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An exterior problem in elastodynamics

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Abstract

A modified kernel is devised which avoids the irregular values which plague the natural integral equations for the exterior problems of harmonic elastic waves in a homogeneous isotropic medium.

1. *Introduction*

The fact that the integral equations which occur naturally in the exterior problem of elastodynamics do not have a unique solution has been known for some time (see, for example, Kupradze[7]). The failure takes place at certain irregular values of the frequency which are related to interior resonance. A similar phenomenon is present for sound waves and it has been found possible to surmount the difficulties by introducing a modified kernel (for a comprehensive overview see Colton and Kress[1]). It is the purpose of this paper to describe modified kernels which are equally efficient at circumventing the irregular frequencies of elastodynamics.

A general formulation for integral equations is given in § 2. The shortcomings of the natural ones are set out in § 3. A modified kernel is proposed in § 4 and its properties elaborated. In particular, it is shown to possess no irregular frequencies for the exterior T-problem. In § 5 it is shown that the same is true for the exterior D-problem. It may be concluded that the modified kernel is a useful weapon in the armoury for tackling the exterior problem by means of integral equations.

2. *The integral equations*

When harmonic elastic waves propagating in an unbounded medium encounter an impenetrable obstacle scattering takes place. The determination of the scattering can be formulated in terms of integral equations and it is the purpose of this section to derive some of them, assuming that there is a single scatterer of finite size.

Let the interior of the obstacle be denoted by V_- while the boundary is the closed surface S. Outside S is a homogeneous elastic medium V_+. Its parameters are the Lamé constants λ, μ and the density ρ. A typical point of V_+ will be signified by $\mathbf{x}$ with Cartesian coordinates (x_1, x_2, x_3). The displacement at a point of V_+ will be called $\mathbf{u}$, with Cartesian components u_j, and the stress tensor is τ_{jk}. With a time dependence $e^{i\omega t}$, the equations to be satisfied in V_+ are

$$\tau_{jk} = \lambda \frac{\partial u_m}{\partial x_m} \delta_{jk} + \mu \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right), \tag{1}$$

$$\frac{\partial \tau_{jk}}{\partial x_k} + \omega^2 \rho u_j = 0 \tag{2}$$

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where δ_{jk} is the standard Kronecker symbol and the summation convention has been employed. In order to ensure that genuinely elastic waves are considered all the quantities μ, $\lambda+2\mu$ and ρ will be taken to be positive.

The excitation in V_+ will be specified by the displacement $\mathbf{u}^i$ and the stress tensor τ^i_{jk}. The total field in V_+ is then $\mathbf{u}^i+\mathbf{u}^s$ where the scattered wave $\mathbf{u}^s$ also satisfies (1) and (2). It must be outgoing at infinity as well. To express this mathematically let

$$a=\{\rho/(\lambda+2\mu)\}^{\frac{1}{2}},\quad b=(\rho/\mu)^{\frac{1}{2}}$$

be the slownesses of P- and S-waves respectively. Then the radiation condition is that, as $R=|\mathbf{x}|\to\infty$,

$$R\{\hat{x}_l\tau^s_{jl}+i\omega b\mu(u^s_j-u^s_l\hat{x}_l\hat{x}_j)+i\omega a(\lambda+2\mu)\hat{x}_j u^s_l\hat{x}_l\}\to 0 \tag{3}$$

where $\hat{\mathbf{x}}$ is a unit vector in the direction of $\mathbf{x}$. It has been shown (see Jones[3]) that (3) implies that $R\mathbf{u}^s$ is bounded as $R\to\infty$ so that this property can be invoked whenever convenient.

The specification of $\mathbf{u}^s$ is completed by imposing boundary conditions on S. Two possibilities will be discussed. In the first it will be assumed that there is no displacement of the surface S; thus $\mathbf{u}^s=-\mathbf{u}^i$ on S. This will be known as the *exterior D-problem*. The second possibility is there is no traction on S i.e. $n_k\tau^s_{jk}=-n_k\tau^i_{jk}$ where $\mathbf{n}$ is the unit normal to S out of V_- into V_+. This will be designated the *exterior T-problem*.

From an alternative point of view the incident field can be ignored and the exterior D-problem regarded as finding a radiating solution in which the displacement takes prescribed values on S. Similarly, in the exterior T-problem, the solution has a prescribed traction on S.

It is known that any solution to the exterior D-problem is unique (see Jones[3]) and the same is true for the T-problem. Any lack of uniqueness that makes its presence felt subsequently is a fault of the mechanism deployed for determining $\mathbf{u}^s$ and not of the basic formulation.

A standard means of tackling the partial differential equations (1) and (2) is to convert them to integral equations. Let

$$g_{jk}(\mathbf{x},\mathbf{y})=\frac{1}{4\pi\rho\omega^2}\frac{\partial^2}{\partial x_j\,\partial x_k}\left(\frac{e^{-i\omega a|\mathbf{x}-\mathbf{y}|}-e^{-i\omega b|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\right)-\frac{e^{-i\omega b|\mathbf{x}-\mathbf{y}|}}{4\pi\mu|\mathbf{x}-\mathbf{y}|}\delta_{jk}. \tag{4}$$

Then g_{jk} satisfies

$$(\lambda+\mu)\frac{\partial^2 g_{jk}}{\partial x_i\,\partial x_j}+\mu\frac{\partial^2 g_{ik}}{\partial x_j\,\partial x_j}+\rho\omega^2 g_{ik}=\delta_{ik}\delta(\mathbf{x}-\mathbf{y}). \tag{5}$$

It is also radiating at infinity in the sense that g_{jk}, with k and $\mathbf{y}$ fixed, satisfies the same condition as u_j in (3).

Introduce now $h_{jk}(\mathbf{x},\mathbf{y})$ which has no singularities except in V_- and is such that

$$(\lambda+\mu)\frac{\partial^2 h_{jk}}{\partial x_i\,\partial x_j}+\mu\frac{\partial^2 h_{ik}}{\partial x_j\,\partial x_j}+\rho\omega^2 h_{ik}=0 \tag{6}$$

in V_+. In addition, it is required that as $R\to\infty$ for fixed $\mathbf{y}$ and k,

$$R\left\{\lambda\frac{\partial h_{mk}}{\partial x_m}\hat{x}_j+\mu\hat{x}_l\left(\frac{\partial h_{lk}}{\partial x_j}+\frac{\partial h_{jk}}{\partial x_l}\right)+i\omega b\mu(h_{jk}-h_{lk}\hat{x}_l\hat{x}_j)+i\omega a(\lambda+2\mu)\hat{x}_j h_{lk}\hat{x}_l\right\}\to 0. \tag{7}$$

Moreover, suppose that $h_{jk}(\mathbf{x},\mathbf{y})=h_{kj}(\mathbf{y},\mathbf{x})$. Then $h_{jk}(\mathbf{x},\mathbf{y})$ also satisfies

$$(\lambda+\mu)\frac{\partial^2 h_{kj}(\mathbf{x},\mathbf{y})}{\partial y_i\,\partial y_j}+\mu\frac{\partial^2 h_{ki}}{\partial y_j\,\partial y_j}+\rho\omega^2 h_{ki}=0 \tag{8}$$

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$$\text{and } |\mathbf{y}|\left\{\lambda\hat{y}_j\frac{\partial h_{km}(\mathbf{x},\mathbf{y})}{\partial y_m}+\mu\hat{y}_l\left(\frac{\partial h_{kl}}{\partial y_j}+\frac{\partial h_{kj}}{\partial y_l}\right)+i\omega b\mu(h_{kj}-h_{kl}\hat{y}_l\hat{y}_j)+i\omega a(\lambda+2\mu)\hat{y}_j h_{kl}\hat{y}_l\right\}\to 0 \tag{9}$$

as $|\mathbf{y}|\to\infty$ for fixed $\mathbf{x}$.

Put $G_{jk}(\mathbf{x},\mathbf{y})=g_{jk}(\mathbf{x},\mathbf{y})+h_{jk}(\mathbf{x},\mathbf{y})$ and consider

$$\int_\Sigma[n_i\tau_{ij}(\mathbf{y})\,G_{kj}(\mathbf{x},\mathbf{y})-\{\lambda n_m u_m\delta_{ij}+\mu(n_j u_i+n_i u_j)\}\frac{\partial}{\partial y_i}G_{kj}(\mathbf{x},\mathbf{y})]\,dS_y$$

where Σ is a closed surface in V_+ and $\mathbf{y}$ is a point of Σ at which the surface element is dS_y. Apply the divergence theorem. Then, by virtue of (1), (2), (5) and (8),

$$\int_\Sigma\{n_i\tau_{ij}(\mathbf{y})\,G_{kj}(\mathbf{x},\mathbf{y})-v_{ij}(\mathbf{y})\frac{\partial}{\partial y_i}G_{kj}(\mathbf{x},\mathbf{y})\}\,dS_y=\begin{cases}-u_k(\mathbf{x}), & \mathbf{x}\text{ inside }\Sigma\\ 0, & \mathbf{x}\text{ outside }\Sigma\end{cases} \tag{10}$$

where
$$v_{ij}=\lambda n_m u_m\delta_{ij}+\mu(n_j u_i+n_i u_j). \tag{11}$$

Select Σ to be composed of S and a large sphere Ω. Let the radius of Ω tend to infinity and suppose that $\mathbf{u}$ is outgoing. Then $|\mathbf{y}|$ becomes infinite with the radius of Ω while $\mathbf{x}$ remains fixed. Hence, by (3), the observation just after it and (9) the contribution of Ω disappears. It follows from (10) that, when $\mathbf{u}$ is a radiating wave,

$$u_k(\mathbf{x})=\int_S\{n_i\tau_{ij}(\mathbf{y})\,G_{kj}(\mathbf{x},\mathbf{y})-v_{ij}(\mathbf{y})\frac{\partial}{\partial y_i}G_{kj}(\mathbf{x},\mathbf{y})\}\,dS_y \tag{12}$$

when $\mathbf{x}$ is in V_+. The change of sign from (10) is accounted for by bringing the normal to Σ into line with that already defined for S.

One choice for h_{jk} is zero so that G_{jk} becomes the same as g_{jk}. The representation (12) then agrees with what has been called Betti's formula (Kupradze[7], Eringen & Sububi[2]). Whether h_{jk} is zero or not, it is straightforward to verify that, on account of (5) and (6), the representation (12) does indeed supply a solution to (1) and (2) in V_+. Furthermore, that solution is outgoing by virtue of (7). Consequently, any radiating solution can be expressed in terms of values of S through the form (12).

Now allow $\mathbf{x}$ to tend to a point of S. By definition, h_{jk} has no singularity on S. Thus only those of g_{jk} are relevant and we may quote results for surface integrals involving g_{jk} from, for example, Jones[4]. For instance, the integral of τ_{ij} is continuous across S, whereas that of v_{ij} has a discontinuity of u_k. In fact

$$\tfrac{1}{2}u_k(\mathbf{x})=\int_S\left\{n_i\tau_{ij}(\mathbf{y})\,G_{kj}(\mathbf{x},\mathbf{y})-v_{ij}(\mathbf{y})\frac{\partial}{\partial y_i}G_{kj}(\mathbf{x},\mathbf{y})\right\}dS_y \tag{13}$$

for $\mathbf{x}$ on S, integrals being interpreted as principal values if necessary. Similarly

$$\tfrac{1}{2}n_i\tau_{ij}(\mathbf{x})=n_i T'_{ij}, \tag{14}$$

where T'_{ij} is obtained from the integral of (12) by applying the differential operator of (1).

In the exterior T-problem $n_i\tau_{ij}$ is prescribed on S. Therefore (13) leads to

$$\tfrac{1}{2}u_k(\mathbf{x})+\int_S v_{ij}(\mathbf{y})\frac{\partial}{\partial y_i}G_{kj}(\mathbf{x},\mathbf{y})\,dS_y=f_k \tag{15}$$

where f_k is known. Then (15) constitutes an integral equation on S to determine the surface displacement. Once the surface displacement is known the displacement

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throughout V_+ can be calculated from (12). If desired, linear combinations of (15) can be arranged so that only v_{ij} (but not u_k explicitly) appears though at the cost of complicating the kernel.

The corresponding integral equation for the D-problem is derived from (14) and is, if $t_j = n_i \tau_{ij}$,

$$\tfrac{1}{2}t_j(\mathbf{x}) - \int_S t_j(\mathbf{y}) \left\{ n_k(\mathbf{x}) \frac{\partial}{\partial x_i} G_{ji}(\mathbf{x}, \mathbf{y}) + \mu n_i(\mathbf{x}) \left(\frac{\partial G_{ij}}{\partial x_k} + \frac{\partial G_{kj}}{\partial x_i} \right) \right\} dS_y = g_k \tag{16}$$

where g_k is known.

3. *Uniqueness*

The question of whether any solution of (15) is unique will now be examined in the case when G_{jk} has its simplest form, namely that in which h_{jk} is identically zero. The difference between any two solutions of (15) will then satisfy

$$\tfrac{1}{2}u_k(\mathbf{x}) + \int_S v_{ij}(\mathbf{y}) \frac{\partial}{\partial y_i} g_{kj}(\mathbf{x}, \mathbf{y})\, dS_y = 0, \tag{17}$$

with v_{ij} given by (11). In terms of any solution of (17) define U_k in V_- and V_+ by

$$U_k(\mathbf{x}) = -\int_S v_{ij}(\mathbf{y}) \frac{\partial}{\partial y_i} g_{kj}(\mathbf{x}, \mathbf{y})\, dS_y. \tag{18}$$

Associate with $\mathbf{U}$ a stress tensor T_{ij} by means of (1). Then an elastic wave has been specified for a medium of parameters λ, μ, ρ occupying V_- and V_+. Allow $\mathbf{x}$ to tend to S from V_-. Because of the limit of the surface integral for $\mathbf{U}$, (17) asserts that $\mathbf{U}_- = \mathbf{0}$ on S where the suffix $-$ indicates a value obtained on S in going from V_-. Likewise $+$ signifies a limit from V_+.

The condition $\mathbf{U}_- = \mathbf{0}$ may imply that $\mathbf{U} \equiv \mathbf{0}$ in V_-. It follows that $T_{ij} \equiv 0$ in V_- and, in particular, $(T_{ij})_- = 0$. The continuity across S of the integral formula for T_{ij} now forces $(T_{ij})_+ = 0$. Invoking the exterior uniqueness theorem for an outgoing wave (Jones [3]) $\mathbf{U} \equiv \mathbf{0}$ in V_+ so that $\mathbf{U}_+ = \mathbf{0}$. Thus there is no discontinuity in $\mathbf{U}$ across S. But, from (18) and (11), the values of $\mathbf{U}$ differ by $\mathbf{u}$. Hence $\mathbf{u}$, and thereby v_{ij} is zero. In other words the solution of (15) is unique even with h_{ik} identically zero.

If, however, $\mathbf{U}_- = \mathbf{0}$ does not require $\mathbf{U}$ to vanish identically in V_-, as when ω is an eigenvalue of the interior D-problem, the above procedure breaks down. Indeed, non-trivial solutions to (17) can be constructed by relating v_{ij} to the stress tensor of the interior eigenoscillation.

Accordingly, the solution of the integral equation for the exterior T-problem is unique when $h_{jk} \equiv 0$ except for a discrete set of values of ω corresponding to the eigensolutions of the interior D-proble. Such values of ω may be termed, following Ursell, *irregular*. The failure of uniqueness at the irregular value is an awkward feature especially for numerical work. It is known to cause trouble in the analogous problem in acoustics and may be expected to give rise to similar difficulties in elastodynamics.

As regards the exterior D-problem the starting point is

$$U_k(\mathbf{x}) = \int_S t_j(\mathbf{y})\, g_{kj}(\mathbf{x}, \mathbf{y})\, \mathrm{d}S_y.$$

The appropriate form of (16) then leads to $(n_i T_{ij})_- = 0$ on S. By following an argument parallel to the foregoing, we may conclude that there is uniqueness except for certain

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irregular values of ω, which now occur when there are eigensolutions for the interior T-problem.

In general, therefore, it must be accepted that choosing h_{jk} to be identically zero leaves the kernel in an unsatisfactory state. The next section will be concerned with picking h_{jk} so that the failing in uniqueness is overcome by the modified kernel.

4. *A modified kernel for the exterior T-problem*

In order to define a suitable h_{jk} some auxiliary functions are introduced. Pick any point of V_- as origin. Let R, θ and ϕ be the spherical polar coordinates of $\mathbf{x}$ with respect to this origin. The notation $z_\nu^{(j)}(R)$ will be adopted for the spherical Bessel functions $j_\nu(R)$, $y_\nu(R)$, $h_\nu^{(1)}(R)$, $h_\nu^{(2)}(R)$ when $j = 1, 2, 3, 4$ respectively. Define

$$p_{\sigma\nu}^{(j)}(\mathbf{x}) = (2-\delta_{\sigma 0})^{\frac{1}{2}} z_\nu^{(j)}(\omega b R)\, P_\nu^\sigma(\cos\theta)\cos\sigma\phi, \tag{19}$$

$$\tilde{p}_{\sigma\nu}^{(j)}(\mathbf{x}) = 2^{\frac{1}{2}} z_\nu^{(j)}(\omega b R)\, P_\nu^\sigma(\cos\theta)\sin\sigma\phi \tag{20}$$

where it is understood that repeated Greek suffixes are not to be summed. P_ν^σ is the customary Legendre function and $\tilde{p}_{0\nu}$ is identically zero. On account of (19), $p_{\sigma\nu}^{(j)}$ satisfies the partial differential equation

$$\nabla^2 p_{\sigma\nu}^{(j)} + \omega^2 b^2 p_{\sigma\nu}^{(j)} = 0 \tag{21}$$

away from the origin; the same equation is obeyed by $\tilde{p}_{\sigma\nu}^{(j)}$.

Two vectors are specified from (19) by

$$\mathbf{p}_{\sigma\nu}^{(j)}(\mathbf{x}) = \mathbf{x} \wedge \operatorname{grad} p_{\sigma\nu}^{(j)}(\mathbf{x}), \tag{22}$$

$$\mathbf{q}_{\sigma\nu}^{(j)}(\mathbf{x}) = -\operatorname{curl} \mathbf{p}_{\sigma\nu}^{(j)}(\mathbf{x}) = \omega^2 b^2 \mathbf{x} p_{\sigma\nu}^{(j)}(\mathbf{x}) + \operatorname{grad}\frac{\partial}{\partial R}(R p_{\sigma\nu}^{(j)}) \tag{23}$$

by virtue of (21). The vectors $\tilde{\mathbf{p}}_{\sigma\nu}^{(j)}$ and $\tilde{\mathbf{q}}_{\sigma\nu}^{(j)}$ are obtained in a similar way from (20).

Another pair of vectors $\mathbf{r}_{\sigma\nu}^{(j)}$, $\tilde{\mathbf{r}}_{\sigma\nu}^{(j)}$ is obtained from

$$\mathbf{r}_{\sigma\nu}^{(j)}(\mathbf{x}) = \operatorname{grad} r_{\sigma\nu}^{(j)}(\mathbf{x}),\ \tilde{\mathbf{r}}_{\sigma\nu}^{(j)}(\mathbf{x}) = \operatorname{grad} \tilde{r}_{\sigma\nu}^{(j)}(\mathbf{x}) \tag{24}$$

where

$$r_{\sigma\nu}^{(j)}(\mathbf{x}) = (2-\delta_{\sigma 0})^{\frac{1}{2}} z_\nu^{(j)}(\omega a R)\, P_\nu^\sigma(\cos\theta)\cos\sigma\phi, \tag{25}$$

$$\tilde{r}_{\sigma\nu}^{(j)}(\mathbf{x}) = 2^{\frac{1}{2}} z_\nu^{(j)}(\omega a R)\, P_\nu^\sigma(\cos\theta)\sin\sigma\phi. \tag{26}$$

Observe that

$$\nabla^2 r_{\sigma\nu}^{(j)} + \omega^2 a^2 r_{\sigma\nu}^{(j)} = 0 \tag{27}$$

except at the origin.

For the definition of h_{jk} we now take, in dyadic notation,

$$\begin{aligned} h_{kj}(\mathbf{x},\mathbf{y})\,\mathbf{i}_j\mathbf{i}_k = {} & \frac{ib}{4\pi\rho\omega}\sum_{\nu=1}^{\infty}\sum_{\sigma=0}^{\nu} d_{\sigma\nu}\{a_{\sigma\nu}\,\mathbf{q}_{\sigma\nu}^{(4)}(\mathbf{y})\,\mathbf{q}_{\sigma\nu}^{(4)}(\mathbf{x}) + \tilde{a}_{\sigma\nu}\,\tilde{\mathbf{q}}_{\sigma\nu}^{(4)}(\mathbf{y})\,\tilde{\mathbf{q}}_{\sigma\nu}^{(4)}(\mathbf{x}) \\ & + \omega^2 b^2 b_{\sigma\nu}\,\mathbf{p}_{\sigma\nu}^{(4)}(\mathbf{y})\,\mathbf{p}_{\sigma\nu}^{(4)}(\mathbf{x}) + \omega^2 b^2 \tilde{b}_{\sigma\nu}\,\tilde{\mathbf{p}}_{\sigma\nu}^{(4)}(\mathbf{y})\,\tilde{\mathbf{p}}_{\sigma\nu}^{(4)}(\mathbf{x})\} \\ & - \frac{ia}{4\pi\rho\omega}\sum_{\nu=0}^{\infty}\sum_{\sigma=0}^{\nu} e_{\sigma\nu}\{c_{\sigma\nu}\,\mathbf{r}_{\sigma\nu}^{(4)}(\mathbf{y})\,\mathbf{r}_{\sigma\nu}^{(4)}(\mathbf{x}) + \tilde{c}_{\sigma\nu}\,\tilde{\mathbf{r}}_{\sigma\nu}^{(4)}(\mathbf{y})\,\tilde{\mathbf{r}}_{\sigma\nu}^{(4)}(\mathbf{x})\} \end{aligned} \tag{28}$$

where

$$d_{\sigma\nu} = \frac{(\nu-\sigma)!\,(2\nu+1)}{(\nu+\sigma)!\,\nu(\nu+1)}, \tag{29}$$

$$e_{\sigma\nu} = (\nu-\sigma)!\,(2\nu+1)/(\nu+\sigma)!. \tag{30}$$

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The only restriction on the constants $a_{\sigma\nu}, \ldots, \tilde{c}_{\sigma\nu}$ is that they are such as to secure convergence of the infinite series. It is easy to check that (28) fulfils the requirements placed on h_{jk} by taking advantage of (19)–(27) which, in particular, imply that

$$\operatorname{curl}\operatorname{curl}\mathbf{p}_{\sigma\nu}^{(j)} = \omega^2 b^2 \mathbf{p}_{\sigma\nu}^{(j)}, \tag{31}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{q}_{\sigma\nu}^{(j)} = \omega^2 b^2 \mathbf{q}_{\sigma\nu}^{(j)} \tag{32}$$

with similar equations for $\tilde{\mathbf{p}}_{\sigma\nu}$ and $\tilde{\mathbf{q}}_{\sigma\nu}$.

Let
$$U_k(\mathbf{x}) = \int_S v_{ij}(\mathbf{y}) \frac{\partial}{\partial y_i} G_{kj}(\mathbf{x}, \mathbf{y})\, dS_y. \tag{33}$$

When $\mathbf{x}$ is in V_-, so that $|\mathbf{x}| < |\mathbf{y}|$, there is the standard expansion

$$\begin{aligned} g_{kj}(\mathbf{x}, \mathbf{y})\, \mathbf{i}_j \mathbf{i}_k = \frac{ib}{4\pi\rho\omega} \Sigma' d_{\sigma\nu} \{ & \mathbf{q}_{\sigma\nu}^{(4)}(\mathbf{y})\, \mathbf{q}_{\sigma\nu}^{(1)}(\mathbf{x}) + \tilde{\mathbf{q}}_{\sigma\nu}^{(4)}(\mathbf{y})\, \tilde{\mathbf{q}}_{\sigma\nu}^{(1)}(\mathbf{x}) \\ & + \omega^2 b^2 \mathbf{p}_{\sigma\nu}^{(4)}(\mathbf{y})\, \mathbf{p}_{\sigma\nu}^{(1)}(\mathbf{x}) + \omega^2 b^2 \tilde{\mathbf{p}}_{\sigma\nu}^{(4)}(\mathbf{y})\, \tilde{\mathbf{p}}_{\sigma\nu}^{(1)}(\mathbf{x}) \} \\ & - \frac{ia}{4\pi\rho\omega} \Sigma'' e_{\sigma\nu} \{ \mathbf{r}_{\sigma\nu}^{(4)}(\mathbf{y})\, \mathbf{r}_{\sigma\nu}^{(1)}(\mathbf{x}) + \tilde{\mathbf{r}}_{\sigma\nu}^{(4)}(\mathbf{y})\, \tilde{\mathbf{r}}_{\sigma\nu}^{(1)}(\mathbf{x}) \}, \end{aligned} \tag{34}$$

where Σ' stands for $\sum_{\nu=1}^{\infty} \sum_{\sigma=0}^{\nu}$ while Σ'' represents $\sum_{\nu=0}^{\infty} \sum_{\sigma=0}^{\nu}$. Put

$$\alpha_{\sigma\nu} = \frac{ibd_{\sigma\nu}}{4\pi\rho\omega} \int_S v_{ij}(\mathbf{y}) \frac{\partial}{\partial y_i} \{\mathbf{q}_{\sigma\nu}^{(4)}(\mathbf{y})\}_j\, dS_y, \tag{35}$$

$$\beta_{\sigma\nu} = \frac{ib^3\omega d_{\sigma\nu}}{4\pi\rho} \int_S v_{ij}(\mathbf{y}) \frac{\partial}{\partial y_i} \{\mathbf{p}_{\sigma\nu}^{(4)}(\mathbf{y})\}_j\, dS_y, \tag{36}$$

$$\gamma_{\sigma\nu} = -\frac{iae_{\sigma\nu}}{4\pi\rho\omega} \int_S v_{ij}(\mathbf{y}) \frac{\partial}{\partial y_i} \{\mathbf{r}_{\sigma\nu}^{(4)}(\mathbf{y})\}_j\, dS_y, \tag{37}$$

and define $\tilde{\alpha}_{\sigma\nu}, \tilde{\beta}_{\sigma\nu}, \tilde{\gamma}_{\sigma\nu}$ in an analogous manner. Then (33), (28) and (34) give for $\mathbf{x}$ in V_-

$$\begin{aligned} \mathbf{U}(\mathbf{x}) = \Sigma' [& \alpha_{\sigma\nu} \{\mathbf{q}_{\sigma\nu}^{(1)}(\mathbf{x}) + a_{\sigma\nu}\, \mathbf{q}_{\sigma\nu}^{(4)}(\mathbf{x})\} + \tilde{\alpha}_{\sigma\nu} \{\tilde{\mathbf{q}}_{\sigma\nu}^{(1)}(\mathbf{x}) + \tilde{a}_{\sigma\nu}\, \tilde{\mathbf{q}}_{\sigma\nu}^{(4)}(\mathbf{x})\} \\ & + \beta_{\sigma\nu} \{\mathbf{p}_{\sigma\nu}^{(1)}(\mathbf{x}) + b_{\sigma\nu}\, \mathbf{p}_{\sigma\nu}^{(4)}(\mathbf{x})\} + \tilde{\beta}_{\sigma\nu} \{\tilde{\mathbf{p}}_{\sigma\nu}^{(1)}(\mathbf{x}) + \tilde{b}_{\sigma\nu}\, \tilde{\mathbf{p}}_{\sigma\nu}^{(4)}(\mathbf{x})\}] \\ & + \Sigma'' [\gamma_{\sigma\nu} \{\mathbf{r}_{\sigma\nu}^{(1)}(\mathbf{x}) + c_{\sigma\nu}\, \mathbf{r}_{\sigma\nu}^{(4)}(\mathbf{x})\} + \tilde{\gamma}_{\sigma\nu} \{\tilde{\mathbf{r}}_{\sigma\nu}^{(1)}(\mathbf{x}) + \tilde{c}_{\sigma\nu}\, \tilde{\mathbf{r}}_{\sigma\nu}^{(4)}(\mathbf{x})\}]. \end{aligned} \tag{38}$$

The representation (38) can be expressed in the form

$$\mathbf{U} = \mathbf{x} \wedge \operatorname{grad} p + \operatorname{curl}(\mathbf{x} \wedge \operatorname{grad} q) + \operatorname{grad} r \tag{39}$$

where, by virtue of (22)–(24),

$$p = \Sigma' [\beta_{\sigma\nu} \{p_{\sigma\nu}^{(1)} + b_{\sigma\nu} p_{\sigma\nu}^{(4)}\} + \tilde{\beta}_{\sigma\nu} \{\tilde{p}_{\sigma\nu}^{(1)} + \tilde{b}_{\sigma\nu}\, \tilde{p}_{\sigma\nu}^{(4)}\}], \tag{40}$$

$$q = -\Sigma' [\alpha_{\sigma\nu} \{p_{\sigma\nu}^{(1)} + a_{\sigma\nu} p_{\sigma\nu}^{(4)}\} + \tilde{\alpha}_{\sigma\nu} \{\tilde{p}_{\sigma\nu}^{(1)} + \tilde{a}_{\sigma\nu}\, \tilde{p}_{\sigma\nu}^{(4)}\}], \tag{41}$$

$$r = \Sigma'' [\gamma_{\sigma\nu} \{r_{\sigma\nu}^{(1)} + c_{\sigma\nu} r_{\sigma\nu}^{(4)}\} + \tilde{\gamma}_{\sigma\nu} \{\tilde{r}_{\sigma\nu}^{(1)} + \tilde{c}_{\sigma\nu}\, \tilde{r}_{\sigma\nu}^{(4)}\}]. \tag{42}$$

Remark that p and q both satisfy (21) whereas r is a solution of (27), the origin being excepted of course.

Let P be the surface of a sphere, with centre the origin, which lies entirely within V_-. Suppose that its radius is R. A key result is

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LEMMA 1. *With* $\mathbf{U}$ *given by* (38)

$$\int_P \mathbf{n}\cdot\{(\lambda+2\mu)(\mathbf{U}^*\operatorname{div}\mathbf{U}-\mathbf{U}\operatorname{div}\mathbf{U}^*)+\mu(\mathbf{U}^*\wedge\operatorname{curl}\mathbf{U}-\mathbf{U}\wedge\operatorname{curl}\mathbf{U}^*)\}\,dS$$
$$=-\frac{8\pi i\mu}{\omega b}\Sigma'\{\omega^2b^2|\alpha_{\sigma\nu}|^2(|a_{\sigma\nu}+\tfrac12|^2-\tfrac14)+\omega^2b^2|\tilde\alpha_{\sigma\nu}|^2(|\tilde a_{\sigma\nu}+\tfrac12|^2-\tfrac14)$$
$$+|\beta_{\sigma\nu}|^2(|b_{\sigma\nu}+\tfrac12|^2-\tfrac14)+|\tilde\beta_{\sigma\nu}|^2(|\tilde b_{\sigma\nu}+\tfrac12|^2-\tfrac14)\}/d_{\sigma\nu}$$
$$-8\pi i\omega\rho\,\Sigma''\{|\gamma_{\sigma\nu}|^2(|c_{\sigma\nu}+\tfrac12|^2-\tfrac14)+|\tilde\gamma_{\sigma\nu}|^2(|\tilde c_{\sigma\nu}+\tfrac12|^2-\tfrac14)\}/e_{\sigma\nu}a,$$

the star indicating a complex conjugate.

Proof. From (39) and (27)

$$\operatorname{div}\mathbf{U}=-\omega^2a^2r \tag{43}$$

while from (21)

$$\operatorname{curl}\mathbf{U}=\omega^2b^2\mathbf{x}\wedge\operatorname{grad}q-\omega^2b^2p\mathbf{x}-\operatorname{grad}\frac{\partial}{\partial R}(Rp), \tag{44}$$

$$\operatorname{curl}\operatorname{curl}\mathbf{U}=\omega^2b^2\{\mathbf{x}\wedge\operatorname{grad}p+\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q)\}. \tag{45}$$

By Stokes' theorem

$$\int_P \mathbf{n}\cdot\operatorname{grad}r^*\wedge\operatorname{curl}\mathbf{U}\,dS=-\int_P r^*\mathbf{n}\cdot\operatorname{curl}\operatorname{curl}\mathbf{U}\,dS$$
$$=-\omega^2b^2\int_P r^*\mathbf{n}\cdot\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q)\,dS$$

from (45) and the fact that $\mathbf{n}$ is parallel to $\mathbf{x}$ on P. Inserting this result and (43), (44) in the surface integral of the lemma, we obtain

$$2iI\int_P\Big[\omega^2\rho r^*\frac{\partial r}{\partial R}+\mu\mathbf{n}.\{\mathbf{x}\wedge\operatorname{grad}p^*+\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q^*)\}$$
$$\wedge\Big\{\omega^2b^2\mathbf{x}\wedge\operatorname{grad}q-\operatorname{grad}\frac{\partial}{\partial R}(Rp)\Big\}\Big]\,dS$$

Now, by Stokes' theorem,

$$\int_P \mathbf{n}\cdot\operatorname{grad}\frac{\partial}{\partial R}(Rp)\wedge\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q^*)\,dS$$
$$=-\int_P\frac{\partial}{\partial R}(Rp)\,\mathbf{n}\cdot\operatorname{curl}\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q^*)\,dS=0$$

since $\operatorname{curl}\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q^*)=\omega^2b^2\mathbf{x}\wedge\operatorname{grad}q^*$. Also

$$\int_P \mathbf{n}\cdot(\mathbf{x}\wedge\operatorname{grad}p^*)\wedge(\mathbf{x}\wedge\operatorname{grad}q)\,dS=\int_P\mathbf{n}\cdot\mathbf{x}\mathbf{x}\wedge\operatorname{grad}p^*\cdot\operatorname{grad}g\,dS$$
$$=R^2\int_P\mathbf{n}\cdot\operatorname{grad}p^*\wedge\operatorname{grad}q\,dS=0$$

by Stokes's theorem. Hence the surface integral reduces to

$$2iI\int_P\Big\{\omega^2\rho r^*\frac{\partial r}{\partial R}-\mu\mathbf{n}\cdot(\mathbf{x}\wedge\operatorname{grad}p^*)\wedge\operatorname{grad}\frac{\partial}{\partial R}(Rp)$$
$$+\omega^2b^2\mu\mathbf{n}\cdot\operatorname{curl}(\mathbf{x}\wedge\operatorname{grad}q^*)\wedge(\mathbf{x}\wedge\operatorname{grad}q)\Big\}\,dS.$$

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The integrand can be converted to a more convenient form by noting that

$$\int_P \mathbf{n}\cdot\operatorname{grad}\frac{\partial}{\partial R}(Rp)\wedge(\mathbf{x}\wedge\operatorname{grad}p^*)\,dS=\int_P\frac{\partial}{\partial R}(Rp)\left\{\omega^2b^2Rp^*+\frac{\partial^2}{\partial R^2}(Rp^*)\right\}dS$$

on a further application of Stokes' theorem. Hence the integral becomes

$$2iI\int_P\left[\omega^2\rho r^*\frac{\partial r}{\partial R}+\mu\frac{\partial}{\partial R}(Rp)\left\{\omega^2b^2Rp^*+\frac{\partial^2}{\partial R^2}(Rp^*)\right\}\right.$$
$$\left.+\mu\omega^2b^2\frac{\partial}{\partial R}(Rq^*)\left\{\omega^2b^2Rq+\frac{\partial^2}{\partial R^2}(Rq)\right\}\right]dS.$$

The orthogonal properties of the Legendre and sinusoidal functions lead via (42) to

$$\int_P r^*\frac{\partial r}{\partial R}dS=4\pi\omega aR^2\,\Sigma''\left[|\gamma_{\sigma\nu}|^2\{j_\nu(\omega aR)+c^*_{\sigma\nu}h^{(1)}_\nu(\omega aR)\}\{j'_\nu(\omega aR)+c_{\sigma\nu}h^{(2)'}_\nu(\omega aR)\}\right.$$
$$\left.+|\tilde\gamma_{\sigma\nu}|^2\{j_\nu(\omega aR)+\tilde c^*_{\sigma\nu}h^{(1)}_\nu(\omega aR)\}\{j'_\nu(\omega aR)+\tilde c_{\sigma\nu}h^{(2)'}_\nu(\omega aR)\}\right]/e_{\sigma\nu}.$$

But

$$\begin{aligned}I\{j_\nu(\omega aR)+c^*_{\sigma\nu}h^{(1)}_\nu(\omega aR)\}\{j'_\nu(\omega aR)+c_{\sigma\nu}h^{(2)'}_\nu(\omega aR)\}&\\=-\tfrac12ic_{\sigma\nu}(j_\nu h^{(2)'}_\nu-h^{(2)}_\nu j'_\nu)+\tfrac12ic^*_{\sigma\nu}(j_\nu h^{(1)'}_\nu-h^{(1)}_\nu j'_\nu)-\tfrac12i|c_{\sigma\nu}|^2(h^{(1)}_\nu h^{(2)'}_\nu-h^{(1)'}_\nu h^{(2)}_\nu)&\\=-(\tfrac12c_{\sigma\nu}+\tfrac12c^*_{\sigma\nu}+|c_{\sigma\nu}|^2)/\omega^2a^2R^2&\end{aligned}$$

on account of the Wronskian relation

$$j_\nu(z)\,h^{(2)'}_\nu(z)-j'_\nu(z)\,h^{(2)}_\nu(z)=-i/z^2.$$

The series involving Σ'' in the statement of the lemma is consequently confirmed.

The other two parts of the integral can be dealt with in a like manner, benefit being drawn from the additional information that

$$\frac{d^2}{dR^2}\{Rx^{(j)}_\nu(\omega bR)\}+\omega^2b^2Rx^{(j)}_\nu(\omega bR)=\nu(\nu+1)\,x^{(j)}_\nu(\omega bR)/R.$$

Accordingly, the proof of the lemma is complete.

By means of the lemma we can demonstrate

THEOREM 1. *Suppose that either*

$$|a_{\sigma\nu}+\tfrac12|>\tfrac12,\quad|b_{\sigma\nu}+\tfrac12|>\tfrac12,\quad\ldots,\quad|\tilde c_{\sigma\nu}+\tfrac12|>\tfrac12$$

for all σ and ν or

$$|a_{\sigma\nu}+\tfrac12|<\tfrac12,\quad|b_{\sigma\nu}+\tfrac12|<\tfrac12,\quad\ldots,\quad|\tilde c_{\sigma\nu}+\tfrac12|<\tfrac12$$

for all σ and ν. Then every solution of the homogeneous integral equation

$$\tfrac12u_k(\mathbf{x})+\int_S v_{ij}(\mathbf{y})\frac{\partial}{\partial y_i}G_{kj}(\mathbf{x},\mathbf{y})\,dS_y=0$$

is also a solution of (17) *which is such that*

$$\alpha_{\sigma\nu}=0,\quad\beta_{\sigma\nu}=0,\quad\ldots,\quad\tilde\gamma_{\sigma\nu}=0$$

for all σ and ν.

Proof. With v_{ij} a solution of the given integral equation, define $\mathbf{U}$ by (33). Then, as in § 3, the integral equation implies that, in V_-, $\mathbf{U}$ is a solution of the interior D-problem.

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Writing T_{ij} for the associated stress tensor we have

$$\int_S n_i(T_{ij}U_j^* - T_{ij}^* U_j)_- \, dS = 0.$$

It follows immediately that

$$\int_P n_i(T_{ij}U_j^* - T_{ij}^* U_j)\, dS = 0.$$

The next step is to substitute for T_{ij} its formula in terms of $\mathbf{U}$. The resulting integrand can be transformed by observing that

$$\operatorname{curl}(\mathbf{U} \wedge \mathbf{U}^*) = \mathbf{U} \operatorname{div} \mathbf{U}^* - \mathbf{U}^* \operatorname{div} \mathbf{U} + (\mathbf{U}^* \cdot \operatorname{grad})\mathbf{U} - (\mathbf{U} \cdot \operatorname{grad})\mathbf{U}^*$$

so that, after an application of Stokes' theorem,

$$\int_P n_i \left(\frac{\partial U_i}{\partial x_j} U_j^* - \frac{\partial U_i^*}{\partial x_j} U_j\right) dS = \int_P \mathbf{n} \cdot (\mathbf{U}^* \operatorname{div} \mathbf{U} - \mathbf{U} \operatorname{div} \mathbf{U}^*)\, dS.$$

Consequently

$$\int_P \mathbf{n} \cdot \{(\lambda + 2\mu)(\mathbf{U}^* \operatorname{div} \mathbf{U} - \mathbf{U} \operatorname{div} \mathbf{U}^*) + \mu(\mathbf{U}^* \wedge \operatorname{curl} \mathbf{U} - \mathbf{U} \wedge \operatorname{curl} \mathbf{U}^*)\}\, dS = 0.$$

This is the same integral as occurs in Lemma 1. But, under either of the sets of conditions in the theorem, all the terms in the series are of the same sign. They can therefore sum to zero only if

$$\alpha_{\sigma\nu} = 0, \quad \beta_{\sigma\nu} = 0, \quad \ldots, \quad \tilde{\gamma}_{\sigma\nu} = 0$$

for all σ and ν. Once these coefficients are zero it follows from (28), (35–(37) that h_{jk} disappears from the integral equation, i.e. v_{ij} satisfies (17). The theorem has now been proved.

It is, of course, obvious from the method of proof that, if the conditions of Theorem 1 hold for a finite number of $a_{\sigma\nu}, \ldots, \tilde{c}_{\sigma\nu}$ and the remaining ones are all zero, the conclusions of Theorem 1 are still valid except that it can be asserted only that $\alpha_{\sigma\nu} = 0, \ldots$ for the coefficients having the same suffixes as the non-zero $a_{\sigma\nu}, \ldots$.

THEOREM 2. *Subject to either of the sets of conditions of Theorem* 1 *the solution of* (15) *is unique.*

Proof. The difference between any two solutions of (15) satisfies the integral equation of Theorem 1. Therefore $\mathbf{U}$ is either identically zero in V_- or an eigenfunction of the interior D-problem. However, because Theorem 1 forces the coefficients $\alpha_{\sigma\nu}, \ldots$, to be zero the representation (38) shows that $\mathbf{U}$ is identically zero in P. By analyticity, it must be identically zero throughout V_-. Once it has been established that $\mathbf{U}$ vanishes at every point of V_- we can proceed as in § 3 to demonstrate that the solution of the homogeneous integral equation must be zero, i.e. the solution of (15) is unique. The proof is finished.

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5. *The exterior D-problem*

Little change needs to be made to the foregoing analysis for the exterior D-problem. We commence from

$$U_k(\mathbf{x}) = \int_S t_j(\mathbf{y})\, G_{kj}(\mathbf{x}, \mathbf{y})\, dS_y,$$

and the principal effect of this alteration is to provide new definitions for the coefficients $\alpha_{\sigma\nu}, \ldots$. The subsequence analysis can then be carried through with the sole modification that the properties of the interior D-problem are replaced by those of the interior T-problem. Thus we can assert

THEOREM 3. *Subject to either of the sets of conditions of Theorem* 1 *the solution of* (16) *is unique.*

6. *Conclusion*

A modified kernel has been constructed for exterior T- and D-problems which ensures that the integral equation possesses at most one solution. The restrictions on the coefficients are very light. The considerable flexibility in their choice offered to the user should make the kernels a valuable tool in the construction of practical solutions. In acoustics Kleinman and Roach [5, 6] showed that the modified kernel could be made optimal according to two different criteria. No doubt there are corresponding results for the kernels of elastodynamics, but they will be left for a future investigation.

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