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J. Appl. Probab. **63,** 1186–1212 (2026)
*doi:*10.1017/jpr.2026.10074

OPTIMAL PERIODIC–CLASSICAL BARRIER STRATEGIES FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

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Abstract

We study a stochastic control problem where the underlying process follows a spectrally negative Lévy process. A controller can continuously increase the process but only decrease it at independent Poisson arrival times. We show the optimality of the periodic–classical barrier strategy, which increases the process whenever it would fall below some lower barrier and decreases it whenever it is observed above a higher barrier. An optimal strategy and the value function are written semi-explicitly using scale functions. Numerical results are also given.

Keywords: Stochastic control; scale functions; periodic observations; spectrally one-sided Lévy process

2020 Mathematics Subject Classification: Primary 60G51
Secondary 93E20; 90B05

1. Introduction

We revisit a two-sided stochastic singular control problem, where the objective is to optimally modify a stochastic process to minimize the expected net present value (NPV) of costs. In this setting, the process can be modified in both directions – increased or decreased. The total cost comprises a running cost, represented as a function f of the controlled process accumulated over time, and control costs (or rewards) proportional to the magnitude of the applied control. Problems of this kind have applications in various fields; see [4, 15] for examples in finance and insurance, and [7, 8] for applications in inventory management.

Specifically, the state process in our problem is modeled by a spectrally negative Lévy process (i.e. Lévy process with only downward jumps). The controller can continuously increase the state process but can only decrease it at independent Poisson arrival times. The restriction on downward control to discrete times distinguishes our problem from classical studies of two-sided stochastic control, such as [5], as well as more recent work like [18], both of which assume that controls in both directions can be applied continuously. Recently, stochastic control problems with random discrete control opportunities have received much attention in the literature; see, for example, [1, 3, 9, 19, 22, 24]. For recent results on Lévy processes observed at Poisson arrival times, see [2, 14].

Received 30 December 2024; accepted 3 January 2026.
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Restricting downward control opportunities to random discrete times can have useful implications in practice. For instance, in inventory management, selling is often more challenging than replenishing. While replenishment from suppliers may occur continuously, selling surplus stock might not be feasible in continuous time but instead requires some waiting period to find a suitable buyer. In such cases, modeling downward control opportunities with random discrete times is appropriate. More generally, the problem we address is applicable in situations where downward control of the stochastic process is subject to tighter constraints.

The motivation for using Poisson arrival times to model control opportunities is two-fold. First, a semi-explicit solution to the problem can be obtained by leveraging the scale functions and the exponential inter-arrival times of a Poisson process. To the best of our knowledge, explicit analytical solutions fail to exist for other types of discrete random times. In such cases numerical methods are typically employed, as the state space must be expanded to ensure the problem remains Markovian. Additionally, insights from the mathematical finance literature suggest that the solution to the constant inter-arrival model, where controls can only be applied at deterministic, uniformly spaced times, can be approximated by the solution to the Poissonian inter-arrival model. For a study on this topic, we refer readers to [13].

The aim of this study is to establish the optimality of a barrier strategy under the Poisson inter-arrival model. To this end, we follow the standard guess-and-verify approach:

(i) The NPV of costs corresponding to the *periodic–classical barrier strategy* is computed semi-explicitly using scale functions. The control cost has been computed in [19] and the running cost is computed using a result from [17].

(ii) The candidate barriers, denoted by (a^*, b^*), are identified using the method proposed in [20] (see Section 3 of this work), rather than through the classical method based on the smooth-fit principle. Nevertheless, for the current problem the equivalence of the two methods is established.

(iii) The existence and uniqueness of (a^*, b^*) is established through a probabilistic argument under a most general condition.

(iv) The optimality of the candidate periodic–classical barrier strategy is established via a verification lemma, using a conventional argument that leverages the analytical properties of the scale functions and fluctuation identities.

While our approach leverages the scale functions of a spectrally one-sided Lévy process, several alternative methods for solving similar singular control problems have been explored in the literature. For instance, Noba and Yamazaki [20] solve a one-sided variation of our problem for general Lévy processes. In another direction, Mordecki and Oliú [18] address a related problem by establishing a connection to Dynkin games, under a different set of assumptions on the function f and the state process.

The remainder of this paper proceeds as follows: Section 2 presents the mathematical formulation of the problem considered. Section 3 introduces the periodic–classical barrier strategies for our problem and states the main result. Section 4 presents a sequence of results that completes the proof of the main result stated in Section 3. Section 5 presents a numerical example. Section 6 concludes this study with a discussion of several directions for potential extensions.

In this paper we use $x+$ and $x-$ to denote the right- and left-hand limits at $x \in \mathbb{R}$, respectively. Additionally, terms like strictly increasing and strictly decreasing indicate a function's

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strict monotonicity, while non-decreasing and non-increasing indicate weak monotonicity. Convexity of a function is always understood in the weak sense.

2. Problem setting

Defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X = (X(t); t \geq 0)$ be a one-dimensional spectrally negative Lévy process. For $x \in \mathbb{R}$, we use $\mathbb{P}_x$ to denote the law of X with initial value x, and $\mathbb{E}_x$ to denote the corresponding expectation operator. When $x = 0$, we drop the subscript and simply write $\mathbb{P}$ and $\mathbb{E}$. The *Laplace exponent* of X is given by

$$\psi(s) := \log \mathbb{E}\left[\mathrm{e}^{sX(1)}\right] = \gamma s + \frac{\sigma^2}{2}s^2 + \int_{(-\infty,0)} (\mathrm{e}^{sz} - 1 - sz\mathbf{1}_{\{z>-1\}})\,\mu(\mathrm{d}z), \quad s \geq 0, \tag{2.1}$$

for some $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and a Lévy measure μ on $(-\infty, 0)$ satisfying $\int_{(-\infty,0)} (1 \wedge z^2)\,\mu(\mathrm{d}z) < \infty$.

The process X has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-\infty,0)} (1 \wedge |z|)\,\mu(\mathrm{d}z) < \infty$. In the case of bounded variation, X admits the form $X(t) = \delta t - S(t)$, $t \geq 0$, where $(S(t); t \geq 0)$ is a subordinator and $\delta := \gamma - \int_{(-1,0)} z\,\mu(\mathrm{d}z)$. We assume that X is not the negative of a subordinator, and therefore $\delta > 0$.

In this paper, we consider the following singular control problem. Let $\mathcal{T}_r := (T(i); i \geq 0)$ be the arrival times of a Poisson process $N_r = (N_r(t); t \geq 0)$ with intensity $r > 0$. The Poisson process N_r and its corresponding arrival times $\mathcal{T}_r$ are independent of X. Let $\mathbb{F} := (\mathcal{F}(t); t \geq 0)$ denote the (completed) filtration generated by (X, N_r). A strategy $\pi = \{(R^\pi(t), L^\pi(t)); t \geq 0\}$ is a pair of non-decreasing, càdlàg, and $\mathbb{F}$-adapted processes, with $R^\pi(0-) = L^\pi(0-) = 0$. Under π, the controlled process $Y^\pi = (Y^\pi(t); t \geq 0)$ is given by

$$Y^\pi(t) := X(t) + R^\pi(t) - L^\pi(t), \quad t \geq 0.$$

Specifically, the process R^π controls the state process in the upward direction and can be activated continuously in time, while the process L^π controls it in the downward direction and can only be activated at the arrival times $\mathcal{T}_r$ of the Poisson process N_r. Mathematically, L^π admits the form

$$L^\pi(t) = \int_{[0,t]} \nu^\pi(s)\,\mathrm{d}N_r(s) = \sum_{i=1}^{\infty} \nu^\pi(T(i))\mathbf{1}_{\{T(i)\leq t\}}$$

for an $\mathbb{F}$-adapted càglàd process $(\nu^\pi(t); t \geq 0)$.

Fix a discount factor $q > 0$ and an initial value $x \in \mathbb{R}$ for the state process. Associated with each admissible strategy π, the NPV of costs is given by

$$v^\pi(x) = \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt} f(Y^\pi(t))\,\mathrm{d}t + \int_{[0,\infty)} \mathrm{e}^{-qt}(C_{\mathrm{U}}\,\mathrm{d}R^\pi(t) + C_{\mathrm{D}}\,\mathrm{d}L^\pi(t))\right].$$

Here, $f\colon \mathbb{R} \to \mathbb{R}$ is a function modeling the running cost, and C_{U} and C_{D} are real numbers representing the unit costs or rewards of control. For our problem to be well-defined, we assume

$$C_{\mathrm{U}} + C_{\mathrm{D}} > 0. \tag{2.2}$$

Assumption (2.2) is standard in the literature, such as in [5].

We impose the following assumptions.

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Assumption 2.1. *We assume that* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfies the following:*

(i) *f is a convex, piecewise continuously differentiable function. Additionally, it is assumed to be slowly or regularly varying as $x \to \infty$ (resp. $x \to -\infty$) if* $\lim_{x\to\infty} |f(x)| = \infty$ *(resp.* $\lim_{x\to-\infty} |f(x)| = \infty$*).*

(ii) *There exists a number* $\bar{a} := \inf\{a \in \mathbb{R}\colon \tilde{f}'(a) := f'(a) + qC_{\mathrm{U}} \geq 0\} \in \mathbb{R}$ *such that the function*

$$\tilde{f}(x) := f(x) + qC_{\mathrm{U}}x, \quad x \in \mathbb{R}, \tag{2.3}$$

is strictly decreasing on $(-\infty, \bar{a})$ and non-decreasing on $(\bar{a}, \infty)$.

(iii) *There exists a number* $\bar{\bar{a}} := \inf\{a \in \mathbb{R}\colon f'(a) - qC_{\mathrm{D}} > 0\} \in \mathbb{R}$.

Here and in the remainder of this paper we use $f'(x)$ to denote the right derivative of f at x, whenever the standard derivative does not exist.

In Assumptions 2.1, the most crucial condition is convexity, which is standard in similar singular control problems (see, for example, [5] and [22]). The growth condition in (i) is imposed for integrability, and it is satisfied by any convex, piecewise continuously differentiable function f that is a polynomial of degree $n \geq 1$ outside a compact interval. The growth behavior of f is summarized in the following remark.

Remark 2.1. By Assumption 2.1, the running cost function f grows at most polynomially, and the derivative satisfies $f'(-\infty) := \lim_{x\to-\infty} f'(x) \in [-\infty, -qC_{\mathrm{U}})$ and $f'(\infty) := \lim_{x\to\infty} f'(x) \in (qC_{\mathrm{D}}, \infty]$. Moreover, by (2.2) and the convexity of f, we have $\bar{a} < \bar{\bar{a}}$.

Assumptions 2.1(ii) and (iii) are imposed to avoid cases where it is optimal not to control the process from below or above. In [5], (ii) is imposed, but (iii) is not explicitly specified. The discussion for cases in which (iii) does not hold is deferred to Remark 4.2.

We also impose the following assumption for integrability. In conjunction with Assumption 2.1(i), it guarantees that $\mathbb{E}[X(1)] = \psi'(0+) \in (-\infty, \infty)$ and $\mathbb{E}_x[\int_0^\infty \mathrm{e}^{-qs}|f(X(s))|\,\mathrm{d}s] < \infty$ for all $x \in \mathbb{R}$.

Assumption 2.2. *There exists $\theta > 0$ such that* $\int_{(-\infty,-1]} \exp(\theta|y|)\,\mu(\mathrm{d}y) < \infty$.

Let Π be the set of *admissible strategies*, consisting of all π such that:

- $\mathbb{E}_x[\int_0^\infty \mathrm{e}^{-qs}|f(Y^\pi(s))|\,\mathrm{d}s] < \infty$, and
- $\mathbb{E}_x[\int_{[0,\infty)} \mathrm{e}^{-qs}\,(\mathrm{d}R^\pi(s) + \mathrm{d}L^\pi(s))] < \infty$.

The objective of this singular control problem is to compute the *value function* (i.e. the optimal cost function)

$$v(x) := \inf_{\pi\in\Pi} v^\pi(x), \quad x \in \mathbb{R},$$

and an associated optimal strategy π^* such that $v(x) = v^{\pi^*}(x)$, if such a strategy exists.

3. Periodic–classical barrier strategies

We now introduce the set of periodic–classical barrier strategies, comprising strategies of the form $\pi_{a,b} := \{(L^{a,b}(t), R^{a,b}(t))\colon t \geq 0\}$, parameterized by two barriers $a < b$. Under $\pi_{a,b}$, the controlled process

$$Y^{a,b}(t) := X(t) + R^{a,b}(t) - L^{a,b}(t), \quad t \geq 0,$$

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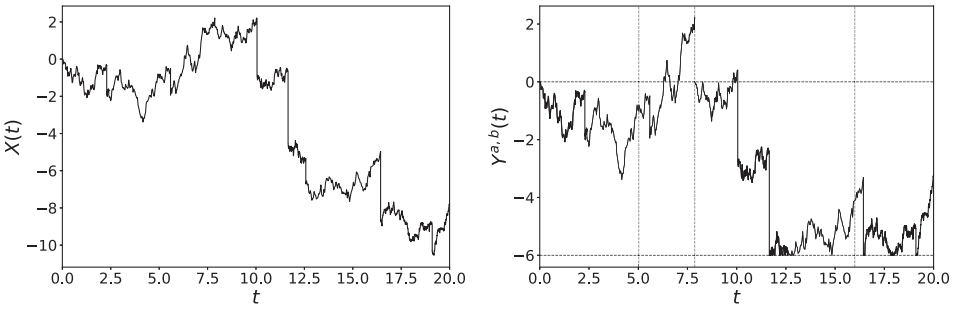


FIGURE 1. Left: A sample path of a spectrally negative Lévy process X. Right: The corresponding controlled process $Y^{a,b}$ with $a=-6$ and $b=0$. Vertical dashed lines indicate the Poisson arrival times $\mathcal{T}_r$. Among the three arrival times displayed, $Y^{a,b}$ exceeds $b=0$ only at the second one, causing downward control to activate there but not at the first or third arrival times.

is continuously reflected from below at the lower barrier a and periodically reflected from above at the upper barrier b. In other words, the controlled process is never allowed to fall below a and is thus reflected upwards whenever it would fall below this barrier. On the other hand, at any decision time in $\mathcal{T}_r$, if the controlled process is observed above b, it is immediately pushed down to b. The process $Y^{a,b}$ has been studied in works such as [19], among others. As described in [19], it can be defined as a concatenation of classical reflected processes with additional downward jumps. An example is provided in Figure 1.

Previous works, such as [5] and [22], have shown the optimality of barrier-type strategies for one- and two-sided stochastic singular control problems by identifying an optimal barrier, or an optimal pair of barriers, according to the smooth-fit principle. For instance, in the continuous-monitoring version of this problem studied in [5], an optimal pair of barriers, denoted by (a^*, b^*), is selected such that the corresponding NPV of costs satisfies a certain smoothness condition. In this paper, we instead seek (a^*, b^*) such that $\mathfrak{C}$ holds:

$$\mathfrak{C}\colon \mathbb{E}_{a^*}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{a^*,b^*}(t))\,\mathrm{d}t\right] = -C_{\mathrm{U}}, \quad \mathbb{E}_{b^*}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{a^*,b^*}(t))\,\mathrm{d}t\right] = C_{\mathrm{D}}.$$

A similar approach was recently applied in [20] to establish the optimality of a single-barrier strategy for the one-sided control problem considered therein. By revisiting the two-sided control problem in [5] through this approach, we can further confirm that the conditions outlined above are satisfied. Hence, although our problem is more complex than the one studied in [5], we expect that this approach will be similarly effective in identifying the optimal barriers in our setting.

The advantage of this approach is that it allows for a probabilistic interpretation. We observe that, intuitively, $\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-qt} f'(Y^\pi(t))\,\mathrm{d}t\big]$ represents the rate of change of the running cost when the controlled process Y^π with initial value x is shifted to the right by an infinitesimal amount. Thus, it is desirable for a strategy π to keep Y^π within the set $\mathcal{C}^\pi$ for as long as possible, where

$$\mathcal{C}^\pi := \left\{x \in \mathbb{R}\colon\ -C_{\mathrm{U}} \le \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^\pi(t))\,\mathrm{d}t\right] \le C_{\mathrm{D}}\right\}.$$

We now present Theorem 3.1, the main result of this paper.

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Theorem 3.1. *There exists a unique pair* (a^*, b^*) *such that* $\mathfrak{C}$ *holds. Furthermore, the strategy* π_{a^*,b^*} *is optimal with* $v_{a^*,b^*}(x) = \inf_{\pi\in\Pi} v^{\pi}(x)$, $x \in \mathbb{R}$, *where* v_{a^*,b^*} *denotes the NPV of costs under the periodic–classical barrier strategy* π_{a^*,b^*}.

The proof of Theorem 3.1 is developed in Section 4 through a series of lemmas and propositions. The existence and uniqueness of (a^*, b^*) are established by first computing, and then analyzing, the expression for $v_{a,b}$ for $a < b$. Specifically, a semi-explicit expression for $v_{a,b}$ is obtained using scale functions, and the pair (a^*, b^*) is identified as the solution to a pair of equations that also admit semi-explicit expressions (see (4.20)). Once existence has been established, the optimality of v_{a^*,b^*} is demonstrated through further analysis of the semi-explicit expression of $v_{a,b}$ with $a = a^*$ and $b = b^*$, in conjunction with a conventional application of Itô's lemma. The complete proof of Theorem 3.1 is presented at the end of Section 4.

4. Solution

As a first step towards Theorem 3.1, we compute the expression of the NPV of costs under $\pi_{a,b}$. To this end, we introduce the *scale functions* of a spectrally negative Lévy process and define several auxiliary functions that are relevant to our subsequent computations.

4.1. Scale functions and fluctuation identities

For $q \geq 0$, the scale function $W^{(q)}\colon \mathbb{R} \to [0, \infty)$ of the spectrally negative Lévy process X is defined as follows. On the negative half-line, $W^{(q)}$ is set to be zero. On the positive half-line, it is a continuous and strictly increasing function that satisfies

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi_q. \tag{4.1}$$

Here, ψ is the Laplace exponent defined in (2.1), and $\Phi_q := \sup\{s \geq 0\colon \psi(s) = q\}$, $q \geq 0$, is its right inverse. We refer the readers to [11, 12] for a comprehensive review of scale functions. In particular, [11, Lemmas 2.3, 2.4, 3.1, and 3.2], which expound the smoothness of the scale functions, are of particular importance.

We define, for $x \in \mathbb{R}$,

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y)\,\mathrm{d}y, \qquad \overline{\overline{W}}^{(q)}(x) := \int_0^x \overline{W}^{(q)}(y)\,\mathrm{d}y.$$

For $x \in \mathbb{R}$ and $r > 0$, we also define the *second scale function*,

$$Z^{(q)}(x, \Phi_{q+r}) := \mathrm{e}^{\Phi_{q+r}x}\left(1 - r\int_0^x \mathrm{e}^{-\Phi_{q+r}z} W^{(q)}(z)\,\mathrm{d}z\right) = r\int_0^\infty \mathrm{e}^{-\Phi_{q+r}z} W^{(q)}(z + x)\,\mathrm{d}z, \tag{4.2}$$

where the second equality holds because (4.1) gives $\int_0^\infty \mathrm{e}^{-\Phi_{q+r}x} W^{(q)}(x)\,\mathrm{d}x = r^{-1}$. Differentiating the second scale function with respect to the first argument, we obtain

$$Z^{(q)'}(x, \Phi_{q+r}) := \frac{\partial}{\partial x} Z^{(q)}(x, \Phi_{q+r}) = \Phi_{q+r} Z^{(q)}(x, \Phi_{q+r}) - r W^{(q)}(x), \quad x > 0.$$

Moreover, for $q \geq 0$ and $x \in \mathbb{R}$, we define

$$Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x), \qquad \overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(y)\,\mathrm{d}y = x + q\overline{\overline{W}}^{(q)}(x).$$

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For $b > a$, $x \in \mathbb{R}$, and $h\colon \mathbb{R} \to \mathbb{R}$, we define

$$\rho^{(q)}_{a,b}(x;h) := \int_a^b W^{(q)}(x-y)h(y)\,\mathrm{d}y, \tag{4.3}$$

$$\rho^{(q,r)}_{a,b}(x;h) := \rho^{(q)}_{a,b}(x;h) + r\int_b^x W^{(q+r)}(x-y)\rho^{(q)}_{a,b}(y;h)\,\mathrm{d}y. \tag{4.4}$$

Additionally, for $x \in \mathbb{R}$, we define

$$W^{(q,r)}_{a,b}(x) := W^{(q)}(x-a) + r\int_b^x W^{(q+r)}(x-y)W^{(q)}(y-a)\,\mathrm{d}y, \tag{4.5}$$

$$Z^{(q,r)}_{a,b}(x) := Z^{(q)}(x-a) + r\int_b^x W^{(q+r)}(x-y)Z^{(q)}(y-a)\,\mathrm{d}y, \tag{4.6}$$

$$\overline{Z}^{(q,r)}_{a,b}(x) := \overline{Z}^{(q)}(x-a) + r\int_b^x W^{(q+r)}(x-y)\overline{Z}^{(q)}(y-a)\,\mathrm{d}y. \tag{4.7}$$

It is convenient to decompose the NPV of costs into the following two components, both of which can be expressed in terms of the scale functions $W^{(q)}$ and $W^{(q+r)}$, as well as the auxiliary functions defined above:

$$v_{a,b}(x) = v^{LR}_{a,b}(x) + v^{f}_{a,b}(x), \quad x \in \mathbb{R},$$

where

$$v^{LR}_{a,b}(x) := C_{\mathrm{D}}\mathbb{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}L^{a,b}(t)\right] + C_{\mathrm{U}}\mathbb{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}R^{a,b}(t)\right],$$

$$v^{f}_{a,b}(x) := \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}f(Y^{a,b}(t))\,\mathrm{d}t\right].$$

To compute $v^{LR}_{a,b}$, we apply the result established in the proof of [19, Lemma 3.1], which expresses $v^{LR}_{0,b}$ for $b > 0$ in terms of the scale functions and the auxiliary functions.

Lemma 4.1 ([19, Lemma 3.1].) *For $a < b$ and $x \in \mathbb{R}$,*

$$\begin{aligned} v^{LR}_{a,b}(x) = {} & \left(\frac{r}{q\Phi_{q+r}}\frac{C_{\mathrm{U}}Z^{(q)}(b-a) + C_{\mathrm{D}}}{Z^{(q)}(b-a,\,\Phi_{q+r})} + \frac{C_{\mathrm{U}}}{\Phi_{q+r}}\right)\left(Z^{(q,r)}_{a,b}(x) - rZ^{(q)}(b-a)\overline{W}^{(q+r)}(x-b)\right) \\ & - rC_{\mathrm{D}}\overline{\overline{W}}^{(q+r)}(x-b) - C_{\mathrm{U}}\left(\overline{Z}^{(q,r)}_{a,b}(x) + \frac{\psi'(0+)}{q} - r\overline{Z}^{(q)}(b-a)\overline{W}^{(q+r)}(x-b)\right). \end{aligned} \tag{4.8}$$

The running cost $v^{f}_{a,b}$ is computed using [17, Proposition 5.1]. Details of the computation are given in Appendix A.1.

Lemma 4.2. *For $a < b$ and $x \in \mathbb{R}$,*

$$\begin{aligned} v^{f}_{a,b}(x) = {} & \frac{1}{q}\left(f(a) + \int_a^\infty f'(y)\frac{Z^{(q)}(b-y,\,\Phi_{q+r})}{Z^{(q)}(b-a,\,\Phi_{q+r})}\,\mathrm{d}y\right)Z^{(q,r)}_{a,b}(x) - \rho^{(q,r)}_{a,b}(x;f) \\ & - r\overline{W}^{(q+r)}(x-b)v^{f}_{a,b}(b) - \int_b^x f(y)W^{(q+r)}(x-y)\,\mathrm{d}y, \end{aligned} \tag{4.9}$$

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where

$$v^f_{a,b}(b) = -\rho^{(q)}_{a,b}(b; f) + \frac{1}{q}\left(f(a) + \int_a^\infty f'(y)\frac{Z^{(q)}(b-y,\,\Phi_{q+r})}{Z^{(q)}(b-a,\,\Phi_{q+r})}\,\mathrm{d}y\right)Z^{(q)}(b-a).$$

Combining (4.8) and (4.9), and applying integration by parts, we obtain Lemma 4.3, with the proof provided in Appendix A.2. Recall $\tilde{f}$ as defined in (2.3).

Lemma 4.3. *For $a < b$ and $x \in \mathbb{R}$,*

$$\begin{aligned} v_{a,b}(x) = {} & \frac{1}{q}\left(\tilde{f}(a) + \frac{\Gamma(a,b)}{Z^{(q)}(b-a,\,\Phi_{q+r})}\right)\left(Z^{(q,r)}_{a,b}(x) - r\overline{W}^{(q+r)}(x-b)Z^{(q)}(b-a)\right) \\ & - r(C_{\mathrm{U}} + C_{\mathrm{D}})\overline{\overline{W}}^{(q+r)}(x-b) - C_{\mathrm{U}}x + r\overline{W}^{(q+r)}(x-b)\rho^{(q)}_{a,b}(b;\tilde{f}) \\ & - \tilde{f}(b)\overline{W}^{(q+r)}(x-b) - \rho^{(q,r)}_{a,b}(x;\tilde{f}) - C_{\mathrm{U}}\frac{\psi'(0+)}{q} - \int_b^x \overline{W}^{(q+r)}(x-y)\tilde{f}'(y)\,\mathrm{d}y, \end{aligned} \tag{4.10}$$

where

$$\Gamma(a,b) := \int_a^\infty \tilde{f}'(y)Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y + \frac{r}{\Phi_{q+r}}(C_{\mathrm{U}} + C_{\mathrm{D}}), \quad a < b. \tag{4.11}$$

4.2. Smoothness of $v_{a,b}$

We now analyze the smoothness of the NPV of costs as computed in (4.10).

Lemma 4.4. *For $a < b$ and $x \in \mathbb{R}\backslash\{a\}$,*

$$\begin{aligned} v'_{a,b}(x) = {} & \frac{\Gamma(a,b)}{Z^{(q)}(b-a,\,\Phi_{q+r})}W^{(q,r)}_{a,b}(x) - \rho^{(q,r)}_{a,b}(x;\tilde{f}') \\ & - \int_b^x W^{(q+r)}(x-y)\tilde{f}'(y)\,\mathrm{d}y - r(C_{\mathrm{U}} + C_{\mathrm{D}})\overline{W}^{(q+r)}(x-b) - C_{\mathrm{U}}. \end{aligned} \tag{4.12}$$

If X is of unbounded variation, for $a < b$ and $x \in \mathbb{R}\backslash\{a\}$,

$$\begin{aligned} v''_{a,b}(x) = {} & \frac{\Gamma(a,b)}{Z^{(q)}(b-a,\,\Phi_{q+r})}\left(\frac{\mathrm{d}}{\mathrm{d}x}W^{(q,r)}_{a,b}(x)\right) \\ & - \left(\int_a^b W^{(q)'}(x-y)\tilde{f}'(y)\,\mathrm{d}y + r\int_b^x W^{(q+r)'}(x-y)\rho^{(q)}_{a,b}(y;\tilde{f}')\,\mathrm{d}y\right) \\ & - \int_b^x W^{(q+r)'}(x-y)\tilde{f}'(y)\,\mathrm{d}y - r(C_{\mathrm{U}} + C_{\mathrm{D}})W^{(q+r)}(x-b). \end{aligned} \tag{4.13}$$

The proof of Lemma 4.4 is provided in Appendix A.3. The derivative $(\mathrm{d}/\mathrm{d}x)W^{(q,r)}_{a,b}(x)$ in (4.13) is well-defined for $x \neq a$ in light of (4.5) and the smoothness of scale functions as in [11, Lemma 2.4]. In the remainder of this paper, we call a function *sufficiently smooth* if it is continuously differentiable on $\mathbb{R}$ when X is of bounded variation and twice continuously differentiable on $\mathbb{R}$ when X is of unbounded variation.

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The following results establish the equivalence between our approach and the classical approach based on the smooth-fit principle.

Proposition 4.1. *For $a < b$, the function $x \mapsto v_{a,b}(x)$ is sufficiently smooth if and only if $\Gamma(a, b) = 0$.*

Proof. By Lemma 4.4, we have

$$v'_{a,b}(a-) = -C_{\mathrm{U}}, \qquad v'_{a,b}(a+) = -C_{\mathrm{U}} + \frac{\Gamma(a, b)}{Z^{(q)}(b-a, \Phi_{q+r})} W^{(q)}(0)$$

in the case that X is of bounded variation, and

$$v''_{a,b}(a-) = 0, \qquad v''_{a,b}(a+) = \frac{\Gamma(a, b)}{Z^{(q)}(b-a, \Phi_{q+r})} W^{(q)'}(0+)$$

when X is of unbounded variation. Now, by applying [11, Lemmas 3.1 and 3.2], we find that the function $x \mapsto v_{a,b}(x)$ is sufficiently smooth if and only if $\Gamma(a, b) = 0$. □

Define the partial derivative of Γ as follows:

$$\begin{aligned}\gamma(a, b) := \frac{\partial}{\partial b}\Gamma(a, b) &= \Phi_{q+r} \int_a^\infty \tilde{f}'(y) Z^{(q)}(b-y, \Phi_{q+r})\, \mathrm{d}y - r\rho^{(q)}_{a,b}(b; \tilde{f}') \\ &= \Phi_{q+r}\Gamma(a, b) - r\big(C_{\mathrm{U}} + C_{\mathrm{D}} + \rho^{(q)}_{a,b}(b; \tilde{f}')\big), \quad a < b. \end{aligned} \tag{4.14}$$

Note that $\gamma(a, b)$ is independent of the values of C_{U} and C_{D}, as the second equality of (4.14) holds by adding and subtracting $r(C_{\mathrm{U}} + C_{\mathrm{D}})$, together with (4.11). This equality is included only to simplify complex expressions that appear later in the paper. The proof of Lemma 4.2 is presented in Appendix A.4.

Lemma 4.5. *For $a < b$ and $x \in \mathbb{R}$, we have*

$$\begin{aligned} v^{f'}_{a,b}(x) &:= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{a,b}(t))\, \mathrm{d}t\right] \\ &= \frac{\gamma(a, b)}{qZ^{(q)}(b-a; \Phi_{q+r})} Z^{(q,r)}_{a,b}(x) - \rho^{(q,r)}_{a,b}(x; \tilde{f}') - \int_b^x W^{(q+r)}(x-y)\tilde{f}'(y)\, \mathrm{d}y \\ &\quad - r\overline{W}^{(q+r)}(x-b)\left(-\rho^{(q)}_{ab}(b; \tilde{f}') + \frac{Z^{(q)}(b-a)}{qZ^{(q)}(b-a; \Phi_{q+r})}\gamma(a, b)\right) - C_{\mathrm{U}}. \end{aligned} \tag{4.15}$$

In particular, at $x = a$ and $x = b$,

$$v^{f'}_{a,b}(a) = \frac{\gamma(a, b)}{qZ^{(q)}(b-a; \Phi_{q+r})} - C_{\mathrm{U}}, \tag{4.16}$$

$$v^{f'}_{a,b}(b) = \frac{Z^{(q)}(b-a)}{qZ^{(q)}(b-a; \Phi_{q+r})}\gamma(a, b) - \rho^{(q)}_{a,b}(b; \tilde{f}') - C_{\mathrm{U}}. \tag{4.17}$$

Proposition 4.2. *For $a < b$, the following statements are equivalent:*

(i) $v^{f'}_{a,b}(a) = -C_{\mathrm{U}}$ *and* $v^{f'}_{a,b}(b) = C_{\mathrm{D}}$.

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(ii) $\Gamma(a, b) = 0$ *and* $\gamma(a, b) = 0$.

(iii) $v_{a,b}$ *is sufficiently smooth, with* $v'_{a,b}(a) = -C_{\mathrm{U}}$ *and* $v'_{a,b}(b) = C_{\mathrm{D}}$.

Proof. For (i) $\Longleftrightarrow$ (ii), first, by (4.16),

$$v^{f'}_{a,b}(a) = -C_{\mathrm{U}} \Longleftrightarrow \gamma(a, b) = 0. \tag{4.18}$$

By (4.14), we have

$$\rho^{(q)}_{a,b}(b; \tilde{f}') = \frac{\Phi_{q+r}\Gamma(a, b) - \gamma(a, b)}{r} - (C_{\mathrm{U}} + C_{\mathrm{D}}). \tag{4.19}$$

Substituting (4.19) into (4.17), we have

$$v^{f'}_{a,b}(b) = -\frac{\Phi_{q+r}}{r}\Gamma(a, b) + \left(\frac{Z^{(q)}(b-a)}{qZ^{(q)}(b-a; \Phi_{q+r})} + \frac{1}{r}\right)\gamma(a, b) + C_{\mathrm{D}}.$$

The equivalence follows from the last equality and (4.18).

For (ii) $\Longleftrightarrow$ (iii), substituting (4.19) in (4.12) with $x = b$,

$$v'_{a,b}(b) = \left(\frac{W^{(q)}(b-a)}{Z^{(q)}(b-a, \Phi_{q+r})} - \frac{\Phi_{q+r}}{r}\right)\Gamma(a, b) + \frac{\gamma(a, b)}{r} + C_{\mathrm{D}}.$$

Thus, $v'_{a,b}(b) = C_{\mathrm{D}}$ is equivalent to $\gamma(a, b) = 0$ when $\Gamma(a, b) = 0$. Finally, Lemma 4.1 shows the equivalence. □

4.3. Selection of barriers

The next step of our analysis involves establishing the existence of (a^*, b^*) satisfying

$$\mathfrak{C}\colon\ v^{f'}_{a^*,b^*}(a^*) = -C_{\mathrm{U}} \quad \text{and} \quad v^{f'}_{a^*,b^*}(b^*) = C_{\mathrm{D}},$$

which is, by Proposition 4.2, equivalent to

$$\mathfrak{C}'\colon\ \Gamma(a^*, b^*) = \gamma(a^*, b^*) = 0. \tag{4.20}$$

Recall $\bar{a} = \inf\{a \in \mathbb{R} : \tilde{f}'(a) \geq 0\}$ as defined in Assumption 2.1(ii). We can immediately eliminate the half-line $[\bar{a}, \infty)$ from consideration for a^*. Indeed, for $a \geq \bar{a}$, we have $\tilde{f}'(y + a) \geq \tilde{f}'(y + \bar{a}) \geq 0$ for all $y \geq 0$ by Assumption 2.1(i). Thus, from (4.11), we have $\Gamma(a, b) > 0$ for $b > a \geq \bar{a}$. We show that, by decreasing the value of a from $\bar{a}$, we reach a^* such that the function $b \mapsto \Gamma(a^*, b)$ starts at $\Gamma(a^*, a^* +) > 0$ and becomes tangent to the x-axis at b^*, where its partial derivative $\gamma(a^*, b^*)$ is zero.

To facilitate the proof of the existence of (a^*, b^*), we analyze the following auxiliary functions.

Lemma 4.6. *For all* $a \in \mathbb{R}$,

$$\Gamma_1(a) := \Gamma(a, a+) = \int_0^\infty \mathrm{e}^{-\Phi_{q+r}y}\tilde{f}'(y + a)\,\mathrm{d}y + \frac{r}{\Phi_{q+r}}(C_{\mathrm{U}} + C_{\mathrm{D}}), \tag{4.21}$$

$$\Gamma_2(a) := \lim_{b\to\infty} \frac{\Gamma(a, b)}{Z^{(q)}(b-a, \Phi_{q+r})} = \int_0^\infty \mathrm{e}^{-\Phi_q y}\tilde{f}'(y + a)\,\mathrm{d}y. \tag{4.22}$$

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The proof of Lemma 4.6 is deferred to Appendix A.5. We note that both $a \mapsto \Gamma_1(a)$ and $a \mapsto \Gamma_2(a)$ are continuous. Moreover, both are strictly increasing on $(-\infty, \bar{a})$ due to the convexity of $\tilde{f}$ and Assumption 2.1(iii). Define their left inverses,

$$\underline{a}_i := \inf\{a \in \mathbb{R} \colon \Gamma_i(a) \geq 0\}, \quad i = 1, 2. \tag{4.23}$$

Observe that $\underline{a}_1 \in [-\infty, \bar{a})$ and $\underline{a}_2 \in (-\infty, \bar{a})$. To establish that $\underline{a}_2 > -\infty$, note that by monotone convergence and the convexity of $\tilde{f}$, we have $\Gamma_2(a) \xrightarrow{a \downarrow -\infty} \int_0^\infty \mathrm{e}^{-\Phi_q z} \tilde{f}'(-\infty)\, \mathrm{d}z \in [-\infty, 0)$. Hence, $\underline{a}_2 > -\infty$. To show that both $\underline{a}_1$ and $\underline{a}_2$ are upper bounded by $\bar{a}$, note that by (2.2) and Assumption 2.1(ii), we have

$$\Gamma_1(\bar{a}) = \int_0^\infty \mathrm{e}^{-\Phi_{q+r} y} \tilde{f}'(y + \bar{a})\, \mathrm{d}y + \frac{r}{\Phi_{q+r}}(C_{\mathrm{U}} + C_{\mathrm{D}}) > 0,$$

which implies $\bar{a} > \underline{a}_1$. Additionally, by Assumption 2.1(iii), $\Gamma_2(\bar{a}) > 0$, which implies $\bar{a} > \underline{a}_2$.

Remark 4.1. The value $\underline{a}_2$ is the optimal barrier in the setting where only upward control is permitted; see [23, Section 7]. This follows from the definition of $\underline{a}_2$ in (4.23), which gives

$$0 = \frac{\Phi_q}{q}\Gamma_2(\underline{a}_2) = \frac{\Phi_q}{q}\int_0^\infty \mathrm{e}^{-\Phi_q y} f'(y + \underline{a}_2)\, \mathrm{d}y + C_{\mathrm{U}} = \mathbb{E}_{\underline{a}_2}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{\underline{a}_2,\infty}(t))\, \mathrm{d}t\right] + C_{\mathrm{U}},$$

where $Y^{\underline{a}_2,\infty}$ denotes the process reflected from below at the level $\underline{a}_2$, and the last equality holds by [11, Theorem 2.8(iii)]. Thus,

$$\mathbb{E}_{\underline{a}_2}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{\underline{a}_2,\infty}(t))\, \mathrm{d}t\right] = -C_{\mathrm{U}}. \tag{4.24}$$

Moreover, under certain conditions in our current setting, an optimal strategy is the barrier strategy that continuously reflects the state process from below at the lower barrier $\underline{a}_2$, as summarized in the following remark.

Remark 4.2. Suppose that Assumption 2.1(iii) fails to hold (thus $f' \leq qC_{\mathrm{D}}$ on $\mathbb{R}$) and all other standing assumptions, including Assumptions 2.1(i) and (ii), hold. Then,

- not activating the downward control $(L^\pi(t); t \geq 0)$ is optimal;
- $v_{\underline{a}_2,\infty}(x) = \inf_{\pi \in \Pi} v^\pi(x)$ for all $x \in \mathbb{R}$, where

$$v_{\underline{a}_2,\infty}(x) := \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt} f(Y^{\underline{a}_2,\infty}(t))\, \mathrm{d}t + C_{\mathrm{U}} \int_{[0,\infty)} \mathrm{e}^{-qt}\, \mathrm{d}R^{\underline{a}_2,\infty}(t)\right].$$

Verifying the above remark is straightforward. The first statement holds since the downward control cost is no less than the reduction in the running cost. The second statement holds due to the first and [23, Theorem 7.1].

As a next step towards establishing the existence of (a^*, b^*), we define

$$\underline{\Gamma}(a) := \inf_{b > a} \Gamma(a, b), \quad a \in \mathbb{R}.$$

The proofs of the following two lemmas are provided in Appendices A.6 and A.7.

Lemma 4.7. (i) $\underline{\Gamma}(\bar{a}) > 0$; (ii) $\underline{\Gamma}(\underline{a}_1 \vee \underline{a}_2) < 0$.

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Lemma 4.8. *The mapping $a \mapsto \underline{\Gamma}(a)$ is continuous and strictly increasing on $(\underline{a}_1 \vee \underline{a}_2, \bar{a})$.*

We are now ready to establish the existence and uniqueness of the candidate barriers.

Proposition 4.3. *There exists a unique pair (a^*, b^*) such that $\mathfrak{C}$ holds.*

Proof. By Lemmas 4.7 and 4.8, there exists a unique root $a^* \in (\underline{a}_1 \vee \underline{a}_2, \infty)$ such that $\underline{\Gamma}(a^*) = 0$. Moreover, because $\Gamma(a^*, a^* +) > 0$ and $\lim_{b\to\infty} \Gamma(a^*, b) = \infty$ in view of the definitions of $\underline{a}_1$ and $\underline{a}_2$, the minimum of $b \mapsto \Gamma(a^*, b)$ is attained at some $b^* \in (a^*, \infty)$. By the continuity of $b \mapsto \gamma(a^*, b)$, it must hold that $\gamma(a^*, b^*) = 0$. Thus, $\mathfrak{C}'$, or equivalently, $\mathfrak{C}$, holds. Moreover, such a b^* is unique. To establish uniqueness, recall that b^* must satisfy $v^{f'}_{a^*,b^*}(b^*) = C_{\mathrm{D}}$, as required by $\mathfrak{C}$. Given the probabilistic expression $v^{f'}_{a^*,b}(b) = \mathbb{E}_b\big[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{a^*,b}(t))\,\mathrm{d}t\big]$, it is clear that the mapping $b \mapsto v^{f'}_{a^*,b}(b)$ is non-decreasing on (a^*, ∞). Further, by applying the argument used in the proof of Lemma 4.7, we see that $b \mapsto v^{f'}_{a^*,b}(b)$ is, in fact, strictly increasing. Hence, there is exactly one $b^* > a^*$ such that $\mathfrak{C}'$ (equivalently $\mathfrak{C}$) holds. □

4.4. Verification

As the final step in completing the proof of Theorem 3.1, we establish the optimality of the periodic–classical barrier strategy π_{a^*,b^*}. This is achieved by applying the conventional verification technique based on Itô's lemma, which has been employed in previous works such as [4, 5, 22], among others.

Let $\mathcal{L}$ be the infinitesimal generator associated with X. When applied to a sufficiently smooth function $h\colon \mathbb{R} \to \mathbb{R}$, the following holds for any $x \in \mathbb{R}$:

$$\mathcal{L}h(x) := \gamma h'(x) + \frac{\sigma^2}{2} h''(x) + \int_{(-\infty,0)} h(x+z) - h(x) - h'(x) z \mathbf{1}_{\{-1<z<0\}}\, \mu(\mathrm{d}z).$$

Additionally, we define an operator $\mathcal{M}$ for a measurable function h,

$$\mathcal{M}h(x) := \inf_{l\geq 0}\{C_{\mathrm{D}} l + h(x-l)\},$$

which has been used to establish the optimality of barrier-type strategies in periodic control problems, as in [17, 22]. The following lemma gives the conditions that are sufficient for the optimality of π_{a^*,b^*}; its proof is provided in Appendix A.8.

Lemma 4.9. *Let $w\colon \mathbb{R} \to \mathbb{R}$ be the NPV of costs under an admissible strategy. Suppose that w has the following properties:*

- *it is sufficiently smooth on $\mathbb{R}$,*
- *it has at most polynomial growth, and*
- *it satisfies $w' \geq -C_{\mathrm{U}}$.*

Further suppose that $(\mathcal{L} - q)w(x) + r(\mathcal{M}w(x) - w(x)) + f(x) \geq 0$, $x \in \mathbb{R}$, and that, for every admissible strategy $\pi \in \Pi$, $\limsup_{t,n\uparrow\infty} \mathbb{E}_x[\mathrm{e}^{-q(t\wedge\tau_n)} w(Y^\pi(t \wedge \tau_n))] \leq 0$, where $\tau_n := \inf\{t \geq 0\colon |Y^\pi(t)| > n\}$. Under these conditions, w coincides with the value function, i.e. $w(x) = v(x) = \inf_{\pi\in\Pi} v^\pi(x)$.

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In the remainder of this section, we show that the conditions in Lemma 4.9 are satisfied for the function v_{a^*,b^*}. As a first step, by Lemma 4.1, v_{a^*,b^*} is sufficiently smooth. Moreover, by Assumption 2.2 in conjunction with the linearity of $x \mapsto v_{a^*,b^*}(x)$ below a^*, the integral component of $\mathcal{L}v_{a^*,b^*}$ is finite on $\mathbb{R}$. Thus, $\mathcal{L}v_{a^*,b^*}$ makes sense everywhere on $\mathbb{R}$.

Next, we establish the related properties for the computation of $\mathcal{M}v_{a^*,b^*}$. We note that, with the condition $\mathfrak{C}'$, the function v_{a^*,b^*} is simplified. By $\mathfrak{C}'$ and (4.12), we have

$$v'_{a^*,b^*}(x) = -\rho^{(q,r)}_{a^*,b^*}(x;\tilde{f}') - \int_{b^*}^{x} W^{(q+r)}(x-y)\tilde{f}'(y)\,\mathrm{d}y - r(C_{\mathrm{U}}+C_{\mathrm{D}})\overline{W}^{(q+r)}(x-b^*) - C_{\mathrm{U}}. \tag{4.25}$$

Proposition 4.4.

(i) $v^{f'}_{a^*,b^*}(x) = v'_{a^*,b^*}(x)$ *for all* $x \in \mathbb{R}$.

(ii) $x \mapsto v_{a^*,b^*}(x)$ *is convex on* $\mathbb{R}$.

(iii) $v'_{a^*,b^*}(x) \geq -C_{\mathrm{U}}$ *for all* $x \in \mathbb{R}$.

Proof.

(i) By Proposition 4.2, we have $v'_{a^*,b^*}(b^*) = C_{\mathrm{D}}$. Combining this with the expression for v'_{a^*,b^*} in (4.25), we get $C_{\mathrm{D}} = -\rho^{(q)}_{a^*,b^*}(b^*;\tilde{f}') - C_{\mathrm{U}}$. Substituting this and $\gamma(a^*, b^*) = 0$ into (4.15), the function $v^{f'}_{a^*,b^*}$ matches (4.25).

(ii) By the convexity of f and the monotonicity of Y^{a^*,b^*} in the starting point, the mapping $x \mapsto v^{f'}_{a^*,b^*}(x)$ is monotone. Hence, by (i), v_{a^*,b^*} is convex.

(iii) By (ii) and as $v'_{a^*,b^*}(x) = -C_{\mathrm{U}}$ for $x \in (-\infty, a^*]$, $v'_{a^*,b^*}(x) \geq -C_{\mathrm{U}}$ for all $x \in \mathbb{R}$. □

As a direct consequence of Proposition 4.4(ii) and the condition $\mathfrak{C}$, $\mathcal{M}v_{a^*,b^*}$ has the following characterization.

Corollary 4.1.

$$\mathcal{M}v_{a^*,b^*}(x) = \begin{cases} v_{a^*,b^*}(x), & x < b^*, \\ C_{\mathrm{D}}(x-b^*) + v_{a^*,b^*}(b^*), & x \geq b^*. \end{cases}$$

Furthermore, standard computations lead to the following characterization of $(\mathcal{L}-q)v_{a^*,b^*}$, with the proof given in Appendix A.9.

Lemma 4.10.

$$(\mathcal{L}-q)v_{a^*,b^*}(x) + f(x) = \begin{cases} \tilde{f}(x) - \tilde{f}(a^*), & x \leq a^*, \\ 0, & a^* < x < b^*, \\ -r(v_{a^*,b^*}(b^*) - v_{a^*,b^*}(x) + C_{\mathrm{D}}(x-b^*)), & x \geq b^*. \end{cases}$$

The polynomial growth of v_{a^*,b^*} can be established using the proof technique in [22, Lemma 4.7]. As the method is nearly identical, the proof of Lemma 4.11 is omitted. The proof of Lemma 4.12 is presented in Appendix A.10.

Lemma 4.11. *The function* $x \mapsto v_{a^*,b^*}(x)$ *is of polynomial growth.*

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Lemma 4.12. *For all admissible strategies* $\pi \in \Pi$,

$$\limsup_{t,n\uparrow\infty} \mathbb{E}_x[\mathrm{e}^{-q(t\wedge\tau_n)} v_{a^*,b^*}(Y^\pi(t\wedge\tau_n))] \le 0.$$

Having verified that v_{a^*,b^*} satisfies the conditions of Lemma 4.9, its optimality is a direct consequence of the lemma. We now complete the proof of Theorem 3.1, as stated in Section 3.

Proof of Theorem 3.1. First, by Proposition 4.2 and Lemmas 4.4 and 4.11, the function v_{a^*,b^*} is sufficiently smooth on $\mathbb{R}$, has polynomial growth, and satisfies $v'_{a^*,b^*} \ge -C_{\mathrm{U}}$. Additionally, by Corollary 4.1 and Lemma 4.10, we have

- For $x \le a^*$, $(\mathcal{L}-q)v_{a^*,b^*}(x) + r(\mathcal{M}v_{a^*,b^*}(x) - v_{a^*,b^*}(x)) + f(x) = \tilde{f}(x) - \tilde{f}(a^*) \ge 0$, where the inequality holds because $a^* < \bar{a}$, and by Assumption 2.1(ii).
- For $x > a^*$, $(\mathcal{L}-q)v_{a^*,b^*}(x) + r(\mathcal{M}v_{a^*,b^*}(x) - v_{a^*,b^*}(x)) + f(x) = 0$.

Hence, $(\mathcal{L}-q)v_{a^*,b^*}(x) + r(\mathcal{M}v_{a^*,b^*}(x) - v_{a^*,b^*}(x)) + f(x) \ge 0$, $x \in \mathbb{R}$. Since v_{a^*,b^*} is the NPV of costs under the admissible strategy π_{a^*,b^*}, by applying Lemma 4.9, we have $v_{a^*,b^*}(x) = v(x)$ for all $x \in \mathbb{R}$. □

5. Numerical example

This section presents a numerical study. Let $f(x) = x^2$ and X be a spectrally negative Lévy process with exponential jumps given by $X(t) = x + t + B(t) - \sum_{n=1}^{N(t)} Z_n$, where $B = (B(t); t \ge 0)$ is a standard Brownian motion, $N = (N(t); t \ge 0)$ is a Poisson process with arrival rate $\lambda = 0.2$, and $\{Z_n\}_{n\ge1}$ is a sequence of independent and identically distributed exponential random variables with rate 1. By [10, Example 3.1], the q-scale function is

$$W^{(q)}(x) = \frac{\mathrm{e}^{\Phi_q x}}{\psi'(\Phi_q)} - \sum_{i=1}^{2} B_{i,q}\mathrm{e}^{-\xi_{i,q}x},$$

where, for all $i = 1, 2$, we have $B_{i,q} = -1/\psi'(-\xi_{i,q})$, with $\xi_{i,q}$ defined such that $\psi(-\xi_{i,q}) = q$ and $-\xi_{i,q} < 0$. Additionally, set $q = 0.05$, $C_{\mathrm{U}} = 200$, $C_{\mathrm{D}} = 200$, and let r, the rate parameter of the Poisson process governing downward control opportunities, vary.

Figure 2 shows the value function v_{a^*,b^*} alongside the NPV of costs $v_{a,b}$ for various barrier strategies where $(a, b) \ne (a^*, b^*)$, with the rate parameter for N_r set to $r = 0.1$. We can confirm in Figure 2 that $v_{a^*,b^*}(x) \le v_{a,b}(x)$ for all $x \in \mathbb{R}$.

The graph on the left in Figure 3 shows the optimal periodic–classical barriers a^* and b^* as functions of arrival rates. These are compared with the classical barriers, which are the optimal barriers of the problem studied in [5], the continuous-monitoring version of our current problem. As the arrival rate r increases, both a^* and b^* converge to the classical barriers as obtained in [5], indicating the convergence to the classical strategy.

Similarly, the graph on the right in Figure 3 shows the value functions v_{a^*,b^*} corresponding to various r values, alongside the classical value function computed in [5]. As r increases, v_{a^*,b^*} decreases and converges pointwise to the classical value function.

6. Concluding remarks

In this work, we establish the optimality of the periodic–classical barrier strategy under a spectrally negative Lévy model, within a singular control framework where the controller

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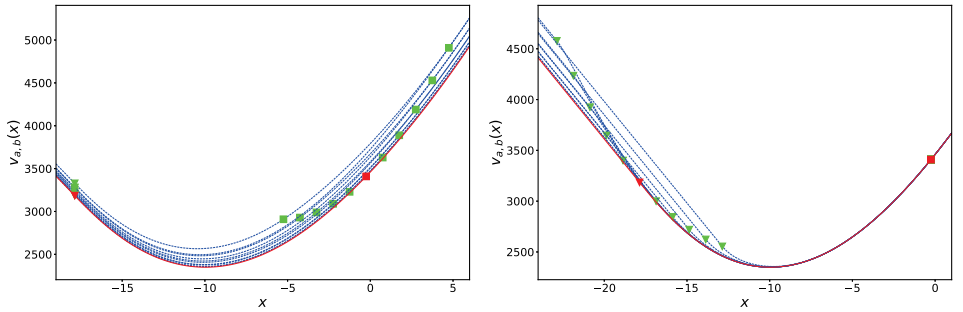


FIGURE 2. Plots of $v_{a,b}$ (for $r = 0.1$) against the initial position x. Left: Plot of $v_{a^*,b}$ (blue dashed) for $b = b^* - 5, b^* - 4, \ldots, b^* - 1, b^* + 1, \ldots, b^* + 5$, with $(a^*, v_{a^*,b}(a^*))$ (lime triangle), $(a^*, v_{a^*,b^*}(a^*))$ (red triangle), $(b, v_{a^*,b}(b))$ (lime square), and $(b^*, v_{a^*,b^*}(b^*))$ (red square). Right: Plot of v_{a,b^*} (blue dashed) for $a = a^* - 5, a^* - 4, \ldots, a^* - 1, a^* + 1, \ldots, a^* + 5$, with $(a, v_{a,b^*}(a))$ (lime triangle), $(a^*, v_{a^*,b^*}(a^*))$ (red triangle), $(b^*, v_{a,b^*}(b^*))$ (lime square), and $(b^*, v_{a^*,b^*}(b^*))$ (red square). In both plots, the value function v_{a^*,b^*} is indicated by red solid curves.

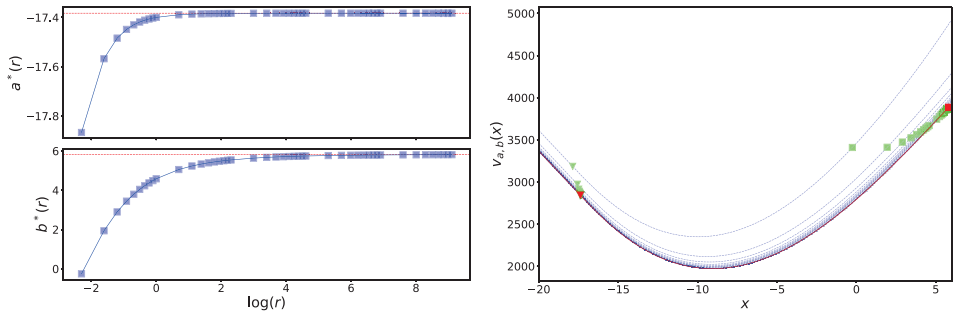


FIGURE 3. Left: Plot of the periodic–classical barriers (a^*, b^*) for $r = 0.1, 0.2, \ldots, 0.9, 1, 2, \ldots, 9, 10, 20, \ldots, 90, 100, 200, \ldots, 900$ (squares) and the classical barriers (the optimal barriers for the problem in [5]; in red dashed lines). Right: Plot of v_{a^*,b^*} (blue dashed) for $r = 0.1, 0.2, \ldots, 0.9, 1, 2, \ldots, 9, 10, 20, \ldots, 90$, with $(a^*, v_{a^*,b^*}(a^*))$ (lime triangles) and $(b^*, v_{a^*,b^*}(b^*))$ (lime squares), alongside the classical value function (the optimal cost function for the problem in [5]; in red solid line), with its lower and upper barriers marked by a red triangle and square, respectively.

can continuously increase the state process but may only decrease it at independent Poisson arrival times. In this concluding section, we discuss several interesting directions for potential extensions and the associated challenges.

One natural direction for extending the model is to restrict upward control opportunities to the arrival times of an independent Poisson process – either with the same or a different rate parameter – as is currently done for downward control. The main challenge with this extension is that the scale function approach used in this work would likely become unsuitable. When both controls are restricted to random discrete times, the expression for the cost function under a barrier strategy is expected to be substantially more complex and technically involved, thereby complicating the subsequent analysis. Another possible extension is to relax the spectrally one-sided assumption on the Lévy model. This presents a similar challenge, as a general Lévy process does not admit a family of scale functions.

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The approach in [20] may be more suitable for both extensions, as it has been successfully applied to establish the optimality of barrier-type strategies under a general Lévy model for various one-sided control problems. However, unlike this study, where the value function is obtained in a semi-explicit form using scale functions, [20] does not derive an explicit expression for the value function. Moreover, the method in [20] has yet to be applied to two-sided control problems, which – based on experience with spectrally one-sided Lévy models – are substantially more challenging than their one-sided counterparts. While we believe that conditions analogous to $\mathfrak{C}$ would play a crucial role in identifying the optimal barrier strategy, establishing the existence of barrier values that satisfy such conditions is expected to require substantial effort. Thus, both extensions pose interesting challenges and are well-suited for future research.

Appendix A. Proofs

The following proposition is used for the proofs of Lemmas 4.2 and 4.5.

Proposition A.1. *Suppose h is a positive, bounded, measurable function on $\mathbb{R}$ with compact support. We have, for $a < b$ and $x \in \mathbb{R}$,*

$$\begin{aligned} v^h_{a,b}(x) &:= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt} h(Y^{a,b}(t))\,\mathrm{d}t\right] \\ &= \frac{\rho^{(q)}_{a,b}(b;h) + v^h_{a,b}(b)}{Z^{(q)}(b-a)} Z^{(q,r)}_{a,b}(x) - \rho^{(q,r)}_{a,b}(x;h) - r\overline{W}^{(q+r)}(x-b) v^h_{a,b}(b) \\ &\quad - \int_b^x W^{(q+r)}(x-y) h(y)\,\mathrm{d}y, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} v^h_{a,b}(b) = -\rho^{(q)}_{a,b}(b;h) &+ \left(\frac{q}{\Phi_{q+r}} \frac{Z^{(q)}(b-a;\Phi_{q+r})}{Z^{(q)}(b-a)}\right)^{-1} \\ &\times \left(-\frac{r}{\Phi_{q+r}} \rho^{(q)}_{a,b}(b;h) + \int_a^\infty h(y) Z^{(q)}(b-y,\Phi_{q+r})\,\mathrm{d}y\right). \end{aligned} \tag{A.2}$$

Furthermore, (A.1) and (A.2) also hold for any function $h\colon \mathbb{R} \to \mathbb{R}$ satisfying Assumption 2.1(i), provided that X satisfies Assumption 2.2.

Proof. The first part of the proposition follows from [17, Proposition 5.1]. For the second part, suppose that h satisfies Assumption 2.1(i) and that X satisfies Assumption 2.2. Let $h^+ := \max(h, 0)$ and $h^- := \max(-h, 0)$ be the positive and negative parts of h, respectively. The positivity and boundedness assumptions on h can be relaxed through the decomposition $h = h^+ - h^-$ and dominated convergence, by noting that $\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-qt} h^+(Y^{a,b}(t))\,\mathrm{d}t\big] < \infty$ and $\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-qt} h^-(Y^{a,b}(t))\,\mathrm{d}t\big] < \infty$. Indeed, the finiteness of these expectations holds because h is of at most polynomial growth and $a \le Y^{a,b}(t) \le Y^{a,\infty}(t)$, where $Y^{a,\infty}$ is the classical reflected process from below at a of the spectrally negative Lévy process X, for which it is known that $\mathbb{E}_x\big[\int_0^\infty \mathrm{e}^{-qt+\theta Y^{a,\infty}(t)}\,\mathrm{d}t\big] < \infty$ for any $\theta \ge 0$. □

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A.1. Proof of Lemma 4.2

Proof. Proposition A.1 can be applied due to Assumption 2.1(i). With (A.2), via integration by parts as shown in the proof of [17, Lemma 6.3], we obtain

$$v^h_{a,b}(b) = -\rho^{(q)}_{a,b}(b;h) + \frac{1}{q}\left(h(a) + \int_a^\infty h'(y)\frac{Z^{(q)}(b-y,\,\Phi_{q+r})}{Z^{(q)}(b-a,\,\Phi_{q+r})}\,\mathrm{d}y\right)Z^{(q)}(b-a).$$

Substituting this into (A.1) and replacing h with f, we have the desired expression. □

A.2. Proof of Lemma 4.3

Proof. Summing the running cost and the control costs, by Lemmas 4.1 and 4.2,

$$\begin{aligned} v_{a,b}(x) = {} & A\big(Z^{(q,r)}_{a,b}(x) - r\overline{W}^{(q+r)}(x-b)Z^{(q)}(b-a)\big) + B - rC_{\mathrm{D}}\overline{\overline{W}}^{(q+r)}(x-b) \\ & - C_{\mathrm{U}}\left(\overline{Z}^{(q,r)}_{a,b}(x) + \frac{\psi'(0+)}{q} - r\overline{W}^{(q+r)}(x-b)\overline{Z}^{(q)}(b-a)\right), \quad x\in\mathbb{R}, \end{aligned} \tag{A.3}$$

where

$$A := \frac{1}{q}\left(f(a) + \int_a^\infty f'(y)\frac{Z^{(q)}(b-y,\,\Phi_{q+r})}{Z^{(q)}(b-a,\,\Phi_{q+r})}\,\mathrm{d}y + \frac{r}{\Phi_{q+r}}\frac{C_{\mathrm{U}}Z^{(q)}(b-a)+C_{\mathrm{D}}}{Z^{(q)}(b-a,\,\Phi_{q+r})} + \frac{qC_{\mathrm{U}}}{\Phi_{q+r}}\right),$$

$$B := -\rho^{(q,r)}_{a,b}(x;f) + r\overline{W}^{(q+r)}(x-b)\rho^{(q)}_{a,b}(b;f) - \int_b^x f(y)W^{(q+r)}(x-y)\,\mathrm{d}y.$$

First, we simplify the expression for A.

Lemma A.1. *For $a<b$,*

$$\int_a^\infty Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y = (Z^{(q)}(b-a,\,\Phi_{q+r}) + r\overline{W}^{(q)}(b-a))/\Phi_{q+r}.$$

Proof. By exchanging the order of integration and the second equality of (4.2),

$$\begin{aligned} \int_a^b Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y &= r\int_a^b\int_0^\infty \mathrm{e}^{-\Phi_{q+r}z}W^{(q)}(z+b-y)\,\mathrm{d}z\,\mathrm{d}y \\ &= r\int_0^\infty \mathrm{e}^{-\Phi_{q+r}z}\int_a^b W^{(q)}(z+b-y)\,\mathrm{d}y\,\mathrm{d}z \\ &= r\int_0^\infty \mathrm{e}^{-\Phi_{q+r}z}\left(\overline{W}^{(q)}(z+b-a) - \overline{W}^{(q)}(z)\right)\,\mathrm{d}z. \end{aligned}$$

By integration by parts and the second equality of (4.2),

$$\int_0^\infty \mathrm{e}^{-\Phi_{q+r}z}\overline{W}^{(q)}(z+b-a)\,\mathrm{d}z = \frac{1}{\Phi_{q+r}}\overline{W}^{(q)}(b-a) + \frac{1}{r\Phi_{q+r}}Z^{(q)}(b-a,\,\Phi_{q+r}).$$

In particular, $\int_0^\infty \mathrm{e}^{-\Phi_{q+r}z}\overline{W}^{(q)}(z)\,\mathrm{d}z = (r\Phi_{q+r})^{-1}Z^{(q)}(0,\,\Phi_{q+r}) = (r\Phi_{q+r})^{-1}$. Thus,

$$\int_a^b Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y = \frac{1}{\Phi_{q+r}}\big(Z^{(q)}(b-a,\,\Phi_{q+r}) + r\overline{W}^{(q)}(b-a) - 1\big).$$

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In addition, $\int_b^\infty Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y = \int_b^\infty \mathrm{e}^{\Phi_{q+r}(b-y)}\,\mathrm{d}y = \int_0^\infty \mathrm{e}^{-\Phi_{q+r}y}\,\mathrm{d}y = (\Phi_{q+r})^{-1}$. By summing these terms, the claim follows. □

By Lemma A.1,

$$\int_a^\infty f'(y)Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y = \int_a^\infty \tilde{f}'(y)Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y - \frac{qC_{\mathrm{U}}}{\Phi_{q+r}}\big(Z^{(q)}(b-a,\,\Phi_{q+r}) + r\overline{W}^{(q)}(b-a)\big).$$

Substituting this into the expression for A, we have

$$A = \frac{1}{q}\left(f(a) + \frac{\Gamma(a,\,b)}{Z^{(q)}(b-a,\,\Phi_{q+r})}\right). \tag{A.4}$$

Second, we write B in terms of $\tilde{f}\colon x \mapsto f(x) + qC_{\mathrm{U}}x$, for $x \in \mathbb{R}$. We have

$$\rho_{a,b}^{(q)}(x;f) = \rho_{a,b}^{(q)}(x;\tilde{f}) - qC_{\mathrm{U}}\rho_{a,b}^{(q)}(x;\,y\mapsto y), \tag{A.5}$$

where, due to integration by parts,

$$\rho_{a,b}^{(q)}(x;\,y\mapsto y) = \overline{W}^{(q)}(x-a)a - \overline{W}^{(q)}(x-b)b + \overline{\overline{W}}^{(q)}(x-a) - \overline{\overline{W}}^{(q)}(x-b). \tag{A.6}$$

In particular, at $x = b$,

$$\rho_{a,b}^{(q)}(b;\,y\mapsto y) = \overline{W}^{(q)}(b-a)a + \overline{\overline{W}}^{(q)}(b-a). \tag{A.7}$$

More generally, we have

$$\rho_{a,b}^{(q,r)}(x;f) = \rho_{a,b}^{(q,r)}(x;\tilde{f}) - qC_{\mathrm{U}}\rho_{a,b}^{(q,r)}(x;\,y\mapsto y), \tag{A.8}$$

where $\rho_{a,b}^{(q,r)}(x;\,y\mapsto y) = \rho_{a,b}^{(q)}(x;\,y\mapsto y) + r\int_b^x W^{(q+r)}(x-y)\rho_{a,b}^{(q)}(y;\,z\mapsto z)\,\mathrm{d}y$.

Applying integration by parts twice, we obtain

$$\int_b^x f(y)W^{(q+r)}(x-y)\,\mathrm{d}y = \int_b^x \tilde{f}'(y)\overline{W}^{(q+r)}(x-y)\,\mathrm{d}y + f(b)\overline{W}^{(q+r)}(x-b) - qC_{\mathrm{U}}\overline{\overline{W}}^{(q+r)}(x-b). \tag{A.9}$$

Substituting (A.5), (A.8), and (A.9) into the expression for B, for $x \in \mathbb{R}$,

$$\begin{aligned} B = {} & -\rho_{a,b}^{(q,r)}(x;\tilde{f}) + qC_{\mathrm{U}}\rho_{a,b}^{(q,r)}(x;\,y\mapsto y) + r\overline{W}^{(q+r)}(x-b)\rho_{a,b}^{(q)}(b;\tilde{f}) \\ & - r\overline{W}^{(q+r)}(x-b)qC_{\mathrm{U}}\rho_{a,b}^{(q)}(b;\,y\mapsto y) - \int_b^x \tilde{f}'(y)\overline{W}^{(q+r)}(x-y)\,\mathrm{d}y \\ & - f(b)\overline{W}^{(q+r)}(x-b) + qC_{\mathrm{U}}\overline{\overline{W}}^{(q+r)}(x-b). \end{aligned} \tag{A.10}$$

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With

$$\overline{W}_{a,b}^{(q,r)}(x) := \overline{W}^{(q)}(x-a) + r\int_b^x W^{(q+r)}(x-y)\overline{W}^{(q)}(y-a)\,\mathrm{d}y,$$
$$\overline{\overline{W}}_{a,b}^{(q,r)}(x) := \overline{\overline{W}}^{(q)}(x-a) + r\int_b^x W^{(q+r)}(x-y)\overline{\overline{W}}^{(q)}(y-a)\,\mathrm{d}y,$$

and using (A.6), we have

$$\rho_{a,b}^{(q,r)}(x; y\mapsto y) = a\overline{W}_{a,b}^{(q,r)}(x) + \overline{\overline{W}}_{a,b}^{(q,r)}(x) - b\overline{W}_{b,b}^{(q,r)}(x) - \overline{\overline{W}}_{b,b}^{(q,r)}(x).$$

In terms of (4.6) and (4.7), we can write

$$\overline{W}_{a,b}^{(q,r)}(x) = \frac{1}{q}\Big(Z_{a,b}^{(q,r)}(x) - 1 - r\overline{W}^{(q+r)}(x-b)\Big),$$
$$\overline{\overline{W}}_{a,b}^{(q,r)}(x) = \frac{1}{q}\Big(\overline{Z}_{a,b}^{(q,r)}(x) - (x-a) - r\overline{W}^{(q+r)}(x-b)(b-a) - r\overline{\overline{W}}^{(q+r)}(x-b)\Big).$$

Moreover, by [16, (6)], we can write $\overline{W}_{b,b}^{(q,r)}(x) = \overline{W}^{(q+r)}(x-b)$. Then, by noticing that $\overline{\overline{W}}_{b,b}^{(q,r)}$ is an anti-derivative of $\overline{W}_{b,b}^{(q,r)}$ and using the identity $\overline{W}_{b,b}^{(q,r)}(x) = \overline{W}^{(q+r)}(x-b)$, we obtain $\overline{\overline{W}}_{b,b}^{(q,r)}(x) = \overline{\overline{W}}^{(q+r)}(x-b)$. Substituting these,

$$\rho_{a,b}^{(q,r)}(x; y\mapsto y) = \frac{1}{q}\Big(aZ_{a,b}^{(q,r)}(x) + \overline{Z}_{a,b}^{(q,r)}(x) - x - r\overline{W}^{(q+r)}(x-b)b - r\overline{\overline{W}}^{(q+r)}(x-b)\Big)$$
$$- \overline{\overline{W}}^{(q+r)}(x-b) - \overline{W}^{(q+r)}(x-b)b. \tag{A.11}$$

Substituting (A.7) and (A.11) in (A.10),

$$B = -\rho_{a,b}^{(q,r)}(x;\tilde{f}) + C_{\mathrm{U}}\Big(aZ_{a,b}^{(q,r)}(x) + \overline{Z}_{a,b}^{(q,r)}(x) - x - r\overline{W}^{(q+r)}(x-b)b - r\overline{\overline{W}}^{(q+r)}(x-b)\Big)$$
$$+ r\overline{W}^{(q+r)}(x-b)\rho_{a,b}^{(q)}(b;\tilde{f}) - r\overline{W}^{(q+r)}(x-b)qC_{\mathrm{U}}\Big(\overline{W}^{(q)}(b-a)a + \overline{\overline{W}}^{(q)}(b-a)\Big)$$
$$- \int_b^x \tilde{f}'(y)\overline{W}^{(q+r)}(x-y)\,\mathrm{d}y - \tilde{f}(b)\overline{W}^{(q+r)}(x-b).$$

Finally, substituting this expression for B and (A.4) into (A.3), and simplifying, we obtain the desired expression. □

A.3. Proof of Lemma 4.4

Proof. By direct computation and the smoothness of the scale functions, we obtain the following results for $x\in\mathbb{R}\backslash\{a, b\}$ and for any continuous and almost everywhere differentiable function $h\colon \mathbb{R}\to\mathbb{R}$:

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$$\frac{\mathrm{d}}{\mathrm{d}x}Z_{a,b}^{(q,r)}(x) = qW_{a,b}^{(q,r)}(x) + rW^{(q+r)}(x-b)Z^{(q)}(b-a), \tag{A.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\rho_{a,b}^{(q)}(x;h) = -W^{(q)}(x-b)h(b) + W^{(q)}(x-a)h(a) + \int_a^b W^{(q)}(x-y)h'(y)\,\mathrm{d}y. \tag{A.13}$$

We further compute $(\mathrm{d}/\mathrm{d}x)\rho_{a,b}^{(q,r)}(x;h)$. Recalling (4.4), by partial integration,

$$\frac{\mathrm{d}}{\mathrm{d}x}\rho_{a,b}^{(q,r)}(x;h) = \frac{\mathrm{d}}{\mathrm{d}x}\rho_{a,b}^{(q)}(x;h) + rW^{(q+r)}(x-b)\rho_{a,b}^{(q)}(b;h) + C, \quad x \in \mathbb{R}\backslash\{a,b\}, \tag{A.14}$$

where, by (A.13) and [16, (6)],

$$\begin{aligned} C &:= r\int_b^x W^{(q+r)}(x-y)\frac{\mathrm{d}}{\mathrm{d}y}\rho_{a,b}^{(q)}(y;h)\,\mathrm{d}y \\ &= -h(b)(W^{(q+r)}(x-b) - W^{(q)}(x-b)) + rh(a)\int_b^x W^{(q+r)}(x-y)W^{(q)}(y-a)\,\mathrm{d}y \\ &\quad + r\int_b^x W^{(q+r)}(x-y)\int_a^b W^{(q)}(y-z)h'(z)\,\mathrm{d}z\,\mathrm{d}y. \end{aligned}$$

Substituting this in (A.14) and using (4.5), for $x \in \mathbb{R}\backslash\{a,b\}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\rho_{a,b}^{(q,r)}(x;h) &= h(a)W_{a,b}^{(q,r)}(x) + rW^{(q+r)}(x-b)\rho_{a,b}^{(q)}(b;h) \\ &\quad - h(b)W^{(q+r)}(x-b) + \rho_{a,b}^{(q,r)}(x;h'). \end{aligned} \tag{A.15}$$

We obtain (4.12) by differentiating (4.10) using (A.12) and (A.15). The statement that (4.12) is defined on $\mathbb{R}\backslash\{a\}$ can be confirmed by inspecting its left- and right-hand limits and using the smoothness of scale functions. To obtain the second derivative of $v_{a,b}$, we directly differentiate (4.12). This gives (4.13), which is again defined on $\mathbb{R}\backslash\{a\}$ by the smoothness of scale functions. □

A.4. Proof of Lemma 4.5

Proof. By (4.3) and the definition of $\tilde{f}$, for $x \in \mathbb{R}$,

$$\rho_{a,b}^{(q,r)}(x;f') = \rho_{a,b}^{(q,r)}(x;\tilde{f}') - qC_{\mathrm{U}}\rho_{a,b}^{(q,r)}(x;1),$$

where, with $\rho_{a,b}^{(q)}(x;1) = \int_a^b W^{(q)}(x-y)\,\mathrm{d}y = (Z^{(q)}(x-a) - Z^{(q)}(x-b))/q$, we can write

$$\rho_{a,b}^{(q,r)}(x;1) = \rho_{a,b}^{(q)}(x;1) + r\int_b^x W^{(q+r)}(x-y)\rho_{a,b}^{(q)}(y;1)\mathrm{d}y = \frac{1}{q}\Big(Z_{a,b}^{(q,r)}(x) - Z_{b,b}^{(q,r)}(x)\Big).$$

Using [16, (6)], we obtain $Z_{b,b}^{(q,r)}(x) = Z^{(q+r)}(x-b)$, and thus

$$\rho_{a,b}^{(q,r)}(x;f') = \rho_{a,b}^{(q,r)}(x;\tilde{f}') - C_{\mathrm{U}}\Big(Z_{a,b}^{(q,r)}(x) - Z^{(q+r)}(x-b)\Big). \tag{A.16}$$

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Moreover, for $x \in \mathbb{R}$, we have

$$\int_b^x W^{(q+r)}(x-y)f'(y)\,\mathrm{d}y = \int_b^x W^{(q+r)}(x-y)\tilde{f}'(y)\,\mathrm{d}y - qC_{\mathrm{U}}\overline{W}^{(q+r)}(x-b). \tag{A.17}$$

Rewriting (A.2) with $h = f'$ by applying Lemma A.1 and (A.16), and simplifying using (4.14), we obtain

$$\begin{aligned} v^{f'}_{a,b}(b) &= -\rho^{(q)}_{a,b}(b;\tilde{f}') + qC_{\mathrm{U}}\overline{W}^{(q)}(b-a) \\ &\quad + \left(\frac{q}{\Phi_{q+r}}\frac{Z^{(q)}(b-a;\Phi_{q+r})}{Z^{(q)}(b-a)}\right)^{-1} \\ &\quad \times \left(-\frac{r}{\Phi_{q+r}}\rho^{(q)}_{a,b}(b;\tilde{f}') + \int_a^\infty \tilde{f}'(y)Z^{(q)}(b-y,\,\Phi_{q+r})\,\mathrm{d}y \right. \\ &\quad \left. - \frac{qC_{\mathrm{U}}}{\Phi_{q+r}}Z^{(q)}(b-a,\,\Phi_{q+r})\right) \\ &= -\rho^{(q)}_{a,b}(b;\tilde{f}') - C_{\mathrm{U}} + \frac{Z^{(q)}(b-a)}{qZ^{(q)}(b-a;\Phi_{q+r})}\gamma(a,b). \end{aligned} \tag{A.18}$$

Substituting (A.16), (A.17), and (A.18) into (A.1) with $h = f'$, and simplifying, we obtain the desired result. □

A.5. Proof of Lemma 4.6

Proof. As $b \mapsto \Gamma(a,b)$ is continuous on (a,∞), the first claim holds by setting $b = a+$ in (4.11) and noting that $Z^{(q)}(a-y,\,\Phi_{q+r}) = \exp(\Phi_{q+r}(a-y))$ for $y > a$.

For the second claim, by the second equality of (4.2) we have two bounds for $Z^{(q)}(\cdot,\,\Phi_{q+r})$. First, with $W_{\Phi_q}(x) = \mathrm{e}^{-\Phi_q x}W^{(q)}(x)$ (see [12, p. 247]), for $x \in \mathbb{R}$,

$$Z^{(q)}(x,\,\Phi_{q+r}) = r\int_0^\infty \mathrm{e}^{-\Phi_{q+r}y}W^{(q)}(x+y)\,\mathrm{d}y = r\mathrm{e}^{\Phi_q x}\int_0^\infty \mathrm{e}^{-(\Phi_{q+r}-\Phi_q)y}W_{\Phi_q}(x+y)\,\mathrm{d}y.$$

Since W_{Φ_q} is increasing,

$$\frac{r\mathrm{e}^{\Phi_q x}W_{\Phi_q}(x)}{\Phi_{q+r}-\Phi_q} \le Z^{(q)}(x,\,\Phi_{q+r}) \le \frac{r\mathrm{e}^{\Phi_q x}W_{\Phi_q}(\infty)}{\Phi_{q+r}-\Phi_q},$$

where $W_{\Phi_q}(\infty) := \lim_{x\to\infty} W_{\Phi_q}(x)$ is well-defined and finite. Thus, for $b > b'$ with $b' > a$ fixed,

$$\begin{aligned} \left|\tilde{f}'(z)\frac{Z^{(q)}(b-z,\,\Phi_{q+r})}{Z^{(q)}(b-a,\,\Phi_{q+r})}\mathbf{1}_{\{a\le z\}}\right| &\le |\tilde{f}'(z)|\frac{\mathrm{e}^{\Phi_q(b-z)}W_{\Phi_q}(\infty)}{\mathrm{e}^{\Phi_q(b-a)}W_{\Phi_q}(b-a)} \\ &\le |\tilde{f}'(z)|\mathrm{e}^{-\Phi_q(z-a)}\frac{W_{\Phi_q}(\infty)}{W_{\Phi_q}(b'-a)}, \end{aligned}$$

which is integrable. Thus, dominated convergence gives

$$\lim_{b\to\infty}\frac{\Gamma(a,b)}{Z^{(q)}(b-a,\,\Phi_{q+r})} = \int_0^\infty \mathrm{e}^{-\Phi_q z}\tilde{f}'(z+a)\,\mathrm{d}z.$$

□

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A.6. Proof of Lemma 4.7

Proof. For (i), by (4.21) and the definition of $\bar{a}$, we have $\Gamma(\bar{a}, \bar{a}+) > 0$. From (4.11), it is easy to see that $b \mapsto \Gamma(\bar{a}, b)$ is strictly increasing on $(\bar{a}, \infty)$. Hence, $\underline{\Gamma}(\bar{a}) > 0$.

For (ii), we show separately that $\underline{\Gamma}(\underline{a}_1 \vee \underline{a}_2) < 0$, first for the case $\underline{a}_2 \geq \underline{a}_1$, and then for the case $\underline{a}_2 < \underline{a}_1$.

Suppose that $\underline{a}_2 \geq \underline{a}_1$. By (4.16) with $a = \underline{a}_2$ and (4.24), we have, for any $b > \underline{a}_2$,

$$\begin{aligned}\frac{\gamma(\underline{a}_2, b)}{qZ^{(q)}(b-\underline{a}_2; \Phi_{q+r})} - C_{\mathrm{U}} &= \mathbb{E}_{\underline{a}_2}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{\underline{a}_2, b}(t))\,\mathrm{d}t\right] \\ &< \mathbb{E}_{\underline{a}_2}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{\underline{a}_2, \infty}(t))\,\mathrm{d}t\right] = -C_{\mathrm{U}},\end{aligned}$$

where the inequality holds because $Y^{\underline{a}_2, b}(t) \leq Y^{\underline{a}_2, \infty}(t)$ for all $t \geq 0$. To see why the inequality is strict, first note that $\underline{a}_2 < \bar{a} < \bar{\bar{a}}$ implies the existence of some $c > \underline{a}_2$ and $\varepsilon > 0$ such that $f'(c) < f'(c+\varepsilon)$. Moreover, since the set $\{t \in [0, \infty)\colon Y^{\underline{a}_2, b}(t) < c,\ Y^{\underline{a}_2, \infty}(t) > c + \varepsilon\}$ has a positive Lebesgue measure with positive probability, the inequality is indeed strict. This inequality implies

$$0 > \gamma(\underline{a}_2, b) = \Phi_{q+r}\Gamma(\underline{a}_2, b) - r\big(C_{\mathrm{U}} + C_{\mathrm{D}} + \rho^{(q)}_{\underline{a}_2, b}(b; \tilde{f}')\big), \quad b > \underline{a}_2. \tag{A.19}$$

Now, since f' is non-decreasing and $f'(\infty) > qC_{\mathrm{D}}$ by Assumption 2.1(iii), for some sufficiently large $\tilde{b}$,

$$\begin{aligned}C_{\mathrm{D}} < \mathbb{E}_{\tilde{b}}\left[\int_0^\infty \mathrm{e}^{-qt} f'(Y^{\underline{a}_2, \tilde{b}}(t))\,\mathrm{d}t\right] &= v^{f'}_{\underline{a}_2, \tilde{b}}(\tilde{b}) \\ &= \frac{Z^{(q)}(\tilde{b}-\underline{a}_2)}{qZ^{(q)}(\tilde{b}-\underline{a}_2; \Phi_{q+r})}\gamma(\underline{a}_2, \tilde{b}) - \rho^{(q)}_{\underline{a}_2, \tilde{b}}(\tilde{b}; \tilde{f}') - C_{\mathrm{U}},\end{aligned}$$

where we used (4.17) for the last equality. This, together with (A.19) with $b = \tilde{b}$, gives $\rho^{(q)}_{\underline{a}_2, \tilde{b}}(\tilde{b}; \tilde{f}') + C_{\mathrm{U}} + C_{\mathrm{D}} < 0$. Substituting this inequality back into (A.19), we have $\Gamma(\underline{a}_2, \tilde{b}) < 0$, and therefore $\underline{\Gamma}(\underline{a}_1 \vee \underline{a}_2) = \underline{\Gamma}(\underline{a}_2) \leq \Gamma(\underline{a}_2, \tilde{b}) < 0$.

Now suppose that $\underline{a}_1 > \underline{a}_2$. By (4.14) and the definition of $\underline{a}_1$, we have $\gamma(\underline{a}_1, \underline{a}_1+) = -r(C_{\mathrm{U}} + C_{\mathrm{D}}) < 0$. Since $\gamma(\underline{a}_1, \cdot)$ is the partial derivative of $\Gamma(\underline{a}_1, \cdot)$, which starts at $\Gamma(\underline{a}_1, \underline{a}_1+) = 0$, the function $\Gamma(\underline{a}_1, \cdot)$ must go below zero. Thus, $0 > \underline{\Gamma}(\underline{a}_1) = \underline{\Gamma}(\underline{a}_1 \vee \underline{a}_2)$. □

A.7. Proof of Lemma 4.8

Proof. By taking the partial derivative of Γ with respect to a, we obtain, for almost every a at which $\tilde{f}'(a)$ exists, $(\partial/\partial a)\Gamma(a, b) = -\tilde{f}'(a)Z^{(q)}(b-a, \Phi_{q+r})$, $a < b$, which is positive for $a < \bar{a}$. For $a' < a \leq \bar{a}$, notice that

$$\begin{aligned}\underline{\Gamma}(a') &\leq \inf_{b>a} \Gamma(a', b) \\ &= \inf_{b>a}\left(\Gamma(a, b) + \int_{a'}^{a} \tilde{f}'(y) Z^{(q)}(b-y, \Phi_{q+r})\,\mathrm{d}y\right) \\ &\leq \inf_{b>a}\left(\Gamma(a, b) + \int_{a'}^{(a+a')/2} \tilde{f}'(y) Z^{(q)}(b-y, \Phi_{q+r})\,\mathrm{d}y\right)\end{aligned}$$

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$$\leq \inf_{b>a}\left(\Gamma(a,b)+Z^{(q)}\left(b-\frac{a+a'}{2},\Phi_{q+r}\right)\int_{a'}^{(a+a')/2}\tilde{f}'(y)\,\mathrm{d}y\right)$$

$$\leq \underline{\Gamma}(a)+Z^{(q)}\left(\frac{a-a'}{2},\Phi_{q+r}\right)\int_{a'}^{(a+a')/2}\tilde{f}'(y)\,\mathrm{d}y<\underline{\Gamma}(a).$$

Hence, $a\mapsto\underline{\Gamma}(a)$ is strictly increasing on $(-\infty,\bar{a})$. Continuity is guaranteed because $\Gamma(a,b)\xrightarrow{b\uparrow\infty}\infty$ for all $a>\underline{a}_2$, by (4.22) and the definition of $\underline{a}_2$ as in (4.23). □

A.8. Proof of Lemma 4.9

Proof. Fix an admissible strategy $\pi\in\Pi$. For a càdlàg process A, denote its jump at time s by $\Delta A(s):=A(s)-A(s-)$ and its continuous part by A^{c}, such that $A(s)=A^{\mathrm{c}}(s)+\sum_{0\leq u\leq s}\Delta A(u)$. By first applying Itô's lemma and then rearranging terms, we obtain

$$\begin{aligned}
w(x)=&-\int_0^{t\wedge\tau_n}\mathrm{e}^{-qs}(\mathcal{L}-q)w(Y^\pi(s-))\,\mathrm{d}s\\
&-\int_{[0,t\wedge\tau_n]}\mathrm{e}^{-qs}w'(Y^\pi(s-))\,\mathrm{d}R^{\pi,c}(s)+C_{\mathrm{D}}\int_{[0,t\wedge\tau_n]}\mathrm{e}^{-qs}v^\pi(s)\,\mathrm{d}N_r(s)\\
&-\int_0^{t\wedge\tau_n}r\mathrm{e}^{-qs}[C_{\mathrm{D}}v^\pi(s)+w(Y^\pi(s-)-v^\pi(s))-w(Y^\pi(s-))]\,\mathrm{d}s\\
&-\sum_{0\leq s\leq t\wedge\tau_n}\mathrm{e}^{-qs}[w(Y^\pi(s-)+\Delta X(s)+\Delta R^\pi(s))-w(Y^\pi(s-)+\Delta X(s))]\\
&-M(t\wedge\tau_n)+\mathrm{e}^{-q(t\wedge\tau_n)}w(Y^\pi(t\wedge\tau_n)),
\end{aligned}$$

where $(M(t\wedge\tau_n);t\geq 0)$ is defined in [21, (A.1)]. Noting that $Y^\pi(s-)$ is bounded almost surely on $[0,t\wedge\tau_n]$, and using [12, Corollary 4.6], we have that the process $M(t\wedge\tau_n)$ is a zero-mean $\mathbb{P}_x$-martingale. Now, since $-C_{\mathrm{U}}\leq w'(x)$ and $(\mathcal{L}-q)w(x)+r(\mathcal{M}w(x)-w(x))+f(x)\geq 0$ for all $x\in\mathbb{R}$,

$$\begin{aligned}
w(x)\leq&-\int_0^{t\wedge\tau_n}\mathrm{e}^{-qs}[(\mathcal{L}-q)w(Y^\pi(s-))+r(\mathcal{M}w(Y^\pi(s-))-w(Y^\pi(s-)))]\,\mathrm{d}s\\
&+C_{\mathrm{D}}\int_{[0,t\wedge\tau_n]}\mathrm{e}^{-qs}v^\pi(s)\,\mathrm{d}N_r(s)+C_{\mathrm{U}}\int_{[0,t\wedge\tau_n]}\mathrm{e}^{-qs}\,\mathrm{d}R^\pi(s)\\
&-M(t\wedge\tau_n)+\mathrm{e}^{-q(t\wedge\tau_n)}w(Y^\pi(t\wedge\tau_n))\\
\leq&\int_0^{t\wedge\tau_n}\mathrm{e}^{-qs}f(Y^\pi(s))\,\mathrm{d}s+C_{\mathrm{D}}\int_{[0,t\wedge\tau_n]}\mathrm{e}^{-qs}v^\pi(s)\,\mathrm{d}N_r(s)+C_{\mathrm{U}}\int_{[0,t\wedge\tau_n]}\mathrm{e}^{-qs}\,\mathrm{d}R^\pi(s)\\
&-M(t\wedge\tau_n)+\mathrm{e}^{-q(t\wedge\tau_n)}w(Y^\pi(t\wedge\tau_n)).
\end{aligned}$$

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Taking expectation, we obtain

$$w(x) \le \mathbb{E}_x\left[\int_0^{t\wedge\tau_n} \mathrm{e}^{-qs} f(Y^\pi(s))\,\mathrm{d}s + C_{\mathrm{D}}\int_{[0,t\wedge\tau_n]} \mathrm{e}^{-qs} v^\pi(s)\,\mathrm{d}N_r(s)\right]$$
$$+ \mathbb{E}_x\left[C_{\mathrm{U}}\int_{[0,t\wedge\tau_n]} \mathrm{e}^{-qs}\,\mathrm{d}R^\pi(s)\right] + \mathbb{E}_x[\mathrm{e}^{-q(t\wedge\tau_n)} w(Y^\pi(t\wedge\tau_n))]. \tag{A.20}$$

We take $t, n \uparrow \infty$ to complete the proof. As assumed in the lemma, for any admissible strategy π, $\limsup_{t,n\uparrow\infty} \mathbb{E}_x[\mathrm{e}^{-q(t\wedge\tau_n)} w(Y^\pi(t\wedge\tau_n))] \le 0$. Taking the limit of the other terms on the right-hand side of (A.20), because π is admissible, dominated convergence gives

$$w(x) \le \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qs} f(Y^\pi(s))\,\mathrm{d}s + \int_{[0,\infty)} \mathrm{e}^{-qt}(C_{\mathrm{U}}\,\mathrm{d}R^\pi(t) + C_{\mathrm{D}}\,\mathrm{d}L^\pi(t))\right] = v^\pi(x).$$

The proof is now complete, as $\pi \in \Pi$ was chosen arbitrarily. □

A.9. Proof of Lemma 4.10

Proof. Applying $\Gamma(a^*, b^*) = 0$ in (4.10), for $x \le a^*$, notice that $v_{a^*,b^*}(x) = (-C_{\mathrm{U}}\psi'(0+) + \tilde{f}(a^*))/q - C_{\mathrm{U}}x$. It follows that $(\mathcal{L} - q)v_{a^*,b^*}(x) + f(x) = \tilde{f}(x) - \tilde{f}(a^*)$, which is non-negative. This shows the statement for $x \le a^*$.

For $x > a^*$, the same argument as in [17, Lemma 7.2] applies, since the form of $v^{LR}_{a^*,b^*}$ in terms of scale functions is identical to that considered in [17, Lemma 7.2]. Therefore, for $x > a^*$, we have

$$(\mathcal{L} - q)v^{LR}_{a^*,b^*}(x) = \begin{cases} 0, & a^* < x < b^*, \\ -r\big(C_{\mathrm{D}}(x - b^*) + v^{LR}_{a^*,b^*}(b^*) - v^{LR}_{a^*,b^*}(x)\big), & x \ge b^*; \end{cases}$$

$$(\mathcal{L} - q)v^{f}_{a^*,b^*}(x) = \begin{cases} -f(x), & a^* < x < b^*, \\ -r\big(v^{f}_{a^*,b^*}(b^*) - v^{f}_{a^*,b^*}(x)\big) - f(x), & x \ge b^*. \end{cases}$$

Summing these, we get the desired expression. □

A.10. Proof of Lemma 4.12

We first establish the following auxiliary result. Define

$$g(x) := \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt} f(X(t))\,\mathrm{d}t\right] = q^{-1}\mathbb{E}[f(X(e_q) + x)], \tag{A.21}$$

where e_q is an independent exponential random variable with rate q.

Lemma A.2. *Suppose f is a convex and slowly or regularly varying function at $+\infty$ (resp. $-\infty$), with $|f(x)| \to \infty$ as $x \to +\infty$ (resp. $x \to -\infty$). Then $\lim_{x\uparrow\infty} |g(x)/f(x)| = 1/q$ (resp. $\lim_{x\downarrow-\infty} |g(x)/f(x)| = 1/q$).*

Proof. We show the claim for $x \to \infty$; the other case can be treated analogously. For such f, there exists some $c \in \mathbb{R}$ such that, for all $\alpha > 0$,

$$u(\alpha) := \lim_{x\to\infty} \frac{f(\alpha x)}{f(x)} = \begin{cases} 1 & \text{if } f \text{ is slowly varying at } \infty, \\ \alpha^c & \text{if } f \text{ is regularly varying at } \infty. \end{cases}$$

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Fix some $1 < a_0 < 2$ and $\varepsilon > 0$. By the uniform convergence theorem for regularly varying functions (see [6, Theorem 1.5.2]), there exists some $x_0 > 0$ such that

$$\sup_{a\in[1,a_0],x>x_0} \left| \frac{f(ax)}{f(x)} - u(a) \right| < \varepsilon.$$

Define

$$x_1 := \inf\{x \in \mathbb{R} \colon |f| \geq 1 \text{ and } |f| \text{ is strictly increasing on } (x, \infty)\} < \infty. \tag{A.22}$$

For all $x > x_0 \vee x_1$, $y \in [0, (a_0 - 1)x]$,

$$1 \leq \left| \frac{f(y+x)}{f(x)} \right| \leq \left| \frac{f(a_0 x)}{f(x)} \right| \leq u(a_0) + \varepsilon.$$

Hence, setting $M_1 := u(a_0) + \varepsilon$, we have

$$\sup_{x>x_0\vee x_1, y\in[0,(a_0-1)x]} \left| \frac{f(y+x)}{f(x)} \right| < M_1 < \infty.$$

Consider the following integral, which depends on the value of x:

$$\mathbb{E}\left[\frac{f(X(e_q)+x)}{f(x)}; X(e_q) \geq 0 \right].$$

We seek an integrable function that bounds the integrand for all sufficiently large x. Fix $x > x_0 \vee x_1$; for $y \in [0, (a_0 - 1)x]$ we have, from the analysis above, that $|f(y+x)/f(x)| \leq M_1$. For $y > (a_0 - 1)x$, $f(y+x)/f(x)$ can be bounded by a polynomial function, as

$$\left| \frac{f(x+y)}{f(x)} \right| < |f(x+y)| < \left| f\left(\frac{y}{a_0 - 1} + y \right) \right| = \left| f\left(y \frac{a_0}{a_0 - 1} \right) \right|,$$

where the first inequality follows as $x > x_1$ and the second inequality follows as $x > x_1$ and $y > (a_0 - 1)x$. Thus, for $y \geq 0$, we have

$$\sup_{x>x_0\vee x_1} \left| \frac{f(y+x)}{f(x)} \right| \leq M_1 \vee f\left(y \frac{a_0}{a_0 - 1} \right) =: h(y),$$

where h is of polynomial growth such that $\mathbb{E}[h(X(e_q))] < \infty$ by Assumption 2.2. Hence, by dominated convergence, we obtain

$$\lim_{x\to\infty} \mathbb{E}\left[\frac{f(X(e_q)+x)}{f(x)}; X(e_q) \geq 0 \right] = \mathbb{P}(X(e_q) \geq 0).$$

Now, we consider the case with the event $\{X(e_q) < 0\}$. Let m denote the minimum value of f if it exists (i.e. if f is not monotone); otherwise, set m to zero. By convexity, for all $y \leq 0$ and $x \geq 0$, $|f(x+y)| \leq |m| \vee |f(y)| \vee |f(x)|$. For $x > x_1$ (as in (A.22)), we have $|f(x+y)/f(x)| \leq |m| \vee |f(y)| \vee 1$.

Thus, there exists some $x_2 > 0$ and a function $\tilde{h} \colon \mathbb{R} \to \mathbb{R}_+$ with $\mathbb{E}[\tilde{h}(X(e_q))] < \infty$ such that, for all $y \in \mathbb{R}$, we have $\sup_{x>x_2} |f(y+x)/f(x)| \leq \tilde{h}(y)$. Applying dominated convergence, we obtain $\lim_{x\to\infty} \mathbb{E}[f(X(e_q)+x)/f(x); X(e_q) < 0] = \mathbb{P}(X(e_q) < 0)$. □

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Proof of Lemma 4.12. Following the same argument as the proof of [23, Lemma 7.5], we have the following upper bound:

$$\mathbb{E}_x[\mathrm{e}^{-q(t\wedge\tau_n)}w(Y^\pi(t\wedge\tau_n))] \le \mathbb{E}_x\left[\mathrm{e}^{-q(t\wedge\tau_n)}\mathbb{E}_{Y^\pi(t\wedge\tau_n)}\left[\int_0^\infty \mathrm{e}^{-qs}f(X(s))\,\mathrm{d}s\right]\right]$$
$$= \mathbb{E}_x[\mathrm{e}^{-q(t\wedge\tau_n)}g(Y^\pi(t\wedge\tau_n))]. \tag{A.23}$$

We show that

$$\text{there exist } M, N < \infty \text{ such that } |g(x)| \le \max(M|f(x)|, N). \tag{A.24}$$

If $|f(x)| \to \infty$ as $x \to +\infty$ and also as $x \to -\infty$, then, by Lemma A.2, (A.24) holds.

Let $|f|$ be bounded as $x \to -\infty$ and slowly or regularly varying at $+\infty$, with $|f|$ unbounded as $x \to +\infty$. Again using Lemma A.2, for some positive $M < \infty$ and $x' \in \mathbb{R}$, $|g(x)| \le M|f(x)|$ for $x > x'$. Since $|f|$ is bounded as $x \to -\infty$, and f is convex, it follows that f is a monotonically increasing function. Consequently, for $x < x'$ we have $g(x') \ge g(x) \ge q^{-1}\mathbb{E}[f(-\infty)]$, which is finite. Thus, we again establish (A.24).

The case where $|f|$ is bounded as $x \to +\infty$ and slowly or regularly varying at $-\infty$, with $|f|$ unbounded as $x \to -\infty$, follows analogously to the previous paragraph.

Finally, Assumption 2.1 excludes the case where f is bounded on $\mathbb{R}$. Hence, in all cases, (A.24) holds.

To complete the proof, we show that the right-hand side of (A.23) limits to 0. Since π is admissible, we have $\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qs}|f(Y^\pi(s))|\,\mathrm{d}s\right] < \infty$, and thus $\lim_{t\uparrow\infty}\mathbb{E}_x[\mathrm{e}^{-qt}f(Y^\pi(t))] = 0$. Then, $\lim_{t\uparrow\infty}\mathbb{E}_x[\mathrm{e}^{-qt}g(Y^\pi(t))] = 0$ follows by the bound in (A.24). This, in conjunction with (A.23), completes the proof of Lemma 4.12. □

Acknowledgements

We thank the anonymous referee and the associate editor for their careful reading of our manuscript and for their valuable suggestions for improvements.

Funding information

K. Yamazaki was supported by JSPS KAKENHI grants JP20K03758, JP24K06844, and JP24H00328. Q. Zhang is supported by the Australian Research Training Program Scholarship.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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