal{N} = 8$ supersymmetry and $SU(8)$ constraints, for example the $L = 3, 5, 6$ operators in (12.43). But it is not easy to use Feynman rules to calculate the 6-point matrix elements of, say, R^4. However, $M_6(S^{1234}S^{5678}h^-h^-h^+h^+)_{R^4}$ can be extracted from the α'-expansion of the closed superstring theory tree amplitude. This may bother you, because there are no continuous global symmetries in string theory, and here we are interested in testing global continuous $E_{7(7)}$. However, the 4d tree-level superstring amplitudes have an accidental global $SU(4) \times SU(4)$ symmetry. This is a consequence of T-duality when 10-dimensional superstring theory is reduced to 4d by compactifying it on a 6-torus. The easiest way to see the $SU(4) \times SU(4)$ symmetry is through the KLT relations: the two open string tree amplitudes on one side of KLT have $SU(4)$ symmetry, so the closed string tree amplitude on the other side of KLT inherits $SU(4) \times SU(4)$; the $SU(4) \times SU(4)$ is enhanced to $SU(8)$ only in the $\alpha' \to 0$ limit [36].

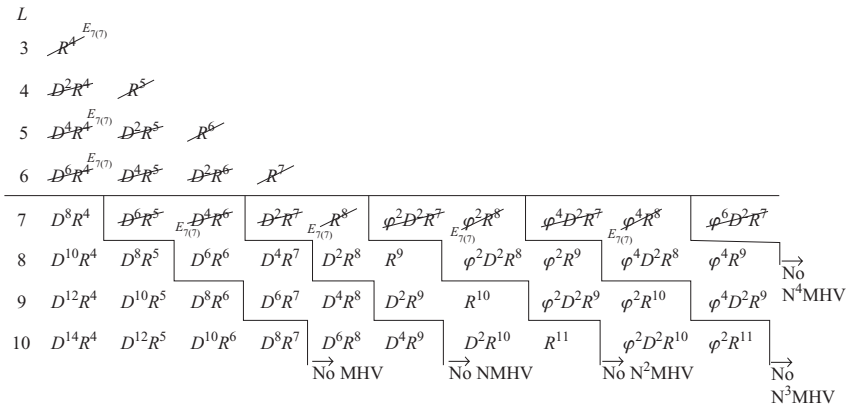
Now, we are going to use the α' contributions from the superstring tree amplitudes, but their $SU(4) \times SU(4)$ is not good enough, we need $SU(8)$. So we *average* the string amplitude over all 35 independent embeddings of $SU(4) \times SU(4)$ into $SU(8)$ to get an $SU(8)$-invariant answer. For example, the leading α'-correction to the closed superstring amplitude is order α'^3. Dimensional analysis implies that this comes from an 8-derivative effective operator with $\mathcal{N} = 8$ supersymmetry and (after averaging) $SU(8)$-invariance. But we know from the previous analysis that there is only one such operator, namely R^4. So after making it $SU(8)$-invariant, the α'^3 contribution from the $SU(8)$-averaged superstring amplitude must be identical (up to an overall constant) to the matrix element of the operator R^4 in $\mathcal{N} = 8$ supergravity [220]![11] The open superstring tree amplitudes are known in the literature [229], so pulling them through KLT, the $M_6(S^{1234}S^{5678}h^-h^-h^+h^+)$ matrix element can be extracted, $SU(8)$-averaged, and then subjected to the single-soft-scalar test (12.45).[12] And R^4 fails this test: so R^4 ***is not compatible with*** $E_{7(7)}$. Hence $E_{7(7)}$ excludes R^4 as a candidate counterterm and *this explains why the 3-loop 4-graviton amplitude is finite.* The on-shell matrix element technique made it possible to show that the 3-loop divergence of $\mathcal{N} = 8$ supergravity is excluded without doing any loop-amplitude calculations [220]. Note that this is *not* a string theory argument – string theory amplitudes were only used as a tool to calculate the relevant field theory matrix elements – so the argument in [220] is field theoretical.

The analysis outlined above can be repeated [221] for D^4R^4 and D^6R^8 and both are shown to be excluded by $E_{7(7)}$. (See also [231] for a string-based argument for the absence of R^4, D^4R^4 and D^6R^8.) This means that *the symmetries of* $\mathcal{N} = 8$ *supergravity exclude the divergences in any amplitudes below 7-loop order* [221]. Explicit calculations of the $L = 5, 6$ amplitudes are not yet available, but as explained here they are expected to yield finite results in 4d.

[11] In the Einstein frame, the effective operator in superstring theory at order α'^3 is actually $e^{-6\phi}R^4$, where ϕ is the dilaton. The presence of the dilaton (which can be identified as a certain linear combination of the 70 scalar scalars in the $\mathcal{N} = 8$ supergravity spectrum) operator breaks $SU(8)$ to $SU(4) \times SU(4)$. The average over embeddings $SU(4) \times SU(4) \to SU(8)$ eliminates this breaking and restores $SU(8)$.

[12] An earlier test [230] did not involve the $SU(8)$-average.

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Overview of exclusion of counterterm operators in $\mathcal{N} = 8$ supergravity in 4d. **Fig. 12.1**

An overview of possible counterterms in 4d $\mathcal{N} = 8$ supergravity is presented in Figure 12.1. Note that at 7-loop order, all but the D^8R^4 operator are excluded. This means that calculation of the 4-graviton 7-loop amplitude can completely settle the question of finiteness at 7-loop order. But this is not known to be the case at 8-loops or higher.

In the following section, we give a brief survey of the current status of the UV behavior of loop amplitudes in supergravity theories.

12.6 Supergravity divergences for various $\mathcal{N}$, L, and D

Current approaches to examining the possible UV divergences of perturbative (super)-gravity can be categorized as follows:

- ***Direct computation.*** The explicit computations of loop amplitudes are made possible by increasingly sophisticated applications of the generalized unitarity method; this in itself advances the technical tools for attacking higher-loop computations in general field theories. For the purpose of exploring UV divergences in supergravity theories, most efforts focus on the 4-graviton amplitude; this is because (as we have seen in the previous section) the lowest counterterm for pure supergravity is of the form $D^{2k}R^4$. See [232] for a discussion on the various details of obtaining multi-loop integrands and extracting the UV divergences.
- ***Symmetry analysis.*** The leading UV divergence of a theory has a corresponding local gauge-invariant operator that must respect all non-anomalous global symmetries of the theory. Analyzing the symmetry properties of operators, one can rule out UV divergences and identify candidate counterterms. In Section 12.5, we took an approach based on the on-shell matrix elements of the candidate operators, and used it to rule out divergences in 4d $\mathcal{N} = 8$ supergravity for $L \leq 6$. Alternatively, one can also use extended-superspace to construct possible invariant operators; for early constructions, see [233, 234]. If the invariant operator can be expressed as a superspace integral over a subset of superspace coordinates, then it is considered a "BPS operator" and it is subject to non-renormalization

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theorems. If it is given as a full superspace integral, then it is non-BPS and expected to receive quantum corrections, thus serving as a candidate counterterm. The distinction between BPS and non-BPS invariants relies on subtle assumptions about the number of supersymmetries that can be linearly realized off-shell. This lies outside the scope of this presentation, so we refer you to [235] and references therein.

- ***Pure spinor formalism.*** The "pure spinor formalism" [236] is a first-quantized approach (in contrast, QFT is a second-quantized approach) to scattering processes in 10d maximal supersymmetric theories. Using the 10d loop-integrand obtained in the pure spinor formalism, one can infer properties of the loop amplitudes in $D \leq 10$ and this can be helpful for assessing potential UV divergences. For more details of this approach to multi-loop amplitudes, see [218].
- ***Role of non-perturbative states?*** The 2^8 massless states of ungauged $\mathcal{N} = 8$ supergravity in 4d match exactly the spectrum of massless states of closed Type IIB superstring theory compactified on a six-torus T^6. At the classical level, $\mathcal{N} = 8$ supergravity in 4d can be viewed as the low-energy ($\alpha' \to 0$) limit of Type IIB superstring theory on T^6. However, it was pointed out in [237] that one cannot obtain *perturbative* $\mathcal{N} = 8$ supergravity in 4d as a consistent truncation of the string spectrum: in the limit $\alpha' \to 0$, keeping the 4d coupling small forces infinite towers of additional states to become light, e.g. Kaluza–Klein states, winding modes, Kaluza–Klein monopoles, and/or wrapped branes. Thus one obtains from string theory not just the spectrum of $\mathcal{N} = 8$ supergravity, but a slew of additional light states. This argument is independent of whether $\mathcal{N} = 8$ supergravity is finite in 4d or not. However, it does mean that even if pure $\mathcal{N} = 8$ supergravity in 4d were to be a well-defined theory of quantum gravity, its UV completion would not be Type IIB string theory.

  A related objection [238][13] to the program of studying finiteness of perturbative supergravity is that even the 4d $\mathcal{N} = 8$ supergravity theory itself contains non-perturbative states, namely BPS black holes, that in certain regions of moduli-space become light enough that they may enter the perturbative expansion [238]. Such contributions would never enter the unitarity method approach to explicit calculation of amplitudes.

As we have noted, there are certainly examples of divergences in various perturbative (non-super)gravity theories in 4d: 1-loop in gravity with generic matter [204, 207], 2-loop in pure gravity [205, 206], and at 1-loop [240] in dilaton–axion gravity whose spectrum was given in (12.16). The first example of a UV divergence in pure supergravity was found at 4-loop order in the 4-graviton amplitude of $\mathcal{N} = 4$ supergravity [241].

We end this chapter by summarizing what explicit computations of supergravity loop amplitudes have revealed so far about the critical dimension D_c of supergravity with various numbers of supersymmetry:

Maximal supergravity (32 supercharges)

Loop-order	1	2	3	4
D_c	8 [82]	7 [242]	6 [213]	$\frac{11}{2}$ [216]

. (12.46)

[13] See also [239].

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Half-maximal supergravity (16 supercharges)

Loop-order	1	2	3	4
D_c	8 [243]	6 [244]	> 4 [245]	≤ 4 [241]

. (12.47)

Half-maximal supergravity with matter (both with 16 supercharges)

Loop-order	1	2	3
D_c	4 [246]	≤ 4 [247]	≤ 4 [247]

. (12.48)

In the above, "≤ 4" indicates an upper bound for the critical dimension and we have given a literature reference for each result.

The absence of a UV divergence for half-maximal supergravity at 3-loops in 4d [245] as well as 2-loops in 5d [244], was *not* anticipated by superspace-based analyses. This unexpected result prompted a conjecture of the existence of an off-shell formalism that preserves the full 16 supersymmetries [248]. This would imply finiteness for half-maximal supergravity with matter at 2-loop in 5d, but explicit calculations [247] have shown that a UV divergence is actually present, thus contradicting the conjecture.

The study of the UV structure of perturbative supergravity theories in diverse dimensions has resulted in some interesting insights into the relation between gravity and gauge theory scattering amplitudes: this includes the color-kinematics duality that is the subject of the next chapter. It is relevant to note that no matter what one thinks of the program to study the UV-behavior of supergravity, these new insights would have been difficult to come by without the effort put into explicit calculations and the lessons learned in the process.

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