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ARTICLE

On the Arakawa lifting Part I: Eichler commutation relations

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Abstract

We investigate the theta correspondence of cusp forms for the dual pair $(O^*(4), \mathrm{Sp}(1,1))$ originally introduced by Tsuneo Arakawa in the non-adelic setting. We call this Arakawa lifting. In this article, reformulating the theta correspondence in the adelic setting, we provide commutation relations of Hecke operators satisfied by the Arakawa lifting at all non-archimedean places, which is referred to as Eichler commutation relations for classical modular forms. Their archimedean analog is also given in terms of reproducing kernels.

1. Introduction

This is the first part in the series of papers, whose aim is to investigate the theta correspondence of cusp forms for the dual pair $(O^*(4), \mathrm{Sp}(1,1))$, which we call Arakawa lifting.

The study of this article has an origin in Eichler [3] about the action of Hecke operators on theta series. Examining the action of Hecke operators is a fundamental tool to study arithmetic properties of automorphic forms. On the other hand, the notion of theta series fits into the theta correspondence or Howe correspondence of automorphic representations of reductive groups in the representation theoretic context. In the spirit of the Langlands philosophy, the Hecke actions via the theta correspondence are preferred to understand in terms of L-morphisms between relevant L-groups, which are nothing but the notion of the Langlands functoriality. This point of view is taken up by Rallis [14], for example. Our aim is to know the Hecke action on the Arakawa lifting explicitly. We can refer to this as "Eichler's commutation relations" in the sense of the classical modular forms and also to "functoriality," following Rallis [14]. This lift is understood in terms of an inner form version of the endoscopic lift for the symplectic group of degree two (cf. [15]). Different from the case of similitude groups $GO^*(4)$ and $\mathrm{GSp}(1,1)$ in our previous paper [5], our commutation relations in this article deal with local Hecke algebras for isometry groups $O^*(4)$ and $\mathrm{Sp}(1,1)$ at all non-archimedean places including ramified ones. As an additional novelty of our results, we should remark that their archimedean analog is given in terms of the reproducing kernels of holomorphic discrete series of $O^*(4)(\mathbb{R})$ and quaternionic discrete series of $\mathrm{Sp}(1,1)(\mathbb{R})$.

Let $\mathrm{Sp}(1,q)$ be the quaternion unitary group of a Hermitian form of signature $(1,q)$. Tsuneo Arakawa, in his unpublished notes, constructed a "theta lifting" from the space of holomorphic cusp

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forms on SL_2 to that of cusp forms on $\mathrm{Sp}(1,q)$. Note that $(\mathrm{SL}_2, \mathrm{Sp}(1,q))$ does not form a dual pair when $q > 1$ and Arakawa obtained the lifting by restricting the theta lifting for the dual pair $(\mathrm{SL}_2, SO(4,4q))$ viewing $\mathrm{Sp}(1,q)$ as a subgroup of $SO(4,4q)$. When $q = 1$, Arakawa showed that the image of the lifting is in the space of cusp forms on $\mathrm{Sp}(1,1)$ generating the quaternionic discrete series representation at the archimedean place.

The second named author, based on the theory of the Fourier expansion developed in [Na2], showed that the image of the lifting is in the space of cusp forms on $\mathrm{Sp}(1,q)$ generating the quaternionic discrete series representation at the archimedean place when $q \geq 1$ [9], [11]. The first and second named authors studied the arithmetic properties of the theta lifting for similitude groups $(GO^*(4), \mathrm{GSp}(1,1))$ [5]–[7]. The second named author investigated a relation between the theta lifting for $(GO^*(4), \mathrm{GSp}(1,1))$ (or $(GO^*(4), \mathrm{GSp}^*(2))$) and that for $(GO(2,2), \mathrm{GSp}(2))$ in terms of the Jacquet–Langlands–Shimizu correspondence [12], [13]. We here note that $\mathrm{GSp}(1,1)$ and $\mathrm{GSp}^*(2)$ are inner forms of $\mathrm{GSp}(2)$.

In this article, we construct a theta lifting $\mathcal{L}$ for the dual pair (G_1, G_2), where $G_1 = O^*(4)$ (resp. $G_2 = \mathrm{Sp}(1,1)$) is the quaternion unitary group of an anti-Hermitian (resp. Hermitian) form of degree 2, and study its basic properties. In particular, we show the functoriality of $\mathcal{L}$ at all non-archimedean places including ramified ones by showing Eichler commutation relations. We also give an archimedean analog of commutation relations, by which we show that $\mathcal{L}$ sends holomorphic cuspidal automorphic forms on G_1 to cuspidal automorphic forms on G_2 generating the quaternionic discrete series representation at the archimedean place. In the forthcoming Parts II and III of the series of the papers, we will investigate the image of $\mathcal{L}$ and an inner product formula for $\mathcal{L}$ based on the results of this article.

To be more precise, let B be a definite quaternion algebra over $\mathbb{Q}$ with the main involution $x \mapsto \overline{x}$. We denote by $B^{(m,n)}$ the set of $m \times n$ matrices whose components are in B. We fix an identification $B \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{M}_2(\mathbb{C})$. Let

$$G_1(\mathbb{Q}) = \left\{ h \in \mathrm{M}_2(B) \mid {}^t\overline{h} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$
$$G_2(\mathbb{Q}) = \left\{ g \in \mathrm{M}_2(B) \mid {}^t\overline{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Then (G_1, G_2) forms a dual pair. We fix an integer $\kappa > 8$ and let σ_κ be the symmetric tensor representation of $\mathrm{M}_2(\mathbb{C})$ on V_κ, the space of homogeneous polynomials in X and Y of total degree κ. Let r be the Weil representation of $G_1(\mathbb{A}) \times G_2(\mathbb{A})$ on $W = \mathcal{S}(B_{\mathbb{A}}^{(2,1)}) \otimes_{\mathbb{C}} V_\kappa$. Here, $\mathcal{S}(B_{\mathbb{A}}^{(2,1)})$ is the space of Schwartz–Bruhat functions on $B_{\mathbb{A}}^{(2,1)}$. With suitable test functions $\varphi_0, \varphi_0^* \in W$, we construct theta kernels $\theta, \theta^* \colon G_1(\mathbb{A}) \times G_2(\mathbb{A}) \to \mathrm{End}(V_\kappa)$ defined by

$$\theta(g_1, g_2) = \sum_{X \in B^{(2,1)}} \left(r(g_1, g_2)\varphi_0\right)(X)$$

and

$$\theta^*(g_1, g_2) = \sum_{X \in B^{(2,1)}} \left(r(g_1, g_2)\varphi_0^*\right)(X).$$

Let $\mathcal{S}_{1,\kappa}$ be the space of V_κ-valued holomorphic cusp forms on $G_1(\mathbb{Q})\backslash G_1(\mathbb{A})$ and $\mathcal{S}_{2,\kappa}$ the space of V_κ-valued cusp forms on $G_2(\mathbb{Q})\backslash G_2(\mathbb{A})$ generating the quaternionic discrete series representation at the archimedean place of $\mathbb{Q}$ (see Section 4.2).

We show that the theta lift

$$\mathcal{L}(f)(g_2) := \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \theta(g_1, g_2) f(g_1) dg_1 \qquad (g_2 \in G_2(\mathbb{A}))$$

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of $f \in \mathcal{S}_{1,\kappa}$ is in $\mathcal{S}_{2,\kappa}$. We also show that the theta lift

$$\mathcal{L}^*(F)(g_1) := \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \theta^*(g_1, g_2)F(g_2)dg_2 \qquad (g_1 \in G_1(\mathbb{A}))$$

of $F \in \mathcal{S}_{2,\kappa}$ is in $\mathcal{S}_{1,\kappa}$ and that $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$ with respect to the Petersson inner products on $\mathcal{S}_{1,\kappa}$ and $\mathcal{S}_{2,\kappa}$, respectively. Note that the proof of these facts for Arakawa's lifting in [10] is based on the study of the lifts of Poincaré series on $\mathrm{SL}(2)$. On the other hand, the method of the proof in this article uses an archimedean analog of Eichler commutation relations:

$$c_{1,\kappa} \int_{G_{1,\infty}} \theta(g_1x, g_2)\omega_{1,\kappa}(x)dx = c_{2,\kappa} \int_{G_{2,\infty}} \omega_{2,\kappa}(y)\theta(g_1, g_2y^{-1})dy, \tag{1}$$

where

$$\omega_{1,\kappa}(g_1) = \sigma_\kappa\left(2^{-1}(-\sqrt{-1}, 1)g_1\begin{pmatrix}\sqrt{-1}\\ 1\end{pmatrix}\right)^{-1} \qquad (g_1 \in G_{1,\infty}),$$

$$\omega_{2,\kappa}(g_2) = (\Delta_2(g_2)\overline{\Delta_2(g_2)})^{-1}\,\sigma_\kappa(\Delta_2(g_2))^{-1} \qquad \left(g_2 \in G_{2,\infty}, \Delta_2(g_2) = 2^{-1}(1,1)g_2\begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

are spherical functions on $G_{1,\infty} = G_1(\mathbb{R})$ and $G_{2,\infty} = G_2(\mathbb{R})$, respectively, and $c_{1,\kappa}$ and $c_{2,\kappa}$ are constants defined in Section 4.2.

We also show the correspondence of the local factors of the standard L-functions at all non-archimedean places for $\mathcal{L}$. Namely, we prove that, if $f \in \mathcal{S}_{1,\kappa}$ is a Hecke eigenform, so is $\mathcal{L}(f)$ and the standard L-function $L(s, \mathcal{L}(f), \mathrm{std})$ of $\mathcal{L}(f)$ coincides with the product of the Riemann zeta function and the standard L-function $L(s, f, \mathrm{std})$ of f. Similar facts for $\mathcal{L}^*$ are also proved. The proof is based on the Eichler commutation relations as follows.

Let p be a finite place of $\mathbb{Q}$. Denote by $\mathcal{H}(G_{1,p}, K_{1,p})$ and $\mathcal{H}(G_{2,p}, K_{2,p})$ the Hecke algebras, where $K_{1,p}$ and $K_{2,p}$ are maximal open compact subgroups of $G_{1,p}$ and $G_{2,p}$ defined in Section 3.4, respectively. Then there exists an algebra homomorphism $\upsilon\colon \mathcal{H}(G_{2,p}, K_{2,p}) \to \mathcal{H}(G_{1,p}, K_{1,p})$ such that

$$\int_{G_{1,p}} \theta(g_1x, g_2)\,\upsilon(\alpha)(x)dx = \int_{G_{2,p}} \theta(g_1, g_2y^{-1})\,\alpha(y)dy$$

holds for $\alpha \in \mathcal{H}(G_{2,p}, K_{2,p})$ and $g_1 \in G_{1,p}, g_2 \in G_{2,p}$.

The article is organized as follows. In Section 2, we prepare the notation and several basic facts on the quaternion algebra used in the later sections. In Section 3, we define two quaternion unitary groups G_1 and G_2. In Section 4, we introduce the space of automorphic forms on G_i and define the standard L-functions attached to Hecke eigenforms. In Section 5, after recalling the definition of the Weil representation of $G_1(\mathbb{A}) \times G_2(\mathbb{A})$, we define the theta kernel $\theta\colon G_1(\mathbb{A}) \times G_2(\mathbb{A}) \to \mathrm{End}(V_\kappa)$ with a suitable test function and introduce the theta lifting $\mathcal{L}$. In Section 6, we define the adjoint $\mathcal{L}^*$ of $\mathcal{L}$. The heart of the article is the following two sections. In Section 7, we show that $\mathcal{L}$ defines a linear mapping from the space of V_κ-valued holomorphic cusp forms on G_1 to the space of V_κ-valued cusp forms on G_2 generating the quaternionic discrete series representation at the archimedean place by an analog (1) of Eichler commutation relations (Theorem 7.1.1). The proof of (1) is quite technical, but we give full details of the proof since the method seems to be a new one. In Section 8, we state the functoriality properties of $\mathcal{L}$ and $\mathcal{L}^*$ as a correspondence of the standard L-functions (Theorem 8.0.1). Note that, in our previous work [5], the functoriality is given as a correspondence of the spinor L-functions. The proof of the functoriality is based on the Eichler commutation relations (Theorem 8.0.2). In Section 9, we give a sketch of the proof of the commutation relations in the

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unramified case, which is quite technical. The detailed account of the proof is given in [8]. In Section 10, we give a full proof of the commutation relations in the ramified case.

2. Notation and preliminaries

2.1.

We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers and the fields of rational numbers, real numbers, and complex numbers, respectively. For a place v of $\mathbb{Q}$, $\mathbb{Q}_v$ denotes the v-adic completion of $\mathbb{Q}$. For a finite place p of $\mathbb{Q}$, let $\mathbb{Z}_p$ be the ring of p-adic integers and $|x|_p$ the p-adic absolute value of $x \in \mathbb{Q}_p^\times$. We denote by $\mathbb{A}$ the adele ring of $\mathbb{Q}$.

For a ring R, denote by $R^{(m,n)}$ the set of $m \times n$ matrices whose components are in R. We also write $\mathrm{M}_m(R)$ for $R^{(m,m)}$. Let X be an algebraic group X over $\mathbb{Q}$. We denote by X_v the group of $\mathbb{Q}_v$-rational points of X for a place v of $\mathbb{Q}$, and by $X_{\mathbb{A}}$ the adelization of X.

For $x \in \mathbb{C}$, we put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$. Let ψ be the additive character of $\mathbb{A}/\mathbb{Q}$ such that $\psi(x_\infty) = \mathbf{e}(x_\infty)$ for $x_\infty \in \mathbb{R}$. For a place v of $\mathbb{Q}$, the restriction of ψ to $\mathbb{Q}_v$ is denoted by ψ_v.

Denote by $X_{\mathrm{unr}}(\mathbb{Q}_p^\times)$ the set of quasi-characters of $\mathbb{Q}_p^\times$ trivial on $\mathbb{Z}_p^\times$. For a prime number p, we write $\mathbb{F}_p$ for the finite field $\mathbb{Z}/p\mathbb{Z}$.

We put

$$\delta(P) = \begin{cases} 1, & \text{if } P \text{ holds,} \\ 0, & \text{if } P \text{ does not hold} \end{cases}$$

for a condition P and $\delta_{i,j} = \delta(i = j)$ for $i, j \in \mathbb{Z}$.

2.2.

Throughout the article, we fix a definite quaternion algebra B over $\mathbb{Q}$ with discriminant d_B. Let $x \mapsto \overline{x}$ be the main involution of B. For a subset X of B, put $X^- = \{\alpha \in X \mid \alpha + \overline{\alpha} = 0\}$.

We write N_r and tr_r for the reduced norm and the reduced trace of $\mathrm{M}_r(B)$ to $\mathbb{Q}$, respectively. We often suppress the subscript r if there is no fear of confusion.

We fix an isomorphism from $B \otimes_{\mathbb{Q}} \mathbb{R}$ to the Hamilton quaternion algebra

$$\mathbb{H} = \mathbb{R} \cdot 1 + \mathbb{R} \cdot i + \mathbb{R} \cdot j + \mathbb{R} \cdot k \qquad (i^2 = j^2 = k^2 = -1,\ ij = -ji = k)$$

and identify $B \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{H}$. Throughout the article, we always distinguish the element i of $\mathbb{H}$ from the imaginary unit $\sqrt{-1} \in \mathbb{C}$. We define an embedding $A\colon \mathbb{H} \to \mathrm{M}_2(\mathbb{C})$ by

$$A(a_0 + a_1 i + a_2 j + a_3 k) = \begin{pmatrix} a_0 + a_1\sqrt{-1} & a_2 + a_3\sqrt{-1} \\ -a_2 + a_3\sqrt{-1} & a_0 - a_1\sqrt{-1} \end{pmatrix} \tag{2}$$

for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The map A extends to an isomorphism of $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ to $\mathrm{M}_2(\mathbb{C})$.

2.3.

The measures on $\mathbb{R}$ and $\mathbb{C}$ are taken to be the usual Lebesgue measures.

We define the measures $d_L\alpha$ on $\mathbb{H}$ and $d_L\beta$ on $\mathbb{H}^-$ by $d_L\alpha = da_0 da_1 da_2 da_3$ $(\alpha = a_0 + a_1 i + a_2 j + a_3 k, a_i \in \mathbb{R})$ and $d_L\beta = db_1 db_2 db_3$ $(\beta = b_1 i + b_2 j + b_3 k, b_i \in \mathbb{R})$, respectively. The invariant measure $d^\times t$ on $\mathbb{H}^1 = \{t \in \mathbb{H} \mid N(t) = 1\}$ is normalized by $\mathrm{vol}(\mathbb{H}^1) = 1$.

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2.4.

For a place v of $\mathbb{Q}$, we define the measures $d\alpha_v$ on B_v and $d\beta_v$ on B_v^- so that they are self-dual with respect to the pairings $(\alpha_v, \alpha_v') \mapsto \psi_v(\mathrm{tr}(\overline{\alpha_v}\alpha_v'))$ and $(\beta_v, \beta_v') \mapsto \psi_v(\mathrm{tr}(\overline{\beta_v}\beta_v'))$, respectively. Note that, for $v = \infty$, $d\alpha_\infty = 4d_L\alpha_\infty$. We define the measure on $B_v^{(m,n)}$ by $dX = \prod_{1\le i\le m,\, 1\le j\le n} dx_{ij}$ $(X = (x_{ij}))$, and the measures $d\alpha$ on $B_{\mathbb{A}}$ and $d\beta$ on $B_{\mathbb{A}}^-$ by the product measures $\prod_{v\le\infty} d\alpha_v$ and $\prod_{v\le\infty} d\beta_v$, respectively.

2.5.

For a place v of $\mathbb{Q}$, we denote by $\mathcal{S}(B_v^{(m,n)})$ the space of Schwartz–Bruhat functions on $B_v^{(m,n)}$.

2.6.

For a positive integer κ, let V_κ be the space of homogeneous polynomials in the two variables X and Y of total degree κ. Then $\{P_r\}_{r=0}^{\kappa}$ forms a basis of V_κ and $\dim V_\kappa = \kappa + 1$, where

$$P_r(X,Y) = X^r\, Y^{\kappa-r} \in V_\kappa \qquad (0 \le r \le \kappa). \tag{3}$$

We define $\sigma_\kappa : \mathbb{H} \to \mathrm{End}(V_\kappa)$ by

$$(\sigma_\kappa(\alpha)P)(X,Y) = P((X,Y)A(\alpha)) \qquad (\alpha \in \mathbb{H}, P \in V_\kappa), \tag{4}$$

where A is defined by (2). Then σ_κ defines an irreducible representation of $\mathbb{H}$ on V_κ. The inner product of V_κ defined by

$$(P,P')_\kappa = \int_{\mathbb{C}^2} P(x,y)\,\overline{P'(x,y)} e^{-2\pi(|x|^2+|y|^2)}\,dxdy \qquad (P, P' \in V_\kappa)$$

is $A(\mathbb{H}^1)$-invariant. Put $\|P\|_\kappa = \sqrt{(P,P)_\kappa}$ for $P \in V_\kappa$. The following formula is often used in this article.

Lemma 2.6.1. *For $\alpha, \beta \in \mathbb{H}$, we have*

$$\int_{\mathbb{H}} \mathbf{e}(\mathrm{tr}(\alpha y) + \sqrt{-1}\overline{y}y)\,\sigma_\kappa(y+\beta)dy = \mathbf{e}(\sqrt{-1}\overline{\alpha}\alpha)\,\sigma_\kappa(\sqrt{-1}\overline{\alpha}+\beta).$$

Proof. Note that the matrix coefficients of $\sigma_\kappa(y)$ $(y \in \mathbb{H})$ are spherical functions on $\mathbb{H} \simeq \mathbb{R}^4$. Then the lemma follows from an argument in Chapter 1, Section 2 in [4]. □

3. Groups

3.1.

In this article, we consider two quaternion unitary groups G_1 and G_2 over $\mathbb{Q}$ defined by

$$G_i(\mathbb{Q}) = \left\{ g \in \mathrm{GL}_2(B) \mid {}^t\overline{g} Q_i g = Q_i \right\},$$

where

$$Q_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The groups G_1 and G_2 are denoted by $O^*(4)$ and $\mathrm{Sp}(1,1)$, respectively, and (G_1, G_2) forms a dual pair.

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Remark 3.1.1. Let H' and Z' be algebraic groups over $\mathbb{Q}$ defined by

$$H'(\mathbb{Q}) = \{(h, \alpha) \in \mathrm{GL}_2(\mathbb{Q}) \times B^\times \mid \det h \cdot N(\alpha) = 1\},$$
$$Z'(\mathbb{Q}) = \{(a1_2, a^{-1}) \mid a \in \mathbb{Q}^\times\} \subset H'(\mathbb{Q}).$$

A homomorphism $H' \ni (h, \alpha) \mapsto \alpha h \in G_1$ defines an isomorphism $H'/Z' \cong G_1$.

3.2.

Let

$$X_i = \begin{cases} \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} & (i = 1), \\ \{z \in \mathbb{H} \mid \mathrm{tr}(z) > 0\} & (i = 2). \end{cases}$$

The group $G_{i,\infty}$ acts transitively on X_i by

$$g \cdot z = (az + b)(cz + d)^{-1} \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{i,\infty}, z \in X_i\right).$$

Let

$$K_{i,\infty} = \{k \in G_{i,\infty} \mid k \cdot z_i^0 = z_i^0\},$$

where

$$z_i^0 = \begin{cases} \sqrt{-1} & (i = 1), \\ 1 & (i = 2). \end{cases}$$

Then $K_{i,\infty}$ is a maximal compact subgroup of $G_{i,\infty}$ and

$$K_{1,\infty} = \left\{ t \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid t \in \mathbb{H}^1,\ 0 \le \theta < 2\pi \right\},$$
$$K_{2,\infty} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{H},\ N(a) + N(b) = 1,\ \mathrm{tr}(\overline{a}b) = 0 \right\}.$$

Note that

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto (a + b, a - b)$$

gives an isomorphism $K_{2,\infty} \cong \mathbb{H}^1 \times \mathbb{H}^1$. We normalize the Haar measure on $K_{i,\infty}$ by $\mathrm{vol}(K_{i,\infty}) = 1$.

3.3.

For a positive integer κ, we define irreducible representations $\tau_{1,\kappa}$ (resp., $\tau_{2,\kappa}$ and $\tau_{2,\kappa}^-$) of $K_{1,\infty}$ (resp. $K_{2,\infty}$) on V_κ as follows:

$$\tau_{1,\kappa}(k) = e^{-\sqrt{-1}\kappa\theta}\, \sigma_\kappa(t) \qquad \left(k = t \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_{1,\infty},\ t \in \mathbb{H}^1, 0 \le \theta < 2\pi\right),$$
$$\tau_{2,\kappa}(k) = \sigma_\kappa(a + b), \quad \tau_{2,\kappa}^-(k) = \sigma_\kappa(a - b) \qquad \left(k = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in K_{2,\infty}\right).$$

Note that $\tau_{2,\kappa}$ is not equivalent to $\tau_{2,\kappa}^-$.

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3.4.

We henceforth fix a maximal order O of B. For a finite place p of $\mathbb{Q}$, let $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For $p|d_B$, denote by $\mathfrak{P}_p$ the maximal ideal of O_p and fix an element Π_p of B_p such that $\mathfrak{P}_p = \Pi_p O_p$. We define a lattice $L_{1,p}$ (resp. $L_{2,p}$) of $B_p^{(2,1)}$ (resp. $B_p^{(1,2)}$) by

$$L_{1,p} = \begin{pmatrix} O_p \\ O_p \end{pmatrix},$$
$$L_{2,p} = \begin{cases} (O_p, O_p) & (p \nmid d_B), \\ (O_p, \mathfrak{P}_p^{-1}) & (p|d_B). \end{cases}$$

Then $L_{1,p}$ and $L_{2,p}$ are maximal O_p-lattices of $B_p^{(2,1)}$ and $B_p^{(1,2)}$, respectively (see [17, Section 3.2]).

Set

$$K_{1,p} = \{k \in G_{1,p} \mid kL_{1,p} = L_{1,p}\},$$
$$K_{2,p} = \{k \in G_{2,p} \mid L_{2,p}k = L_{2,p}\}.$$

Then $K_{i,p}$ is a maximal open compact subgroup of $G_{i,p}$.

Set

$$K_{i,f} = \prod_{p<\infty} K_{i,p}.$$

3.5.

We normalize the Haar measures on $G_{1,\infty}$ and $G_{2,\infty}$ by

$$\int_{G_{1,\infty}} \varphi(g)dg = \int_{\mathbb{R}} db \int_0^\infty a^{-2}d^\times a \int_{K_{1,\infty}} dk\, \varphi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k\right) \quad (\varphi \in L^1(G_{1,\infty}))$$

and

$$\int_{G_{2,\infty}} \varphi(g)dg = \int_{\mathbb{H}^-} d_L\beta \int_0^\infty a^{-6}d^\times a \int_{K_{2,\infty}} dk\, \varphi\left(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k\right) \quad (\varphi \in L^1(G_{2,\infty})),$$

where note that we have normalized the measures dk at the infinite place so that $\mathrm{vol}(K_{i,\infty}) = 1$. In addition, the Haar measure on $G_{i,p}$ is normalized by $\mathrm{vol}(K_{i,p}) = 1$.

4. Automorphic forms

4.1.

Let κ be an integer greater than 4. We define an $\mathrm{End}(V_\kappa)$-valued spherical function $\omega_{i,\kappa}$ on $G_{i,\infty}$ by

$$\omega_{i,\kappa}(g) = \begin{cases} \sigma_\kappa(\Delta_1(g))^{-1} & (i = 1), \\ N(\Delta_2(g))^{-1}\sigma_\kappa(\Delta_2(g))^{-1} & (i = 2) \end{cases}$$

for $g \in G_{i,\infty}$, where

$$\Delta_1(g) = \frac{1}{2}(-\sqrt{-1}, 1)g\begin{pmatrix} \sqrt{-1} \\ 1 \end{pmatrix} \in (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})^\times \qquad (g \in G_{1,\infty}),$$
$$\Delta_2(g) = \frac{1}{2}(1, 1)g\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{H}^\times \qquad (g \in G_{2,\infty}).$$

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It is easily checked that

$$\omega_{i,\kappa}(kgk') = \tau_{i,\kappa}(k')^{-1}\omega_{i,\kappa}(g)\tau_{i,\kappa}(k)^{-1} \qquad (g \in G_{i,\infty}, k, k' \in K_{i,\infty}). \tag{5}$$

Note that $\omega_{i,\kappa}$ is integrable on $G_{i,\infty}$.

4.2.

Let $\mathcal{S}_{i,\kappa}$ be the space of smooth functions $f\colon G_i(\mathbb{Q})\backslash G_i(\mathbb{A}) \to V_\kappa$ satisfying the following conditions:

(1) For $g \in G_i(\mathbb{A})$, $k_f \in K_{i,f}$, and $k_\infty \in K_{i,\infty}$, we have

$$f(gk_f k_\infty) = \tau_{i,\kappa}(k_\infty)^{-1} f(g). \tag{6}$$

(2) $\|f(g)\|_\kappa$ is bounded on $G_i(\mathbb{A})$.
(3) For $g \in G_i(\mathbb{A})$, we have

$$f(g) = c_{i,\kappa} \int_{G_{i,\infty}} \omega_{i,\kappa}(x_\infty) f(gx_\infty^{-1}) dx_\infty, \tag{7}$$

where

$$c_{i,\kappa} = \begin{cases} \dfrac{\kappa - 1}{2\pi} & (i = 1), \\ \dfrac{\kappa(\kappa-1)}{8\pi^2} & (i = 2). \end{cases} \tag{8}$$

Note that $f \in \mathcal{S}_{i,\kappa}$ is cuspidal; namely,

$$f_0(g_1) = \int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g_1\right) db = 0 \qquad (g_1 \in G_1(\mathbb{A}))$$

if $i = 1$ (this follows from the boundedness and the Fourier expansion of f) and

$$f_0(g_2) = \int_{B^-\backslash B^-_{\mathbb{A}}} f\left(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} g_2\right) d\beta = 0 \qquad (g_2 \in G_2(\mathbb{A}))$$

if $i = 2$ (see [2, Proposition 3.1]).

We define an inner product $\langle , \rangle_i$ on $\mathcal{S}_{i,\kappa}$ by

$$\langle f, f' \rangle_i = \int_{G_i(\mathbb{Q})\backslash G_i(\mathbb{A})} (f(g_i), f'(g_i))_\kappa dg_i \qquad (f, f' \in \mathcal{S}_{i,\kappa}),$$

where dg_i is the right-invariant measure on $G_i(\mathbb{Q})\backslash G_i(\mathbb{A})$ induced by the product measure $\prod_v dg_v$ on $G_i(\mathbb{A})$.

Remark 4.2.1. We can see $f \in \mathcal{S}_{1,\kappa}$ as a V_κ-valued holomorphic elliptic modular form.

Remark 4.2.2. It is known that $f \in \mathcal{S}_{2,\kappa}$ generates the quaternionic discrete series representation at the archimedean place of $\mathbb{Q}$ (see [10]). For a general theory of such automorphic forms, we refer to [1], [2], [9], and [10].

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4.3.

For a finite place p of $\mathbb{Q}$, let $\mathcal{H}(G_{i,p}, K_{i,p})$ be the Hecke algebra: By definition, $\mathcal{H}(G_{i,p}, K_{i,p})$ is the space of compactly supported bi- $K_{i,p}$-invariant functions on $G_{i,p}$ and the product is defined by

$$\left(\varphi_p * \varphi'_p\right)(g_p) = \int_{G_{i,p}} \varphi_p(g_p x_p^{-1}) \varphi'_p(x_p) dx_p \qquad (\varphi_p, \varphi'_p \in \mathcal{H}(G_{i,p}, K_{i,p}), g_p \in G_{i,p}).$$

The Hecke algebra $\mathcal{H}(G_{i,p}, K_{i,p})$ acts on $\mathcal{S}_{i,\kappa}$ by

$$\left(f * \varphi_p\right)(g) = \int_{G_{i,p}} f(g x_p^{-1}) \varphi_p(x_p) dx_p \qquad (f \in \mathcal{S}_{i,\kappa}, \varphi_p \in \mathcal{H}(G_{i,p}, K_{i,p}), g \in G_i(\mathbb{A})).$$

We say that $f \in \mathcal{S}_{i,\kappa}$ is a *Hecke eigenform* if

$$f * \varphi_p = \lambda_{i,p}^f(\varphi_p)\, f \tag{9}$$

holds for any finite place p of $\mathbb{Q}$ and any $\varphi_p \in \mathcal{H}(G_{i,p}, K_{i,p})$ with a $\mathbb{C}$-algebra homomorphism $\lambda_{i,p}^f$ from $\mathcal{H}(G_{i,p}, K_{i,p})$ to $\mathbb{C}$. The following result is a straightforward consequence of [16, Section 5.2].

Lemma 4.3.1. *Suppose that $f \in \mathcal{S}_{i,\kappa}$ is a Hecke eigenform and let p be a finite place of $\mathbb{Q}$.*

(1) *There exists a unique bi- $K_{i,p}$-invariant function $\omega_{i,p}^f$ on $G_{i,p}$ satisfying $\omega_{i,p}^f(1) = 1$ and*

$$\int_{G_{i,p}} \omega_{i,p}^f(g x^{-1}) \varphi_p(x) dx = \lambda_{i,p}^f(\varphi_p) \omega_{i,p}^f(g) \qquad (g \in G_{i,p})$$

for any $\varphi_p \in \mathcal{H}(G_{i,p}, K_{i,p})$.

(2) *We have*

$$\int_{K_{i,p}} f(g k_p x) dk_p = \omega_{i,p}^f(x_p) f(g x_p^{-1} x)$$

for $g, x \in G_i(\mathbb{A})$, where $x_p \in G_{i,p}$ is the p-component of x.

4.4.

In this and the next sections, we assume that $f \in \mathcal{S}_{i,\kappa}$ is a Hecke eigenform and define the local factor $L_p(s, f, \mathrm{std})$ of the standard L-function attached to f at a finite place p of $\mathbb{Q}$.

In this section, we suppose that B is unramified at p (namely, $p \nmid d_B$). We identify B_p with $\mathrm{M}_2(\mathbb{Q}_p)$ by an isomorphism $B_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$ such that $\mathcal{O}_p$ maps to $\mathrm{M}_2(\mathbb{Z}_p)$ and such that the main involution of B_p corresponds to

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto J^{-1} {}^t X J = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\mathrm{tr}(X)$ and $N(X)$ are the trace and the determinant of $X \in \mathrm{M}_2(\mathbb{Q}_p)$, respectively. We identify $G_{i,p}$ and $K_{i,p}$ with

$$\left\{ g \in \mathrm{GL}_4(\mathbb{Q}_p) \mid {}^t g \mathcal{Q}_i g = \mathcal{Q}_i \right\},$$

and

$$\left\{ k \in \mathrm{GL}_4(\mathbb{Z}_p) \mid {}^t k \mathcal{Q}_i k = \mathcal{Q}_i \right\},$$

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respectively. Here,

$$Q_1 = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

For $\chi = (\chi_1, \chi_2) \in X_{\mathrm{unr}}(\mathbb{Q}_p^\times)^2$, we define the spherical function $\omega_{i,\chi}$ on $G_{i,p}$ by

$$\omega_{i,\chi}(g) = \int_{K_{i,p}} \eta_{i,\chi}(kg)dk.$$

Here,

$$\eta_{1,\chi}(g) = \chi_1(a_1)\chi_2(a_2)|a_1|_p \left(g = \begin{pmatrix} a_1 & * & * & * \\ 0 & a_2 & * & * \\ 0 & 0 & a_2^{-1} & * \\ 0 & 0 & 0 & a_1^{-1} \end{pmatrix} k \in G_{1,p},\ a_1, a_2 \in \mathbb{Q}_p^\times, k \in K_{1,p} \right)$$

and

$$\eta_{2,\chi}(g) = \chi_1(a_1)\chi_2(a_2)|a_1|_p^2|a_2|_p \left(g = \begin{pmatrix} a_1 & * & * & * \\ 0 & a_2 & * & * \\ 0 & 0 & a_2^{-1} & 0 \\ 0 & 0 & * & a_1^{-1} \end{pmatrix} k \in G_{2,p},\ a_1, a_2 \in \mathbb{Q}_p^\times, k \in K_{2,p} \right)$$

(for the definition of $X_{\mathrm{unr}}(\mathbb{Q}_p^\times)$, see Section 2.1).

Lemma 4.4.1 [16, Section 5.4, Theorem 2]. *There exists* $\chi = (\chi_1, \chi_2) \in X_{\mathrm{unr}}(\mathbb{Q}_p^\times)^2$ *such that* $\omega_{i,p}^f = \omega_{i,\chi}$.

We define the local factor of the standard L-function attached to f at p by

$$L_p(s, f, \mathrm{std}) = \begin{Bmatrix} 1 & (i=1) \\ \zeta_p(s) & (i=2) \end{Bmatrix} \cdot \prod_{j=1}^{2} L_p(s, \chi_j) L_p(s, \chi_j^{-1}),$$

where $\zeta_p(s) = (1-p^{-s})^{-1}$ and $L_p(s, \chi_j^{\pm 1}) = (1-\chi_j^{\pm 1}(p)p^{-s})^{-1}$. Note that the definition of $L_p(s, f, \mathrm{std})$ does not depend on the choice of χ.

4.5.

In this section, we suppose that B is ramified at p (namely, $p|d_B$).

Let $X_{\mathrm{unr}}(B_p^\times)$ be the set of homomorphisms from $B_p^\times$ to $\mathbb{C}^\times$ trivial on $\mathcal{O}_p^\times$. For $\chi \in X_{\mathrm{unr}}(B_p^\times)$, we define the spherical function $\omega_{i,\chi}$ on $G_{i,p}$ by

$$\omega_{i,\chi}(g) = \int_{K_{i,p}} \eta_{i,\chi}(kg)dk.$$

Here,

$$\eta_{1,\chi}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix} k \right) = \chi(\alpha)|N(\alpha)|_p^{1/2} \qquad (b \in \mathbb{Q}_p, \alpha \in B_p^\times, k \in K_{1,p})$$

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and

$$\eta_{2,\chi}\left(\begin{pmatrix}1&\beta\\0&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\overline{\alpha}^{-1}\end{pmatrix}k\right)=\chi(\alpha)|N(\alpha)|_p^{3/2}\qquad(\beta\in B_p^-,\alpha\in B_p^\times,k\in K_{2,p}).$$

Lemma 4.5.1 [16, Section 5.4, Theorem 2]. *There exists $\chi\in X_{\mathrm{unr}}(B_p^\times)$ such that $\omega_{i,p}^f=\omega_{i,\chi}$.*

We define the local factor of the standard L-function attached to f at p by

$$L_p(s,f,\mathrm{std})=\begin{Bmatrix}1&(i=1)\\\zeta_p(s)&(i=2)\end{Bmatrix}\cdot\left(1-\chi(\Pi_p)p^{-s-1/2}\right)^{-1}\left(1-\chi^{-1}(\Pi_p)p^{-s-1/2}\right)^{-1}.$$

4.6.

We define the standard L-function attached to a Hecke eigenform $f\in\mathcal{S}_{i,\kappa}$ by

$$L(s,f,\mathrm{std})=\prod_{p<\infty}L_p(s,f,\mathrm{std}).\tag{10}$$

5. The Arakawa lifting

5.1.

Let v be a place of $\mathbb{Q}$. We define the Weil representation r_v of $G_{1,v}\times G_{2,v}$ on the space

$$W_v=\begin{cases}\mathcal{S}(B_v^{(2,1)})&(v<\infty),\\\mathcal{S}(B_v^{(2,1)})\otimes\mathrm{End}(V_\kappa)&(v=\infty)\end{cases}$$

as follows: For $\varphi\in W_v$ and $X\in B_v^{(2,1)}$,

$$\begin{aligned}r_v(g,1_2)\varphi(X)&=\varphi({}^t\overline{g}X)\qquad(g\in G_{1,v}),\\r_v\left(1_2,\begin{pmatrix}\alpha&0\\0&\overline{\alpha}^{-1}\end{pmatrix}\right)\varphi(X)&=|N(\alpha)|_v^2\,\varphi(X\alpha)\qquad(\alpha\in B_v^\times),\\r_v\left(1_2,\begin{pmatrix}1&\beta\\0&1\end{pmatrix}\right)\varphi(X)&=\psi_v\left(-\frac{1}{2}\mathrm{tr}(\beta{}^t\overline{X}JX)\right)\varphi(X)\qquad(\beta\in B_v^-),\\r_v\left(1_2,\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)\varphi(X)&=\int_{B_v^{(2,1)}}\psi_v\left(-\mathrm{tr}({}^t\overline{Y}JX)\right)\varphi(Y)dY.\end{aligned}$$

To give another realization of the Weil representation, we define an intertwining operator I_v from W_v to

$$W_v'=\begin{cases}\mathcal{S}(B_v^{(1,2)})&(v<\infty),\\\mathcal{S}(B_v^{(1,2)})\otimes\mathrm{End}(V_\kappa)&(v=\infty)\end{cases}$$

by a partial Fourier transform

$$I_v\varphi(x_1,x_2)=\int_{B_v}\psi_v\left(\mathrm{tr}(\overline{y}x_2)\right)\varphi\begin{pmatrix}x_1\\y\end{pmatrix}dy\qquad(\varphi\in W_v,x_1,x_2\in B_v).$$

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The inverse of I_v is given by

$$I_v^{-1}\phi\begin{pmatrix}x_1\\x_2\end{pmatrix}=\int_{B_v}\psi_v\left(-\mathrm{tr}(\overline{y}x_2)\right)\phi(x_1,y)dy\qquad(\phi\in W_v',x_1,x_2\in B_v).$$

We set

$$r_v'(g_1,g_2)=I_v\circ r_v(g_1,g_2)\circ I_v^{-1}\qquad(g_i\in G_{i,v}),$$

which defines a smooth representation of $G_{1,v}\times G_{2,v}$ on W_v'. A direct calculation shows the following.

Lemma 5.1.1. *For $\phi\in W_v'$ and $X'\in B_v^{(1,2)}$, we have*

$$\begin{aligned}
r_v'\left(\begin{pmatrix}\alpha&0\\0&\overline{\alpha}^{-1}\end{pmatrix},1_2\right)\phi(X')&=|N(\alpha)|_v^2\,\phi(\overline{\alpha}X')\qquad(\alpha\in B_v^\times),\\
r_v'\left(\begin{pmatrix}1&b\\0&1\end{pmatrix},1_2\right)\phi(X')&=\psi_v\left(-b\,\overline{X'}Q{}^tX'\right)\phi(X')\qquad(b\in\mathbb{Q}_v),\\
r_v'\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},1_2\right)\phi(X')&=\int_{B_v^{(1,2)}}\psi_v\left(-\mathrm{tr}(\overline{X'}Q{}^tY)\right)\phi(Y)dY,\\
r_v'(1_2,g)\phi(X')&=\phi(X'g)\qquad(g\in G_{2,v}).
\end{aligned}$$

5.2.

Let $\mathcal{S}(B_{\mathbb{A}}^{(2,1)})$ and $\mathcal{S}(B_{\mathbb{A}}^{(1,2)})$ be the restricted tensor products of $\mathcal{S}(B_v^{(2,1)})$ and $\mathcal{S}(B_v^{(1,2)})$ over the places v of $\mathbb{Q}$ with respect to $\{\varphi_{0,p}\}_{p<\infty}$ and $\{\phi_{0,p}\}_{p<\infty}$, respectively. Here, $\varphi_{0,p}$ and $\phi_{0,p}$ denote the characteristic functions of $L_{1,p}$ and $L_{2,p}$, respectively (for the definitions of $L_{1,p}$ and L_2,p, see Section 3.4). We define smooth representations r and r' of $G_1(\mathbb{A})\times G_2(\mathbb{A})$ on $W=\mathcal{S}(B_{\mathbb{A}}^{(2,1)})\otimes\mathrm{End}(V_\kappa)$ and $W'=\mathcal{S}(B_{\mathbb{A}}^{(1,2)})\otimes\mathrm{End}(V_\kappa)$ as restricted tensor products of r_v and r_v', respectively:

$$r=\otimes_v r_v,\ r'=\otimes_v r_v'.$$

Then $r'(g_1,g_2)=I\circ r(g_1,g_2)\circ I^{-1}\ \ (g_i\in G_i(\mathbb{A}))$, where

$$I\varphi(x_1,x_2)=\int_{B_{\mathbb{A}}}\psi\left(\mathrm{tr}(\overline{y}x_2)\right)\varphi\begin{pmatrix}x_1\\y\end{pmatrix}dy\qquad(\varphi\in W,x_1,x_2\in B_{\mathbb{A}}).$$

We define test functions $\varphi_0=\otimes_v\varphi_{0,v}\in W$ and $\phi_0=\otimes_v\phi_{0,v}\in W'$, where

$$\begin{aligned}
\varphi_{0,\infty}(X)&=\mathbf{e}(\sqrt{-1}\,{}^t\overline{X}X)\,\sigma_\kappa((1,-\sqrt{-1})\overline{X})\qquad(X\in\mathbb{H}^{(2,1)}),\\
\phi_{0,\infty}(X')&=\mathbf{e}(\sqrt{-1}\,\overline{X'}\,{}^tX')\,\sigma_\kappa((1,1)\,{}^t\overline{X'})\qquad(X\in\mathbb{H}^{(1,2)}).
\end{aligned}$$

The following results are proved in a straightforward manner.

Lemma 5.2.1. *For a place v of $\mathbb{Q}$, we have*

$$\phi_{0,v}=I_v\varphi_{0,v}.$$

Lemma 5.2.2. (1) *For a finite place p, we have*

$$\begin{aligned}
r_p(k_{1,p},k_{2,p})\varphi_{0,p}&=\varphi_{0,p},\\
r_p'(k_{1,p},k_{2,p})\phi_{0,p}&=\phi_{0,p}
\end{aligned}$$

for $k_{i,p}\in K_{i,p}$.

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(2) *We have*

$$\left(r_\infty(k_{1,\infty},k_{2,\infty})\varphi_{0,\infty}\right)(X)=\tau_{2,\kappa}(k_{2,\infty})^{-1}\circ\varphi_{0,\infty}(X)\circ\tau_{1,\kappa}(k_{1,\infty}),$$
$$\left(r'_\infty(k_{1,\infty},k_{2,\infty})\phi_{0,\infty}\right)(X')=\tau_{2,\kappa}(k_{2,\infty})^{-1}\circ\phi_{0,\infty}(X')\circ\tau_{1,\kappa}(k_{1,\infty})$$

for $k_{i,\infty}\in K_{i,\infty}$, $X\in\mathbb{H}^{(2,1)}$, *and* $X'\in\mathbb{H}^{(1,2)}$.

5.3.

We define the theta kernel $\theta\colon G_1(\mathbb{A})\times G_2(\mathbb{A})\to\mathrm{End}(V_\kappa)$ by

$$\theta(g_1,g_2)=\sum_{X\in B^{(2,1)}}\left(r(g_1,g_2)\varphi_0\right)(X)\qquad(g_i\in G_i(\mathbb{A})).$$

By the Poisson summation formula, we have

$$\theta(g_1,g_2)=\sum_{X'\in B^{(1,2)}}\left(r'(g_1,g_2)\phi_0\right)(X').$$

Lemma 5.3.1. *We have*

$$\theta\left(\gamma_1 g_1 k_{1,f}\,k_{1,\infty},\gamma_2 g_2 k_{2,f}\,k_{2,\infty}\right)=\tau_{2,\kappa}(k_{2,\infty})^{-1}\circ\theta(g_1,g_2)\circ\tau_{1,\kappa}(k_{1,\infty})$$

for $\gamma_i\in G_i(\mathbb{Q})$, $g_i\in G_i(\mathbb{A})$, $k_{i,f}\in K_{i,f}$, *and* $k_{i,\infty}\in K_{i,\infty}$.

Proof. This follows from Lemma 5.2.2. □

5.4.

We define the theta lift $\mathcal{L}(f)$ of $f\in\mathcal{S}_{1,\kappa}$ by

$$\mathcal{L}(f)(g_2)=\int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})}\theta(g_1,g_2)f(g_1)dg_1\qquad(g_2\in G_2(\mathbb{A})).\tag{11}$$

Here, the Haar measure dg_1 on $G_1(\mathbb{A})$ is normalized by $\mathrm{vol}(G_1(\mathbb{Q})\backslash G_1(\mathbb{A}))=1$. The integral (11) is absolutely convergent.

Theorem 5.4.1 [11]. *We have* $\mathcal{L}(f)\in\mathcal{S}_{2,\kappa}$.

The theta lifting $\mathcal{L}\colon\mathcal{S}_{1,\kappa}\to\mathcal{S}_{2,\kappa}$ was first introduced essentially by Arakawa in his unpublished notes. The second named author of this article, based on Arakawa's notes, developed Arakawa's work to the lifting from elliptic modular forms to automorphic forms on $\mathrm{Sp}(1,q)$ generating the quaternionic discrete series representations at the archimedean place [11].

Specifying some arithmetic group defined by a global maximal lattice, Arakawa and the second named author proved Theorem 5.4.1 by showing that the lifts of Poincaré series on G_1 satisfy (7) under the assumption $\kappa>6$. In Section 7, we give another proof of Theorem 5.4.1 by using an archimedean analog of Eichler commutation relations under the assumption $\kappa>8$.

We call $\mathcal{L}(f)$ the *Arakawa lift* of f. The arithmetic properties of the Arakawa lifting $\mathcal{L}$ (in particular the Fourier expansion of $\mathcal{L}(f)$) have been studied in [5], [6], and [7]. In [12] and [13], the second named author treated the case where the involved representation at the archimedean place belongs to a more general discrete series. He also studied a relation between the theta lifting for $(GO^*(4),\mathrm{GSp}(1,1))$ (or $(GO^*(4),\mathrm{GSp}^*(2))$) and that for $(GO(2,2),\mathrm{GSp}(2))$ in terms of the Jacquet–Langlands–Shimizu correspondence.

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6. The adjoint of the Arakawa lifting

6.1.

Define $\varphi_0^* \in W$ and $\phi_0^* \in W'$ by

$$\varphi_0^* = \varphi_{0,\infty}^* \otimes \bigotimes_{p<\infty} \varphi_{0,p}$$
$$\phi_0^* = \phi_{0,\infty}^* \otimes \bigotimes_{p<\infty} \phi_{0,p},$$

where

$$\varphi_{0,\infty}^*(X) = \mathbf{e}\left(\sqrt{-1}{}^t\overline{X}X\right)\sigma_\kappa((1,\sqrt{-1})X) \qquad (X \in \mathbb{H}^{(2,1)}),$$
$$\phi_{0,\infty}^*(X') = \mathbf{e}\left(\sqrt{-1}X'{}^t\overline{X'}\right)\sigma_\kappa((1,-1){}^tX') \qquad (X' \in \mathbb{H}^{(1,2)}).$$

The proofs of the following results are straightforward and omitted.

Lemma 6.1.1. (1) *We have* $\phi_0^* = I\varphi_0^*$.
(2) *For* $v, v' \in V_\kappa$, *we have*

$$(\varphi_0(X)v, v')_\kappa = (v, \varphi_0^*(X)v')_\kappa \qquad (X \in B_{\mathbb{A}}^{(2,1)}),$$
$$(\phi_0(x_1,x_2)v, v')_\kappa = (v, \phi_0^*(x_1,-x_2)v')_\kappa \qquad (x_1,x_2 \in B_{\mathbb{A}}).$$

(3) *For* $k_{1,\infty} \in K_{1,\infty}$ *and* $k_{2,\infty} \in K_{2,\infty}$, *we have*

$$\left(r(k_{1,\infty},k_{2,\infty})\varphi_0^*\right)(X) = \tau_{1,\kappa}(k_{1,\infty})^{-1} \circ \varphi_0^*(X) \circ \tau_{2,\kappa}^-(k_{2,\infty}) \qquad (X \in B_{\mathbb{A}}^{(2,1)}),$$
$$\left(r'(k_{1,\infty},k_{2,\infty})\phi_0^*\right)(X') = \tau_{1,\kappa}(k_{1,\infty})^{-1} \circ \phi_0^*(X') \circ \tau_{2,\kappa}^-(k_{2,\infty}) \qquad (X' \in B_{\mathbb{A}}^{(1,2)}).$$

Lemma 6.1.2. *We have*

$$(r(g_1,g_2)\varphi_0(X)v, v')_\kappa = (v, r(g_1,g_2^-)\varphi_0^*(X)v')_\kappa \qquad (g_i \in G_i(\mathbb{A}), X \in B_{\mathbb{A}}^{(2,1)}, v, v' \in V_\kappa),$$

where

$$g_2^- = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \qquad \text{for } g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_2(\mathbb{A}).$$

6.2.

Set

$$\theta^*(g_1,g_2) = \sum_{X \in B^{(2,1)}} \left(r(g_1,g_2^-)\varphi_0^*\right)(X) \in \mathrm{End}(V_\kappa) \qquad (g_i \in G_i(\mathbb{A})).$$

By the Poisson summation formula, we have

$$\theta^*(g_1,g_2) = \sum_{X' \in B^{(1,2)}} \left(r'(g_1,g_2^-)\phi_0^*\right)(X').$$

The following facts are direct consequences of Lemmas 6.1.1(3) and 6.1.2.

Lemma 6.2.1. (1) *We have*

$$\theta^*\left(\gamma_1 g_1 k_{1,f}\, k_{1,\infty}, \gamma_2 g_2 k_{2,f}\, k_{2,\infty}\right) = \tau_{1,\kappa}(k_{1,\infty})^{-1} \circ \theta^*(g_1,g_2) \circ \tau_{2,\kappa}(k_{2,\infty})$$

for $\gamma_i \in G_i(\mathbb{Q})$, $g_i \in G_i(\mathbb{A})$, $k_{i,f} \in K_{i,f}$, *and* $k_{i,\infty} \in K_{i,\infty}$.

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(2) *We have*

$$(\theta(g_1,g_2)v,v')_\kappa = (v,\theta^*(g_1,g_2)v')_\kappa \qquad (g_i \in G_i(\mathbb{A}), v, v' \in V_\kappa).$$

For $F \in \mathcal{S}_{2,\kappa}$, we define

$$\mathcal{L}^*(F)(g_1) = \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} \theta^*(g_1,g_2)\, F(g_2) dg_2 \qquad (g_1 \in G_1(\mathbb{A})). \tag{12}$$

Here, the Haar measure dg_2 on $G_2(\mathbb{A})$ is normalized by $\mathrm{vol}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A})) = 1$.

Theorem 6.2.2. *Suppose that* $\kappa > 8$.

(1) *For* $F \in \mathcal{S}_{2,\kappa}$, *we have* $\mathcal{L}^*(F) \in \mathcal{S}_{1,\kappa}$.
(2) *We have*

$$\langle \mathcal{L}(f), F\rangle_2 = \langle f, \mathcal{L}^*(F)\rangle_1$$

for $f \in \mathcal{S}_{1,\kappa}$ *and* $F \in \mathcal{S}_{2,\kappa}$.
(3) *For* $F \in \mathcal{S}_{2,\kappa}$ *and* $g_2 \in G_2(\mathbb{A})$,

$$\mathcal{L}(\mathcal{L}^*(F))(g_2) = \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} dg_1 \int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})} dy\, \theta(g_1,g_2)\circ\theta^*(g_1,y)\, F(y). \tag{13}$$

The proof of (1) is given in the next section. The second assertion follows from Lemma 6.2.1. The third assertion is easily verified.

In a forthcoming paper, we will investigate $\mathcal{L}(\mathcal{L}^*(F))$ by using (13).

7. An archimedean analog of Eichler commutation relations

7.1.

The aim of this section is to prove Theorems 5.4.1 and 6.2.2(1) by using the following archimedean analog of Eichler commutation relations.

Theorem 7.1.1. *Suppose that* $\kappa > 8$. *For* $(g_1,g_2) \in G_1(\mathbb{A}) \times G_2(\mathbb{A})$, *we have*

$$c_{1,\kappa} \int_{G_{1,\infty}} \theta(g_1x, g_2)\omega_{1,\kappa}(x)dx = c_{2,\kappa} \int_{G_{2,\infty}} \omega_{2,\kappa}(y)\theta(g_1, g_2y^{-1})dy \tag{14}$$

and

$$c_{1,\kappa} \int_{G_{1,\infty}} \omega_{1,\kappa}(x)\theta^*(g_1x^{-1}, g_2)dx = c_{2,\kappa} \int_{G_{2,\infty}} \theta^*(g_1, g_2y)\omega_{2,\kappa}(y)dy, \tag{15}$$

where the Haar measures dx *and* dy *on* $G_{1,\infty}$ *and* $G_{2,\infty}$, *respectively, are given in Section 3.5.*

7.2.

We first prove Theorem 5.4.1 assuming (14). We can show Theorem 6.2.2(1) assuming (15) in a similar manner and omit its proof. Throughout this section, we fix an integer $\kappa > 4$. To simplify the notation, we write ω_1 and ω_2 for $\omega_{1,\kappa}$ and $\omega_{2,\kappa}$, respectively.

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Let $f \in S_{1,\kappa}$ and put $F = \mathcal{L}(f)$. The condition (6) for $F = \mathcal{L}(f)$ follows from Lemma 5.3.1. We have

$$\begin{aligned}
F(g_2) &= \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \theta(g_1, g_2) f(g_1) dg_1 \\
&= \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \theta(g_1, g_2) \left(c_{1,\kappa} \int_{G_{1,\infty}} \omega_1(x) f(g_1 x^{-1}) dx \right) dg_1 \\
&= \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \left(c_{1,\kappa} \int_{G_{1,\infty}} \theta(g_1 x, g_2) \omega_1(x) dx \right) f(g_1) dg_1 \\
&= \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \left(c_{2,\kappa} \int_{G_{2,\infty}} \omega_2(y) \theta(g_1, g_2 y^{-1}) dy \right) f(g_1) dg_1 \\
&= c_{2,\kappa} \int_{G_{2,\infty}} \omega_2(y) \left(\int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \theta(g_1, g_2 y^{-1}) f(g_1) dg_1 \right) dy \\
&= c_{2,\kappa} \int_{G_{2,\infty}} \omega_2(y) F(g_2 y^{-1}) dy,
\end{aligned}$$

proving (7) for $F = \mathcal{L}(f)$.

It now remains to show that F is cuspidal. We have

$$\begin{aligned}
F_0(g_2) &= \int_{B^-\backslash B^-_{\mathbb{A}}} F\left(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} g_2 \right) d\beta \\
&= \int_{B^-\backslash B^-_{\mathbb{A}}} \left(\int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \sum_{X \in B^{(2,1)}} r\left(g_1, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} g_2 \right) \varphi_0(X) f(g_1) dg_1 \right) d\beta \\
&= \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \sum_{X \in B^{(2,1)}} \left(\int_{B^-\backslash B^-_{\mathbb{A}}} \psi\left(-\frac{1}{2} \mathrm{tr}(\beta {}^t\overline{X} J X) \right) d\beta \right) r(g_1, g_2) \varphi_0(X) f(g_1) dg_1 \\
&= \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} \sum_{X \in B^{(2,1)},\, {}^t\overline{X} J X = 0} r(1, g_2) \varphi_0({}^t\overline{g_1} X) f(g_1) dh.
\end{aligned}$$

Observe that

$$\left\{ X \in B^{(2,1)} \mid {}^t\overline{X} J X = 0 \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \cup \left\{ {}^t\overline{\gamma} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid \gamma \in N_1(\mathbb{Q})\backslash G_1(\mathbb{Q}) \right\},$$

where

$$N_1(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Q} \right\}.$$

It follows that

$$\begin{aligned}
&F_0(g_2) \\
&= (r(1, g_2)\varphi_0)(0) \int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} f(g_1) dg_1 \\
&\quad + \int_{N_1(\mathbb{A})\backslash G_1(\mathbb{A})} (r(1, g_2)\varphi_0) \left({}^t\overline{g_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \left(\int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g_1 \right) db \right) dg_1 \\
&= 0,
\end{aligned}$$

since f is cuspidal. This shows the cuspidality of F and Theorem 5.4.1 has been proved.

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7.3.

To prove Theorem 7.1.1, we need some preparations. Define $\Phi_\kappa : \mathbb{H}^\times \to \mathrm{End}(V_\kappa)$ by

$$\Phi_\kappa(x) = N(x)^{-1}\sigma_\kappa(x)^{-1} \qquad (x \in \mathbb{H}^\times).$$

Note that $\omega_2(g) = \Phi_\kappa\left(2^{-1}(1,1)g\begin{pmatrix}1\\1\end{pmatrix}\right)$ for $g \in G_{2,\infty}$. The following is easily verified.

Lemma 7.3.1. (1) *For $a \in \mathbb{R}^\times$ and $x \in \mathbb{H}^\times$, we have $\Phi_\kappa(ax) = a^{-\kappa-2}\Phi_\kappa(x)$.*
(2) *For $x, y \in \mathbb{H}^\times$, we have $\Phi_\kappa(xy) = \Phi_\kappa(y)\Phi_\kappa(x)$.*
(3) *For $x \in \mathbb{H}^\times$, we have $\sigma_\kappa(\overline{x}) = N(x)^{\kappa+1}\Phi_\kappa(x)$.*

We define a map $M_\kappa : \mathbb{H} \to \mathrm{End}(V_\kappa)$ by

$$M_\kappa(\xi) = \int_{\mathbb{H}^-} \mathbf{e}(-\mathrm{tr}(\xi\beta))\Phi_\kappa(1+\beta)d_L\beta \qquad (\xi \in \mathbb{H}). \tag{16}$$

It is easily verified that $M_\kappa(\xi) = M_\kappa(x)$ for $\xi = t + x \quad (t \in \mathbb{R}, x \in \mathbb{H}^-)$ and that

$$M_\kappa(\alpha\xi\alpha^{-1}) = \sigma_\kappa(\alpha)M_\kappa(\xi)\sigma_\kappa(\alpha)^{-1} \qquad (\alpha \in \mathbb{H}^\times, \xi \in \mathbb{H}).$$

For $\xi \in \mathbb{H}$, there exists a $u \in \mathbb{H}^1$ such that $\xi = u(s+it)u^{-1} \quad (s,t \in \mathbb{R}, t \geq 0)$. Then $M_\kappa(\xi) = \sigma_\kappa(u)M_\kappa(it)\sigma_\kappa(u)^{-1}$. The following fact is proved in [1, Lemma 1.2]. Recall that we have introduced $P_r(X,Y) = X^rY^{\kappa-r} \in V_\kappa$ at (3).

Proposition 7.3.2. *For $t \geq 0$, we have*

$$M_\kappa(it)P_r = \delta_{r\kappa}\,\lambda(\kappa)\,t^{\kappa-1}e^{-4\pi t}\,P_\kappa \qquad (0 \leq r \leq \kappa),$$

where

$$\lambda(\kappa) = \frac{2^{2\kappa-1}\pi^{\kappa+1}}{\kappa!}.$$

In particular, $M_\kappa(0) = 0$.

Lemma 7.3.3. *For $a, b \in \mathbb{C}$, we have*

$$\sigma_\kappa(a+ib)\,P_r = (a+b\sqrt{-1})^r(a-b\sqrt{-1})^{\kappa-r}\,P_r \qquad (0 \leq r \leq \kappa). \tag{17}$$

Proof. This follows from the definition (4) and

$$A(a+bi) = \begin{pmatrix} a+b\sqrt{-1} & 0 \\ 0 & a-b\sqrt{-1}\end{pmatrix}$$

(for the definition of A, see (2)). □

The following formula is well-known.

Lemma 7.3.4. *For $s, t \in \mathbb{R}$ with $t > 0$, we have*

$$\int_{\mathbb{R}} (t+\sqrt{-1}x)^{-\kappa}\,e(sx)dx = \begin{cases} \dfrac{(2\pi)^\kappa}{(\kappa-1)!}\,s^{\kappa-1}\mathbf{e}(\sqrt{-1}st) & (s > 0), \\ 0 & (s \leq 0). \end{cases}$$

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7.4.

Define $\phi_{\omega_1}, \phi^*_{\omega_1} : \mathbb{H}^{(1,2)} \to \mathrm{End}(V_\kappa)$ and $\varphi_{\omega_2}, \varphi^*_{\omega_2} : \mathbb{H}^{(2,1)} \to \mathrm{End}(V_\kappa)$ by

$$\begin{aligned}
\phi_{\omega_1}(X') &= \int_{G_{1,\infty}} r'(g_1, 1)\phi_{0,\infty}(X')\,\omega_1(g_1)dg_1 \\
\phi^*_{\omega_1}(X') &= \int_{G_{1,\infty}} \omega_1(g_1)\, r'(g_1^{-1}, 1)\phi^*_{0,\infty}(X')dg_1, \\
\varphi_{\omega_2}(X) &= \int_{G_{2,\infty}} \omega_2(g_2)\, r(1, g_2^{-1})\varphi_{0,\infty}(X)dg_2 \\
\varphi^*_{\omega_2}(X) &= \int_{G_{2,\infty}} r(1, g_2^-)\varphi^*_{0,\infty}(X)\,\omega_2(g_2)dg_2
\end{aligned}$$

for $X' \in \mathbb{H}^{(1,2)}$ and $X \in \mathbb{H}^{(2,1)}$. The following facts are easily verified.

Lemma 7.4.1. (1) *We have*

$$\phi_{\omega_1}(0,0) = \phi^*_{\omega_1}(0,0) = 0,\ \varphi_{\omega_2}\begin{pmatrix}0\\0\end{pmatrix} = \varphi^*_{\omega_2}\begin{pmatrix}0\\0\end{pmatrix} = 0.$$

(2) *For $\alpha \in \mathbb{H}^1$, $X \in \mathbb{H}^{(2,1)}$, and $X' \in \mathbb{H}^{(1,2)}$, we have*

$$\begin{aligned}
\phi_{\omega_1}(X'\alpha) &= \sigma_\kappa(\alpha)^{-1}\,\phi_{\omega_1}(X'), \\
I_\infty^{-1}\phi_{\omega_1}(X\alpha) &= \sigma_\kappa(\alpha)^{-1}\, I_\infty^{-1}\phi_{\omega_1}(X), \\
\varphi_{\omega_2}(X\alpha) &= \sigma_\kappa(\alpha)^{-1}\,\varphi_{\omega_2}(X), \\
\phi^*_{\omega_1}(X'\alpha) &= \phi^*_{\omega_1}(X')\,\sigma_\kappa(\alpha), \\
I_\infty^{-1}\phi^*_{\omega_1}(X\alpha) &= I_\infty^{-1}\phi^*_{\omega_1}(X)\sigma_\kappa(\alpha), \\
\varphi^*_{\omega_2}(X\alpha) &= \varphi_{\omega_2}(X)\sigma_\kappa(\alpha).
\end{aligned}$$

One of the keys to the proof of Theorem 7.1.1 is the following results.

Proposition 7.4.2. *Let $x_1, x_2 \in \mathbb{H}$.*

(1) *We have*

$$\phi_{\omega_1}(x_1, x_2) = 2^{\kappa-2}\pi^{-1}\,\kappa\cdot\delta(\mathrm{tr}(\overline{x_1}x_2) > 0)\,(\mathrm{tr}(\overline{x_1}x_2))^{\kappa-1}\,\mathbf{e}(\sqrt{-1}\mathrm{tr}(\overline{x_1}x_2))\Phi_\kappa(x_1 + x_2).$$

(2) *Suppose that $\overline{x_1}x_2 = s - it$ $(s, t \in \mathbb{R}, t \geq 0)$. Then we have*

$$\varphi_{\omega_2}\begin{pmatrix}x_1\\x_2\end{pmatrix}\sigma_\kappa(x_1 + \sqrt{-1}x_2)P_r = \delta_{r\kappa}\cdot 2^{2\kappa}\pi\,\kappa^{-1}\cdot t^{\kappa-1}e^{-4\pi t}\cdot P_\kappa \qquad (0 \leq r \leq \kappa).$$

(3) *We have*

$$I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \frac{c_{2,\kappa}}{c_{1,\kappa}}\,\varphi_{\omega_2}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

Proposition 7.4.3. *Let $x_1, x_2 \in \mathbb{H}$.*

(1) *We have*

$$\phi^*_{\omega_1}(x_1, x_2) = 2^{\kappa-2}\pi^{-1}\,\kappa\cdot\delta(\mathrm{tr}(\overline{x_1}x_2) < 0)\,(-\mathrm{tr}(\overline{x_1}x_2))^{\kappa-1}\,\mathbf{e}(-\sqrt{-1}\mathrm{tr}(\overline{x_1}x_2))\Phi_\kappa(\overline{x_1 - x_2}).$$

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(2) *Suppose that* $\overline{x_1}x_2 = s - it$ $(s, t \in \mathbb{R}, t \geq 0)$. *If* $t = 0$, *we have* $\varphi^*_{\omega_2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. *If* $t > 0$, *we have*

$$\begin{aligned}&\varphi^*_{\omega_2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} P_r \\ &= \delta_{r\kappa} \cdot 2^{2\kappa}\pi\,\kappa^{-1} \cdot t^{\kappa-1}e^{-4\pi t}\,(\overline{x_1}x_1 + \overline{x_2}x_2 + 2t)^{-\kappa} \cdot \sigma_\kappa(x_1 + \sqrt{-1}x_2)\,P_\kappa \qquad (0 \leq r \leq \kappa).\end{aligned}$$

(3) *We have*

$$I_\infty^{-1}\phi^*_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{c_{2,\kappa}}{c_{1,\kappa}}\,\varphi^*_{\omega_2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The proof of Proposition 7.4.2 is given in Sections 7.5–7.7. We remark that the assumption $\bar{x_1}x_2 = s - it$ in the second assertion of both propositions is due to the fact that any element in pure quaternions is conjugate to an element in $\mathbb{R}i$, as we have used before Proposition 7.3.2. We omit the proof of Proposition 7.4.3, since it is quite similar to that of Proposition 7.4.2 and slightly easier.

7.5.

In this section, we prove Proposition 7.4.2(1). Let $x_1, x_2 \in \mathbb{H}$. By (5) and Lemma 5.2.2, we have

$$\begin{aligned}&\phi_{\omega_1}(x_1, x_2) \\ &= \int_{G_{1,\infty}} r'(g_1, 1)\phi_{0,\infty}(x_1, x_2)\omega_1(g_1)dg_1 \\ &= \int_{\mathbb{R}} db \int_0^\infty a^{-2}d^\times a \int_{K_{1,\infty}} dk\, r'\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)k, 1\right)\phi_{0,\infty}(x_1, x_2) \\ &\quad \omega_1\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)k\right) \\ &= \int_{\mathbb{R}} db \int_0^\infty a^{-2}d^\times a\,\mathbf{e}(-b\,\mathrm{tr}(\overline{x_1}x_2))\,a^4\,\phi_{0,\infty}(ax_1, ax_2)\left(\frac{a + a^{-1} - \sqrt{-1}a^{-1}b}{2}\right)^{-\kappa} \\ &= 2^\kappa\sigma_\kappa(\overline{x_1 + x_2})\int_0^\infty a^{2\kappa+2}\mathbf{e}(\sqrt{-1}a^2(\overline{x_1}x_1 + \overline{x_2}x_2))d^\times a \int_{\mathbb{R}}\left(a^2 + 1 - \sqrt{-1}b\right)^{-\kappa}\mathbf{e}(-b\,\mathrm{tr}(\overline{x_1}x_2))db.\end{aligned}$$

By Lemma 7.3.4, we see that the integral over $\mathbb{R}$ is equal to

$$\frac{(2\pi)^\kappa}{(\kappa-1)!}\,\delta(\mathrm{tr}(\overline{x_1}x_2) > 0)\,(\mathrm{tr}(\overline{x_1}x_2))^{\kappa-1}\,\mathbf{e}(\sqrt{-1}(a^2+1)\mathrm{tr}(\overline{x_1}x_2)).$$

Thus, we obtain

$$\begin{aligned}\phi_{\omega_1}(x_1, x_2) = \frac{(4\pi)^\kappa}{(\kappa-1)!}\,\delta(\mathrm{tr}(\overline{x_1}x_2) > 0)\,\sigma_\kappa(\overline{x_1 + x_2})\,(\mathrm{tr}(\overline{x_1}x_2))^{\kappa-1}\,\mathbf{e}(\sqrt{-1}\,\mathrm{tr}(\overline{x_1}x_2)) \\ \int_0^\infty a^{2\kappa+2}\,\mathbf{e}(\sqrt{-1}a^2(\overline{x_1}x_1 + \overline{x_2}x_2 + \mathrm{tr}(\overline{x_1}x_2)))\,d^\times a.\end{aligned}$$

Since the last integral is equal to

$$\frac{1}{2}\int_0^\infty a^{\kappa+1}\mathbf{e}(\sqrt{-1}N(x_1 + x_2)a)d^\times a = \frac{\kappa!}{2}(2\pi N(x_1 + x_2))^{-\kappa-1},$$

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we have

$$\phi_{\omega_1}(x_1,x_2) = 2^{\kappa-2}\pi^{-1}\,\kappa\cdot\delta(\mathrm{tr}(\overline{x_1}x_2)>0)\,(\mathrm{tr}(\overline{x_1}x_2))^{\kappa-1}\,\mathbf{e}(\sqrt{-1}\mathrm{tr}(\overline{x_1}x_2))\Phi_\kappa(x_1+x_2),$$

which completes the proof of Proposition 7.4.2(1). Here, we used Lemma 7.3.1(3).

7.6.

In this section, we prove Proposition 7.4.2(2).

Lemma 7.6.1. *Let $x_1, x_2 \in \mathbb{H}$. We have*

$$\varphi_{\omega_2}\begin{pmatrix}x_1\\x_2\end{pmatrix} = 2^{\kappa+1}\int_0^\infty a^\kappa(a+1)^{1-\kappa}\mathbf{e}(\sqrt{-1}(N(x_1)+N(x_2))a)\,M_\kappa(-(a+1)\overline{x_1}x_2)d^\times a \cdot\sigma_\kappa(\overline{x_1}-\sqrt{-1}\overline{x_2}).$$

Proof. By (5) and Lemma 5.2.2, we have

$$\begin{aligned}\varphi_{\omega_2}\begin{pmatrix}x_1\\x_2\end{pmatrix} &= \int_{K_{2,\infty}}dk\int_0^\infty a^6d^\times a\int_{\mathbb{H}^-}d_L\beta\;\omega_2\left(k\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\begin{pmatrix}1&\beta\\0&1\end{pmatrix}\right)\\ &\qquad r\left(1,\begin{pmatrix}1&-\beta\\0&1\end{pmatrix}\begin{pmatrix}a^{-1}&0\\0&a\end{pmatrix}k^{-1}\right)\varphi_{0,\infty}\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ &= \int_0^\infty a^6d^\times a\int_{\mathbb{H}^-}d_L\beta\,\Phi_\kappa\left(\frac{a}{2}(1+a^{-2}+\beta)\right)\mathbf{e}(\mathrm{tr}(\beta\overline{x_1}x_2))a^{-4}\,\varphi_{0,\infty}\begin{pmatrix}a^{-1}x_1\\a^{-1}x_2\end{pmatrix}\\ &= \int_0^\infty a^{-2}d^\times a\int_{\mathbb{H}^-}d_L\beta\left(\frac{a^{-1}}{2}\right)^{-\kappa-2}\Phi_\kappa\left(1+a^2+\beta\right)\mathbf{e}(\mathrm{tr}(\beta\overline{x_1}x_2))\,\varphi_{0,\infty}\begin{pmatrix}ax_1\\ax_2\end{pmatrix}\\ &= 2^{\kappa+2}\int_0^\infty a^{2\kappa}d^\times a\int_{\mathbb{H}^-}d_L\beta\,\Phi_\kappa\left(1+a^2+\beta\right)\\ &\qquad\mathbf{e}\left(\mathrm{tr}(\beta\overline{x_1}x_2)+\sqrt{-1}a^2(N(x_1)+N(x_2))\right)\sigma_\kappa(\overline{x_1}-\sqrt{-1}\overline{x_2}).\end{aligned}$$

Changing the variable β into $(1+a^2)\beta$, we obtain

$$\begin{aligned}\varphi_{\omega_2}\begin{pmatrix}x_1\\x_2\end{pmatrix} &= 2^{\kappa+2}\int_0^\infty a^{2\kappa}(a^2+1)^{1-\kappa}\mathbf{e}\left(\sqrt{-1}a^2(N(x_1)+N(x_2))\right)d^\times a\\ &\qquad\int_{\mathbb{H}^-}\mathbf{e}(\mathrm{tr}((a^2+1)\beta\overline{x_1}x_2)))\Phi_\kappa(1+\beta)d_L\beta\cdot\sigma_\kappa(\overline{x_1}-\sqrt{-1}\overline{x_2})\\ &= 2^{\kappa+1}\int_0^\infty a^\kappa(a+1)^{1-\kappa}\mathbf{e}\left(\sqrt{-1}(N(x_1)+N(x_2))a\right)M_\kappa(-(a+1)\overline{x_1}x_2)d^\times a\\ &\qquad\cdot\sigma_\kappa(\overline{x_1}-\sqrt{-1}\overline{x_2})\end{aligned}$$

and the lemma has been proved. □

We now suppose that $\overline{x_1}x_2 = s-it\ \ (s,t\in\mathbb{R}, t\geq 0)$. Since

$$\begin{aligned}(\overline{x_1}-\sqrt{-1}\overline{x_2})(x_1+\sqrt{-1}x_2) &= N(x_1)+N(x_2)+\sqrt{-1}(\overline{x_1}x_2-\overline{x_2}x_1)\\ &= N(x_1)+N(x_2)-2\sqrt{-1}it,\end{aligned}$$

we have

$$\sigma_\kappa(\overline{x_1}-\sqrt{-1}\overline{x_2})\,\sigma_\kappa(x_1+\sqrt{-1}x_2)\,P_r = (N(x_1)+N(x_2)+2t)^r\,(N(x_1)+N(x_2)-2t)^{\kappa-r}\,P_r$$

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by Lemma 7.3.3. We also have

$$M_\kappa(-(a+1)\overline{x_1}x_2)P_r = M_\kappa(i(a+1)t)P_r = \delta_{r\kappa}\,\lambda(\kappa)(a+1)^{\kappa-1}t^{\kappa-1}e^{-4\pi(a+1)t}P_\kappa \qquad (0 \le r \le \kappa)$$

by Proposition 7.3.2. We thus have

$$\begin{aligned}
&\varphi_{\omega_2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\sigma_\kappa(x_1+\sqrt{-1}x_2)\,P_r \\
&= \delta_{r\kappa}\,2^{\kappa+1}\lambda(\kappa)\,(N(x_1)+N(x_2)+2t)^\kappa\,t^{\kappa-1}e^{-4\pi t}\int_0^\infty a^\kappa e^{-2\pi(N(x_1)+N(x_2)+2t)a}d^\times a\,P_\kappa \\
&= \delta_{r\kappa}\,2^{2\kappa}\pi\kappa^{-1}\,t^{\kappa-1}e^{-4\pi t}\,P_\kappa,
\end{aligned}$$

which completes the proof of Proposition 7.4.2(2).

7.7.

In this section, we prove Proposition 7.4.2(3). Observe that $\varphi_{\omega_2}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$ and $I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$in view of Propositions 7.4.2(1) and (2) and Lemma 7.4.1(1). We hereafter assume that $x_1 \neq 0$.

Lemma 7.7.1. *We have*

$$\begin{aligned}
&I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 2^{2\kappa-1}\pi^{-1}\kappa\int_0^\infty a^\kappa(a+1)^{1-\kappa}\,\mathbf{e}(\mathrm{tr}(-\overline{x_1}x_2+\sqrt{-1}N(x_1))a)\,M_\kappa((a+1)\overline{x_2}x_1)\,d^\times a\,\cdot\,\sigma_\kappa(\overline{x_1}).
\end{aligned}$$

Proof. We put $c(\kappa) = 2^\kappa\pi^{-1}\kappa$. By Proposition 7.4.2(1), we have

$$\begin{aligned}
&I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 4\int_{\mathbb{H}}\mathbf{e}(-\mathrm{tr}(\overline{y}x_2))\phi_{\omega,\infty}(x_1,y)d_Ly \\
&= c(\kappa)\int_{\mathbb{H}}\delta(\mathrm{tr}(\overline{x_1}y)>0)\,(\mathrm{tr}(\overline{x_1}y))^{\kappa-1}\,\mathbf{e}(-\mathrm{tr}(\overline{x_2}y)+\sqrt{-1}\,\mathrm{tr}(\overline{x_1}y))\Phi_\kappa(x_1+y)d_Ly.
\end{aligned}$$

Changing the variable y into x_1y and using Lemma 7.3.1(2), we obtain

$$\begin{aligned}
&I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= c(\kappa)\int_{\mathbb{H}}\delta(\mathrm{tr}(y)>0)\,(\mathrm{tr}(y))^{\kappa-1}\,\mathbf{e}(\mathrm{tr}(-\overline{x_2}x_1y+\sqrt{-1}N(x_1)y))\Phi_\kappa(1+y)d_Ly\cdot N(x_1)^{\kappa+1}\Phi_\kappa(x_1) \\
&= c(\kappa)\int_{\mathbb{R}}da\int_{\mathbb{H}^-}d_L\beta\,\delta(\mathrm{tr}(a+\beta)>0)\,(\mathrm{tr}(a+\beta))^{\kappa-1}\,\mathbf{e}(\mathrm{tr}((-\overline{x_2}x_1+\sqrt{-1}N(x_1))(a+\beta))) \\
&\qquad \Phi_\kappa(1+a+\beta)\,\sigma_\kappa(\overline{x_1}) \\
&= c(\kappa)\int_0^\infty(2a)^{\kappa-1}\,\mathbf{e}(\mathrm{tr}((-\overline{x_2}x_1+\sqrt{-1}N(x_1)))a)da\int_{\mathbb{H}^-}\mathbf{e}(-\mathrm{tr}(\overline{x_2}x_1\beta))\,\Phi_\kappa(1+a+\beta)d_L\beta\cdot\sigma_\kappa(\overline{x_1}).
\end{aligned}$$

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Changing the variable β into $(1+a)\beta$, we obtain

$$\begin{aligned}
&I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= c(\kappa)\, 2^{\kappa-1} \int_0^\infty a^{\kappa-1}(a+1)^{1-\kappa}\mathbf{e}(\mathrm{tr}(-\overline{x_2}x_1 + \sqrt{-1}N(x_1))a)da \\
&\qquad \int_{\mathbb{H}^-} \mathbf{e}(-\mathrm{tr}((a+1)\overline{x_2}x_1\beta))\, \Phi_\kappa(1+\beta)d_L\beta \cdot \sigma_\kappa(\overline{x_1}) \\
&= 2^{2\kappa-1}\pi^{-1}\kappa \int_0^\infty a^\kappa (a+1)^{1-\kappa}\, \mathbf{e}(\mathrm{tr}(-\overline{x_1}x_2 + \sqrt{-1}N(x_1))a)\, M_\kappa((a+1)\overline{x_2}x_1)\, d^\times a \cdot \sigma_\kappa(\overline{x_1})
\end{aligned}$$

and we are done. □

In view of Lemma 7.4.1(2) and Propositions 7.4.2(1) and (2), it suffices to show the following lemma to complete the proof of Proposition 7.4.2(3).

Lemma 7.7.2. *Suppose that* $\overline{x_1}x_2 = s - it$ $(s, t \in \mathbb{R},\ t \geq 0)$. *Then*

$$I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sigma_\kappa(x_1 + \sqrt{-1}x_2)\, P_r = \delta_{r\kappa} \cdot 2^{2\kappa-2} t^{\kappa-1} e^{-4\pi t}\, P_\kappa \quad (r = 0, \ldots, \kappa).$$

Proof. Put $c'(\kappa) = 2^{2\kappa-1}\pi^{-1}\kappa$. By Lemmas 7.7.1 and 7.3.3 and Proposition 7.3.2, we have

$$\begin{aligned}
&I_\infty^{-1}\phi_{\omega_1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sigma_\kappa(x_1 + \sqrt{-1}x_2)\, P_r \\
&= c'(\kappa) \int_0^\infty a^\kappa (a+1)^{1-\kappa}\, \mathbf{e}(\mathrm{tr}(-\overline{x_1}x_2 + \sqrt{-1}N(x_1)a)\, M_\kappa((a+1)\overline{x_2}x_1)\, d^\times a \\
&\qquad \cdot \sigma_\kappa(N(x_1) + \sqrt{-1}\overline{x_1}x_2)\, P_r \\
&= c'(\kappa) \int_0^\infty a^\kappa (a+1)^{1-\kappa}\, \mathbf{e}(2(-s + \sqrt{-1}N(x_1))a)\, M_\kappa(i(a+1)t)\, d^\times a \\
&\qquad \cdot (N(x_1) + t + \sqrt{-1}s)^r (N(x_1) - t + \sqrt{-1}s)^{\kappa-r} P_r \\
&= \delta_{r\kappa} \cdot c'(\kappa)\, \lambda(\kappa) \cdot (N(x_1) + t + \sqrt{-1}s)^\kappa \\
&\qquad \int_0^\infty a^\kappa (a+1)^{1-\kappa}((a+1)t)^{\kappa-1}\, e^{4\pi\sqrt{-1}(-sa+\sqrt{-1}N(x_1)a) - 4\pi(a+1)t} d^\times a \cdot P_\kappa \\
&= \delta_{r\kappa} \cdot \frac{2^{4\kappa-2}\pi^\kappa}{(\kappa-1)!} t^{\kappa-1} e^{-4\pi t} (N(x_1) + t + \sqrt{-1}s)^\kappa \int_0^\infty a^\kappa e^{-4\pi(N(x_1)+t+\sqrt{-1}s)a} d^\times a \cdot P_\kappa \\
&= \delta_{r\kappa} \cdot 2^{2\kappa-2} t^{\kappa-1} e^{-4\pi t} \cdot P_\kappa.
\end{aligned}$$

□

7.8.

We define the norm of $X = (x_{ij}) \in \mathbb{H}^{(m,n)}$ by

$$\|X\| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n N(x_{ij})}.$$

The aim of this section is to prove the following estimates, which are needed in the proof of Theorem 7.1.1.

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Proposition 7.8.1. *Suppose that $\kappa > 4$ and let D be a compact subset of $G_{1,\infty} \times G_{2,\infty}$.*

(1) *There exists a positive constant C_1 such that*

$$\left|\langle r'(g_1, g_2)\phi_{\omega_1}(X')P, P'\rangle_\kappa\right| \leq C_1\|P\|_\kappa\|P'\|_\kappa \cdot (1 + \|X'\|)^{-\kappa-2} \tag{18}$$

holds for any $X' \in \mathbb{H}^{(1,2)}$, $(g_1, g_2) \in D$, and $P, P' \in V_\kappa$.

(2) *There exists a positive constant C_2 such that*

$$\left|\langle r(g_1, g_2)\varphi_{\omega_2}(X)P, P'\rangle_\kappa\right| \leq C_2\|P\|_\kappa\|P'\|_\kappa \cdot (1 + \|X\|)^{-\kappa} \tag{19}$$

holds for any $X \in \mathbb{H}^{(2,1)}$, $(g_1, g_2) \in D$, and $P, P' \in V_\kappa$.

To prove the proposition, we need some preparations.

Lemma 7.8.2. *Let $x_1, x_2 \in \mathbb{H}$.*

(1) *We have $|\mathrm{tr}(\overline{x_1}x_2)| \leq N(x_1) + N(x_2)$, and the equality holds only when $x_2 = \pm x_1$.*
(2) *Suppose that $\overline{x_1}x_2 = s - it$ $(s, t \in \mathbb{R},\ t \geq 0)$. Then we have $t \leq \frac{1}{2}(N(x_1) + N(x_2))$, and the equality holds only when $x_2 = -x_1 i$.*
(3) *Under the same assumption as in (2), $x_1 + \sqrt{-1}x_2$ is invertible in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ if and only if $x_2 \neq -x_1 i$. In this case, $\overline{x_1} - \sqrt{-1}\overline{x_2}$ is invertible and we have*

$$(x_1 + \sqrt{-1}x_2)^{-1} = (N(x_1) + N(x_2) - 2\sqrt{-1}it)^{-1}(\overline{x_1} - \sqrt{-1}\overline{x_2}),$$
$$(\overline{x_1} - \sqrt{-1}\overline{x_2})^{-1} = (x_1 + \sqrt{-1}x_2)(N(x_1) + N(x_2) - 2\sqrt{-1}it)^{-1}.$$

Proof. The first assertion is an immediate consequence of the Cauchy–Schwarz inequality and the inequality of arithmetic and geometric means. The second one follows from the first one and the fact that $t = \frac{1}{2}\mathrm{tr}(\overline{x_1}x_2 i)$. The third one follows from

$$(\overline{x_1} - \sqrt{-1}\overline{x_2})(x_1 + \sqrt{-1}x_2) = N(x_1) + N(x_2) - 2\sqrt{-1}it$$

and

$$A(N(x_1) + N(x_2) - 2\sqrt{-1}it) = \begin{pmatrix} N(x_1) + N(x_2) + 2t & 0 \\ 0 & N(x_1) + N(x_2) - 2t \end{pmatrix},$$

which is invertible in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{M}_2(\mathbb{C})$ if and only if $x_2 \neq -x_1 i$ in view of the second assertion (for the definition of A, see (2)). □

Lemma 7.8.3. *Let $X = {}^t(x_1, x_2) \in \mathbb{H}^{(2,1)}$ with $\overline{x_1}x_2 = s - it$ $(s, t \in \mathbb{R}, t \geq 0)$.*

(1) *If $x_2 \neq -x_1 i$, we have*

$$\varphi_{\omega_2}(X)\sigma_\kappa(\overline{x_1} - \sqrt{-1}\overline{x_2})^{-1}P_r = \delta_{r\kappa} \cdot 2^{2\kappa}\pi\kappa^{-1} \cdot t^{\kappa-1}e^{-4\pi t}(\|X\|^2 + 2t)^{-\kappa}\, P_\kappa. \tag{20}$$

(2) *If $x_2 = -x_1 i \neq 0$, we have*

$$\varphi_{\omega_2}(X)\sigma_\kappa(x_1)P_r = \delta_{r\kappa} \cdot 2\pi\kappa^{-1} \cdot \|X\|^{2\kappa-2}e^{-2\pi\|X\|^2}\, P_\kappa. \tag{21}$$

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Proof. First suppose that $x_2 \neq -x_1 i$. By Lemma 7.8.2, $\overline{x_1} - \sqrt{-1}\overline{x_2}$ is invertible in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. By Lemmas 7.3.3 and 7.8.2(3) and Proposition 7.4.2(2), we have

$$\begin{aligned}
&\varphi_{\omega_2}(X)\sigma_\kappa(\overline{x_1} - \sqrt{-1}\overline{x_2})^{-1} P_r \\
&= \varphi_{\omega_2}(X)\sigma_\kappa(x_1 + \sqrt{-1}x_2)\sigma_\kappa(\|X\|^2 - 2\sqrt{-1}it)^{-1} P_r \\
&= (\|X\|^2 + 2t)^{-r}(\|X\|^2 - 2t)^{-\kappa+r}\varphi_{\omega_2}(X)\sigma_\kappa(x_1 + \sqrt{-1}x_2)\, P_r \\
&= \delta_{r\kappa} \cdot 2^{2\kappa}\pi\kappa^{-1} \cdot t^{\kappa-1}e^{-4\pi t}(\|X\|^2 + 2t)^{-\kappa}\, P_\kappa,
\end{aligned}$$

which proves (20).

Next suppose that $x_2 = -x_1 i \neq 0$. Then $\overline{x_1}x_2 = -N(x_1)i = -\frac{1}{2}\|X\|^2 i$. Note that, if $r \neq \kappa$, we have

$$\sigma_\kappa(\overline{x_1} - \sqrt{-1}\overline{x_2})\sigma_\kappa(x_1)\, P_r = N(x_1)^\kappa\, \sigma_\kappa(1 - \sqrt{-1}\, i)\, P_r = 0$$

by Lemma 7.3.3. It follows from Lemma 7.6.1 that

$$\varphi_{\omega_2}(X)\sigma_\kappa(x_1)\, P_r = 0 \qquad (0 \leq r \leq \kappa - 1).$$

Since $\sigma_\kappa(x_1 + \sqrt{-1}x_2)\, P_\kappa = \sigma_\kappa(x_1)\, \sigma_\kappa(1 - \sqrt{-1}\, i)\, P_\kappa = 2^\kappa\, \sigma_\kappa(x_1)\, P_\kappa$ by Lemma 7.3.3, we have

$$\varphi_{\omega_2}(X)\sigma_\kappa(x_1)\, P_\kappa = 2\pi\kappa^{-1} \cdot \|X\|^{2\kappa-2} e^{-2\pi\|X\|^2}\, P_\kappa$$

by Proposition 7.4.2, which proves (21). □

For $z \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, define $(s_{a,b}(z))_{a,b=0}^{\kappa} \in \mathbb{C}^{(\kappa+1,\kappa+1)}$ by

$$\sigma_\kappa(z)P_b = \sum_{a=0}^{\kappa} P_a\, s_{a,b}(z) \qquad (0 \leq b \leq \kappa).$$

Then

$$\sum_{j=0}^{\kappa} s_{a,j}(z)s_{j,b}(w) = s_{a,b}(zw) \qquad (z, w \in \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C},\ 0 \leq a, b \leq \kappa). \tag{22}$$

We see that $x \mapsto N(x)^{-\kappa/2}\left|s_{a,b}(x)\right|$ is bounded on $\mathbb{H}^\times$, from which the following two lemmas are deduced.

Lemma 7.8.4. *There exists a positive constant c such that*

$$|(\Phi_\kappa(x)P_a, P_b)_\kappa| \leq c\, N(x)^{-\kappa/2-1} \tag{23}$$

for $x \in \mathbb{H}^\times$ and $0 \leq a, b \leq \kappa$.

Lemma 7.8.5. *There exists a positive constant c' such that*

$$|s_{a,b}(x + \sqrt{-1}y)| \leq c'\,(N(x) + N(y))^{\kappa/2} \tag{24}$$

holds for any $x, y \in \mathbb{H}$ and $0 \leq a, b \leq \kappa$.

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We now prove Proposition 7.8.1. It suffices to show the following estimates: There exist positive constants C_1, C_2 such that the inequalities

$$|(\phi_{\omega_1}(X')P_a, P_b)_\kappa| \le C_1 (1 + \|X\|)^{-\kappa-2}, \tag{25}$$

$$|(\varphi_{\omega_2}(X)P_a, P_b)_\kappa| \le C_2 (1 + \|X\|)^{-\kappa} \tag{26}$$

hold for any $X' \in \mathbb{H}^{(1,2)}$ and $X \in \mathbb{H}^{(1,2)}$.

We first show (25). Let $X' = (x_1, x_2) \in \mathbb{H}^{(1,2)}$ and $0 \le a, b \le \kappa$. By Proposition 7.4.2 and Lemma 7.8.4, we have

$$\left|(\phi_{\omega_1}(X')P_a, P_b)_\kappa\right| \le c_1 \cdot \delta(\mathrm{tr}(\overline{x_1}x_2) > 0)\, (\mathrm{tr}(\overline{x_1}x_2))^{\kappa-1}\, e^{-2\pi \mathrm{tr}(\overline{x_1}x_2)}\, N(x_1 + x_2)^{-\kappa/2-1}$$

with a positive constant c_1. Observe that, if $\mathrm{tr}(\overline{x_1}x_2) > 0$, we have $N(x_1 + x_2) > \|X'\|^2$ and $\mathrm{tr}(\overline{x_1}x_2) \le \|X'\|^2$ by Lemma 7.8.2. It follows that

$$\left|(\phi_{\omega_1}(X')P_a, P_b)_\kappa\right| \le c_1 \cdot \delta(\mathrm{tr}(\overline{x_1}x_2) > 0)\|X'\|^{\kappa-4}$$

and hence

$$\mathrm{Sup}_{\|X'\|\le 1} \left|(\phi_{\omega_1}(X')P_a, P_b)_\kappa\right| < \infty \tag{27}$$

under the assumption $\kappa > 4$. On the other hand, since $t \mapsto t^{\kappa-1}e^{-2\pi t}$ is bounded on $\mathbb{R}_{>0}$, we have

$$\left|(\phi_{\omega_1}(X')P_a, P_b)_\kappa\right| \le c_2 \cdot \delta(\mathrm{tr}(\overline{x_1}x_2) > 0)\|X'\|^{-\kappa-2} \tag{28}$$

with a positive constant c_2. The estimate (25) follows from (27) and (28).

We next show (26). Let $X = {}^t(x_1, x_2) \in \mathbb{H}^{(2,1)}$ and $0 \le a, b \le \kappa$. In view of Lemma 7.4.1(2), we may (and do) suppose that $\overline{x_1}x_2 = s - it \quad (s, t \in \mathbb{R}, t \ge 0)$.

We first consider the case $x_2 \ne -x_1 i$. Then $\xi = \overline{x_1} - \sqrt{-1}\overline{x_2}$ is invertible in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ by Lemma 7.8.2. By (22) and Lemma 7.8.3(1), we have

$$\begin{aligned}
\varphi_{\omega_2}(X)P_a &= \sum_{j=0}^{\kappa} s_{j,a}(1)\varphi_{\omega_2}(X)P_j \\
&= \sum_{j=0}^{\kappa} \left(\sum_{r=0}^{\kappa} s_{j,r}(\xi^{-1}) s_{r,a}(\xi) \right) \varphi_{\omega_2}(X)P_j \\
&= \sum_{r=0}^{\kappa} s_{r,a}(\xi)\, \varphi_{\omega_2}(X)\sigma_\kappa(\xi)^{-1}P_r \\
&= 2^{2\kappa}\pi\kappa^{-1} \cdot s_{\kappa,a}(\xi)\, t^{\kappa-1}e^{-4\pi t}\, (\|X\|^2 + 2t)^{-\kappa}\, P_\kappa.
\end{aligned}$$

By Lemma 7.8.5, there exists a positive constant c_3 such that

$$|(\varphi_{\omega_2}(X)P_a, P_b)_\kappa| \le c_3\, t^{\kappa-1}e^{-4\pi t}\, \|X\|^\kappa\, (\|X\|^2 + 2t)^{-\kappa} \tag{29}$$

holds for any $X \in \mathbb{H}^{(2,1)}$. Since

$$t^{\kappa-1}e^{-4\pi t}\, \|X\|^\kappa\, (\|X\|^2 + 2t)^{-\kappa} \le \left(\frac{t}{\|X\|^2 + 2t}\right)^{\kappa-1} \frac{\|X\|^\kappa}{\|X\|^2 + 2t} \le 2^{-\kappa+1}\|X\|^{\kappa-2},$$

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we have

$$\mathrm{Sup}_{\|X\|\le 1}|(\varphi_{\omega_2}(X)P_a, P_b)_\kappa| < \infty. \tag{30}$$

We also have

$$|(\varphi_{\omega_2}(X)P_a, P_b)_\kappa| \le c_3 c_4 \, \|X\|^{-\kappa}, \tag{31}$$

where $c_4 = \mathrm{Sup}_{t>0}\, t^{\kappa-1}e^{-4\pi t}$. The estimate (26) in the case $x_2 \neq -x_1 i$ follows from (30) and (31).

We next consider the remaining case where $x_2 = -x_1 i$ and $x_1 \neq 0$. A similar argument as above and Lemma 7.8.3(2) show that

$$(\varphi_{\omega_2}(X)P_a, P_b)_\kappa = 2\pi\kappa^{-1} \cdot s_{\kappa,a}(x_1^{-1}) \, \|X\|^{2\kappa-2} \, e^{-2\pi\|X\|^2}(P_\kappa, P_b)_\kappa.$$

Since $|s_{\kappa,a}(x_1^{-1})| \, \|X\|^\kappa$ is bounded, we have

$$|(\varphi_{\omega_2}(X)P_a, P_b)_\kappa| \le c_5 \cdot \|X\|^{\kappa-2} e^{-2\pi\|X\|^2}$$

with a positive constant c_5, which implies (26) in this case.

7.9.

In this section, we complete the proof of (14).

Let

$$\Psi_{\omega_1}(X') = \phi_{\omega_1}(X'_\infty) \prod_{p<\infty} \phi_{0,p}(X'_p) \qquad (X' = (X'_v)_{v\le\infty} \in B_{\mathbb{A}}^{(1,2)}),$$

$$\Phi_{\omega_2}(X) = \varphi_{\omega_2}(X_\infty) \prod_{p<\infty} \varphi_{0,p}(X_p) \qquad (X = (X_v)_{v\le\infty} \in B_{\mathbb{A}}^{(2,1)}).$$

Proposition 7.8.1 immediately implies the following.

Proposition 7.9.1. *Suppose that $\kappa > 4$ and let $(g_1, g_2) \in G_1(\mathbb{A}) \times G_2(\mathbb{A})$.*

(1) *The sum*

$$\sum_{X' \in B^{(1,2)}} r'(g_1, g_2)\Psi_{\omega_1}(X') \tag{32}$$

is absolutely convergent and coincides with the integral

$$\int_{G_{1,\infty}} \theta(g_1 x, g_2)\omega_1(x)dx.$$

(2) *The sum*

$$\sum_{X \in B^{(2,1)}} r(g_1, g_2)\Phi_{\omega_2}(X) \tag{33}$$

is absolutely convergent and coincides with the integral

$$\int_{G_{1,\infty}} \omega_2(y)\theta(g_1, g_2 y^{-1})dy.$$

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We now complete the proof of (14). In view of Corollary 2.6 in Chapter VII of [18] and Proposition 7.8.1, under the assumption that $\kappa > \dim_{\mathbb{Q}} B^{(1,2)} = 8$, we can apply the Poisson summation formula to the sum (32) and obtain

$$\begin{aligned}\int_{G_{1,\infty}} \theta(g_1 x, g_2)\omega_1(x)dx &= \sum_{X' \in B^{(1,2)}} r'(g_1, g_2)\Psi_{\omega_1}(X') \\ &= \sum_{X \in B^{(2,1)}} I^{-1}(r'(g_1, g_2)\Psi_{\omega_1})(X).\end{aligned}$$

By Lemma 5.2.1 and Proposition 7.4.2, we have $I^{-1}(r'(g_1, g_2)\Psi_{\omega_1}) = \frac{c_{2,\kappa}}{c_{1,\kappa}} \cdot r(g_1, g_2)\Phi_{\omega_2}$. We thus have

$$\begin{aligned}c_{1,\kappa} \int_{G_{1,\infty}} \theta(g_1 x, g_2)\omega_1(x)dx &= c_{2,\kappa} \sum_{X \in B^{(2,1)}} r(g_1, g_2)\Phi_{\omega_2}(X) \\ &= c_{2,\kappa} \int_{G_{2,\infty}} \omega_2(y)\theta(g_1, g_2 y^{-1})dy,\end{aligned}$$

proving (14).

8. The correspondence of automorphic L-functions for the Arakawa lifting

In the remaining part of the article, we will show the following correspondence of automorphic L-functions for the Arakawa lifting and its adjoint by using the Eichler commutation relations given in Section 9.3 for the unramified case and in Section 10.3 for the ramified case.

Theorem 8.0.1. (1) *If $f \in \mathcal{S}_{1,\kappa}$ is a Hecke eigenform, then so is $\mathcal{L}(f)$ and*

$$L(s, \mathcal{L}(f), \mathrm{std}) = \zeta(s)L(s, f, \mathrm{std}). \tag{34}$$

Here, $\zeta(s)$ stands for the Riemann zeta function. For the definitions of the standard L-functions, see (10).

(2) *If $F \in \mathcal{S}_{2,\kappa}$ is a Hecke eigenform, then so is $\mathcal{L}^*(F)$ and*

$$L(s, F, \mathrm{std}) = \zeta(s)L(s, \mathcal{L}^*(F), \mathrm{std}). \tag{35}$$

We only prove the first assertion of Theorem 8.0.1 since the second assertion is similarly proved. The key of the proof of Theorem 8.0.1 is the following Eichler commutation relations.

Theorem 8.0.2. *For a finite place p of $\mathbb{Q}$, there exists a $\mathbb{C}$-algebra homomorphism*

$$\upsilon \colon \mathcal{H}(G_{2,p}, K_{2,p}) \to \mathcal{H}(G_{1,p}, K_{1,p})$$

such that the equality

$$\int_{G_{2,p}} \theta(g_1, g_2 y^{-1})\alpha(y)dy = \int_{G_{1,p}} \theta(g_1 x, g_2)\upsilon(\alpha)(x)dx \tag{36}$$

holds for $\alpha \in \mathcal{H}(G_{2,p}, K_{2,p})$.

Corollary 8.0.3. *If $f \in \mathcal{S}_{1,\kappa}$ is a Hecke eigenform, so is $\mathcal{L}(f)$ and its Hecke eigenvalues are given by*

$$\lambda_{2,p}^{\mathcal{L}(f)}(\alpha) = \lambda_{1,p}^{f}(\upsilon(\alpha)) \tag{37}$$

for any $\alpha \in \mathcal{H}(G_{2,p}, K_{2,p})$. (For the definitions of $\lambda_{2,p}^{\mathcal{L}(f)}$ and $\lambda_{1,p}^{f}$, see (9).)

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We will give an explicit construction of υ and the proof of Theorem 8.0.1(1) for the unramified (resp. ramified) case in Section 9 (resp. in Section 10).

9. Eichler commutation relations at unramified places

9.1.

In this section, we consider the unramified case; namely, we assume that p does not divide the discriminant d_B of B. Throughout this section, we often drop the subscript p. As in Section 4.4, we let

$$\begin{aligned}
G_1 &= \left\{ g \in \mathrm{GL}_4(\mathbb{Q}_p) \mid {}^t g \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \right\}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
K_1 &= G_1 \cap \mathrm{GL}_4(\mathbb{Z}_p), \\
G_2 &= \left\{ g \in \mathrm{GL}_4(\mathbb{Q}_p) \mid {}^t g \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\}, \\
K_2 &= G_2 \cap \mathrm{GL}_4(\mathbb{Z}_p).
\end{aligned}$$

Define subgroups N_1 and M_1 of G_1 by

$$\begin{aligned}
N_1 &= \left\{ n_1(b) := \begin{pmatrix} 1_2 & b1_2 \\ 0 & 1_2 \end{pmatrix} \mid b \in \mathbb{Q}_p \right\}, \\
M_1 &= \left\{ d_1(A) := \begin{pmatrix} A & 0 \\ 0 & J^{-1}{}^t A^{-1} J \end{pmatrix} \mid A \in \mathrm{GL}_2(\mathbb{Q}_p) \right\}.
\end{aligned}$$

Then $G_1 = N_1 M_1 K_1$. We also define subgroups N_2 and M_2 of G_2 by

$$\begin{aligned}
N_2 &= \left\{ n_2(b,c,d) := \left(\begin{array}{cc|cc} 1 & 0 & b & c \\ 0 & 1 & c & d \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right) \mid b, c, d \in \mathbb{Q}_p \right\}, \\
M_2 &= \left\{ d_2(A) := \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \mid A \in \mathrm{GL}_2(\mathbb{Q}_p) \right\}.
\end{aligned}$$

Then $G_2 = N_2 M_2 K_2$.

The Weil representation r' given in Lemma 5.1.1 is reformulated as follows: For $\phi \in \mathcal{S}(\mathbb{Q}_p^{(2,4)})$ and $x, y \in \mathbb{Q}_p^{(2,2)}$,

$$\begin{aligned}
r'(1,g)\phi(x,y) &= \phi((x,y)g) \qquad (g \in G_2), \\
r'(d_1(A),1)\phi(x,y) &= |\det A|^2 \, \phi({}^t Ax, {}^t Ay) \qquad (A \in \mathrm{GL}_2(\mathbb{Q}_p)), \\
r'(n_1(b),1)\phi(x,y) &= \psi(-b \operatorname{Tr}(J^{-1}{}^t x J y))\phi(x,y) \qquad (b \in \mathbb{Q}_p).
\end{aligned}$$

9.2.

In this section, we prepare several facts on Hecke algebras.

For $i = 1, 2$, let $\mathcal{T}_{G_1}^i$ and $\mathcal{T}_{G_2}^i$ be the characteristic functions of

$$T_{G_1}^i = \{ g_1 \in G_1 \mid p g_1 \in \mathrm{M}_4(\mathbb{Z}_p), \operatorname{rank}_{\mathbb{F}_p}(p g_1) = i \}$$

and

$$T_{G_2}^i = \{ g_2 \in G_2 \mid p g_2 \in \mathrm{M}_4(\mathbb{Z}_p), \operatorname{rank}_{\mathbb{F}_p}(p g_2) = i \},$$

respectively. Here, for $X \in \mathrm{M}_r(\mathbb{Z}_p)$, we write $\operatorname{rank}_{\mathbb{F}_p}(X)$ for the rank of $(X \bmod p\,\mathrm{M}_r(\mathbb{Z}_p))$ in $\mathrm{M}_r(\mathbb{F}_p)$. Then $\mathcal{T}_{G_1}^i \in \mathcal{H}(G_1, K_1)$ and $\mathcal{T}_{G_2}^i \in \mathcal{H}(G_2, K_2)$.

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Lemma 9.2.1. *We have*

$$\mathcal{H}(G_1, K_1) = \mathbb{C}[\mathcal{T}^1_{G_1}, \mathcal{T}^2_{G_1}],$$
$$\mathcal{H}(G_2, K_2) = \mathbb{C}[\mathcal{T}^1_{G_2}, \mathcal{T}^2_{G_2}].$$

Proof. This follows from [16, Section 5.1, Theorem 1]. □

For $a \in \mathbb{Q}_p$, put

$$\nu_1(a) = d_1 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_1,$$

$$\nu_2(a) = d_2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix} \in G_2.$$

We put

$$\Lambda_i = \{(b, c, d) \in (\mathbb{Z}_p/p\mathbb{Z}_p)^3 \mid \operatorname{rank}_{\mathbb{F}_p} \begin{pmatrix} b & c \\ c & d \end{pmatrix} = i\}$$

for $i = 1, 2$. Note that $|\Lambda_1| = p^2 - 1$ and $|\Lambda_2| = p^3 - p^2$.

The proofs of the following facts are straightforward and we omit it.

Lemma 9.2.2. *We have*

$$T^1_{G_1} = \bigsqcup_{a,\, b \in \mathbb{Z}_p/p\mathbb{Z}_p} n_1(b)\nu_1(a) d_1 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_1 \sqcup \bigsqcup_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} n_1(b) d_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_1$$
$$\sqcup \bigsqcup_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \nu_1(a) d_1 \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} K_1 \sqcup d_1 \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} K_1$$

and

$$T^2_{G_1} = \bigsqcup_{a \in \mathbb{Z}_p/p^2\mathbb{Z}_p} \nu_1(a) d_1 \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} K_1 \sqcup d_1 \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix} K_1 \sqcup \bigsqcup_{b \in \mathbb{Z}_p/p^2\mathbb{Z}_p} n_1(b) d_1 \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} K_1$$
$$\sqcup d_1 \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} K_1 \sqcup \bigsqcup_{a \in \mathbb{Z}_p^\times/p\mathbb{Z}_p} \nu_1(p^{-1}a)\, K_1 \sqcup \bigsqcup_{b \in \mathbb{Z}_p^\times/p\mathbb{Z}_p} n_1(p^{-1}b)\, K_1,$$

where $\mathbb{Z}_p^\times/p\mathbb{Z}_p$ *means* $(\mathbb{Z}_p - p\mathbb{Z}_p)/p\mathbb{Z}_p$.

Lemma 9.2.3. *We have*

$$T^1_{G_2} = \bigsqcup_{a,\, c \in \mathbb{Z}_p/p\mathbb{Z}_p,\, b \in \mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(b, c, 0)\nu_2(a) d_2 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_2$$
$$\sqcup \bigsqcup_{c \in \mathbb{Z}_p/p\mathbb{Z}_p,\, d \in \mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(0, c, d) d_2 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} K_2 \sqcup \bigsqcup_{(b,c,d) \in \Lambda_1} n_2(p^{-1}b, p^{-1}c, p^{-1}d) K_2$$
$$\sqcup \bigsqcup_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \nu_2(a) d_2 \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} K_2 \sqcup d_2 \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} K_2.$$

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and

$$
\begin{aligned}
T^2_{G_2} = & \bigsqcup_{b,c,d\in\mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(b,c,d)d_2\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{a,b\in\mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(b,0,0)\nu_2(a)d_2\begin{pmatrix} p & 0\\ 0 & p^{-1}\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{d\in\mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(0,0,d)d_2\begin{pmatrix} p^{-1} & 0\\ 0 & p\end{pmatrix}K_2 \sqcup d_2\begin{pmatrix} p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{a,m\in\mathbb{Z}_p/p\mathbb{Z}_p,\, d\in\mathbb{Z}_p^\times/p\mathbb{Z}_p,\, l\in\mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(p^{-1}a^2d+l,\, p^{-1}ad+m,\, p^{-1}d)\nu_2(a)d_2\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{b\in\mathbb{Z}_p^\times/p\mathbb{Z}_p,\, c\in\mathbb{Z}_p/p\mathbb{Z}_p,\, d\in\mathbb{Z}_p/p^2\mathbb{Z}_p} n_2(p^{-1}b,c,d)d_2\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{a\in\mathbb{Z}_p/p\mathbb{Z}_p,\, b\in\mathbb{Z}_p^\times/p\mathbb{Z}_p} n_2(p^{-1}b,0,0)\nu_2(a)d_2\begin{pmatrix} 1 & 0\\ 0 & p^{-1}\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{d\in\mathbb{Z}_p^\times/p\mathbb{Z}_p} n_2(0,0,p^{-1}d)d_2\begin{pmatrix} p^{-1} & 0\\ 0 & 1\end{pmatrix}K_2\\
& \sqcup \bigsqcup_{a\in\mathbb{Z}_p^\times/p\mathbb{Z}_p,\, c,m\in\mathbb{Z}_p/p\mathbb{Z}_p} n_2(p^{-2}(ac+pm),p^{-1}c,0)\nu_2(p^{-1}a)K_2\\
& \sqcup \bigsqcup_{(b,c,d)\in\Lambda_2} n_2(p^{-1}b,p^{-1}c,p^{-1}d)K_2.
\end{aligned}
$$

Recall that the spherical functions $\omega_{1,\chi}\colon G_1 \to \mathbb{C}$ and $\omega_{2,\Xi}\colon G_2 \to \mathbb{C}$ for $\chi = (\chi_1,\chi_2), \Xi = (\Xi_1,\Xi_2) \in X_{\mathrm{unr}}(\mathbb{Q}_p^\times)^2$ are defined in Section 4.4. We have

$$
\begin{aligned}
(\omega_{1,\chi} * \varphi)(g_1) &:= \int_{G_1} \omega_{1,\chi}(g_1x^{-1})\varphi(x)dx = \lambda_{1,\chi}(\varphi)\omega_{1,\chi}(g_1) \qquad (g_1\in G_1, \varphi\in\mathcal{H}(G_1,K_1)),\\
(\omega_{2,\Xi} * \Phi)(g_2) &:= \int_{G_2} \omega_{2,\Xi}(g_2y^{-1})\Phi(y)dy = \lambda_{2,\Xi}(\Phi)\omega_{2,\Xi}(g_2) \qquad (g_2\in G_2, \Phi\in\mathcal{H}(G_2,K_2))
\end{aligned}
$$

with $\lambda_{1,\chi} \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(G_1,K_1),\mathbb{C})$ and $\lambda_{2,\Xi} \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(G_2,K_2),\mathbb{C})$ (cf. [16, Section 5.3]). Using Lemmas 9.2.2 and 9.2.3, we get the following.

Lemma 9.2.4. *We have*

$$
\begin{aligned}
\lambda_{1,\chi}(\mathcal{T}^1_{G_1}) &= p(\chi_1+\chi_2+\chi_1^{-1}+\chi_2^{-1}),\\
\lambda_{1,\chi}(\mathcal{T}^2_{G_1}) &= p(\chi_1\chi_2+\chi_1\chi_2^{-1}+\chi_1^{-1}\chi_2+\chi_1^{-1}\chi_2^{-1})+2p-2,\\
\lambda_{2,\Xi}(\mathcal{T}^1_{G_2}) &= p^2(\Xi_1+\Xi_2+\Xi_1^{-1}+\Xi_2^{-1})+p^2-1,\\
\lambda_{2,\Xi}(\mathcal{T}^2_{G_2}) &= p^3(\Xi_1\Xi_2+\Xi_1\Xi_2^{-1}+\Xi_1^{-1}\Xi_2+\Xi_1^{-1}\Xi_2^{-1})+p^2(p-1)(\Xi_1+\Xi_2+\Xi_1^{-1}+\Xi_2^{-1})+2p^3-2p^2,
\end{aligned}
$$

where we write, for $i = 1, 2$, χ_i and Ξ_i for $\chi_i(p)$ and $\Xi_i(p)$, respectively.

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9.3.

Denote by $\phi_0 \in \mathcal{S}(\mathbb{Q}_p^{(2,4)})$ the characteristic function of $\mathbb{Z}_p^{(2,4)}$ and set

$$J^i_{G_1}(x,y) = \int_{T^i_{G_1}} r'(g_1,1)\phi_0(x,y)dg_1,$$
$$J^i_{G_2}(x,y) = \int_{T^i_{G_2}} r'(1,g_2)\phi_0(x,y)dg_2$$

for $x, y \in \mathbb{Q}_p^{(2,2)}$ and $i = 1, 2$.

We now state the main result of this section. A sketch of its proof is given in Section 9.5.

Proposition 9.3.1. *For $x, y \in \mathbb{Q}_p^{(2,2)}$, we have*

$$J^1_{G_2}(x,y) = (p^2-1)\phi_0(x,y) + pJ^1_{G_1}(x,y) \tag{38}$$

and

$$J^2_{G_2}(x,y) = (p^2-p)J^1_{G_1}(x,y) + p^2J^2_{G_1}(x,y). \tag{39}$$

This proposition immediately implies the following result.

Corollary 9.3.2. *The algebra homomorphism $\upsilon\colon \mathcal{H}(G_2,K_2) \to \mathcal{H}(G_1,K_1)$ given by*

$$\upsilon(\mathcal{T}^1_{G_2}) = (p^2-1)\,\mathcal{T}^0_{G_1} + p\,\mathcal{T}^1_{G_1},$$
$$\upsilon(\mathcal{T}^2_{G_2}) = (p^2-p)\,\mathcal{T}^1_{G_1} + p^2\mathcal{T}^2_{G_1}$$

satisfies (36) (and hence (37)), where $\mathcal{T}^0_{G_1}$ denotes the characteristic function of K_1.

9.4.

In this section, assuming Corollary 9.3.2, we prove the following correspondence of the local factors of the standard L-functions.

Proposition 9.4.1. *Let $f \in \mathcal{S}_{1,\kappa}$ be a Hecke eigenform.*

(1) *The Arakawa lifting $\mathcal{L}(f)$ of f is an eigenfunction under the action of $\mathcal{H}(G_2,K_2)$.*
(2) *We have*

$$L_p(s,\mathcal{L}(f),\mathrm{std}) = \zeta_p(s)L_p(s,f,\mathrm{std}).$$

Proof. The first assertion is a direct consequence of Lemma 9.2.1 and Corollary 9.3.2. To show the second, we take $\chi = (\chi_1,\chi_2) \in X_{\mathrm{unr}}(\mathbb{Q}_p^\times)^2$ such that $\omega^f_{1,p} = \omega_{1,\chi}$ and $\Xi = (\Xi_1,\Xi_2) \in X_{\mathrm{unr}}(\mathbb{Q}_p^\times)^2$ such that $\omega^{\mathcal{L}(f)}_{2,p} = \omega_{2,\Xi}$ (see Lemma 4.4.1). By abuse of notation, we write χ_i and Ξ_i for $\chi_i(p)$ and $\Xi_i(p)$ ($i = 1, 2$), respectively. In view of Lemma 9.2.4, we have

$$\lambda^f_{1,p}(\mathcal{T}^1_{G_1}) = p\left(\chi_1+\chi_2+\chi_1^{-1}+\chi_2^{-1}\right), \tag{40}$$
$$\lambda^f_{1,p}(\mathcal{T}^2_{G_1}) = p\left(\chi_1\chi_2+\chi_1\chi_2^{-1}+\chi_1^{-1}\chi_2+\chi_1^{-1}\chi_2^{-1}\right)+2p-2, \tag{41}$$
$$\lambda^{\mathcal{L}(f)}_{2,p}(\mathcal{T}^1_{G_2}) = p^2\left(\Xi_1+\Xi_2+\Xi_1^{-1}+\Xi_2^{-1}\right)+p^2-1, \tag{42}$$
$$\lambda^{\mathcal{L}(f)}_{2,p}(\mathcal{T}^2_{G_2}) = p^3\left(\Xi_1\Xi_2+\Xi_1\Xi_2^{-1}+\Xi_1^{-1}\Xi_2+\Xi_1^{-1}\Xi_2^{-1}\right) + p^2(p-1)\left(\Xi_1+\Xi_2+\Xi_1^{-1}+\Xi_2^{-1}\right)+2p^3-2p^2. \tag{43}$$

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By (37), Corollary 9.3.2, (40), and (41), we have

$$\begin{aligned}\lambda_{2,p}^{\mathcal{L}(f)}(\mathcal{T}_{G_2}^1) &= \lambda_{1,p}^f(\upsilon(\mathcal{T}_{G_2}^1))\\ &= (p^2-1)\,\lambda_{1,p}^f(\mathcal{T}_{G_1}^0) + p\,\lambda_{1,p}^f(\mathcal{T}_{G_1}^1)\\ &= p^2\left(\chi_1+\chi_2+\chi_1^{-1}+\chi_2^{-1}\right)+p^2-1\end{aligned}$$

and

$$\begin{aligned}\lambda_{2,p}^{\mathcal{L}(f)}(\mathcal{T}_{G_2}^2) &= \lambda_{1,p}^f(\upsilon(\mathcal{T}_{G_2}^2))\\ &= (p^2-p)\,\lambda_{1,p}^f(\mathcal{T}_{G_1}^1) + p^2\,\lambda_{1,p}^f(\mathcal{T}_{G_1}^2)\\ &= (p^2-p)\left(\chi_1+\chi_2+\chi_1^{-1}+\chi_2^{-1}\right)\\ &\quad + p^3\left(\chi_1\chi_2+\chi_1\chi_2^{-1}+\chi_1^{-1}\chi_2+\chi_1^{-1}\chi_2^{-1}\right)+2p^3-2p^2.\end{aligned}$$

Comparing the above formulas with (42) and (43), we obtain

$$\begin{aligned}\Xi_1+\Xi_2+\Xi_1^{-1}+\Xi_2^{-1} &= \chi_1+\chi_2+\chi_1^{-1}+\chi_2^{-1},\\ \Xi_1\Xi_2+\Xi_1\Xi_2^{-1}+\Xi_1^{-1}\Xi_2+\Xi_1^{-1}\Xi_2^{-1} &= \chi_1\chi_2+\chi_1\chi_2^{-1}+\chi_1^{-1}\chi_2+\chi_1^{-1}\chi_2^{-1}.\end{aligned}$$

Hence, we have

$$\prod_{i=1}^{2}(1-\Xi_i p^{-s})^{-1}(1-\Xi_i^{-1}p^{-s})^{-1} = \prod_{i=1}^{2}(1-\chi_i p^{-s})^{-1}(1-\chi_i^{-1}p^{-s})^{-1},$$

which immediately implies the proposition. □

9.5.

In this section, we give a sketch of the proof of Proposition 9.3.1. The detail of the proof is given in [8, Section 12]. The following fact is easily verified.

Lemma 9.5.1. *For $u_1, u_2 \in \mathrm{GL}_2(\mathbb{Z}_p)$ and $x, y \in \mathbb{Q}_p^{(2,2)}$, we have*

$$\begin{aligned}J_{G_1}^i(u_1xu_2, y) &= J_{G_1}^i(x, u_1^{-1}y\,{}^tu_2),\\ J_{G_2}^i(u_1xu_2, y) &= J_{G_2}^i(x, u_1^{-1}y\,{}^tu_2).\end{aligned}$$

Thus, we may (and do) assume $x = \begin{pmatrix} p^\alpha & 0\\ 0 & p^\beta\end{pmatrix}$ $(\alpha, \beta \in \mathbb{Z},\ \alpha \le \beta)$ and put

$$\begin{aligned}J_{G_1}^i(y;\alpha,\beta) &= J_{G_1}^i\left(\begin{pmatrix} p^\alpha & 0\\ 0 & p^\beta\end{pmatrix}, y\right),\\ J_{G_2}^i(y;\alpha,\beta) &= J_{G_2}^i\left(\begin{pmatrix} p^\alpha & 0\\ 0 & p^\beta\end{pmatrix}, y\right)\end{aligned}$$

to simplify the notation. We have $J_{G_1}^i(y;\alpha,\beta) = J_{G_2}^i(y;\alpha,\beta) = 0$ if $\beta \le -2$. Henceforth, we suppose that $\beta \ge -1$.

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We denote by σ the characteristic function of $\mathbb{Z}_p$ and set

$$[i_1, i_2, i_3, i_4](y) = \prod_{k=1}^{4} \sigma(p^{-i_k} y_k) \qquad \left(y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{Q}_p^{(2,2)} \right).$$

Proposition 9.3.1 is a straightforward consequence of the following four lemmas, whose proofs are technical and omitted (for the full proofs, see [8, Section 12]).

Lemma 9.5.2. *Let* $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{Q}_p^{(2,2)}$.

(1) *If* $\beta < -1$, *we have*

$$J^1_{G_1}(y; \alpha, \beta) = 0.$$

(2) *If* $\alpha = \beta = -1$, *we have*

$$J^1_{G_1}(y; \alpha, \beta) = 0.$$

(3) *If* $\alpha = 0$ *and* $\beta = -1$, *we have*

$$J^1_{G_1}(y; \alpha, \beta) = p^{-1}[0, 0, -1, -1](y)\, \sigma(p^{-1} y_2 - y_3).$$

(4) *If* $\alpha \geq 1$ *and* $\beta = -1$, *we have*

$$J^1_{G_1}(y; \alpha, \beta) = p^{-1}[0, 1, -1, -1](y).$$

(5) *If* $\alpha = \beta = 0$, *we have*

$$\begin{aligned} J^1_{G_1}(y; \alpha, \beta) = {} & p^{-1}[-1, -1, -1, -1](y)\, \sigma(y_2 - y_3) \sigma(p \det y) \\ & + [0, 0, 0, 0](y). \end{aligned}$$

(6) *If* $\alpha \geq 1$ *and* $\beta = 0$, *we have*

$$\begin{aligned} J^1_{G_1}(y; \alpha, \beta) = {} & [0, 0, 0, 0](y) + p^{-1}[-1, 0, -1, 0](y) - p^{-1}[0, 0, -1, 0](y) \\ & + p^{-1}[0, 0, -1, -1](y) + p^2[1, 1, 0, 0](y). \end{aligned}$$

(7) *If* $\alpha \geq \beta \geq 1$, *we have*

$$\begin{aligned} J^1_{G_1}(y; \alpha, \beta) = {} & p^2[0, 0, 0, 0](y)\, \sigma(p^{-1} \det y) + p^{-1}[-1, -1, -1, -1](y)\, \sigma(p \det y) \\ & + p^3[1, 1, 1, 1](y) + [0, 0, 0, 0](y). \end{aligned}$$

Lemma 9.5.3. *Let* $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{Q}_p^{(2,2)}$.

(1) *If* $\beta < -1$, *we have*

$$J^2_{G_1}(y; \alpha, \beta) = 0.$$

(2) *If* $\alpha = \beta = -1$, *we have*

$$J^2_{G_1}(y; \alpha, \beta) = p^{-2}[-1, -1, -1, -1](y)\, \sigma(p^{-1}(y_2 - y_3)).$$

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(3) *If* $\alpha = 0$ *and* $\beta = -1$, *we have*

$$J^2_{G_1}(y;\alpha,\beta) = p^{-2}[-1,0,-1,-1](y)\,\sigma(p^{-1}y_2 - y_3).$$

(4) *If* $\alpha \geq 1$ *and* $\beta = -1$, *we have*

$$J^2_{G_1}(y;\alpha,\beta) = [1,1,-1,-1](y) + p^{-2}[-1,1,-1,-1](y).$$

(5) *If* $\alpha = \beta = 0$, *we have*

$$\begin{aligned} J^2_{G_1}(y;\alpha,\beta) = {} & p^{-2}[-1,-1,-1,-1](y)\,\sigma(y_2 - y_3) \\ & + p[0,0,0,0](y)\,\sigma(p^{-1}(y_2 - y_3)) - [0,0,0,0](y). \end{aligned}$$

(6) *If* $\alpha \geq 1$ *and* $\beta = 0$, *we have*

$$\begin{aligned} J^2_{G_1}(y;\alpha,\beta) = {} & [0,0,-1,-1](y)\,\sigma(\det y) + p[1,1,0,0](y) + p[0,1,0,0](y) \\ & + p^{-2}[-1,0,-1,-1](y) - 2[0,0,0,0](y). \end{aligned}$$

(7) *If* $\alpha \geq \beta \geq 1$, *we have*

$$\begin{aligned} J^2_{G_1}(y;\alpha,\beta) = {} & p\,[0,0,0,0](y)\,\sigma(p^{-1}\det y) + [-1,-1,-1,-1](y)\,\sigma(\det y) \\ & + (p^4 + p^2)[1,1,1,1](y) + p^{-2}[-1,-1,-1,-1](y) \\ & + (p-2)[0,0,0,0](y). \end{aligned}$$

Lemma 9.5.4. *Let* $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{Q}_p^{(2,2)}$.

(1) *If* $\beta < -1$, *we have*

$$J^1_{G_2}(y;\alpha,\beta) = 0.$$

(2) *If* $\alpha = \beta = -1$, *we have*

$$J^1_{G_2}(y;\alpha,\beta) = 0.$$

(3) *If* $\alpha = 0$ *and* $\beta = -1$, *we have*

$$J^1_{G_2}(y;\alpha,\beta) = [0,0,-1,-1](y)\,\sigma(p^{-1}y_2 - y_3).$$

(4) *If* $\alpha \geq 1$ *and* $\beta = -1$, *we have*

$$J^1_{G_2}(y;\alpha,\beta) = [0,1,-1,-1](y).$$

(5) *If* $\alpha = \beta = 0$, *we have*

$$\begin{aligned} J^1_{G_2}(y;\alpha,\beta) = {} & [-1,-1,-1,-1](y)\,\sigma(y_2 - y_3)\sigma(p\,\det y) \\ & + (p^2 + p - 1)[0,0,0,0](y). \end{aligned}$$

(6) *If* $\alpha \geq 1$ *and* $\beta = 0$, *we have*

$$\begin{aligned} J^1_{G_2}(y;\alpha,\beta) = {} & p^3\,[1,1,0,0](y) + (p^2 + p - 1)[0,0,0,0](y) \\ & + [0,0,-1,-1](y) - [0,0,-1,0](y) + [-1,0,-1,0](y). \end{aligned}$$

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(7) *If* $\alpha \geq \beta \geq 1$, *we have*

$$J^1_{G_2}(y;\alpha,\beta) = p^3\,[0,0,0,0](y)\,\sigma(p^{-1}\det y) + [-1,-1,-1,-1](y)\,\sigma(p\det y) + p^4[1,1,1,1](y) + (p^2+p-1)[0,0,0,0](y).$$

Lemma 9.5.5. *Let* $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in \mathbb{Q}_p^{(2,2)}$.

(1) *If* $\beta < -1$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = 0.$$

(2) *If* $\alpha = \beta = -1$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = [-1,-1,-1,-1](y)\,\sigma(p^{-1}(y_2-y_3)).$$

(3) *If* $\alpha = 0$ *and* $\beta = -1$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = (p-1)\,[0,0,-1,-1](y)\,\sigma(p^{-1}y_2-y_3) + [-1,0,-1,-1](y)\,\sigma(p^{-1}y_2-y_3).$$

(4) *If* $\alpha \geq 1$ *and* $\beta = -1$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = (p-1)\,[0,1,-1,-1](y) + p^2[1,1,-1,-1](y) + [-1,1,-1,-1](y).$$

(5) *If* $\alpha = \beta = 0$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = p^3\,[0,0,0,0](y)\,\sigma(p^{-1}(y_2-y_3)) + (p-1)[-1,-1,-1,-1](y)\,\sigma(y_2-y_3)\sigma(p\,\det y) + [-1,-1,-1,-1](y)\,\sigma(y_2-y_3) - p[0,0,0,0](y).$$

(6) *If* $\alpha \geq 1$ *and* $\beta = 0$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = p^2\,[0,0,-1,-1](y)\,\sigma(\det y) + p^4\,[1,1,0,0](y) + p^3\,[0,1,0,0](y) + (p-1)\,[-1,0,-1,0](y) + (-p+1)\,[0,0,-1,0](y) + (p-1)\,[0,0,-1,-1](y) + [-1,0,-1,-1](y) + (-p^2-p)\,[0,0,0,0](y).$$

(7) *If* $\alpha \geq \beta \geq 1$, *we have*

$$J^2_{G_2}(y;\alpha,\beta) = [-1,-1,-1,-1](y)\,(p^2\sigma(\det y)+1) + p^4\,[0,0,0,0](y)\,\sigma(p^{-1}\det y) + (p-1)[-1,-1,-1,-1](y)\,\sigma(p\det y) + (p^6+p^5)\,[1,1,1,1](y) + [-1,-1,-1,-1](y) + (p^3-p^2-p)[0,0,0,0](y).$$

10. Eichler commutation relations at ramified places

10.1.

In this section, we consider the ramified case; namely, we assume that p divides d_B. Throughout this section, we often drop the subscript p. Recall that $\mathcal{O}$ is the maximal order of B and $\mathfrak{P}$ is the maximal ideal of $\mathcal{O}$. We fix an element Π of $\mathcal{O}$ such that $\mathfrak{P} = \Pi\mathcal{O}$. Define $\mathrm{ord}_\Pi : B^\times \to \mathbb{Z}$ such that $\mathrm{ord}_\Pi(\alpha) = n$ for $\alpha \in \Pi^n\mathcal{O}^\times$. We fix a prime element π of $\mathbb{Q}_p$.

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Let

$$G_1 = \left\{ g \in \mathrm{GL}_2(B) \mid {}^t\overline{g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$
$$K_1 = G_1 \cap \mathrm{GL}_2(O)$$
$$G_2 = \left\{ g \in \mathrm{GL}_2(B) \mid {}^t\overline{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$
$$K_2 = G_2 \cap \begin{pmatrix} O & \mathfrak{P}^{-1} \\ \mathfrak{P} & O \end{pmatrix}.$$

Define subgroups of G_1 by

$$N_1 = \left\{ n_1(b) := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Q}_p \right\},$$
$$M_1 = \left\{ d_1(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix} \mid \alpha \in B^\times \right\}.$$

Then $G_1 = N_1 M_1 K_1$.

We also define subgroups of G_2 by

$$N_2 = \left\{ n_2(\beta) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in B^- \right\},$$
$$M_2 = \left\{ d_2(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}^{-1} \end{pmatrix} \mid \alpha \in B^\times \right\}.$$

Then $G_2 = N_2 M_2 K_2$.

The Weil representation r' given in Lemma 5.1.1 is reformulated as follows: For $\phi \in \mathcal{S}(B^{(1,2)})$ and $X \in B^{(1,2)}$,

$$r'(1, g)\phi(X) = \phi(Xg) \qquad (g \in G_2),$$
$$r'(d_1(\alpha), 1)\phi(X) = |N(\alpha)|^2 \, \phi(\overline{\alpha} X) \qquad (\alpha \in B^\times),$$
$$r'(n_1(b), 1)\phi(x, y) = \psi(-b\, \overline{X} Q \, {}^t X)\phi(X) \qquad (b \in \mathbb{Q}_p).$$

10.2.

In this section, we prepare several facts on Hecke algebras.

For $n \in \mathbb{Z}$, we put

$$X_n = \{\beta \in B^- \mid \mathrm{ord}_\Pi(\beta) \geq n\},$$
$$X_n^0 = \{\beta \in B^- \mid \mathrm{ord}_\Pi(\beta) = n\} = X_n \setminus X_{n+1},$$
$$\mathbf{d}_1(n) = d_1(\Pi^n),$$
$$\mathbf{d}_2(n) = d_2(\Pi^n).$$

Note that

$$|X_n / \pi X_n| = p^3,$$
$$|X_n / X_{n+1}| = \begin{cases} p & (n \text{ is even}), \\ p^2 & (n \text{ is odd}). \end{cases}$$

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Let $\mathcal{T}^1_{G_1} \in \mathcal{H}(G_1, K_1)$ (resp. $\mathcal{T}^1_{G_2} \in \mathcal{H}(G_2, K_2)$) be the characteristic function of $T^1_{G_1} = K_1\mathbf{d}_1(1)K_1$ (resp. $T^1_{G_2} = K_2\mathbf{d}_2(1)K_2$).

Lemma 10.2.1. *We have*

$$\mathcal{H}(G_1, K_1) = \mathbb{C}[\mathcal{T}^1_{G_1}],$$
$$\mathcal{H}(G_2, K_2) = \mathbb{C}[\mathcal{T}^1_{G_2}].$$

Proof. This follows from [16, Section 5.1, Theorem 1]. □

The following facts are easily verified.

Lemma 10.2.2. *We have*

$$T^1_{G_1} = \bigsqcup_{b \in \mathbb{Z}_p/\pi\mathbb{Z}_p} n_1(b)\mathbf{d}_1(1)K_1 \sqcup \mathbf{d}_1(-1)K_1.$$

Lemma 10.2.3. *We have*

$$T^1_{G_2} = \bigsqcup_{\beta \in X_{-1}/\pi X_{-1}} n_2(\beta)\mathbf{d}_2(1)K_2 \sqcup \bigsqcup_{\beta \in X^0_{-2}/X_{-1}} n_G(\beta)K \sqcup \mathbf{d}_2(-1)K_2.$$

Recall that the spherical functions $\omega_{1,\chi}\colon G_1 \to \mathbb{C}$ and $\omega_{2,\Xi}\colon G_2 \to \mathbb{C}$ for $\chi, \Xi \in X_{\mathrm{unr}}(B^\times)$ are defined in Section 4.5. Then we have

$$(\omega_{1,\chi} * \varphi)(g_1) := \int_{G_1} \omega_{1,\chi}(hx^{-1})\varphi(x)dx = \lambda_{1,\chi}(\varphi)\omega_{1,\chi}(g_1) \qquad (g_1 \in G_1, \varphi \in \mathcal{H}(G_1, K_1)),$$
$$(\omega_{2,\Xi} * \Phi)(g_2) := \int_{G_2} \omega_{2,\Xi}(g_2y^{-1})\Phi(y)dy = \lambda_{2,\Xi}(\Phi)\omega_{2,\Xi}(g_2) \qquad (g_2 \in G_2, \Phi \in \mathcal{H}(G_2, K_2))$$

with $\lambda_{1,\chi} \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(G_1, K_1), \mathbb{C})$ and $\lambda_{2,\Xi} \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(G_2, K_2), \mathbb{C})$ (cf. [16, Section 5.4]). Using Lemmas 10.2.2 and 10.2.3, we get the following.

Lemma 10.2.4. *We have*

$$\lambda_{1,\chi}(\mathcal{T}^1_{G_1}) = p^{1/2}(\chi(\Pi) + \chi^{-1}(\Pi)),$$
$$\lambda_{2,\Xi}(\mathcal{T}^1_{G_2}) = p^{3/2}(\Xi(\Pi) + \Xi^{-1}(\Pi)) + p - 1.$$

10.3.

Let $\phi_0 \in W'$ be the characteristic function of $(\mathcal{O}, \mathfrak{P}^{-1})$ and set

$$J_{G_1}(X) = \int_{T^1_{G_1}} r'(g_1, 1)\phi_0(X)dg_1,$$
$$J_{G_2}(X) = \int_{T^1_{G_2}} r'(1, g_2)\phi_0(X)dg_2$$

for $X \in B^{(1,2)}$. We now state the main results of this section, whose proof is given in Sections 10.4–10.6.

Proposition 10.3.1. *We have*

$$J_{G_2}(X) = (p-1)\phi_0(X) + pJ_{G_1}(X) \qquad (X \in B^{(1,2)}). \tag{44}$$

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Corollary 10.3.2. *The $\mathbb{C}$-algebra homomorphism $\upsilon\colon \mathcal{H}(G_2, K_2) \to \mathcal{H}(G_1, K_1)$ given by*

$$\upsilon(\mathcal{T}^1_{G_2}) = (p-1)\mathcal{T}^0_{G_1} + p\,\mathcal{T}^1_{G_1} \qquad (X \in B^{(1,2)})$$

satisfies (36) (and hence (37)), where $\mathcal{T}^0_{G_1}$ denotes the characteristic function of K_1.

We now prove the following result assuming Corollary 10.3.2.

Proposition 10.3.3. *We have*

$$L_p(s, \mathcal{L}(f), \mathrm{std}) = \zeta_p(s) L_p(s, f, \mathrm{std}). \tag{45}$$

Proof. We take $\chi \in X_{\mathrm{unr}}(B^\times)$ such that $\omega^f_{1,p} = \omega_{1,\chi}$ and $\Xi \in X_{\mathrm{unr}}(B^\times)$ such that $\omega^{\mathcal{L}(f)}_{2,p} = \omega_{2,\Xi}$ (see Lemma 4.5.1). An argument similar to the proof of Proposition 9.4.1 shows

$$\Xi(\Pi) + \Xi^{-1}(\Pi) = \chi(\Pi) + \chi^{-1}(\Pi)$$

and hence

$$(1 - \Xi(\Pi)p^{-s})^{-1}\,(1 - \Xi^{-1}(\Pi)p^{-s})^{-1} = (1 - \chi(\Pi)p^{-s})^{-1}\,(1 - \chi^{-1}(\Pi)p^{-s})^{-1}.$$

The proposition immediately follows from this. □

10.4.

The proof of Proposition 10.3.1 is reduced to the following explicit formulas for $J_{G_1}(X)$ and $J_{G_2}(X)$. We denote by σ the characteristic function of O.

Lemma 10.4.1. *For $x, y \in B$, we have*

$$J_{G_1}(x,y) = \begin{cases} 0 & (\mathrm{ord}_\Pi(x) \le -2), \\ p^{-1}\delta(\mathrm{tr}(\bar{x}y) \in \mathbb{Z}_p)\,\sigma(\pi y) & (\mathrm{ord}_\Pi(x) = 0, -1), \\ p^2\,\sigma(y) + p^{-1}\sigma(\pi y) & (\mathrm{ord}_\Pi x \ge 1). \end{cases}$$

Lemma 10.4.2. *For $x, y \in B$, we have*

$$J_{G_2}(x,y) = \begin{cases} 0 & (\mathrm{ord}_\Pi(x) \le -2), \\ \delta(\mathrm{tr}(\bar{x}y) \in \mathbb{Z}_p)\,\sigma(\pi y) & (\mathrm{ord}_\Pi(x) = -1), \\ \delta(\mathrm{tr}(\bar{x}y) \in \mathbb{Z}_p)\,\sigma(\pi y) + (p-1)\sigma(\Pi y) & (\mathrm{ord}_\Pi(x) = 0), \\ p^3\,\sigma(y) + (p-1)\sigma(\Pi y) + \sigma(\pi y) & (\mathrm{ord}_\Pi x \ge 1). \end{cases}$$

To prove these results, we need the following facts.

Lemma 10.4.3 [12, Lemmas A.2 and A.3]. *We have*

$$\begin{aligned} [O : \pi O] &= p^4, \\ [O : \Pi O] &= p^2, \\ [O^- : \pi O^-] &= p^3, \\ [O^- : (\Pi O)^-] &= p, \\ [(\Pi O)^- : \pi O^-] &= p^2. \end{aligned}$$

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Lemma 10.4.4 [5, Lemma 8.2]. *For $x \in B$, we have*

$$\sum_{\beta \in X_{-1}/X_0} \sigma(x+\beta) = \sigma(\Pi x),$$
$$\sum_{\beta \in X_{-1}/\pi X_{-1}} \sigma(\Pi^{-1}(x+\beta)) = \delta(\mathrm{tr}(x) \in p\mathbb{Z}_p)\sigma(\Pi x),$$
$$\sum_{\beta \in X_{-2}/X_{-1}} \sigma(\Pi(x+\beta)) = \delta(\mathrm{tr}(x) \in \mathbb{Z}_p)\sigma(\pi x).$$

10.5. The proof of Lemma 10.4.1

Let $x, y \in B$. By Lemma 10.2.2, we have

$$\begin{aligned} J_{G_1}(x,y) &= \sum_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} r'\left(n_1(b)\mathbf{d}_1(1), 1\right)\ \phi_0(x,y) + r'\left(\mathbf{d}_1(-1), 1\right)\ \phi_0(x,y) \\ &= p^{-2} \sum_{b \in \mathbb{Z}_p/p\mathbb{Z}_p} \psi(-b\,\mathrm{tr}(\overline{x}y))\ \phi_0(\overline{\Pi}x, \overline{\Pi}y) + p^2\ \phi_0(\overline{\Pi}^{-1}x, \overline{\Pi}^{-1}y) \\ &= p^{-1}\ \delta(\mathrm{tr}(\overline{x}y) \in \mathbb{Z}_p)\ \sigma(\Pi x)\ \sigma(\pi y) + p^2\ \sigma(\Pi^{-1}x)\ \sigma(y). \end{aligned}$$

We have $J_{G_1}(x,y) = 0$ if $\mathrm{ord}_\Pi(x) \le -2$. If $\mathrm{ord}_\Pi(x) = -1$ or 0, we have

$$J_{G_1}(x,y) = p^{-1}\ \delta(\mathrm{tr}(\overline{x}y) \in \mathbb{Z}_p)\ \sigma(\pi y).$$

If $\mathrm{ord}_\Pi(x) \ge 1$, we have

$$J_{G_1}(x,y) = p^{-1}\ \sigma(\pi y) + p^2\ \sigma(y),$$

which completes the proof of Lemma 10.4.1.

10.6. The proof of Lemma 10.4.2

Lemma 10.6.1. *For $x, y \in B$, we have*

$$J_{G_2}(x,y) = J_1(x,y) + J_2(x,y) + J_3(x,y),$$

where

$$\begin{aligned} J_1(x,y) &= \sum_{\beta \in X_{-1}/\pi X_{-1}} \sigma(x\Pi)\ \sigma(y + x\beta), \\ J_2(x,y) &= \sum_{\beta \in X^0_{-2}/X_{-1}} \sigma(x)\ \sigma(\Pi(y + x\beta)), \\ J_3(x,y) &= \sigma(\Pi^{-1}x)\ \sigma(\pi y). \end{aligned}$$

Proof. By Lemma 10.2.3, we have

$$\begin{aligned} &J_{G_2}(x,y) \\ &= \sum_{\beta \in X_{-1}/\pi X_{-1}} r'(1, n_2(\beta)\mathbf{d}_2(1))\phi_0(x,y) \\ &\quad + \sum_{\beta \in X^0_{-2}/X_{-1}} r'(1, n_2(\beta))\phi_0(x,y) + r'(1, \mathbf{d}_2(-1))\phi_0(x,y) \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{\beta\in X_{-1}/\pi X_{-1}} \phi_0((x,y)n_2(\beta)\mathbf{d}_2(1)) + \sum_{\beta\in X_{-2}^0/X_{-1}} \phi_0((x,y)n_2(\beta)) + \phi_0((x,y)\,\mathbf{d}_2(-1))\\
&= \sum_{\beta\in X_{-1}/\pi X_{-1}} \phi_0(x\Pi, (y+x\beta)\overline{\Pi}^{-1}) + \sum_{\beta\in X_{-2}^0/X_{-1}} \phi_0(x, y+x\beta) + \phi_0(x\Pi^{-1}, y\overline{\Pi})\\
&= \sum_{\beta\in X_{-1}/\pi X_{-1}} \sigma(x\Pi)\,\sigma(\Pi(y+x\beta)\overline{\Pi}^{-1})) + \sum_{\beta\in X_{-2}^0/X_{-1}} \sigma(x)\,\sigma(\Pi(y+x\beta))\\
&\quad + \sigma(x\Pi^{-1})\,\sigma(\Pi y\overline{\Pi})\\
&= \sum_{\beta\in X_{-1}/\pi X_{-1}} \sigma(x\Pi)\,\sigma(y+x\beta) + \sum_{\beta\in X_{-2}^0/X_{-1}} \sigma(x)\,\sigma(\Pi(y+x\beta)) + \sigma(\Pi^{-1}x)\,\sigma(\pi y).
\end{aligned}
$$

□

We now prove Lemma 10.4.2.

It is easy to see that $J_{G_2}(x,y) = 0$ if $\mathrm{ord}_\Pi(x) \le -2$.

Suppose that $\mathrm{ord}_\Pi(x) = -1$. Then $J_2(x,y) = J_3(x,y) = 0$ and

$$
\begin{aligned}
J_1(x,y) &= \sum_{\beta\in X_{-1}/\pi X_{-1}} \sigma(y+x\beta)\\
&= \sum_{\beta\in X_{-1}/\pi X_{-1}} \sigma(\Pi^{-1}(x^{-1}y+\beta))\\
&= \delta(\mathrm{tr}(x^{-1}y)\in p\mathbb{Z}_p)\,\sigma(\Pi x^{-1}y)\\
&= \delta(\mathrm{tr}(\bar{x}y)\in\mathbb{Z}_p)\,\sigma(\pi y).
\end{aligned}
$$

Suppose that $\mathrm{ord}_\Pi(x) = 0$. Then $J_3(x,y) = 0$. We have

$$
\begin{aligned}
J_1(x,y) &= \sum_{\beta\in X_{-1}/\pi X_{-1}} \sigma(y+x\beta)\\
&= \sum_{\beta\in X_{-1}/\pi X_{-1}} \sigma(x^{-1}y+\beta)\\
&= \sum_{\beta'\in X_{-1}/\pi X_0}\ \sum_{\beta''\in X_0/\pi X_1} \sigma(x^{-1}y+\beta'+\beta'')\\
&= |X_0/X_1| \sum_{\beta'\in X_{-1}/\pi X_0} \sigma(x^{-1}y+\beta')\\
&= p\,\sigma(\Pi x^{-1}y)\\
&= p\,\sigma(\Pi y)
\end{aligned}
$$

and

$$
\begin{aligned}
J_2(x,y) &= \sum_{\beta\in X_{-2}^0/X_{-1}} \sigma(\Pi(x^{-1}y+\beta))\\
&= \delta(\mathrm{tr}(x^{-1}y)\in\mathbb{Z}_p)\,\sigma(\pi x^{-1}y) - \sigma(\Pi x^{-1}y)\\
&= \delta(\mathrm{tr}(\bar{x}y)\in\mathbb{Z}_p)\,\sigma(\pi y) - \sigma(\Pi y).
\end{aligned}
$$

Suppose that $\mathrm{ord}_\Pi(x) \ge 1$. We have

$$
\begin{aligned}
J_1(x,y) &= |X_{-1}/\pi X_{-1}|\,\sigma(y) = p^3\,\sigma(y),\\
J_2(x,y) &= |X_-^0/X_{-1}|\,\sigma(\Pi y) = (p-1)\,\sigma(\Pi y),\\
J_3(x,y) &= \sigma(\pi y),
\end{aligned}
$$

which completes the proof of Lemma 10.4.2.

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