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REPORT OF THE FIXED-INTEREST WORKING GROUP

BY K. S. FELDMAN, B. BERGMAN, A. J. G. CAIRNS, G. B. CHAPLIN, G. D. GWILT, P. R. LOCKYER AND F. B. TURLEY

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ABSTRACT

Actuarial models of the market in conventional British Government Stocks, also known as the Gilt-Edged market, are reviewed and contrasted with the methods which have been developed, during the last twenty years, by financial economists.

Following the Treasury's announcement in May 1995 (regarding the taxation of institutional bond holdings), the so-called 'coupon effect' has largely disappeared and gilt prices can now be fitted very closely by using the same simple discounting functions for both income and capital flows. A new model suitable for the calculation of yield indices is proposed and is contrasted with the model currently underlying the FTSE Actuaries Government Securities (FTSEAGS) Yield Indices. A number of new possible applications of the reformulated yield indices, such as forward pricing, asset/liability matching and stochastic simulation, are discussed. An analogous model for index-linked gilts leads to applications involving the forward market in the retail prices index.

A survey of professional users of the FTSEAGS Indices is described, and a revised presentation for the published yield indices is suggested. A summary of current statutory references to the indices is presented.

KEYWORDS

Gilts; FTSEAGS Indices; Bond Modelling; Forward Rates; Interest-Rate Derivatives; Matching

CONTACT ADDRESS

K. S. Feldman, Ph.D., F.I.A., Robert Fleming & Co Ltd, 25 Copthall Avenue, London EC2R 7DR, U.K. Tel: +44(0)171-638-5858; Fax: +44(0)171-628-0053; E-mail: keith.s.feldman@flemings.com

1. INTRODUCTION

1.1 This is the Report of the Fixed-Interest Working Group, set up in early 1995 under the auspices of the Joint Investment Committee of the Institute of Actuaries and the Faculty of Actuaries, under the chairmanship of the first-named author. The report is also addressed to the FTSE Actuaries Bond Indices Committee, which is organised in conjunction with *FTSE International Limited.* The Bond Indices Committee is responsible, *inter alia*, for taking decisions concerning the formulation and content of the published indices. The terms of reference of the Working Group suggested that the following points be addressed:

(a) the problem with the present formula for the redemption-yield curve used in the FTSE Actuaries Government Securities (FTSEAGS) Yield Indices, which

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allows multiple optima, and hence 'catastrophic' changes in the yield curve on occasion;

(b) whether the present format of the FTSEAGS Yield Indices, based on redemption-yield curves for separate coupon bands, is still a satisfactory method for publication, or whether some alternative formulation, based, for example, on a yield surface, or on a par-yield curve, or on a structure of zero-coupon interest rates, would be more appropriate; and

(c) modelling the changing shape of the term structure of interest rates, as represented by the parameters of the yield curves or in some other way, and the stochastic variability of yields or parameters; in particular to consider how the various theoretical models (e.g. Cox-Ingersoll-Ross, Heath-Jarrow-Morton, etc.) can be applied to United Kingdom conventional and index-linked gilts.

1.2 The above issues, it soon transpired, are interconnected, and some months after the Working Group was set up we had to consider the impact of the Inland Revenue's (1995) proposals for significant changes to the taxation of gilts and bonds. This was seen to be a prerequisite for the development of an official market in the individual coupons and redemption proceeds of particular issues ('strips') and in repurchase agreements ('repos').

1.3 Of profound relevance to the Working Group was the fact that the intended taxation basis for gilts, from April 1996, at least as far as institutional investors were concerned, would treat income and capital gains in a symmetrical manner — although index-linked issues would continue to be free of tax on the gain attributed to changes in the Retail Prices Index (RPI).

1.4 The very significant consequence of the removal of the taxation distinction between capital gains and income was the elimination of a large proportion of the tax arbitrages which still remained in the market (after the introduction of the taxation/relief of accrued interest in February 1986). This, together with the proposed development of official markets in repos and strips, meant that the gilt market would, from April 1996, be largely 'arbitrage-free', unless private investors (who continued to be able to disregard the effect of tax on changes in capital values) exerted a very strong influence on the prices of certain issues. Most work by financial economists took, as a premise, the arbitrage-free nature of markets, and this change in the taxation basis meant that a large volume of work on interest rate modelling would instantly become much more relevant to the gilt market. A comprehensive review of such work is given in Vetzal (1994).

1.5 The paper describing the current gilt indices, Dobbie & Wilkie (1978), was discussed at the Institute in November 1977. At that time there was no market in interest rate swaps (the concept itself had probably not been envisaged), and it was illegal to buy or write options on British Government Securities. Although gilt market players would make policy moves in order to benefit from anticipated changes in the shape of the yield curve, a considerable amount of

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activity, at that time, was devoted to the purchase or sale of gilt coupons in order to create a tax advantage. The resulting price anomalies would then, together with discrepancies caused by the yield curve plays already referred to, give rise to further anomaly switching activity (largely inspired by actuaries).

1.6 Much of the tax-related activity in the gilt market has now been eliminated, and we have, therefore, addressed the uses to which the dynamics of interest rate movements are now being put. Inevitably this leads to a consideration of forward rate agreements and other derivative contracts. For the indices to have any chance of remaining useful over the next twenty years, we believe that they must facilitate these newer applications.

1.7 It was thought to be helpful to our deliberations to ask users about their own current applications of the FTSEAGS Indices, together with their suggestions for any changes. Section 2 is devoted to an analysis of the responses. This section also contains a summary of current statutory references to the FTSEAGS Indices.

1.8 Section 3 reviews some of the different approaches to bond modelling taken by actuaries in the U.K. It begins with a review of the current model underlying the FTSEAGS Yield Indices, and goes on to discuss other formulations proposed by actuaries over the years. This is followed by a review of various models proposed by financial economists.

1.9 The Working Group's proposed model for the FTSEAGS Yield Indices is discussed in Section 4, with several possible applications of the model being examined in Section 5.

1.10 Section 6 concludes the paper, with our suggestion for the yield indices' information which, we believe, should be published.

1.11 The ground covered by the Working Group during the course of our deliberations was both diverse and detailed. In presenting our results, we were faced with the dilemma of balancing our desire to present our work in as accessible a manner as was possible with the simultaneous need to substantiate our review and recommendations with adequate background detail and mathematical rigour. Our decision has been to present a fairly high level overview of our work and conclusions in the body of this paper, whilst consigning the more mathematical and background work which was carried out to appendices or to one of the two accompanying supplementary papers.

1.12 This paper has been written with the aim of encouraging maximum participation at the sessional meetings by the many potential readers who might have an interest in, or contribution to make on, the subject of recent changes in the gilt market and the consequences for the FTSEAGS Yield Indices.

1.13 The first of the two accompanying supplementary papers, entitled 'Descriptive Bond-Yield and Forward-Rate Models for the British Government Securities' Market', by A. J. G. Cairns, goes into the detail of the formulation and examination of the descriptive forward-rate model for the FTSEAGS Yield Indices being put forward by the Working Group.

1.14 The second supplementary paper, entitled 'A Review of Term-Structure Models and their Applications', by G. B. Chaplin, reviews, in some detail, the

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types of term-structure models developed in the world of financial economics. It concludes that the category of models known as 'descriptive' models are the most appropriate for the purpose of calculating yield indices. The paper then goes on to illustrate the fitting of a descriptive polynomial forward-rate model to the gilt market, with the objective of determining how many parameters are significant and necessary for a 'reasonably good' description of the market.

1.15 The Working Group would particularly welcome the profession's input, at the sessional meeting discussions, on the proposed changes to the yield indices and on the model which is being suggested.

2. Current Professional Uses for the FTSEAGS Indices

2.1 *Views of the Users*

2.1.1 In order to gain an understanding of who the current users of the FTSEAGS Indices are and the uses to which the indices are being put, we undertook a survey, during 1995, covering a broad range of professional organisations and commercial undertakings.

2.1.2 Most responses came from people working in one of the following areas: fund management; investment arms of life offices; consulting actuaries; and traditional life office actuarial roles.

2.1.3 The users of the FTSEAGS Indices tend to range from those with just a high level interest in the indices to those requiring intricate details of the very essence of the indices. Most of the users appear to be actuaries (working in either life or pensions environments), fund managers, or certain government bodies. Many respondents described their uses of the indices as falling under either (or both) of the 'research' or 'marketing' headings.

2.1.4 The higher level uses for the indices tended to be amongst the following:

(a) simply to get 'a feel' for the level of interest rates prevailing at any particular point in time;
(b) product pricing, particularly annuity pricing;
(c) valuations, whether of assets or liabilities;
(d) asset share calculations; and
(e) fulfilling statutory requirements.

2.1.5 Those users requiring a more detailed knowledge of the indices tended to operate exclusively in investment environments. The most common use was for benchmarking for performance measurement purposes, whilst, in several cases, the indices were used for the purposes of portfolio construction, presumably to create index funds. For these purposes, detailed knowledge of durations and convexities is needed, and, since such information is not published, users have to evaluate such properties themselves.

2.1.6 From the responses, it was evident that certain of the indices are less well used than others.

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2.1.7 Amongst the conventional gilt indices, the irredeemable price index appeared to be only infrequently used.

2.1.8 The low-coupon yield indices are the least commonly used of the three available coupon bands (namely high, medium and low). This may be due, in part, to the fact that there do not appear to be any legislative references to the low-coupon indices, unlike the case with the medium and high-coupon indices. The changes to the taxation of gilts and bonds, introduced by the 1996 Finance Act, have arguably reduced the attractiveness of low-coupon instruments to non-tax-exempt corporate investors (with the exceptions, of course, being the two gilts which continue to be taxed as per the previous regime, namely the Funding $3\frac{1}{2}\%$ 1999/04 and the Treasury $5\frac{1}{2}\%$ 2008/12). This is because individual investors, unlike corporate holders, continue to enjoy tax-free capital gains, and would, therefore, tend to bid up the prices of low-coupon bonds, making them relatively less appealing to corporate investors. (A similar advantage would be enjoyed by foreign holders of gilts whose tax jurisdictions continue to exempt any capital gains from tax.) Several respondents who had initially indicated some use for the low-coupon yield indices were contacted, and, after consideration, tended to conclude that their use of the low-coupon yield indices might cease as a result of these changes in the market.

2.1.9 Turning now to the index-linked gilt indices, perhaps, not unexpectedly, the yield indices incorporating an underlying inflation assumption of 10% p.a. are not currently much in favour. Amongst the indices with an underlying 5% inflation assumption, there tends to be less call for the 'up to 5 years' classification than for the 'over 5 years' classification. This may well reflect the relatively small number of index-linked gilts with terms of less than five years.

2.1.10 Dobbie & Wilkie (1978) envisaged a number of uses for the indices. The responses to our questionnaire indicated that the indices are being used broadly as envisaged. An exception is probably in the area of bond trading, or issuing, where pricing is more likely to be relative to a benchmark gilt than to the indices themselves.

2.2 *Inadequacies of the Indices and Potential Improvements*

2.2.1 As might have been expected, it was those respondents working in an investment environment who tended to highlight some inadequacies of the indices. These are the people who need to know and understand the technical properties of the indices, and found, in the case of the price indices, the lack of published coupon, duration and convexity information, in particular, to be a drawback.

2.2.2 The consistency between the published gilt price information and the FTSEAGS Indices in the *Financial Times* is sometimes called into question. Changes to the price source used in the calculation of the indices, which were introduced in June 1997, should lead to significant improvements.

2.2.3 Many respondents pointed out that the value of the price indices could be enhanced by the inclusion of total return indices.

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2.2.4 It was interesting to note that no mention was made in the responses, of the occasional 'catastrophic jumps' in the yield indices which do occur. A 'catastrophic jump', in the mathematical sense, is said to have occurred when the yield indices change noticeably from one publication date to the next, without the underlying gilt yields having changed very much. This phenomenon was one of the main *raisons d'être* of the Working Group. One respondent might have alluded to this problem by indicating that, before being prepared to make more use of the indices, she would need to understand why they sometimes move as they do!

2.2.5 The withdrawal of the Corporate Bond Index, at the end of 1994, has been a disappointment for a number of people, who now find that they have to refer to indices produced by investment banks, with a consequent loss of the air of independence of such indices. In fact, the primary reason for the cessation of the published corporate bond indices in the *Financial Times* appears to have been the difficulty in obtaining prices from an adequate number of market makers for the indices to be labelled as independent. Calls, therefore, for the reintroduction of an enhanced set of corporate indices, sub-divided by sector and credit rating, are unlikely to be answered, but, in any event, were not deemed to be within the Working Group's remit.

2.2.6 Other aspects of the indices which respondents, to varying degrees, felt would improve their usefulness included:

(a) splitting the 5-15 year term band in the conventional gilt price indices into two equal term bands (though several respondents stressed that the current 5-15-year band should still be published in addition), and including a 10-year yield index;
(b) increasing the number of term bands currently published for index-linked gilts, from the current 'up to 5 years' and 'over 5 years' split; it was recognised that the rather low number of index-linked gilt issues might not justify too detailed a term split;
(c) the publication of zero-coupon, or implied forward, rates;
(d) clearer presentation of the indices and preferably a more direct comparability between the price and yield indices (it should be recognised, however, that while the price indices are based directly on actual gilts, the yield indices are based on a model of the underlying gilts); and
(e) calculation of the index-linked gilt yields at additional assumed inflation rates (the popular choices being 3% and 4%). The calculation and publication of break-even inflation rates was suggested. It should be pointed out that the current procedure of publishing index-linked gilt yields on two bases, namely 5% and 10% inflation, enables good approximations to be made for real yields with underlying inflation up to about 15%, by interpolation. (See Wilkie, 1984.)

2.2.7 We have focussed most of our attention on the yield indices, as per our brief. Section 6 contains our suggestion for a revised presentation of the FTSEAGS Yield Indices, taking into account the results of the survey and the

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added flexibility afforded by the mathematical model proposed in Section 4. Nevertheless, the results of our entire survey have been passed on to the Bond Indices Committee of *FTSE International Limited*, and many of the user requirements/suggestions are shortly to be implemented.

2.3 *Statutory References to the Indices*

The Working Group was very keen to identify the statutory references to the FTSEAGS Indices in order to avoid unforeseen difficulties as a result of any changes proposed. With the help of the Government Actuary's Department and the Inland Revenue, the following statutory references to the indices (not guaranteed to be exhaustive) were identified:

(a) Insurance Companies Regulations (1994), where the 15-year medium-coupon yield index is used in the calculation of the maximum future interest rate assumption for the valuation of insurance liabilities;

(b) State Scheme Pensions (Actuarial Tables) Regulations 1992, where the 15 and 20-year high-coupon yield indices are required in the calculation of market-level indicators when determining state scheme pensions;

(c) Finance Act 1995, where the instructions to determine maximum and minimum permissible income withdrawals from protected rights and non-protected rights make reference to the 15-year high-coupon yield index;

(d) Guidance Note (GN) 27, drawn up by the U.K. actuarial profession in connection with the minimum funding requirement (MFR) instituted by the Pensions Act 1995, makes reference to two yield indices for the purposes of calculating the value of pensioner liabilities and the market value adjustment on non-pensioner liabilities; the two indices concerned are the 15-year medium-coupon conventional yield and the over-five-year index-linked yield (with underlying 5% inflation); and

(e) Income and Corporation Taxes Act (ICTA) 1988, which makes reference to the 15-year high-coupon yield index for the purposes of the 'floor' calculation, i.e. the minimum investment return to assume for determining whether taxable pensions business profits and overseas life assurance business profits arise.

3. Review of Approaches to Bond Modelling

3.1 *Types of Term-Structure Models*

3.1.1 Models of the term structure of the yield curve can be divided into three main categories, namely, 'descriptive', 'equilibrium' and 'evolutionary' models.

3.1.2 Descriptive models simply aim to describe the yield curve in terms of a mathematical equation (such as a polynomial). Such formulations are motivated more by their flexibility, in terms of the variety of yield curve shapes they can support, than by an underlying economic theory.

3.1.3 Equilibrium models, on the other hand, start with assumptions about economic variables, and use these to drive the entire yield curve. Since such

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models are frequently chosen with their mathematical tractability in mind, their ability to fit observed term structures is sometimes limited.

3.1.4 Evolutionary models take the initial term structure as input, and then allow it to evolve into the future.

3.1.5 Actuarial approaches to modelling the gilt market have tended to concentrate on descriptive models. Financial economists have focussed their attention on equilibrium models, initially, and latterly on evolutionary models. Whether such models could have been applied successfully to the gilt market, as it existed prior to May 1995, is debatable.

3.1.6 A key factor dictating the appropriate choice of model is the purpose to which it is to be put. Descriptive models are particularly well suited for statistical analysis (e.g. yield indices and cheap/dear considerations). Equilibrium models and, all the more so, evolutionary models are used for the valuation of interest-rate contingent claims and other derivative products.

3.1.7 When discussing term-structure models, the terms 'arbitrage-free' and 'no-arbitrage' arise, sometimes with different, even contradictory, meanings. The evolutionary approach is 'self-consistent' in that, as the initial term structure is input to the model, no arbitrages are present between actual bond prices and those priced off the model (though, if the market itself exhibits elements of arbitrage opportunities, this will, of course, be reflected in the opening position of the model). Clearly, descriptive and equilibrium models *are* likely to exhibit these initial modelling arbitrages, as they only attempt to *model* the opening term structure.

3.1.8 Term-structure models, themselves, may not be deemed arbitrage-free if, for example, they permit interest rates to become negative, as, under such circumstances, investors would make risk-free profits by borrowing cash and then burying it.

3.1.9 Nevertheless, such term-structure models are frequently labelled as arbitrage-free, provided it is not possible to make risk-free profits by replicating one portfolio, priced off the model, with a different combination of assets, also priced off the model — and, in this game, burying the cash (to beat negative interest rates) is not permitted!

3.1.10 In arriving at its recommended descriptive model for the yield indices (see Section 4.2), the Working Group focussed on the desirability of a parsimonious (see ¶3.4.7) model with enough parameters to fit the data well, rather than aiming to achieve an exact fit to all gilt prices (this being both impractical and undesirable).

3.2 *The Current FTSEAGS Yield Indices Model*

3.2.1 The approach currently used in the construction of the FTSEAGS Yield Indices was developed by Dobbie & Wilkie (1978) (referred to as D&W). They proposed the descriptive model:

$$y(t) = A + Be^{-Ct} + De^{-Ft}$$

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where $y(t)$ is the gross redemption yield (compounded semi-annually) on a bond which matures at time t. This curve is fitted to high, medium and low-coupon stocks separately, in recognition of the existence of the so-called 'coupon effect' which was evident at the time the indices were proposed. The actual definition of the coupon bands varies from time to time, depending on the prevailing distribution of gilt coupons. Under the previous regime, low-coupon stocks were sought after by high-rate tax payers. Non-tax payers, such as pension funds, were able to take advantage of the resulting lower demand for high-coupon stocks and obtain higher yields. It was, therefore, normal to find that the high-coupon yield curve lay above the medium-coupon yield curve, which, in turn, was above the low-coupon yield curve. Under the new tax regime, however, it is no longer advantageous for institutional tax payers to buy low-coupon stocks, and it is now common for the three yield curves calculated using the D&W model to be very close together and occasionally in the opposite order. Figures 3.1 and 3.2 illustrate the extent to which the redemption yields of gilts have literally moved into line following the announcement of the new tax regime for gilts and bonds. The solid line shown in these figures is the par-yield curve fitted by the formulation described in Section 4.2. Convertible gilts and those with split redemption dates, have been excluded.

3.2.2 D&W used a simple least-squares approach to estimate the parameters. This involves taking observed gross redemption yields for individual stocks and finding the values of the five parameters which minimise the weighted least squares function. Weights are proportional to the amount of nominal stock for each issue. The fitted curve is linear in three parameters (A, B and D), and it is straightforward to estimate these three, given the values of the remaining two parameters (C and F). The least-squares (or objective) function is non-linear in C and F. The optimal values of C and F, therefore, must be found using numerical minimisation algorithms.

3.2.3 Not long after the introduction of the D&W model, it was found that the objective function could have more than one local minimum (and that this is a regular rather than a rare feature). The minimisation algorithm used attempts to track the same minimum from one trading day to the next. However, from time to time the current local minimum disappears and the fitted minimum jumps from one location to another. This can happen when the market is relatively quiet, and the jump in the shape of the curve can be noticeable. Such an event is referred to as a 'catastrophic jump'. The possibility of such jumps has gradually undermined the credibility of the yield indices. This then, together with the aforementioned taxation changes in 1996 and the accompanying evolution of the gilt market, has led to the need for a review of the construction of the yield indices.

3.2.4 The accompanying paper by Cairns (1998) includes some discussion of the D&W model, and illustrates the problem of multiple minima and catastrophic jumps.

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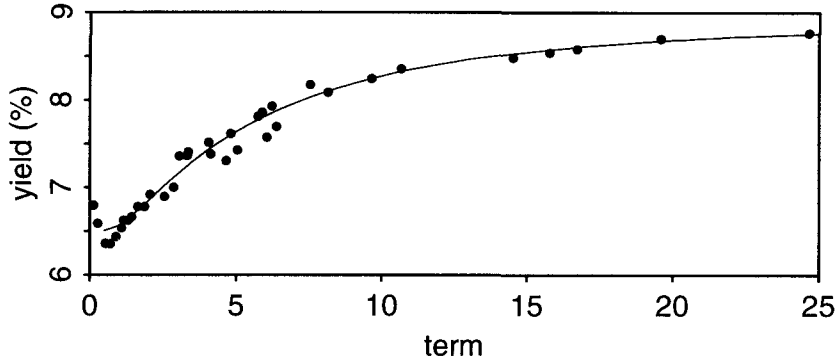


Figure 3.1. Gilt gross redemption yields, 1 January 1993 (old tax regime)

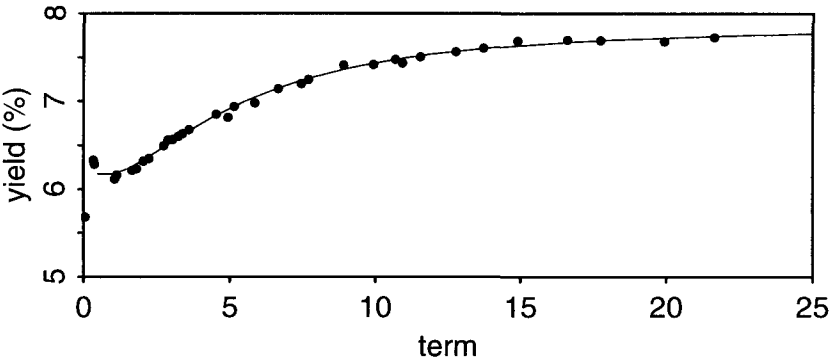


Figure 3.2. Gilt gross redemption yields, 1 January 1996 (new tax regime)

3.3 *Review of other Descriptive Yield-Curve and Price-Surface Models*

3.3.1 Computational methods for fitting yield curves to the gilt market can be traced back to the work of Pepper (1963). In the years following the publication of that paper, market practitioners constructed proprietary curves using a number of variations of Pepper's formulae. In the early years electronic computers were not readily available, and the analysis was performed manually. By charting the deviations from the curve of each gilt issue, it was possible to identify price anomalies, and the resulting switching activity became important commercially, both to the brokers who transacted the business and to the institutions who earned the arbitrage profits.

3.3.2 The methods employed usually involved polynomial regression, with the independent variable being the time to maturity (or occasionally the duration), and the dependent variable the gross redemption yield. The attractions

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to high taxpayers of low running yield gilts pushed up their prices, resulting in redemption yields for these issues well below the curve. By the early 1970s the taxation effect often dominated that caused by maturity (or duration) differences. This led actuaries to explore new formulations, e.g. Clarkson (1978) and Feldman (1977), although the Bank of England had used some of Clarkson's ideas as early as 1973, see Burman *et al.* (1973). Further details are given in Phillips (1996), Chapter 6. More recently, the Bank of England has developed a yield surface model using splines and developing the approach of Clarkson (Mastronikola, 1991).

3.3.3 Denoting the price by P and the coupon by g, Clarkson's model starts with a relationship between the running yield on the stock $r = g/P$ and the capital gain to maturity $1/P - 1$. He showed that, if prices are 'in equilibrium under switching action', then:

$$\frac{\partial}{\partial r}\frac{1}{P} < 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial r^2}\frac{1}{P} \leq 0$$

the strict inequality in the second partial differential corresponding to the 'curvature' caused by taxation effects; if the second derivative is zero, prices will be linearly related to coupon for a fixed maturity date. These equations characterise the relationship between the prices and running yields of issues having the same redemption date, and do not, by themselves, contain any information about term structure.

3.3.4 Practitioners have not always considered it necessary to ensure the strict inequality in the second derivative, and the model outlined in Appendix A is a simplified example of one of a series of price surfaces developed by P. Phillips; his formulae have kindly been made available to the Working Group. This method equates the 'fitted' price to the present value of the various components of return, where different functions are used to discount the income and capital components. Unlike Clarkson's model, the same function of life is used to represent the complete spectrum of maturities. Phillips is continuing to test models of this form in a research programme aimed at fitting historical prices over a very long time period.

3.4 *Alternative Modelling Approaches*

3.4.1 Chaplin (1998), which accompanies this paper, reviews the history of term-structure modelling from the perspective of financial economic theory. Models are classified into 'equilibrium', 'evolutionary' and 'descriptive' categories, as outlined in Section 3.1.

3.4.2 The Vasicek (1977) (equilibrium) model is discussed in detail. The driving factor is the risk-free money market rate, and the model is fitted to money rate history to estimate the model parameters. Based on these values, an assumption of zero 'market price of risk', and the knowledge only of today's

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money rate, the predicted yield curve is calculated. Unsurprisingly, there is little relationship between the predicted and actual curves. (The market price of risk is related to the additional yield required on risky investments over and above the risk-free rate, to compensate for the risk taken.)

3.4.3 One advantage of the Vasicek model is its mathematical tractability. Chaplin (1998) describes how options can be priced in this environment by means of a closed form solution. He then applies this method to valuing the embedded option on double maturity dated bonds.

3.4.4 The Richard (1978) equilibrium model is a two-factor extension of the Vasicek model, and is equally tractable mathematically (unique among two-factor models). It is potentially of great interest to actuaries, being theoretically based on inflation and real yield expectations.

3.4.5 The Hull & White (1990) model is discussed briefly as an example of an evolutionary model. The primary objective of such models is the pricing of derivatives, given the current yield curve.

3.4.6 Chaplin (1998) goes on to discuss polynomial models as an example of descriptive models. This approach is very flexible, since it regards the underlying curve (be it spot or forward rates) merely as an element in function space (i.e. from a mathematical point of view, the yield curve is viewed as an unknown, and fairly general, function of time). As such, it can be approximated to an arbitrarily high degree of accuracy by taking a weighted sum of functions which span function space. A class of orthonormal polynomials is introduced as spanning functions.

3.4.7 By transforming the time axis, the approach can be made parsimonious (i.e. efficient in terms of the number of parameters required). A straight line (the first two elements from the spanning set) in transformed time (called rescaled duration) maps into a 'typical' yield curve as a function of time.

3.4.8 Chaplin (1998) considers practical and theoretical questions relating to the application of the descriptive polynomial model outlined above. His paper discusses the impact of repo rates (i.e. the rate of interest payable on cash deposited as collateral against the loan of a bond) on bond pricing. The short history of an open repo market in the U.K. and the difficulty in obtaining reliable estimates of appropriate longer-term rates, together with a desire to keep the treatment relatively simple, have led to an exposition ignoring repo rates. In the longer term, however, it is likely that this will become an increasingly important influence on bond pricing.

3.4.9 Benchmark status is also shown to be a significant factor in the pricing of bonds — effectively driving the yields of these bonds below the underlying yield curve. Fitting a model which is too highly parameterised will tend to reflect these non-term-structure effects in the term structure.

3.4.10 The paper concludes that the descriptive approach is most appropriate for our purposes, and that the polynomial (or similar) class of models is flexible and parsimonious.

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4. PROPOSED MODEL FOR THE FTSEAGS YIELD INDICES

4.1 *Forward-Rate Models*

4.1.1 The concept of the 'force of interest' δ will be one with which all actuaries are familiar.

4.1.2 The ideal situation expounded in the textbook on compound interest (McCutcheon & Scott, 1989) is that, at each future instant s, there will be a force of interest $\delta(s)$, varying with time, and applying throughout a homogeneous market where each investor uses the same basis of valuation.

4.1.3 Such markets do exist in practice (see ¶¶4.1.5 and 4.1.8). Clearly, in these cases the curve $\delta(s)$ will vary from day to day. In recognition of this, we need to amend our notation to $f(t, t + s)$, to represent the curve of $\delta(s)$ as measured at time t. The term $f(t, t + s)$ is the force of interest, or, in the language of the financial economists, the *instantaneous forward rate* of interest, at time $t + s$, inherent in market prices of bonds (or in our case, gilt prices) at time t. The curve $f(t, t + s)$, for $0 \leq s < \infty$, is known as the instantaneous forward-rate curve at time t.

4.1.4 We are not saying, at time t, that we actually anticipate earning an instantaneous rate of interest of $f(t, t + s)$ on cash when time $t + s$ eventually arrives. We can, however, enter into a forward contract at time t which will fix the rate of interest at time $t + s$ at $f(t, t + s)$. A forward contract which specifies any other rate of interest will admit arbitrage opportunities.

4.1.5 Following the recent changes, as described in Section 1, the gilt market has been reduced to a more homogeneous set of cash flows, by which we mean that cash flows, be they coupons or capital payments, at some time T in the future, will be valued equally by the market, irrespective of the originating gilt. Exley, Mehta & Smith (1997) refer to this as the 'Law of One Price'. Indeed, coupons from strippable gilts (see Cairns 1998, Appendix A), which are due on the same date, will be fully interchangeable or 'fungible'.

4.1.6 Given this homogeneity, it is reasonable to hypothesise that the same set of instantaneous forward rates will apply to all gilts, except those few issues which have features which set them apart from the rest of the market, such as, for example, split redemption dates or benchmark status. The challenge, hence, becomes one of calculating the underlying instantaneous forward rate function.

4.1.7 If we could get to the underlying $f(t, t + s)$ function, we would be able, at time t, to value a payment of one, due at some time $t + T$ in the future as:

$$\exp\left(-\int_0^T f(t, t+s)ds\right).$$

It follows that this would also enable us to calculate the prices (and redemption yields) of all gilts, both at present and in the future. The prices derived for $t = 0$ would, obviously, be very close to current market prices, provided the curve

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$f(0, s)$ fits current market prices well, whilst the forward prices derived for some future point t (> 0) would be consistent with current prices and the current forward-rate curve. It would also be possible to calculate the price of such stocks as a stepped interest debenture, which is not directly possible from a knowledge of redemption yields alone.

4.1.8 Earlier experimentation by one member of the Working Group (before the changes to the gilt market were announced in 1995) concluded that, if one were to limit one's investigation to the fairly homogeneous market of high-coupon gilts, a reasonably good fit of prices could be obtained by using the formula:

$$f(t, t + s) = b_0(t) + b_1(t).e^{-c_1(t).s}$$

the $b_i(t)$ and $c_1(t)$ being the fitted parameters at time t.

4.1.9 Other formulae, such as polynomials and variants of the above, proved less successful.

4.1.10 It may reasonably be asked why this apparently obvious approach (that of using instantaneous forward rates in place of redemption yields) was not adopted in the 1970s, when the existing approach to the yield indices was developed. It is suggested that a combination of the following might have swayed the decision to stick to redemption yields:

(a) An obvious extension to having an index of one redemption yield, as existed prior to the current indices, was to have several. (This extension was suggested in ¶70 of Haycocks & Plymen, 1964.)
(b) Actuarial thought, at the time, tended to be based on the concept of constant interest rates, and would have been inimical to a varying $f(t, t + s)$. Methods of coping with the problem were based on an initial (fixed) rate of interest and a lower (also fixed) reinvestment rate. (See Gwilt, 1982.)
(c) The use of computers was not as widespread as is now the case, making the thought of fitting a curve to an invisible set of forward rates rather daunting.
(d) At least some of the 'coupon effect' influencing redemption yields was due to the distorting impact of tax on the prices of lower coupon stocks. A pure instantaneous forward rate approach would not have described the whole market.
(e) Indices are largely there to reflect market practices and conventions, and not to lead them. Hence, suggestions at the time, which we believe were, in fact, made, for the indices to involve zero-coupon yields and/or instantaneous forward rates, would have been deemed to be rather premature.

4.1.13 Arguably, the above reservations have now largely disappeared. Today, inexpensive and easy access to powerful computing capabilities is readily available. Furthermore, the combined impact of the recent changes to the taxation of gilts and the increasing use and acceptance of models using instantaneous forward rates of interest, and, indeed, the development of a market in forward

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rates of interest, makes the time right, in the opinion of the Working Group, for the existing redemption-yield models underlying the FTSEAGS Yield Indices to be replaced by a more modern approach.

4.2 *A Forward-Rate Model for the FTSEAGS Yield Indices*

4.2.1 The Working Group investigated a number of possible models for forward rates, building up from that described in ¶4.1.8. These included the models described by Nelson & Siegel (1987) and Svensson (1994), the polynomial model described in the accompanying paper by Chaplin (1998) and the *restricted exponential model* described in the accompanying paper by Cairns (1998). The form of the model settled upon was:

$$f(t, t + s) = b_0(t) + b_1(t)e^{-c_1 s} + b_2(t)e^{-c_2 s} + b_3(t)e^{-c_3 s} + b_4(t)e^{-c_4 s}$$

or, for convenience:

$$f(t, t + s) = b_0 + b_1 e^{-c_1 s} + b_2 e^{-c_2 s} + b_3 e^{-c_3 s} + b_4 e^{-c_4 s}. \qquad (4.1)$$

The forward-rate curve is then used to define the discounting function $P(t, t + s)$, which is applied in the valuation of both income and capital (that is, assuming that the coupon effect no longer exists).

4.2.2 The use of this model is described in detail in Cairns (1998). It can be noted that this model can also be written down in the polynomial form:

$$f(t, t + s) = b_0 + b_1 x + b_2 x^{c_2/c_1} + b_3 x^{c_3/c_1} + b_4 x^{c_4/c_1}$$

where $x = x(s) = \exp(-c_1 s)$. Chaplin (1998) uses the alternative form:

$$f(t, t+s) = \sum_{i=0}^{n} \alpha_i \sum_{j=0}^{i} OP(i, j) T^j$$

where $T = 1 - \exp(-c_1 s)$ (thus mapping the interval $[0, \infty)$ to $[0, 1)$) and the OP are sets of orthonormal polynomial coefficients. For equivalence with the restricted exponential model, it is necessary that $c_k = k.c_1$ for $k = 2$, 3 and 4.

4.2.3 The exponential model described in equation (4.1) contains nine parameters. However, the exponential coefficients c are regarded as constants (hence we use the name *restricted exponential* model) and are not fitted on a daily basis. The values of c are (0.2, 0.4, 0.8, 1.6) (see also ¶4.2.7). This leaves the five linear parameters in b to be estimated on a daily basis. The reasons for this choice of model are as follows:

(a) There is considerable flexibility in the range of shapes which the forward-rate curve can take. An alternative (*unrestricted* exponential model) involved setting b_3 and b_4 to zero and optimising over the five parameters b_0, b_1, b_2, c_1

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and c_2. This alternative had a much more restricted range of shapes (a maximum of one turning point instead of three in the four-exponential curve).

(b) The unrestricted exponential model gives rise to multiple optima and catastrophic jumps (as do the models of Nelson & Siegel (1987) and Svensson (1994)). The restricted exponential model is believed to be practically free of such problems (although this is proved only for a zero-coupon bond market).

(c) Restricted exponential models can evolve within a no-arbitrage framework, in such a way that the forward-rate curve remains within the same family. For example, if the initial forward-rate curve is of the form given in equation (4.1), then the Hull & White (1990) model will retain the same form as it evolves through time (possibly with a one-off addition of a small number of extra exponential terms). (See Bjørk & Christensen, 1997.) This aspect is discussed further in Cairns (1998), Section 5.2.

4.2.4 Five degrees of freedom are proposed on the following grounds:

(a) There were a number of dates between 1992 and 1996 where five parameters provided a significantly better fit than four parameters.

(b) This gives sufficient flexibility for the development of the forward-rate curve in the future without the need to vary the number of parameters on a discretionary or automatic basis (which would result in unnecessary and undesirable catastrophic jumps).

4.2.5 The proposed forward-rate curve is fitted using a Bayesian technique which finds an estimate for b which is very close to the maximum likelihood estimate and an equivalent least squares estimate. The Bayesian method was chosen for two reasons:

(a) It allows us to prevent the forward-rate curve from going negative without the need for manual intervention.

(b) It allows us to make precise probability statements about the sizes of errors in estimated forward rates, spot rates, prices and yields.

4.2.6 Other approaches to estimation include maximum likelihood and weighted least squares (for example, Svensson, 1994; and Chaplin, 1998).

4.2.7 Cairns (1998) discusses which values of c should be used. It is noted that a wide range of possible values can be used without materially affecting the goodness of fit. Certain sets of values are rejected on the grounds that the fitted long or short rates are rather more volatile than is reasonable. Cairns (1998) settles upon $c = (0.2, 0.4, 0.8, 1.6)$, whereas Chaplin (1998) uses $c = (0.23, 0.46, 0.69, 0.92)$ to equally good effect.

4.2.8 With only two parameters ($b_2 = b_3 = b_4 = 0$), the root-mean-squared error (RMSE) is often less than 10 basis points. Moving from two to three parameters ($b_3 = b_4 = 0$) gives a significant improvement, bringing the RMSE down to 5 basis points on average. The value in the addition of one or two

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further parameters is less clear cut: on some dates the improvement is significant; while on others it is not (Cairns, 1998; and Chaplin, 1998). It was found that fitting price data which included bonds with a duration of under one year often required one or more additional parameters.

4.2.9 The aims of the Working Group required that both the model and the parameters should remain stable and avoid catastrophic jumps. This provided the reasoning behind the use of a 5-parameter model: three or four parameters often provide just as good a fit; but on a significant number of dates five parameters were necessary.

4.2.10 The modelling approach is stable, and allows one to calculate any bond's sensitivity to each parameter in the model. In hedging (or trading) bond portfolios driven by cheap/dear model analysis, all the model factors have to be hedged. For example, a 6-parameter model would require six risk factors to be hedged out and at least six bonds in the alternative portfolio. This approach is rarely adopted in practice.

4.3 *Index-Linked Issues*

4.3.1 If we can compute the conventional market coefficients, b_j for fixed c_j in equation (4.1), we can also postulate the existence of a function of exactly the same form, with the same c_j, but different linear coefficients, which describes the instantaneous forward rates of change in the RPI. The formulation of this curve corresponding to equation (4.1) is therefore:

$$\begin{aligned} r(t,t'+s') &= r_0 + r_1 e^{-c_1 s'} + r_2 e^{-c_2 s'} + r_3 e^{-c_3 s'} + r_4 e^{-c_4 s'} \\ &= r_0 + r_1 e^{-c_1(s+\Delta t)} + r_2 e^{-c_2(s+\Delta t)} + r_3 e^{-c_3(s+\Delta t)} + r_4 e^{-c_4(s+\Delta t)} \\ &= r_0' + r_1' e^{-c_1 s} + r_2' e^{-c_2 s} + r_3' e^{-c_3 s} + r_4' e^{-c_4 s} \qquad (r_0' = r_0,\ r_i' = r_i e^{-c_i \Delta t}) \\ &= r(t,t+s) \end{aligned}$$

where, t' represents the time of the last known RPI (effective approximately mid-month) and s' the time from then until future time $t+s$. (Clearly, Δt = length of time from the date of the last published RPI to 'today'.) Cairns (1998), Section 7, goes into more detail on the formulation of this model, and shows, in Section 7.2, that it is possible to fit the prices of index-linked gilts to a satisfactory degree of accuracy using this computed forward RPI function.

4.3.2 By definition, a third function of exactly the same form, with the same c_j, but with a third set of linear coefficients, will represent instantaneous real forward rates of interest:

$$g(t,t+s) = a_0 + a_1 e^{-c_1 s} + a_2 e^{-c_2 s} + a_3 e^{-c_3 s} + a_4 e^{-c_4 s}$$

where:

$$a_i = b_i - r_i' \quad \text{for } i = 0,\ldots,4.$$

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4.3.3 It has been customary to quote redemption yields of index-linked gilts on the assumption of a uniform increase in the RPI over the remaining life of each issue. This yield is sensitive to the assumed rate of increase in the RPI, particularly for the shorter stocks. The calculation proceeds by computing the nominal cash flows at the assumed RPI figure, and then converting the resulting nominal redemption yield to a real yield by eliminating the inflation figure. The dependence of the resulting yield on the RPI assumption is due to the existence of at least one item of cash flow which is fixed in nominal terms — namely the next coupon payment (the next two, if the xd period is sufficiently short). In our proposed presentation there is no need to make an assumption about the future level of the RPI; forward levels of inflation are a by-product of the price fitting process itself. The detailed formulae are given in Cairns (1998, Appendix E).

4.3.4 The taxation basis of index-linked stocks is similar to that of conventional issues, except that net institutional investors pay tax (or earn relief) on the difference between the cost price, adjusted by the proportionate change in the RPI during the holding period, and the sale price. Private investors continue to be exempt from taxes on the gains they make on both conventional and index-linked stocks. One should not, because of these differences between the taxation of conventional and index-linked gilts, treat the implied forward rates of RPI as forecasts or expectations. The very shortest dated index-linked stocks are often held by significant numbers of private individuals, and it can be difficult to fit the prices of these issues with the same RPI function as is used for the remaining longer-dated issues.

4.3.5 With so few index-linked stocks, there is a danger in using too many parameters to describe this function, but Cairns (1998) utilises the published annual change in the RPI to constrain the short end of this function.

5. Applications of the Proposed Yield Curve Model

5.1 *Present Values*

5.1.1 The simplicity of the forward rate representation (equation (4.1)) allows us (after an elementary integration) to write the logarithm of the 'fitted' price of a zero-coupon bond of any maturity, as a linear combination of exponential terms (see Appendix C, ¶C.1.1). It is our contention that, to a high degree of accuracy, in current conditions, the present value of a payment due at any given date is independent of its provenance, whether it be coupon or principal.

5.1.2 Thus, the value of coupon paying bonds can be synthesised from the present value of the individual coupons and redemption proceeds, using the same formula for all components. Such a simple approach would not have provided an accurate representation of the full range of gilt prices prior to the 1995 tax changes, when a model similar to that outlined in Appendix A would have been required. For a strippable bond, we know that, if the bond price differs significantly from the sum of the prices of its components, arbitrageurs will

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exploit any anomaly, either by synthesising or decomposing that bond until prices come into line.

5.1.3 A parallel development for index-linked gilts, when combined with the forward rates computed for conventional gilts, leads to a formula for the instantaneous forward rates of change of the RPI. The implied forward rates, computed in this way, are by no means purely of academic interest. The forward market in RPI, although still illiquid, is already in existence, and an investor can contract, for example, to pay or receive the difference between 6-month LIBOR and the change in RPI over the preceding half year.

5.1.4 Traditional actuarial theory is based on applying a uniform yield basis across the maturity spectrum. Commutation functions are tabulated at a number of different interest rate levels, but the same rate is used throughout each table. The statutory references listed in Section 2.3 all refer to a single fixed rate to be used for each type of calculation.

5.1.5 If the profession is persuaded to move to a market-based approach, as recommended by Exley, Mehta & Smith (1997), accurate values of nominal or real liabilities can easily be obtained by using our proposed formulation, expressed in terms of a continuously varying interest rate and price inflation curves.

5.1.6 Our proposed formulation would form a useful starting point for the interest rate assumptions for the pricing of insurance products such as annuities. Conventional annuities would require the discounting of anticipated future payments using appropriate zero-coupon nominal yields. For RPI annuities, future real payments might be discounted using appropriate zero-coupon real yields. The pricing of LPI annuities is discussed by van Bezooyen, Exley & Smith (1997). Inputs to their suggested formulation include nominal and real market-implied instantaneous forward rates of interest, both of which would be available from our model.

5.1.7 The investment community, too, should find that the formulation being proposed enhances their ability to price (and appraise) the increasing number of non-conventional (in terms of their repayment schedules) new bond issues, which are now becoming a feature of the U.K. corporate bond market.

5.2 *Matching*

5.2.1 If liabilities are to be valued accurately in a manner consistent with market prices, and in conjunction with a portfolio of matching assets (which is easily accomplished with our proposed formulation), it will be important to ensure that, as the yield curve changes its level and/or shape, the value of the assets and liabilities always move together.

5.2.2 It is, of course, possible to attempt to match liabilities by using a portfolio of assets with the same present value, duration, convexity (and higher moments). Tilley & Mueller (1991) (referred to as T&M) find, however, that disappointing results are often realised in practice if this method is adopted. They explain that this is due to the very slow convergence of the Taylor series

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expansion (from $t = 0$) of a shock to the term structure, when expressed in terms of these parameters.

5.2.3 T&M illustrate their suggested method (which is tantamount to expanding this shock from the very long, instead of the very short end) by showing how 50-year annuities certain can be hedged. This requires the synthesis of bonds of longer duration than any which exist in the market. This may require the maintenance of short positions in some securities, and now that there is an established gilt repo market, this is feasible for sterling liabilities.

5.2.4 Appendix B illustrates the results of a similar matching problem for the month of September 1992, during which there was a complete reorientation of the yield curve, following sterling's exit from the European exchange rate mechanism.

5.3 *Forward Rate Agreements (FRAs)*

5.3.1 In our idealised world, where borrowing and lending can be carried out at the same central rates determined by the structure of gilt prices, we can calculate the forward price of any zero-coupon bond. This is readily seen to be the current price of that bond divided by the price of a zero-coupon bond maturing at the required forward date, using the convention that zero-coupon bonds mature with a value of unity. Our parameterised description of the gilt market can, therefore, easily be used to appraise the bid or offer price of any forward rate agreement.

5.3.2 To calculate the fair forward sale price of a coupon-paying bond, the same calculation is performed, but the present value of the coupons due (prior to the forward date) must be subtracted from the price of that bond, before dividing by the price of the zero.

5.4 *Stochastic Simulation*

5.4.1 In the last few years there has been an explosive growth in research into stochastic models of the world's bond markets by both academics and practitioners. The motivation for much of this work has been the growth in the demand for hedging products such as interest rate derivatives and options (or, to use a more general description, contingent claims). Our work has not been directed to an analysis of option theory, although we have relied on many concepts first developed in that field. These developments require a unique set of forward interest rates as their starting point; our formulation is ideally suited to such applications.

5.4.2 The value of a contingent claim can, except when certain technical restrictions do not apply, be shown to be the expected present value of all possible claim values, using a modified probability distribution known as the equivalent martingale measure. The range of possible claim values must be generated from a stochastic model, and a particularly suitable technique is often Monte Carlo simulation. In practice, one of the most important aspects of the pricing process is the calibration of the model to the prices of quoted contracts

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such as caps and collars. We have not attempted to address these issues in this work.

5.4.3 We do, however, believe, just like our colleagues in options analysis, that stochastic simulations should be performed in an arbitrage-free framework. If the arbitrage-free requirement is not imposed, strategies will be found which generate unlimited profits without risk. This undesirable feature must be removed for the analysis to progress. An actuarial approach on how this may be achieved is given in Tilley (1992).

5.4.4 The seminal work by Heath, Jarrow & Morton (1992) (referred to as HJM) demonstrates how the forward-rate curve must evolve in order to remain arbitrage free (in the sense described in ¶3.1.9), and this is discussed briefly in Appendix C. One drawback to the simplest (Gaussian) implementation of HJM is that negative interest rates will occur, just as they do in some of the earlier models, such as Ho & Lee (1986) and Hull & White (1990). Also outlined in Appendix C is some recent work by Flesaker & Hughston (1996), which describes a method of avoiding negative interest rates in the implementation of HJM simulations.

5.4.5 For the pricing of complex contingent claims, which depend on the relative movements of different parts of the forward-rate curve, all possible changes in shape of the curve must be considered, and full implementation of an HJM model will be required. For the evaluation of options on just one bond, however, closed form analytic solutions are available from simpler underlying models — see Hull (1997), Chapter 16. However, as input to these formulae, the forward price of the bond at the maturity date of the option is always required. This is computed most easily from a parameterised description of the initial yield curve.

6. Conclusion

6.1 This paper has covered a fairly wide range of topics in relation to the modelling of the gilt market. The view of the Working Group is that the conventional gilt market can now be characterised, to a high degree of accuracy, by a single discounting function which is applicable to both the income and capital elements of each stock; the influence of private investors, with their favourable tax status, appears to be small. This simple formulation could not have been used with any confidence prior to the May 1995 changes in the taxation basis of the market. The prices of Funding 3½% 99/04 and of Treasury 5½% 08/12 will not, because of their special tax treatment, fit this formulation. There will also be a residual price error for the strippable gilts and benchmark gilts, typically ranging between 1 and 10 basis points of yield, as these appear to command a price premium, and a premium or discount for some issues arising out of special features which they may possess.

6.2 The discounting function, and the underlying forward rates from which it is constructed, can be described by just five parameters. If these five parameters are known on a particular day, the 'fitted' price can be computed for any coupon

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and maturity date. Thus, redemption yields, par yields and zero-coupon yields can be calculated anywhere on the curve.

6.3 The publication of these five parameters would be of assistance to users of the yield indices, although it would be possible to compute them from, for example, fitted zero-coupon yields or (with some additional difficulty) from par yields. There are numerous applications of this formulation, such as mark to market valuations, forward pricing, stochastic simulation, and matching.

6.4 The application of the same discounting factors to income and capital still leads to small redemption yield differences between stocks with the same date, but different coupons, if the yield curve is not flat. This can be thought of as a consequence of different durations rather than as a manifestation of taxation effects. Obviously, if, in the future, it proves impossible to continue to fit all stocks accurately by using the same discounting factors for income and capital, it will be necessary either to use models similar to those described in Sections 3.2 and 3.3 to characterise the structure of the gilt market, or to accept that the gilt indices describe only those stocks which are of interest to gross (or lightly taxed) investors.

6.5 An analogous formulation can be proposed for index-linked gilts, which leads to a characterisation of the forward market in RPI if the conventional market discounting factors (described above) are used. There are some residual difficulties at the short end of the index-linked market, presumably because of the extra appeal which the shortest index-linked stocks have for private high-rate taxpayers.

6.6 We suggest that a change to the new model is justified for the following reasons:

(a) it is consistent with the new tax structure — the current model is not;
(b) it achieves a much better fit to prices on many dates;
(c) although we can prove that catastrophic jumps would not occur if the market consisted only of zero-coupon bonds, we believe that our methods, when applied to coupon-paying bonds, will be practically free of such problems; and
(d) the model facilitates the publication of concise, but nevertheless complete, information on the gilt market, in the *Financial Times*. Actuaries can then do accurate market-value-based liability calculations if they so wish.

6.7 The Working Group believes that the presentation of the FTSEAGS Yield Indices, as set out in Table 6.1, would enhance their usefulness to today's users, whilst simultaneously satisfying many of the points raised in our survey. Of course references in legislation to the current indices would require amending. The numbers in the table were based on gilt closing prices on 31 December 1996. For comparison purposes, the existing FTSEAGS Yield Indices, as published in the *Financial Times*, are presented in Table 6.2 (reproduced with the kind permission of FTSE International Limited). It can be seen that the nominal par yields are not very different from the existing medium-coupon redemption yields.

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Table 6.1. Proposed FTSEAGS yield indices presentation

	Nominal yields					Real yields				
Term (yrs)	5	10	15	20	25	5	10	15	20	25
Par	7.29	7.55	7.63	7.65	7.66	3.65	3.67	3.70	3.72	3.73
Zero	7.33	7.63	7.72	7.75	7.75	3.36	3.53	3.61	3.66	3.69
Forward	7.78	7.96	7.86	7.80	7.77	3.61	3.75	3.80	3.82	3.82
Params	7.61	2.42	–7.44	7.86	–5.19	3.79	–0.46	–0.64	2.27	–3.55
						3.82	2.87	–6.80	5.59	–1.63
						RPI adj.	100.48			

Table 6.2. Current FTSEAGS yield indices presentation

	Conventional gilts				Index-linked gilts		
Coupon band	Term					Infl. 5%	Infl. 10%
	5 yrs	15 yrs	20 yrs	Irred.	Up to 5 yrs	3.17	2.49
Low	7.25	7.64	7.70	7.74	Over 5 yrs	3.53	3.33
Medium	7.31	7.62	7.65				
High	7.38	7.56	7.56				

6.8 The yields quoted in Tables 6.1 and 6.2 are on a semi-annual basis. In particular, it should be noted that we suggest that the zero-coupon yields and forward yields be published on a semi-annual basis, thereby making them consistent and more readily comparable with the par yields and, of course, with the quoted yields on actual gilts. Practitioners will have no difficulty converting any of the yields in Table 6.1 to an annual basis, or to a continuously compounding basis, as dictated by their individual needs.

6.9 Regarding the 'nominal yields' in Table 6.1, 'par' refers to gross redemption yields of gilts standing at par, which is, of course, the same as the running yield, 'zero' refers to yields on zero-coupon bonds, and 'forward' to instantaneous forward rates. The parameters quoted are b_0 to b_4 as per the formulation described in Section 4.2, multiplied by 100. It will be clear from equation (4.1) that the sum of the b_i is the modelled (continuously compounding) instantaneous forward rate on 'dead-short' gilts (not to be confused with bank base rates), whilst b_0, itself, is the market implied 'ultra-long' (continuously compounding) instantaneous forward rate.

6.10 The interpretation of the 'real yields' in Table 6.1 requires further explanation. The par yield on an index-linked gilt of a given term is the coupon rate at which an index-linked gilt of the same term will stand at par taking into account the fitted forward-inflation curve. (Details of the calculation are given in Appendix D.)

6.11 Using continuously compounding yields, the real zero-coupon bond yield is the average effective nominal rate of interest (i.e. the zero-coupon nominal yield) minus the average implied rate of inflation. For maturity in T years, the rate of inflation is averaged over T years (to the nearest day) from 'today', rather than from the date of the last known RPI (which, typically, will be

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four to eight weeks earlier). Similarly, the real forward rate (continuously compounding) is the difference between the nominal forward rate T years forward and the instantaneous forward rate of inflation T years from today. Both rates are then converted to semi-annual rates for publication.

6.12 The first set of parameters quoted is a_0 to a_4 as per the real-yield formulation in ¶4.3.2, multiplied by 100. The second set of parameters quoted is r_0' to r_4' as per the inflation model described in ¶4.3.1, multiplied by 100. The 'RPI adj.' is the model-implied increase in the RPI, expressed as a percentage, from the last published figure (effective approximately mid-month) to the date of the calculation of the indices. Any future increase in the level of the RPI, derived from the inflation model, needs to be multiplied by the 'RPI adj.' to derive the full increase in RPI since its last known value. This adjustment, however, should not be seen as an accurate predictor of the actual increase in RPI.

6.13 The Working Group looks forward to hearing the views of the profession regarding the above proposals.

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APPENDIX A

THE PHILLIPS IDENTITIES

A.1 In Section 3.3 we outlined a class of price surface models which attempts to explain gilt prices P_k in terms of their outstanding lives and coupons. As an illustration, we can use the following formulation for the fitted price $\hat{P}_k$:

$$\hat{P}_k = \sum_{j=1}^{N} g_{kj} V_1\{t_{kj}\} + 100\, V_2\{T_k\} \tag{A.1}$$

where, for stock k, g_{kj} are the N individual coupons due at times t_{kj} and the stock matures at time T_k. A function x is used to transform the maturity axis; in place of t we use $x(t) = e^{-\xi t}$, for a fixed parameter ξ chosen by experience.

A.2 The properties of the discounting functions V lead to a number of identities when they are expressed as polynomials in x:

$$V(x) = \sum_{m=0}^{M} a_m x^m \tag{A.2}$$

with $x = 1$ representing the current date, and $x = 0$ the infinitely far future, we have $V(0) = 0$ and $V(1) = 1$. The choice of M depends on the closeness of fit desired.

From the boundary conditions: $\sum_{m=0}^{M} a_m = 1$ and $a_0 = 0$. (A.3)

A.3 The above constraints are applied to both V functions. In addition, if V_1 is expressed in terms of a force of interest $\delta(t)$, with natural logarithms denoted by ln, we have:

$$\ln V_1(n) = -\int_0^n \delta(t)\, dt. \tag{A.4}$$

Differentiation with respect to n leads to:

$$\frac{1}{V_1(n)} \frac{dV_1(n)}{dx} \frac{dx}{dn} = -\delta(n) \tag{A.5}$$

and hence, because $dx/dn = -\xi x$:

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$$V_1(n)\delta(n) = \xi \sum_{m=1}^{M} m\, a_m x^m \tag{A.6}$$

but when $n = 0$, $x = 1$, so that:

$$\delta(0) = \xi \sum_{m=1}^{M} m\, a_m . \tag{A.7}$$

Letting n tend to infinity in equations A.2 and A.6, we find that $\delta(\infty) = \xi$.

A.4 A further relationship can be developed in terms of the running yields on the irredeemables. The value of a continuous unit annuity is:

$$\int_0^{\infty} V_1(n)dn = A_{\infty}. \tag{A.8}$$

The yield on this annuity is, therefore:

$$\frac{1}{A_{\infty}} = y_{\infty}. \tag{A.9}$$

Using equations A.2 and A.8 and the relationship $dx/dn = -\xi x$, we find that:

$$\xi / y_{\infty} = \sum_{m=1} a_m / m. \tag{A.10}$$

A.5 The parameters a_m for each of the functions V_k ($k = 1, 2$) are constrained by equation (A.3). V_1 is constrained, in addition, by equations (A.7) and (A.10), which affect the short and long maturities, respectively. The fitting process finds the values of these parameters which minimise the percentage price errors, subject to these constraints, and with the weight of each issue inversely proportional to its duration. These weights are selected to reduce the errors in the redemption yields associated with the fitted prices.

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APPENDIX B

MATCHING

B.1 In this appendix we illustrate how the forward rate formulation can be used in conjunction with the matching system devised by Tilley & Mueller (1991) (T&M). The example described uses data at the start of September 1992 and compares the effectiveness of the matching algorithm by looking at the results at the start of October 1992. These dates have been chosen to span the period during which sterling's membership of the ERM was terminated; this was accompanied by a significant reorientation of the yield curve.

B.2 T&M show that, by expanding a shock to the term structure as a Taylor series about $t = 0$, the change in the present value of any series of cash flows can be written as a weighted sum of risk factors defined by the following series of moments:

$$\text{Moment } (k) = \sum_t t^k C(t) P(0,t) \quad (k = 0, 1, 2 \ldots) \tag{B.1}$$

where $C(t)$ is the cash flow at time t and the summation is over all non-zero cash flows. $P(0, t)$ is the appropriate discounting function, being the price of a zero coupon bond maturing at time t. Note that, in order to remain consistent with T&M's notation, we have departed from the convention of Section 4.1, in which t is used to denote a forward date, and s the time *then* remaining to the date of the cash flow under consideration.

B.3 Moment (0) is the present value of the aggregate asset or liability, while higher moments (after normalising by this present value) represent its duration and convexity. If asset and liability cash flows do not correspond precisely, approximate matching is still possible by ensuring that the moments of the assets and liabilities are the same. The problem is to ensure that, whatever happens to the discount factors $P(., t)$, the present values of assets and liabilities should move together. Obviously, if sufficient moments are used a good match will be obtained. Unfortunately, although duration is important, T&M found that very little improvement to the match is obtained by taking the next three moments. They found, however, that very satisfactory results can be obtained with new risk functions defined by:

$$\text{Risk factor } (k) = \sum_t t\tau^{(k-1)} C(t) P(0,t) \quad (k = 1, 2, 3 \ldots) \tag{B.2}$$

where:

$$\tau = t / (t + t^*) \tag{B.3}$$

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and the present value/cash flow functions are the same as in equation (B.1). A value of 5 years for t^* provides a satisfactory scale factor, as 5 years is mapped on to 0.5 in the new space. The duration can be calculated from either equation (B.1) or (B.2) by setting $k = 1$.

B.4 This transformation maps $t = [0, \infty)$ on to $t = [0, 1)$, and has the property that the higher order risk factors calculated from equation (B.2) give a much improved balance of weights than in the untransformed power series of moments; the resulting expansion converges much more rapidly. For undated gilts, the sum of the infinite series in equation (B.2) cannot be evaluated as easily using the $t/(t+t^*)$ transformation than if an exponential formula is used instead. Accordingly, we recommend an alternative transformation:

$$\tau = 1 - \exp(-q\,t) \tag{B.4}$$

where q is chosen so that, at $t = 5$, this function, like T&M's, takes the value 0.5, i.e. $q = 0.1386$, and again $t = [0, \infty)$ maps on to $[0, 1)$. This function is raised to the power of $k - 1$ in the definition of the kth risk factor.

B.5 The values of $P(0, t)$ can be found from the published model, as proposed by the Working Group. For matching problems, the moments defined in equation (B.2) should be calculated for all the gilts and for the required liability. Linear programming methods can then be used to select a matching gilt portfolio, which may need to include short positions for some liability profiles.

B.6 A unit 50-year level annuity (with 100 semi-annual payments from 1 March 1993 to 1 September 2042) was chosen for the matching target liability. Its present value and higher moments were computed from the discounting factors $P(0, t)$ in exactly the same way as were the asset moments. Four stocks were selected arbitrarily, but with the following attributes: prices close to par; a good spread of maturity dates; and with market prices fairly close to their fitted values. Portfolio weights were computed such that the duration and convexity of the assets and liabilities were the same. The exercise was repeated using the T&M risk factor corresponding to the second order term. Portfolio optimisation was not attempted. The results are shown in Tables B.1 and B.2, where negative weights correspond to short positions.

B.7 Using the conventional risk factor approach, it can be seen, from Table B.1, that, during September 1992, the market value of the matching portfolio fell by 4.2% (10.82 to 10.37), although the poor fit of the 8.75% 2017 implied a 5.6% decline (10.78 to 10.18) in the present value of the cash flows emanating from this portfolio. The present value of the 50-year annuity, however, fell by only 2.5% (10.78 to 10.51); the suggested fund did not, therefore, lead to a particularly satisfactory match. Utilising the third order term produced a bizarre set of weights (both positive and negative values of a much higher order of magnitude).

B.8 Use of the T&M risk factors (Table B.2) resulted in a more balanced set of weights. The market value of the recommended portfolio fell by 1.5% (10.82

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Table B.1. Using conventional risk factors

1 September 1992						Conventional risk factors			Factors weighted by nominal		
Nominal Stock				Market price (incl accrd)	Present value (from fitted model)	Duration x PV	Second order term	Market value (incl accrd)	Present value (from fitted model)	Duration x PV	Second order term
3.20	Treasury	8.50%	1994	98.67	98.64	134.2	187	3.15	3.15	4.29	–5.99
–9.98	Treasury	9.75%	2002	100.79	100.79	669.1	5,663	–10.06	–10.06	–66.80	–565.47
1.24	Conversion	9.00%	2011	98.66	98.43	919.3	12,831	1.22	1.22	11.37	158.66
17.25	Treasury	8.75%	2017	95.79	95.46	1,002.4	16,818	16.53	16.47	172.94	2,901.47
	Parameters of fund and of 'matching' annuity							10.82	10.78	121.79	2,500.65
	1 October 1992										
3.20	Treasury	8.50%	1994	101.87	101.94			3.26	3.26	Present value of annuity	
–9.98	Treasury	9.75%	2002	105.99	105.91			–10.58	–10.57	recomputed with revised	
1.24	Conversion	9.00%	2011	99.12	99.18			1.23	1.23	model coefficients	
17.25	Treasury	8.75%	2017	95.44	94.31			16.47	16.27		
	Fund parameters							10.37	10.18		10.51

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Table B.2. Using Tilley & Mueller risk factors

1 September 1992						Tilley & Mueller factors			Factors weighted by nominal		
Nominal Stock				Market price (incl accrd)	Present value (from fitted model)	Duration x PV	Second order term	Market value (incl accrd)	Present value (from fitted model)	Duration x PV	Second order term
–0.49	Treasury	8.50%	1994	98.67	98.64	134.2	23.6	–0.49	–0.49	–0.66	–0.12
–2.41	Treasury	9.75%	2002	100.79	100.79	669.1	447.7	–2.43	–2.43	–16.12	–10.79
4.55	Conversion	9.00%	2011	98.66	98.43	919.3	736.5	4.49	4.48	41.85	33.53
9.65	Treasury	8.75%	2017	95.79	95.46	1,002.4	833.0	9.24	9.21	96.73	80.38
	Parameters of fund and of 'matching' annuity							10.82	10.78	121.79	103.01
	1 October 1992										
–0.49	Treasury	8.50%	1994	101.87	101.94			–0.50	–0.50	Present value of annuity	
–2.41	Treasury	9.75%	2002	105.99	105.91			–2.55	–2.55	recomputed with revised	
4.55	Conversion	9.00%	2011	99.12	99.18			4.51	4.52	model coefficients	
9.65	Treasury	8.75%	2017	95.44	94.31			9.21	9.10		
	Fund parameters							10.66	10.56		10.51

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to 10.66), although its present value declined by 2% (10.78 to 10.56). This was, fortuitously, a little less than the fall in the present value of the 50-year annuity, calculated above at 2.5%, clearly a more satisfactory match than in the previous example. In other circumstances, the portfolio in the first illustration resulting from equating convexity and higher order terms *could* result in larger profits or smaller losses, but only because the matching would be less satisfactory than that of the T&M portfolio. Of course, only three different issues are required to equate the present values and two higher moments of a cash flow stream. If, for example, the 2011 issue is excluded, the holding of 8.75% 2017 is increased to 13.1 from 9.65, but the broad results remain unaltered.

B.9 It may seem surprising that it is possible to match liabilities which are longer than any assets available in the market. This depends, to some extent, on the fact that forward rates change only slowly in the distant future; a property which suggested the exponential form of the functions chosen. The inclusion of undated issues in the fitting process helps to ensure the integrity of the very long end of the curve.

B.10 The computations described can easily be performed using Excel or a similar system on a personal computer, although the use of macros is recommended for the evaluation of the risk factors. Very frequent rebalancing of the matching portfolio should not be necessary if the T&M method is utilised.

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APPENDIX C

STOCHASTIC PROJECTION OF FORWARD INTEREST RATES

BY K. S. FELDMAN

ABSTRACT

The general Heath, Jarrow & Morton formulation for the arbitrage-free evolution of forward rates is described in Section C.1. Two alternative special cases are discussed: (a) in which the volatility functions remain fixed; and (b) where the functions alter in a manner which eliminates the possibility of negative spot and forward interest rates. The method of Principal Components is used to quantify the historical values of the functions in (a) for the conventional gilt market and, in Section C.2, for the index-linked market. The method of computing the functions required for method (b) is outlined. Section C.3 contains a glossary of the terms used in the theory of contingent claims.

C.1 *Conventional Issues — Forward Pricing and Simulation*

C.1.1 As is demonstrated in Cairns (1998, ¶3.1.11), a straightforward integration of the instantaneous forward-rate function $f(0,T)$ gives a formula for the (fitted) price $P(0,T)$ today ($t = 0$), of a unit redemption zero-coupon bond maturing at time T:

$$f(0,T) = b_0 + \sum_{j=1}^{4} b_j \exp(-c_j T) \tag{C.1.1a}$$

$$P(0,T) = \exp\left\{-\left[b_0 T + \sum_{j=1}^{4} b_j (1 - \exp(-c_j T))/c_j\right]\right\}. \tag{C.1.1b}$$

where $c_1 = 0.2$, $c_2 = 0.4$, $c_3 = 0.8$, $c_4 = 1.6$.

C.1.2 In this section, unless otherwise stated, we shall use the term 'bond' to imply one with a zero coupon. The time T_1 forward price, today, of a bond with a maturity date T_2 is, quite simply:

$$\tilde{P}(T_1, T_2) = P(0, T_2)/P(0, T_1) \tag{C.1.2}$$

which should be distinguished from $P(T_1, T_2)$, the realised price of the T_2 maturity bond at time T_1. One can, of course, contract, today, to buy or sell the T_2 maturity bond at the forward date T_1. Use of the fitted yield curve will enable market participants to evaluate, very easily, whether the forward price bid or offered is fair, that is, close to $\tilde{P}$. Obviously market rates will vary, by reasonably stable spreads, from the theoretical results.

C.1.3 Equation (C.1.2) is derived by noting that a unit deposit today will

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amount to $1/P(0, T_1)$ at time T_1. The forward price of a coupon paying bond can be found by using equation (C.1.2) repeatedly, adding in a factor $g\tilde{P}(T_1, T_n)$ for each coupon payment g due at time T_n, and still cum dividend at time $T_1 < T_n$. It can be seen that, provided forward rates are given at 5 dates (in a 5-parameter model), the structure is described uniquely, and all the forward rates can be derived. The uniqueness of the solution is by virtue of equation (C.1.1a) being linear in the coefficients b_j.

C.1.4 A forward rate of interest, in continuously compounded form, can be calculated from the forward price (expressed as a fraction of the maturity value) of a zero-coupon bond, by taking its natural logarithm and dividing by $(T_1 - T_2)$. The familiar semi-annual rate is $2\{\exp(r/2) - 1\}$, where r is the continuously compounded rate. Forward rates should not necessarily be treated as central forecasts, as even the simplest stochastic models lead us to conclude that the expected short rate should differ from its implied forward rate. These rates can be traded at prices which are a consequence of arbitrage considerations — if the prices were significantly different from those implied by this analysis, an arbitrage profit could be made. Holding bonds can be risky (not all funds are matched), and investors often require a risk premium — which varies in time, and can cause additional differences between yield expectations and forward yields.

C.1.5 Generalising the relationship between equations (C.1.1) to time t, rather than just to time 0, we can write:

$$f(t, T) = -\partial \ln P(t, T)/\partial T. \qquad \text{(C.1.3)}$$

C.1.6 Both functions f and P in this equation are stochastic variables; they are unknown at time $t > 0$, but we would hope to be able to model their behaviour by a relationship such as:

$$df(t, T) = \alpha(t, T)\,dt + \sigma(t, T)\,dW(t) \qquad \text{(C.1.4)}$$

that is, the instantaneous change in the function f depends on a trend term α and a volatility function σ (see Hull, Chapter 17). The term $dW(t)$ is a random variable whose magnitude, in practical simulations, will be obtained by multiplying a unit normal variate by the square root of the selected time interval dt.

C.1.7 In the Ho & Lee (1986) model, $\sigma(t, T)$ is assumed to be constant; in the Hull & White (1990) formulation it is represented by $\sigma \exp\{-k(T - t)\}$, where k depends on the rate of mean reversion. In both cases, future values of both f and P for all maturities can be expressed analytically in terms of the initial forward curve $f(0, T)$, the parameters σ and k, and the integral of the random variates (in practice, the sum of a set of random numbers). Analytical solutions to a variety of contingent claims problems can then be obtained as modifications to the Black & Scholes formula.

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C.1.8 Unfortunately, these simple models do not permit the shape of the forward rate curves to assume the degrees of variation seen in practice. The Ho & Lee model, for example, allows only parallel shifts. The Hull & White formula admits larger changes at the shorter end, but rotations of the curve cannot arise. In order to accommodate the behaviour of observed yield curves, the stochastic term in equation (C.1.4) must be replaced by the sum of such terms, with two or more σ functions each multiplied by the same number of independent random variates dW.

C.1.9 Mathematically this generalisation is trivial; instead of σdW being comprised of scalar quantities, it can be considered to be the inner product of two vectors. In practice, however, the extension from a single to two or three dimensions can cause an explosive expansion of the run times of some computer applications. It may be helpful to think of the various dimensions as representing the degrees of freedom of the yield curve: level; slope; and curvature, respectively.

C.1.10 Also, in order for the evolution of the forward rates f to remain arbitrage free, Heath, Jarrow & Morton (1992) (HJM) have shown, even in the most general case when the σ are stochastic, that the drift term α must satisfy an equation:

$$\alpha(t, T) = \sigma(t, T)\{\Omega(t, T) - \phi(t)\} \qquad \text{(C.1.5)}$$

where:

$$\Omega(t,T) = \int_t^T \sigma(t,v)\,dv \qquad \text{(C.1.6)}$$

represents the instantaneous volatility of a bond maturing at time T; $\phi(t)$ is an arbitrary function which is connected with the price of risk, mentioned in ¶C.1.4. In equation (C.1.5) it must be remembered that, in general, the quantities are vectors, and the product is an inner product. In practice, this means that each column in the σ and Ω vectors are handled separately, and the various products are then added together to create the scalar drift term α.

C.1.11 HJM showed that the valuation of contingent claims can proceed without reference to the function ϕ, just as equity options can be valued without reference to the trend (or drift) μ in the equity price, which cancels in the usual Black & Scholes derivation. Thus for options calculations, ϕ can be chosen to suit the application. For *risk-neutral* valuations (see Glossary, Appendix C.3), which may not always be the fastest way to proceed, ϕ is set to zero. The resulting bond prices will then be expressed as multiples of an accumulating unit bank deposit. Using the *terminal measure*, ϕ is set equal to $\Omega(t, T')$, where T' is no earlier than the maturity date of the longest bond required in that application. The corresponding *numeraire* then becomes the T'-maturity bond, with all other bond prices calculated as multiples of the price of that bond.

C.1.12 If we have known and deterministic σ functions, α and Ω can be

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computed by integration (numerically), and equation (C.1.4) is then applied successively, in order to obtain the new forward rate curve at a subset of points on the yield curve. This process must be repeated for each of a large number of scenarios. In each scenario the natural logarithms of the bond prices are computed by integrating the instantaneous forward rates, f ; bond prices are then found by exponentiation. It should be noted that the yield curves which result from the evolution described cannot, in general for $t > 0$, be described *exactly* by any set of the b_j of equation (C.1.1), although a very close fit is anticipated, but see ¶4.2.3(c).

C.1.13 We have computed the volatility structure $\sigma(\tau)$ from the yield curve history. Note that we now use τ to denote the time to maturity $(T - t)$, because we hope to find that a reasonable fit to the successive changes in the forward rates can be obtained without demanding that the volatility structure itself evolve over time. This has been accomplished using the method of principal components (eigenvectors), which attempts to approximate the observed changes in terms of a linear combination of a small number of independent factors. Changes in the $f(t, T)$ between observations (i.e. successive values of t) for a finite range of maturities, τ (12 values between zero and infinity, inclusive), were used. The method utilises the variance-covariance matrix derived from these observations, and is described in many standard statistical textbooks, for example Lawley & Maxwell (1963).

C.1.14 We are attempting to describe how movements in the fs are connected across the yield curve, in the general case where the simple models mentioned in ¶C.1.7 are inadequate. The results, in Tables C.1.1 and C.1.2, represent the analysis derived from monthly movements between January 1992 and November 1996 in the conventional market. These vectors are arranged to be orthogonal ($\Sigma x(i)\,y(i)$ is zero, where $x \neq y$ represent any two of the three eigenvectors computed), so that we can think of the processes as independent. The eigenvalues, or factor scores, are independent unit variates which we assume are distributed normally. Apart from the existence of a 5 standard deviation observation following the quite exceptional circumstances of sterling's exit from the ERM in September 1992, we found no reason to reject the hypothesis that the variates are normally distributed. Autocorrelations of each set of eigenvalues with

Table C.1.1. Eigenvectors $\sigma\,(i, \tau)$ for instantaneous forward rate projections

Term — τ	0	1	1.5	2.25	3.38	5.06	7.59
First eigenvector	0.215%	0.453%	0.412%	0.340%	0.264%	0.185%	0.065%
Second	–0.072%	0.141%	0.206%	0.258%	0.277%	0.269%	0.262%
Third	0.425%	–0.017%	0.003%	0.029%	0.003%	–0.076%	–0.113%
Term — τ	11.39	17.09	25.63	38.44	Undated		
First eigenvector	–0.093%	–0.221%	–0.278%	–0.290%	–0.291%		
Second	0.278%	0.300%	0.312%	0.314%	0.314%		
Third	–0.054%	0.033%	0.077%	0.087%	0.088%		

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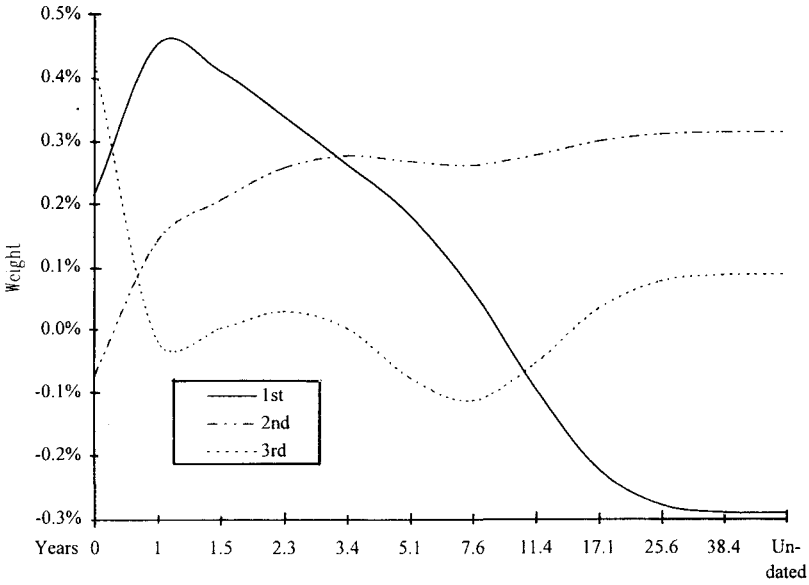


Figure C.1. Eigenvectors for instantaneous nominal forward yield projections

their one-month lagged values (for both the conventional and index-linked markets) are given in Tables C.2.3 and C.2.4.

C.1.15 The fitted monthly change in the forward rate at each term shown is (apart from the drift) obtained by calculating the inner product of the appropriate lines in the two tables $\{\Sigma \sigma(i, \tau) dW(i,t)$, where the summation is for $i = 1, 3\}$. Thus, using just the first three eigenvectors, and ignoring the drift, we find that the approximate change in the instantaneous forward rate at $\tau = 11.39$ years during September 1992 was $(-0.093\% \times -5.42 + 0.278\% \times 2.88 - 0.054\% \times -1.42) = 1.393\%$. This can be compared with the actual change of 1.336%, as measured by the fitted curves at this duration. At terms other than those shown, interpolation methods (probably best based on curve fitting) can be used.

C.1.16 For simulation work we would not expect a repeat of the historical drift terms, and drift functions appropriate to the application would be chosen to produce an arbitrage-free evolution, as described in ¶¶C.1.10 and C.1.11. This is sufficient for the calculation of option prices or for quantifying the magnitude of the variability (but not the mean) of portfolio values in actuarial simulations.

C.1.17 Because the eigenvectors are, by definition, orthogonal, rotations of these vectors can provide equally valid descriptions of the yield curve movements. Experiments using different observation dates can give apparently very different eigenvectors. Indeed, changing the coefficients c_j in the power series in equation (C.1.1) resulted in very different eigenvectors. Figures C.3 and

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Table C.1.2. Eigenvalues $dW(i, t)$: factor scores quantifying changes in instantaneous nominal forward rates during month shown; these have a mean of 0, and variance 1

t	i = 1st	2nd	3rd	i = 1st	2nd	3rd	i = 1st	2nd	3rd
		1992			1993			1994	
Jan	–0.47	–1.04	–0.95	–1.09	0.12	–1.49	0.15	0.22	–0.05
Feb	–0.58	0.13	0.48	0.14	–1.52	–1.25	0.60	2.27	–0.18
Mar	1.80	0.49	–1.97	0.46	–0.23	–0.11	0.81	2.51	0.55
Apr	–1.81	–0.84	2.54	1.17	0.45	–0.47	0.87	1.07	–0.61
May	–0.83	0.02	1.59	–0.21	0.19	0.20	0.71	1.72	–0.66
Jun	0.40	0.31	–1.37	0.15	–1.97	0.02	0.57	–0.58	–0.73
Jul	0.91	–0.08	0.46	–0.30	–1.04	0.74	0.12	–0.60	2.50
Aug	0.87	0.26	0.29	0.19	–1.87	0.25	–0.05	0.20	–1.09
Sep	–5.42	2.88	–1.42	0.34	–0.12	–1.13	0.60	0.58	0.03
Oct	–2.35	–1.47	–2.33	–0.16	–0.59	–0.93	–0.15	–0.08	1.27
Nov	1.04	0.17	–0.95	0.44	–1.40	–1.30	–0.85	–0.09	1.36
Dec	–0.24	–0.98	–1.46	–0.73	–1.19	0.48	0.70	0.21	–0.82
		1995			1996				
Jan	–0.26	–0.34	0.43	–0.41	0.81	–0.16			
Feb	0.02	0.15	0.17	0.30	1.80	–0.76			
Mar	–0.46	–0.01	–0.53	0.75	0.11	–0.99			
Apr	0.01	–0.55	0.93	–0.10	–0.53	–0.53			
May	–1.10	–1.18	–0.39	0.12	0.31	1.49			
Jun	0.40	1.70	0.96	–0.67	–0.25	–0.88			
Jul	–0.79	–0.32	–0.01	0.02	–0.09	0.80			
Aug	–0.35	–0.45	0.50	–0.32	0.16	0.22			
Sep	–0.42	0.63	1.07	0.08	–0.86	0.17			
Oct	–0.31	–0.48	–0.70	0.70	–0.52	–0.36			
Nov	–0.57	–1.35	–0.65						
Dec	–0.12	–0.32	–0.13						

C.5 contrast the effectiveness of the approximation of using just the first three eigenvectors to characterise the changes in forward rates. Note, however, the different vertical axes on these charts. Using all the monthly changes, the first eigenvector accounted for 44.7% of the variance in observed rates, the first two accounted for 83.1% and the first three for 93.8%. Eigenvalue autocorrelation coefficients are given in Table C.2.3.

C.1.18 One consequence of the form of equation (C.1.4) is that there is a non-zero probability that the function f will become negative for certain values of t and T. Whilst this is occasionally acceptable for real rates of interest when inflation is high, in the U.K. this should, many would argue, be avoided for nominal rates. For equity simulations, it is usual to work with logarithms of prices (which avoids this problem), but this is just not possible with interest rates. If logarithms are used, the problem manifests itself at the other extreme; the interest rates can explode, leading to zero bond prices. Direct use of logarithms is not, therefore, available as a solution to these difficulties.

C.1.19 It is sometimes claimed that, for option pricing, for example, negative

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rates can be ignored. However, a method of avoiding negative or exploding interest rates has been published by Hughston (1994) and Flesaker & Hughston (1996), using transformations of the σ and Ω structures referred to above. This work and some recent advances are included in Hughston (1996). Hughston prefers to work with bond prices rather than forward rates, so that the bond price volatility Ω is more fundamental to the methodology, with σ obtained by differentiation.

C.1.20 In this positive interest rate environment it is shown that the realised price, at time t, of a bond maturing at time b, can be written as an expression involving positive martingales M, together with the initial price and forward interest rate structure:

$$P(t,b) = \left\{ P(0,T) - \int_b^T M(t,s)\,\partial_s P(0,s)ds \right\} \Big/ \left\{ P(0,T) - \int_t^T M(t,s)\,\partial_s P(0,s)ds \right\} \quad \text{(C.1.7)}$$

where $M(0, s) = 1$ for all maturity dates s, ∂_s denotes the partial derivative with respect to s, and $M(t, s)$ tends to 0 as s tends to ∞ for all t. We note, from equation (C.1.3), that $\partial_s P(t, s) = - P(t, s)\, f(t, s)$ for any t. The choice of the reference bond with maturity date T is arbitrary, but, of course, $t \le b \le T$. The M are scalar, but stochastic, and for some applications it may be helpful to set $T = \infty$, which has the effect of simplifying equation (C.1.7), since $P(0, \infty) = 0$. Some experimentation with equation (C.1.7) will show that $P(t, b)$ always takes values between 0 (for an infinite maturity date b) and unity, at redemption when $t = b$, and is a decreasing function of b.

C.1.21 Differentiation of equation (C.1.7) with respect to b (that is, along the yield curve at duration b for a particular observation time t), making use of equation (C.1.7) with b set to T, we find that the *realised* values of $M(t, b)$ satisfy:

$$\partial_b P(t, b) / P(t, T) = M(t, b)\, \partial_b P(0, b) / P(0, T) \quad \text{(C.1.8)}$$

or, if T in equation (C.1.7) is set to ∞, we find that:

$$M(t, b) = K(t)\, \partial_b P(t, b) / \partial_b P(0, b) \quad \text{(C.1.9)}$$

where $K(t)$ represents an indeterminate function, since it will cancel from both the numerator and denominator of equation (C.1.7) when $T = \infty$.

C.1.22 All the P and f values can be computed from the parameterised history of the gilt market, and equation (C.1.8) (or (C.1.9) with K set to unity) can be used to calculate the Ms at a number of maturity dates $(b_i + t)$ for the observation dates t. Hughston recommends that the M be expressed as exponential martingales (analogous to equity prices in equity option analysis when they are expressed relative to a money market numeraire):

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$$M(t,s) = \exp\left\{\int_0^t \omega(u,s)dW^T(u) - \tfrac{1}{2}\int_0^t \omega(u,s)^2\,du\right\}. \qquad \text{(C.1.10)}$$

Thus logarithms of the changes in the Ms from one observation to the next can then be used in a principal components analysis, as in ¶C.1.13. The eigenvectors provide the ω, the eigenvalues the dW^T (in equation (C.1.10)).

C.1.23 We will generally expect to be able to write $\omega(u, s) = \omega(s - u)$, since fluctuations will depend on the outstanding term to maturity $s - u$. The dW^T are vectors, each element of which is a unit variate; the T reminds us that the formulation is dependent on the choice of the terminal bond. Stochastic simulation then proceeds by applying unit random variates to the ω in order to generate new Ms by integration, prices, in turn, being generated by equation (C.1.7). It will be observed that all the elements of equation (C.1.7), apart from the M, are known from the initial yield curve.

C.2 *Index-Linked Gilts*

C.2.1 As is described in Cairns (1998, Section 7) and in ¶¶4.3.1 and 4.3.2, the fitting process proceeds by assuming the existence of a function which reflects the instantaneous forward rate of change in the RPI, and then combining this function with that already calculated for instantaneous nominal forward rates.

C.2.2 In the fitting process, index-linked prices are first adjusted by subtracting the present value (using the nominal factors already computed for the conventional market) of the first (or first two) coupon payments due; at least one payment is always known with certainty. Prices and cash flows can then be normalised by dividing by a factor which represents the increase in the RPI from the base month to the date to which the most recently published RPI figure refers.

C.2.3 Although the principles are quite straightforward, in practice adjustments must be made for the eight-month lag between the month to which the index refers and the coupon to which it is applied. There is also a delay of up to about six weeks between the data collection date and the publication date the following month. Full details of the analysis are given in Cairns (1998, Appendix E).

Table C.2.1 Eigenvectors $\sigma(i, \tau)$ for instantaneous forward real rate projections

Term — τ	0	1	1.5	2.25	3.38	5.06	7.59
First eigenvector	0.723%	0.203%	0.131%	0.088%	0.069%	0.060%	0.051%
Second	–0.225%	0.251%	0.322%	0.335%	0.273%	0.162%	0.069%
Third	–0.007%	–0.055%	0.011%	0.042%	–0.020%	–0.135%	–0.158%

Term — τ	11.39	17.09	25.63	38.44	Undated
First eigenvector	0.044%	0.040%	0.038%	0.038%	0.038%
Second	0.035%	0.038%	0.044%	0.046%	0.046%
Third	–0.030%	0.126%	0.204%	0.221%	0.222%

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Table C.2.2. Eigenvalues $dW(i, t)$: factor scores quantifying changes in the real instantaneous forward rates during month shown

t	i = 1st	2nd	3rd	i = 1st	2nd	3rd	i = 1st	2nd	3rd
		1992			1993			1994	
Jan	–0.40	–1.04	–1.07	–1.02	–1.22	0.09	0.55	–0.14	0.81
Feb	–0.69	–0.51	0.90	–0.34	–1.00	0.03	0.49	0.96	0.16
Mar	–0.67	0.90	–2.44	0.73	–0.28	–0.36	0.95	0.93	0.24
Apr	2.74	–0.69	1.06	–0.32	1.75	0.26	–0.42	1.79	1.02
May	0.54	0.28	2.47	1.29	–0.64	–0.61	0.88	1.01	–0.50
Jun	–0.99	1.16	0.55	0.48	–0.57	–1.26	–0.60	0.16	–1.08
Jul	0.71	1.30	1.02	0.72	–1.34	–0.14	1.25	0.22	0.26
Aug	0.45	0.67	–0.98	–0.16	–0.52	–1.20	–1.89	0.12	0.12
Sep	–2.58	–0.64	4.40	–0.56	0.48	–0.57	0.34	0.44	0.33
Oct	–1.71	–4.70	0.15	–0.52	–0.49	0.04	0.81	–0.23	0.02
Nov	–0.52	2.07	1.34	–0.84	–0.08	–0.94	0.28	–0.04	1.42
Dec	–0.95	0.58	–2.33	0.96	–1.68	0.08	–0.32	0.58	–0.69
		1995			1996				
Jan	–0.38	–0.03	–0.54	0.26	0.08	0.61			
Feb	–0.40	0.35	0.40	0.39	0.66	0.04			
Mar	–0.62	0.60	0.08	–0.64	0.42	–0.16			
Apr	–0.04	–0.36	0.14	–0.99	0.20	–0.62			
May	0.15	–1.60	0.15	1.97	–0.14	0.78			
Jun	1.37	–0.04	0.07	–0.72	0.33	0.12			
Jul	–0.35	–0.46	0.56	0.84	–1.02	0.10			
Aug	–0.50	–0.12	–0.35	–1.14	0.06	0.49			
Sep	1.97	–0.36	0.70	0.01	–0.24	–0.30			
Oct	0.14	0.33	–0.10	0.02	–0.24	–0.39			
Nov	–2.20	0.57	0.09						
Dec	–1.38	0.60	–0.37						

C.2.4 Analysis of real yields follows exactly the same development as for nominal yields, as described in Section C.1. Tables C.2.1 and C.2.2 present the results of our principal component analysis for the instantaneous forward real rates. The shape of the eigenvector functions is consistent with the observation made in Exley, Mehta & Smith (1997), ¶7.2.24, that there is very little curvature at the long end of the volatility curves. We found that the first eigenvector accounted for 43.3% of the variance in observed rates, the first two accounted for 75% and the first three, 89.7%.

C.2.5 Tables C.2.3 and C.2.4 give the correlation coefficients between each set of the above eigenvalues and these values lagged by one month. The data do not, therefore, exhibit any significant levels of autocorrelation.

Table C.2.3. Nominal rates

	1st lagged	2nd lagged	3rd lagged
1st	0.08	0.18	0.28
2nd	–0.05	0.20	–0.21
3rd	–0.03	0.15	0.00

Table C.2.4. Real rates

	1st lagged	2nd lagged	3rd lagged
1st	–0.01	0.25	0.13
2nd	0.05	0.04	–0.16
3rd	–0.25	–0.04	–0.16

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Table C.2.5. Correlation coefficients between eigenvalues of nominal and real forward rates

		Nominal forward rates		
	Component	1st	2nd	3rd
Real	1st	0.25	0.09	0.73
forward	2nd	0.57	0.45	0.01
rates	3rd	–0.57	0.42	0.26

C.2.6 There are, as one would expect, significant correlations between the eigenvalues which characterise movements in the conventional and index-linked markets. It is possible to ensure that the random variates selected for simulation runs exhibit these correlations by a Cholesky decomposition of the correlation matrix, see Wilkie (1995) Appendix B.4.

C.2.7 Figures C.1 to C.5 illustrate data relevant to nominal interest rates, whereas the remaining figures give the corresponding data for real interest rates. Figure C.9 contrasts implied forward RPI at the start of September 1992 with that at the start of October 1992.

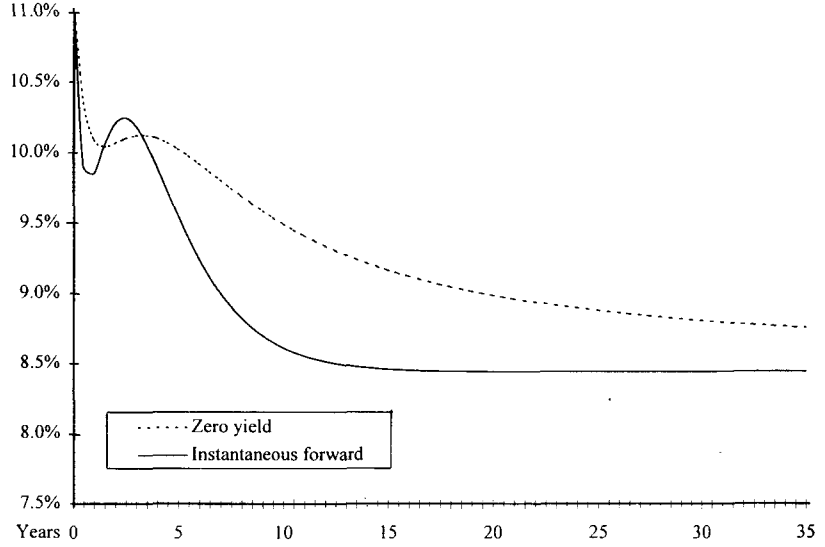


Figure C.2. Fitted nominal yield curve, 1 Sep 1992 (semi-annual convention)

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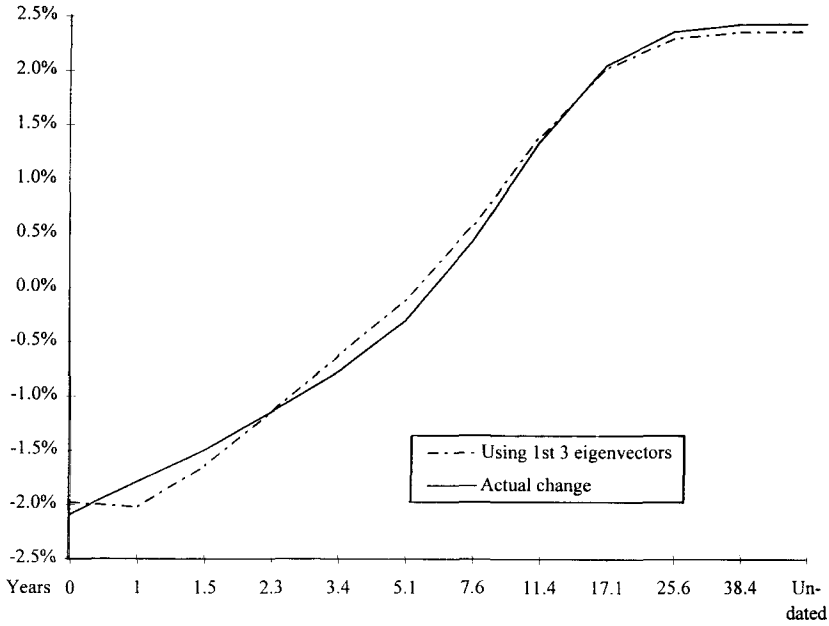


Figure C.3. Change in instantaneous nominal forward yields, Sep-Oct 1992

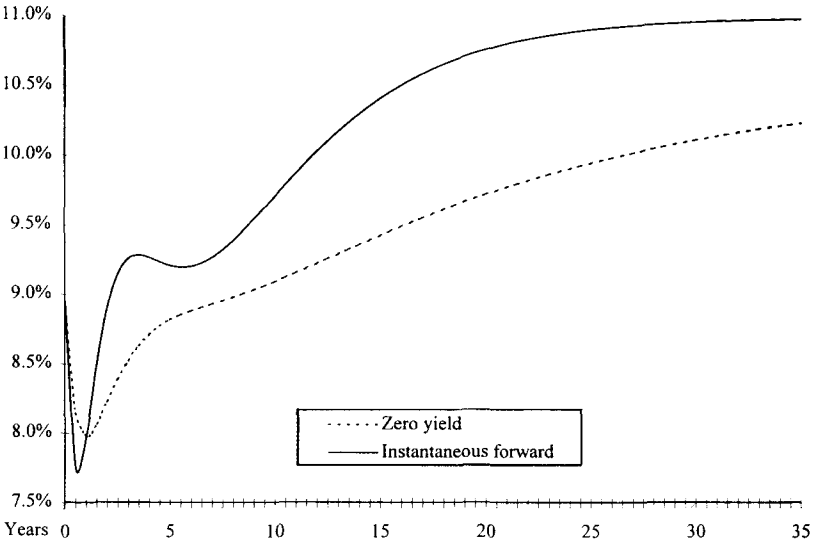


Figure C.4. Fitted nominal yield curve, 1 Oct 1992 (semi-annual convention)

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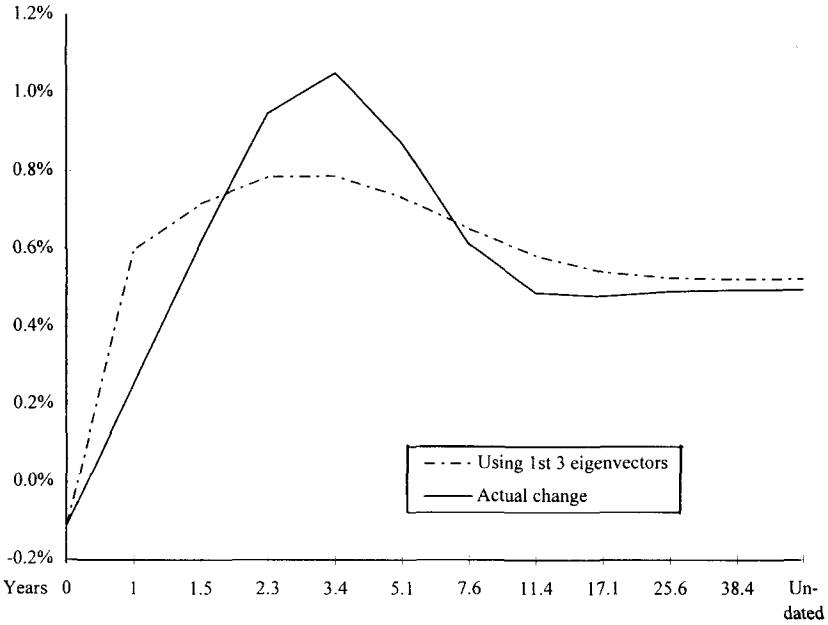


Figure C.5. Change in instantaneous nominal forward yields, Feb-Mar 1994

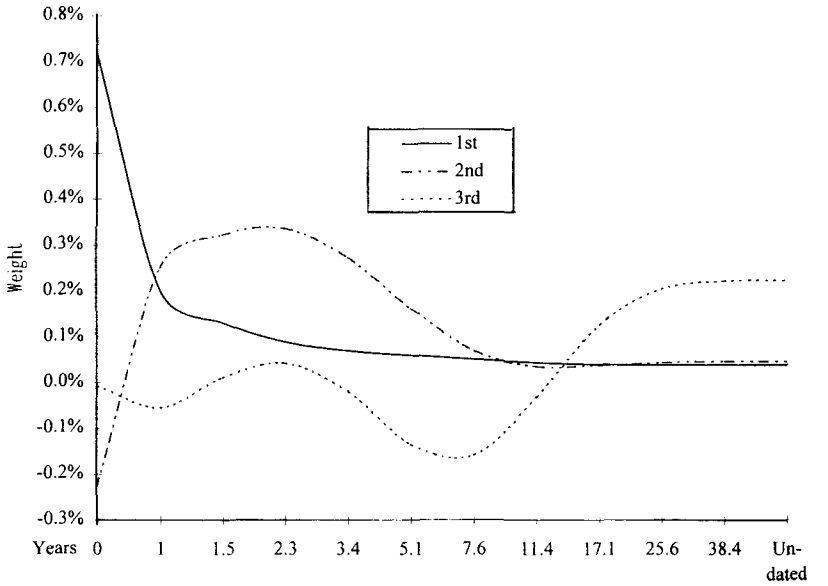


Figure C.6. Eigenvectors for instantaneous real forward yield projections

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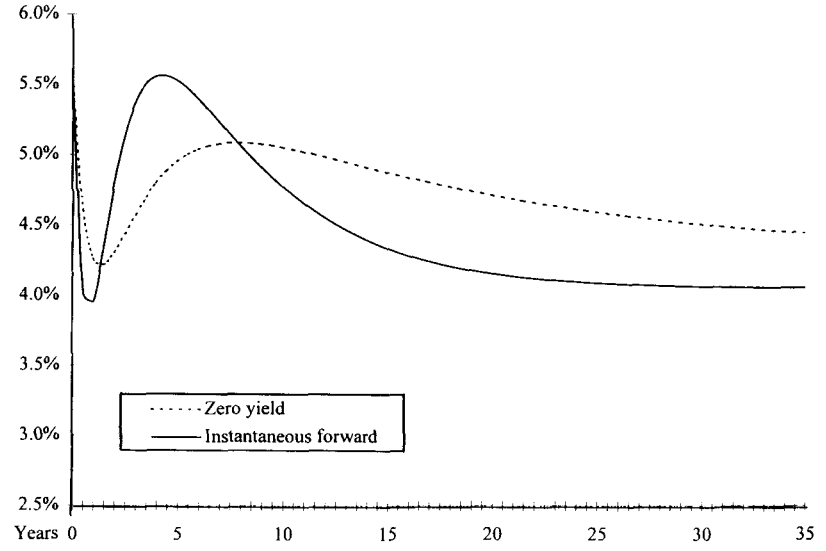


Figure C.7. Fitted real yield curve, 1 Sep 1992 (semi-annual convention)

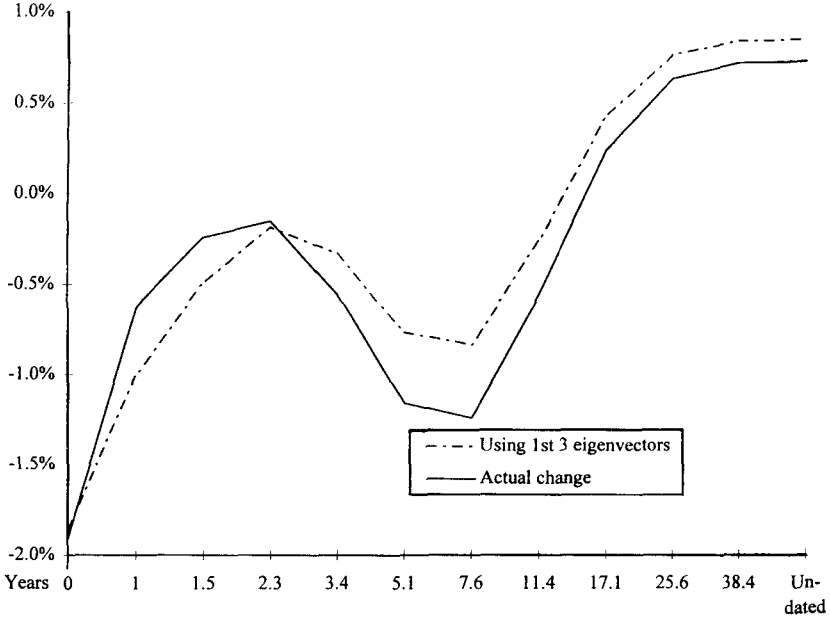


Figure C.8. Change in instantaneous real forward yields, Sep-Oct 1992

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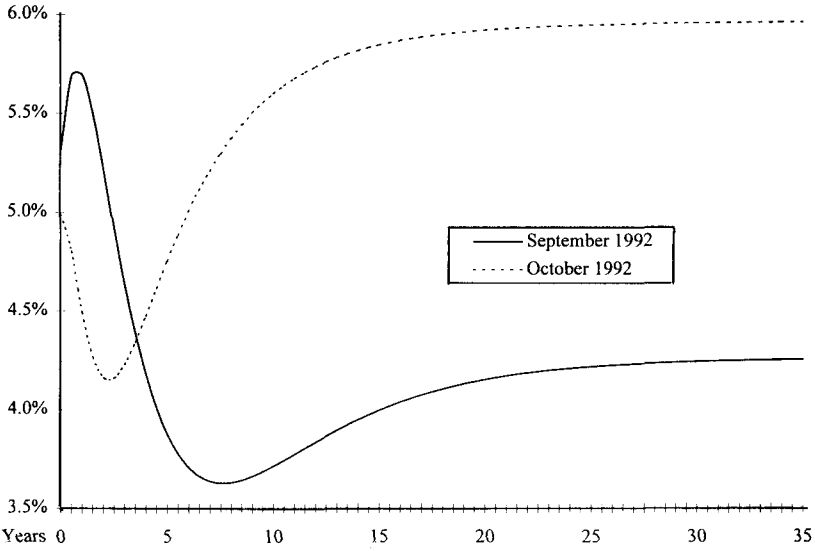


Figure C.9. Implied forward RPI

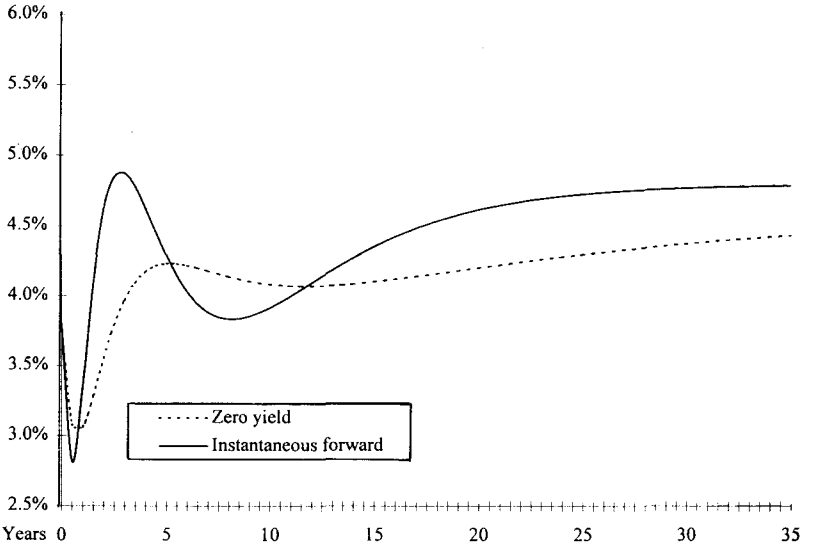


Figure C.10. Fitted real yield curve, 1 Oct 1992 (semi-annual convention)

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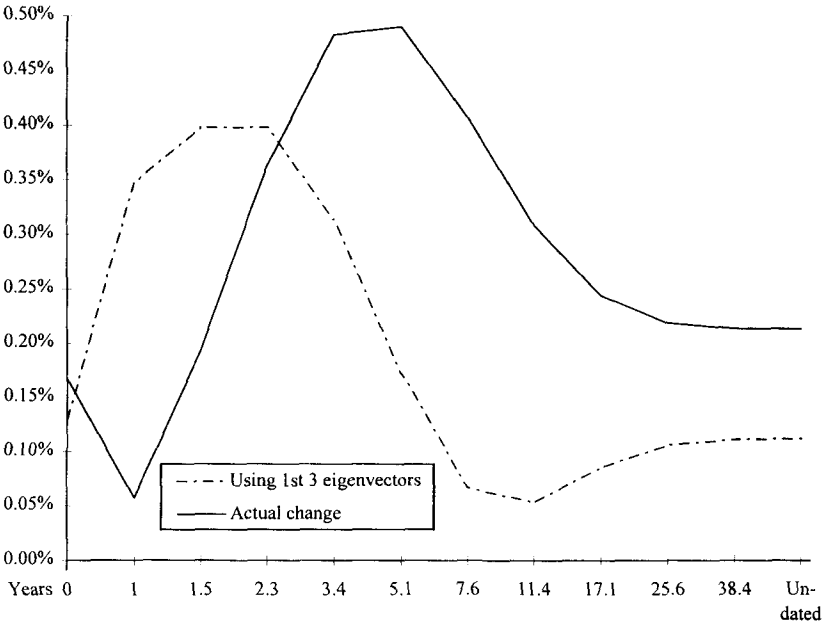


Figure C.11. Change in instananeous real forward yields, Feb-Mar 1994

C.3 *Glossary* (with acknowledgements to Neftci (1996))

Drift

In a stochastic process such as $dS_t = \mu\, dt + \sigma\, dz$, where dt represents the time increment and dz a random variate, μ is referred to as the drift, and σ as the volatility.

Exponential Martingale

The function $\xi_t = \exp\{\int_0^t \Lambda_u\, dW_u - \frac{1}{2}\int_0^t \Lambda_u^2\, du\}$ takes the value of unity at $t = 0$, and this is also its expected value when $t > 0$. W is a Wiener process and Λ must be square integrable. In the general case Λ and W are vectors, and the integrals are over implied inner products. It can be shown that ξ (a scalar quantity) is a *martingale,* and is referred to as an *exponential martingale*. The function ξ has a central role in Girsanov's Theorem. Differentiation (using *Ito's lemma*) yields: $d\xi_t = \xi_t\, \Lambda_t\, dW_t$.

Ito's Lemma

The starting point for the formulation of differential equations is the theory of

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finance. *Ito's lemma* can be thought of as the fundamental rule for differentiating stochastic variables. Assume that the evolution of a random process S_t at time t is represented by dS_t = (a function of S_t) $dt + \sigma_t\, dz$, where dz is a normal variate with variance dt, and σ_t represents the volatility of S at time t. A function $F(S_t, t)$ of the process S and time t can be expanded by a Taylor series. It is, however, important to retain the second order term in dz^2, which, from the definition of a Wiener process, has an expected value of dt. This yields Ito's lemma: $dF = F_s\, dS_t + F_t\, dt + 1/2\, F_{ss}\, \sigma_t^2\, dt$, where the subscripts of F denote partial differentiation with respect to that subscript.

Martingale

A stochastic process X_s is said to be a *martingale,* under the probability measure P, if $E^P\{X_s \mid I_t\} = X_t$ for all $s > t$, where E is the expectation (using this probability measure P) at time t, and I_t denotes all the information (price history) known at time t. There are some additional technical constraints on X.

Numeraire

These are the units for accounting purposes. Many problems in option pricing can be simplified by a change in *numeraire.* In the theory of finance, *risk-neutrality* has assumed considerable significance, and under this measure the *numeraire* corresponds to accounting units equal to the value of an accumulating bank account whose initial value is unity.

Risk Neutrality

It can be shown that the valuation of contingent claims (options) can proceed without knowing the true *drift* of the underlying asset. For many such applications, especially when the rate of interest can be assumed to remain fixed, it may be convenient to assume that the *drift* is such that investors obtain the same expected return from holding risky assets as they do from holding bank deposits. This approach is known as *risk-neutral* options valuation. Option prices can be calculated in terms of the bank account as *numeraire*, by calculating expectations of the various payoffs. Current values are then obtained by dividing by the accumulated value of the bank deposit (equivalent to discounting at the fixed rate of interest), since it is the ratio of asset prices which are *martingales.* When interest rates vary stochastically, the formulae for the accumulation of the bank deposit become complex, and it is often simpler to utilise the *terminal measure* in order to compute the value of bond options.

Terminal Measure or T'-Forward Measure

In the HJM formulation (see equation (C.1.5)), the drift term can be set to zero for a particular bond whose maturity date is T', by setting the arbitrary function ϕ in that equation equal to $\Omega\ (t,\ T')$, the volatility at time t of the T' maturity bond. This makes the *numeraire* correspond to units of the price of this T' maturity bond. By making T' longer than the maturity date of any bond required

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in a particular application, the prices of all relevant bonds (and option values) in each simulation can be found rather more readily than from a *risk-neutral* study. The ratios of all (non-coupon paying) bonds and options are *martingales* in the HJM formulation, so that option prices can be obtained by multiplying the expectations (under the appropriate probability measure) of their maturity values by the *current* price of the T' bond. This follows from the *martingale* property of any defined or contingent cash flow, $c(t)$ payable at time t: $\mathrm{E}\{c(t)/P(t, T')\} = c(0)/P(0, T')$.

Wiener Process

Formally, a *Wiener* process W_t, relative to a family of information sets $\{I_t\}$, is a stochastic process such that:

(a) the pair I_t, W_t is a square integrable *martingale* with $W_0 = 0$ and $\mathrm{E}\{(W_t - W_s)^2\} = t - s$, where $s \leq t$; and
(b) the trajectories of W_t are continuous over t.

It can be shown that this is the same as Brownian motion: $(W_t - W_s)$ is normally distributed with zero mean and variance $(t - s)$; changes in the W over disjoint intervals are independent.

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APPENDIX D

CALCULATION OF INDEX-LINKED PAR YIELDS

D.1 Here we show how the index-linked par yields in Table 6.1 are calculated.

D.2 Par yields are calculated only for stocks which have precisely $n/2$ years to maturity where n is an integer. In this description we assume that the month of the first coupon payment is 8 months after the last known RPI. The only alternative is that this could be 7 months, which would require a small modification to the projected inflation given below.

D.3 Let $RPI1$ be the value of the last known RPI and let $RPI0$ be the value of the index 6 months earlier. $RPI0$ is thus the base RPI for the par-yield stock about to be issued. Let $RFAC=RPI1/RPI0$. If the coupon rate is g p.a., then the amounts of the future coupon and redemption payments will be:

$$\frac{g}{2}\times RFAC \qquad \text{at } s=0.5 \text{ years from today}$$

$$\frac{g}{2}\times RFAC\times ILC(2) \qquad \text{at } s=1$$

$$\frac{g}{2}\times RFAC\times ILC(3) \qquad \text{at } s=1.5$$

$$\vdots$$

$$\vdots$$

$$\left(1+\frac{g}{2}\right)\times RFAC\times ILC(n) \qquad \text{at } s=n/2$$

where $ILC(m)=\exp\left(\int_0^{(m-1)/2} r(t,t'+s')ds'\right)$ gives the amount of indexation implied by index-linked gilts market prices for a payment at time $m/2$ in excess of the known amount of indexation $RFAC$. t' is the time of the last known RPI and s' runs over the period from the last known RPI until the relevant month for indexation of a payment (that is 8 months before the mth coupon payment). Implicitly $ILC(1)=1$. $r(t,t'+s')$ is the implied forward inflation curve as observed at time t where $t'=t-\Delta t$ is the time of the last known RPI and $s'=0$ corresponds to the time of the last known RPI.

D.4 These payments are multiplied by the zero-coupon prices $P(t,t+0.5)$, ..., $P(t,t+n/2)$ implied by the fixed-interest coupon bond market and then summed. Thus, for a given g, the price P of a new index-linked issue would be:

$$P=\sum_{i=1}^{n}\frac{g}{2}\times RFAC\times ILC(i)\times P(t,t+i/2)+RFAC\times ILC(n)\times P(t,t+n/2).$$

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Setting P=100 allows us to derive the appropriate par yield:

$$g = 200\frac{(1 - RFAC \times ILC(n) \times P(t,t+n/2))}{\left(RFAC \times \sum_{i=1}^{n} ILC(i) \times P(t,t+i/2)\right)}.$$

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