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PARALLEL BODIES AND NORMALS OF CONVEX BODIES

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Dedicated to Erwin Lutwak on the occasion of his 65th birthday.

Abstract. The primary aim of this paper is to determine how two convex bodies are related if their respective normal bundles are equal or, in a certain sense, close to each other. In connection with this problem some pertinent results concerning parallel bodies and the Steiner point are also discussed.

§1. *Introduction and notation.* Non-empty compact convex subsets of the n-dimensional Euclidean space $\mathbf{R}^n$ ($n \geq 2$) will be called *convex bodies* in $\mathbf{R}^n$, and the set of all convex bodies in $\mathbf{R}^n$ will be denoted by $\mathcal{K}^n$. If $K \in \mathcal{K}^n$ and $r \geq 0$ then $K[r]$ will denote the *parallel body* of K. It is defined as the Minkowski sum $K + rB^n$ where B^n denotes the unit ball in $\mathbf{R}^n$ (centered at the origin o of $\mathbf{R}^n$). Whenever the notation $K[r]$ is used it is assumed that $r \geq 0$. If h_K denotes the support function of K then $K[r]$ can also be defined as the convex body with the support function $h_K + r$. Throughout this paper the support function is understood to be a function on the unit sphere $S^{n-1} = \partial B^n$. Clearly, if $K[r_1] = K[r_2]$ and $x \geq -r_1$, $x \geq -r_2$ it follows that $K[r_1 + x] = K[r_2 + x]$.

Two convex bodies K_1 and K_2 will be said to be *parallel*, denoted by $K_1 \approx K_2$, if there is an $r \geq 0$ such that either $K_1[r] = K_2$ or $K_2[r] = K_1$. This can also be expressed by saying that $K_1 \approx K_2$ holds exactly if there exist two non-negative numbers x and y such that $K_1[x] = K_2[y]$. The relation $\approx$ is apparently reflexive, symmetric and transitive. It is therefore an equivalence relation in $\mathcal{K}^n$, and $\mathcal{K}^n$ can be subdivided into its equivalence classes, that is, into disjoint subsets such that their union is $\mathcal{K}^n$ and that any two convex bodies are in the same subset exactly if they are parallel. These equivalence classes will be called the *parallel classes* of $\mathcal{K}^n$, and we let $\mathcal{P}^n$ denote the collection of all parallel classes of $\mathcal{K}^n$.

Note. From now on whenever the subscript ν is used it is always supposed to range over the set $\{1, 2\}$. Furthermore, K, K_ν, K'_ν will always denote convex bodies in $\mathbf{R}^n$, and Π, Π_ν, Π'_ν parallel classes of $\mathcal{K}^n$.

Now let u be a unit vector in $\mathbf{R}^n$, and let $H(u)$ denote the support plane of K that is orthogonal to u and such that u points into the half-space bounded by $H(u)$ that does not contain K. It is known (cf. Schneider [4, §2.2]) that for

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almost all u (considered as points on the unit sphere S^{n-1} endowed with the Lebesgue measure) the support set $K \cap H(u)$ consists of exactly one point. A direction u of this kind will be called a *regular direction* of K, and the set of all regular directions of K will be denoted by $\mathcal{R}(K)$. If $u \in \mathcal{R}(K)$ then the directed line that passes through the support point $K \cap H(u)$, is orthogonal to $H(u)$, and inherits its direction from u will be called a *normal* of K, and the set of all normals of K, denoted by $\mathbf{N}(K)$, will be called the *normal bundle* of K.

A function f defined on $\mathcal{K}^n$ or $\mathcal{K}^n \times \mathcal{K}^n$ will be said to be *invariant* if $K_1 \approx K_2$ implies $f(K_1) = f(K_2)$ or, respectively, $K_1 \approx K_2$ and $K_1' \approx K_2'$ implies $f(K_1, K_1') = f(K_2, K_2')$. Clearly, invariant functions are constant on every parallel class Π (or on $\Pi_1 \times \Pi_2$). Thus invariant functions can also be interpreted as functions on $\mathcal{P}^n$ (or on $\mathcal{P}^n \times \mathcal{P}^n$) and if this interpretation is meant and $K \in \Pi$ we write $f(\Pi)$ instead of $f(K)$ (or $f(\Pi_1, \Pi_2)$ instead of $f(K_1, K_2)$). Functions defined on $\mathcal{P}^n$ or $\mathcal{P}^n \times \mathcal{P}^n$ will be called *class functions*. The following proposition shows that the normal bundle is an invariant function.

PROPOSITION 1. *If* $K_1 \approx K_2$ *then* $\mathbf{N}(K_1) = \mathbf{N}(K_2)$.

Proof. It is evidently sufficient to prove that whenever $K \in \mathcal{K}^n$ and $r \geq 0$ then $\mathbf{N}(K[r]) = \mathbf{N}(K)$. Since it is obvious that the regular directions of K are the same as the regular directions of $K[r]$, all one has to show is that if $u \in \mathcal{R}(K)$ and N, N_r are normals of direction u of, respectively, K and $K[r]$ then $N = N_r$. Let H and H_r be the corresponding support planes of K and $K[r]$ belonging to the direction u, and p, p_r the corresponding support points. Since the distance between H and H_r is r, the sphere of radius r centered at p has one point in common with H_r, and because of the regularity of u this must be the point p_r. The equality $N = N_r$ now follows immediately from the fact that N and N_r have the same direction and pass through the same point p. □

Since this proposition shows that the relationship between convex bodies and normal bundles is not one-to-one it appears to be more appropriate to study the relationship between parallel classes and normal bundles. In the following section the role of the support function and the mean width of convex bodies for the study of parallel classes is discussed. Section 3 concerns the definition of a deviation measure for parallel classes. It is based on the L_2-distance for support functions. Section 4 deals with translation properties of parallel classes and a certain minimal property of the Steiner point. Section 5 is concerned with the introduction of a deviation measure for normal bundles. The main result of the present article is the theorem in §6 that exhibits sharp upper bounds for the deviation of two parallel classes if the deviation of the corresponding normal bundles is given.

§2. *The mean width and the support function.* For the study of parallel classes the mean width of a convex body plays an important role. It is defined by

$$\bar{w}(K) = \frac{1}{\sigma_n} \int_{S^{n-1}} (h_K(u) + h_K(-u))\, d\sigma(u)$$

where $\sigma(u)$ refers to the surface area measure on S^{n-1}, and σ_n denotes the surface area measure of B^n.

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The following proposition shows that if the respective mean widths of K_1 and K_2 are known and $K_1 \approx K_2$ then it is easy to find suitable values for x and y such that $K_1[x] = K_2[y]$.

PROPOSITION 2. *If* $K_1[x] = K_2[y]$ *then* $x - y = \frac{1}{2}(\bar{w}(K_2) - \bar{w}(K_1))$. *Conversely, if* $K_1 \approx K_2$ *and* $x \geq 0$, $y \geq 0$ *are such that* $x - y = \frac{1}{2}(\bar{w}(K_2) - \bar{w}(K_1))$ *then* $K_1[x] = K_2[y]$. *In particular,* $K_1 \approx K_2$ *holds if and only if* $K_1[\bar{w}(K_2)/2] = K_2[\bar{w}(K_1)/2]$*; and if* K_1 *is, with respect to mean width, smaller than* K_2 *and* $K_1 \approx K_2$ *then* $K_2 = K_1[(\bar{w}(K_2) - \bar{w}(K_1))/2]$.

Proof. If $K_1[x] = K_2[y]$ then $K_1 + xB^n = K_2 + yB^n$, and since the mean width is additive under Minkowski addition this implies that $\bar{w}(K_1) + 2x = \bar{w}(K_2) + 2y$, proving the first part of the proposition. To prove the second part one may assume that $\bar{w}(K_1) \leq \bar{w}(K_2)$. Since $K_1 \approx K_2$ there exist $u \geq 0$ and $v \geq 0$ such that $K_1[u] = K_2[v]$, and by the part already proved this implies that $u - v = \frac{1}{2}(\bar{w}(K_2) - \bar{w}(K_1)) = x - y \geq 0$. Since $K_1[u - v] = K_2$ we have $K_1[x - y] = K_2$, and therefore $K_1[x] = K_2[y]$. □

The following facts are also worth mentioning although they will not be used later on. A straightforward application of Blaschke's selection theorem shows that every parallel class $\Pi(K)$ contains an element, say C, of smallest mean width. This body is then evidently also the unique smallest convex body in its parallel class with respect to set inclusion. Calling C the *kernel* of K we see that two convex bodies are parallel exactly if they have the same kernel, and that every parallel class is of the form $\{C[r] : r \geq 0\}$. It is, however, not a simple matter to actually determine the kernel of a given convex body.

The support function h_K of K, considered as a function of K, is obviously not invariant, but a slight modification of h_K, defined by

$$\tilde{h}_K = h_K - \tfrac{1}{2}\bar{w}(K),$$

is invariant. Indeed, if $K \approx L$ then $K[\bar{w}(L)/2] = L[\bar{w}(K)/2]$ and it follows immediately that $\tilde{h}(K) = \tilde{h}(L)$. The modified support function $\tilde{h}_K(u)$ can be visualized as the signed distance between the two support planes of direction u belonging to K and $\frac{1}{2}\bar{w}(K)B^n$. Because of its invariance we may consider this modified support function as a class function and use the notation $\tilde{h}_\Pi$.

§3. *A metric for parallel classes.* There are many possible ways to introduce a metric on $\mathcal{K}^n$. For our purpose the most suitable version is the metric based on the L_2-norm for functions on S^{n-1}.

As usual, if Φ and Ψ are two real-valued square integrable functions that are defined almost everywhere on S^{n-1} their inner product $\langle \Phi, \Psi \rangle$ is given by

$$\langle \Phi, \Psi \rangle = \int_{S^{n-1}} \Phi(u)\Psi(u)\, d\sigma(u).$$

Accordingly, we let

$$\|\Phi\| = \langle \Phi, \Phi \rangle^{1/2}$$

denote the norm of Φ.

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Note. If $p \in \mathbf{R}^n$ then $|p|$ denotes the length of the vector p (or the absolute value of p if $n = 1$). The inner product of two vectors p, q in $\mathbf{R}^n$ will be denoted by $p \cdot q$. If F is a vector-valued function on S^{n-1} we simply write $\|F\|$ instead of $\||F|\|$. The letter u that appears in mathematical expressions always denotes a fixed or variable unit vector (also referred to as a direction or a point of S^{n-1}). Functions on S^{n-1} or $\mathcal{R}(K)$ are denoted with the letter u either indicated (as in the above integral) or omitted (as in the above norm and inner product).

We now define the distance between K_1 and K_2 by

$$d(K_1, K_2) = \frac{1}{\sqrt{\sigma_n}} \|h_{K_1} - h_{K_2}\|.$$

It is obvious that d is a metric on $\mathcal{K}^n$ and that $d(K_1[x], K_2[x]) = d(K_1, K_2)$. This does not mean, however, that d is invariant as defined before, since in general $d(K_1[x], K_2[y])$ is for $x \neq y$ different from $d(K_1, K_2)$. The following proposition exhibits the relationship between $d(K_1, K_2)$ and $d(K_1', K_2')$ for any two pairs K_ν, K_ν' with $K_\nu \approx K_\nu'$. In fact, it shows that the function $d(K_1, K_2)^2 - \frac{1}{4}(\bar{w}(K_1) - \bar{w}(K_2))^2$ is invariant.

PROPOSITION 3. *If $K_1 \approx K_1'$ and $K_2 \approx K_2'$ then*

$$d(K_1', K_2')^2 - \tfrac{1}{4}(\bar{w}(K_1') - \bar{w}(K_2'))^2 = d(K_1, K_2)^2 - \tfrac{1}{4}(\bar{w}(K_1) - \bar{w}(K_2))^2.$$

Hence, if the bodies K_1, K_2 are given then among all pairs K_1', K_2' with $K_\nu' \approx K_\nu$, the distance $d(K_1', K_2')$ attains its minimum exactly if $\bar{w}(K_1') = \bar{w}(K_2')$.

Proof. Replacing if necessary K_ν' with some $K_\nu'[r]$ we may assume that $K_\nu' = K_\nu[x_\nu]$, with $x_\nu \geq 0$. Letting $x_1 - x_2 = t$ we find that

$$\begin{aligned}
\sigma_n d(K_1[x_1], K_2[x_2])^2 &= \|h_{K_1} + t - h_{K_2}\|^2 \\
&= \|h_{K_1}\|^2 + \|h_{K_2}\|^2 + \|t\|^2 - 2\langle h_{K_1}, h_{K_2}\rangle \\
&\quad + 2\langle h_{K_1} - h_{K_2}, t\rangle \\
&= \|h_{K_1} - h_{K_2}\|^2 + \sigma_n t^2 + \sigma_n(\bar{w}(K_1) - \bar{w}(K_2))t \\
&= \sigma_n d(K_1, K_2)^2 + \sigma_n\left(t + \frac{1}{2}(\bar{w}(K_1) - \bar{w}(K_2))\right)^2 \\
&\quad - \frac{\sigma_n}{4}(\bar{w}(K_1) - \bar{w}(K_2))^2
\end{aligned}$$

and the desired conclusion follows. □

The following proposition shows that from this metric on $\mathcal{K}^n$ one can derive in a natural way a metric on $\mathcal{P}^n$. This metric will then be used as a measure for the mutual deviation of two parallel classes or, as one may also say, as a measure for the deviation of two convex bodies if parallel convex bodies are considered to be equal.

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PROPOSITION 4. *If $\Pi_1, \Pi_2 \in \mathcal{P}^n$ and δ is defined by*

$$\delta(\Pi_1, \Pi_2) = \inf\{d(X_1, X_2) : X_1 \in \Pi_1, X_2 \in \Pi_2\}$$

then δ has the following properties.

(a) *If $K_\nu \in \Pi_\nu$ then*

$$\delta(\Pi_1, \Pi_2)^2 = d(K_1, K_2)^2 - \tfrac{1}{4}(\bar{w}(K_1) - \bar{w}(K_2))^2.$$

In particular, $\delta(\Pi_1, \Pi_2) = d(K_1, K_2)$ holds if and only if $\bar{w}(K_1) = \bar{w}(K_2)$.

(b) $\delta(\Pi_1, \Pi_2) = (1/\sqrt{\sigma_n})\|\tilde{h}_{\Pi_1} - \tilde{h}_{\Pi_2}\|$.

(c) *δ is a metric on $\mathcal{P}^n$.*

Proof. (a) is an immediate consequence of Proposition 3. To prove (b) select $K_\nu \in \Pi_\nu$ so that $\bar{w}(K_1) = \bar{w}(K_2)$. Then it follows from (a) and the definition of the modified support function that

$$\begin{aligned}\sqrt{\sigma_n}\delta(\Pi_1, \Pi_2) &= \sqrt{\sigma_n} d(K_1, K_2) = \|h_{K_1} - h_{K_2}\| \\ &= \|(h_{K_1} - \bar{w}(K_1)) - (h_{K_2} - \bar{w}(K_2))\| = \|\tilde{h}_{\Pi_1} - \tilde{h}_{\Pi_2}\|.\end{aligned}$$

For the proof of (c) we first note that evidently $\delta(\Pi_1, \Pi_2) = \delta(\Pi_2, \Pi_1)$. Furthermore, if $\delta(\Pi_1, \Pi_2) = 0$ and if one selects $K_\nu \in \Pi_\nu$ so that $\bar{w}(K_1) = \bar{w}(K_2)$ then, according to (a), $d(K_1, K_2) = \delta(\Pi_1, \Pi_2) = 0$ and therefore $K_1 = K_2$, which implies $\Pi_1 = \Pi_2$. Finally, to prove the triangle inequality assume that three parallel classes Π_μ ($\mu = 1, 2, 3$) are given and that corresponding convex bodies $K_\mu \in \Pi_\mu$ are selected so that $\bar{w}(K_1) = \bar{w}(K_2) = \bar{w}(K_3)$; then

$$d(K_1, K_3) \le d(K_1, K_2) + d(K_2, K_3)$$

and, again using part (a), one obtains

$$\delta(\Pi_1, \Pi_3) \le \delta(\Pi_1, \Pi_2) + \delta(\Pi_2, \Pi_3). \qquad \square$$

§4. *Translations and the Steiner point of parallel classes.* If $p \in \mathbf{R^n}$ it is obvious that $K_1 \approx K_2$ implies $K_1 + p \approx K_2 + p$. Hence, if $\Pi \in \mathcal{P}^n$ one may define $\Pi + p$ as the parallel class containing the bodies $K + p$ where $K \in \Pi$.

The Steiner point $s(K)$ of a convex body is usually defined by

$$s(K) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_K(u) u \, d\sigma(u),$$

where the integration is to be performed over each component of $h_K(u)u$ separately and κ_n denotes the volume of B^n. Since evidently $s(K_1 + K_2) = s(K_1) + s(K_2)$ and $S(B^n) = o$ it follows that the Steiner point is an invariant function mapping of $\mathcal{K}^n$ onto $\mathbf{R}^n$. Consequently it can be viewed as a class function, and for any $\Pi \in \mathcal{P}^n$ we may denote it by $s(\Pi)$. Since $\int_{S^n-1} u \, d\sigma(u) = o$ the Steiner point of a parallel class can evidently be expressed in terms of the

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modified support function in the form

$$s(\Pi) = \frac{1}{\kappa_n} \int_{S^{n-1}} \tilde{h}_\Pi(u) u \, d\sigma(u).$$

The Steiner point has a remarkable minimal property with respect to translations of parallel classes that is formulated in the following proposition.

PROPOSITION 5. *If Π_1^o, Π_2^o are two parallel classes having the same Steiner point, and if Π_1, Π_2 are respective translates of Π_1^o, Π_2^o then*

$$\delta(\Pi_1, \Pi_2)^2 = \delta(\Pi_1^o, \Pi_2^o)^2 + \frac{1}{n}|s(\Pi_1) - s(\Pi_2)|^2.$$

Hence, the minimum of $\delta(\Pi_1, \Pi_2)$, considered as a function of Π_1 and Π_2, is $\delta(\Pi_1^o, \Pi_2^o)$, and it is attained exactly if the respective Steiner points of Π_1 and Π_2 coincide.

Proof. First observe that for $u = (u_1, \ldots, u_n) \in S^{n-1}$ and any fixed $p \in \mathbf{R}^n$ one obtains that $\|p \cdot u\|^2 = \kappa_n |p|^2$, $\tilde{h}_{P+p}(u) = \tilde{h}_P + p \cdot u$, and, according to the above definition of the Steiner point, $\langle (\tilde{h}_{\Pi_1^o}(u) - \tilde{h}_{\Pi_2^o}(u)), u_k \rangle = 0$ (for $k = 1, \ldots, n$). Using this we find

$$\begin{aligned}
\sigma_n \delta(\Pi_1, \Pi_2)^2 &= \|\tilde{h}_{\Pi_1^o}(u) - \tilde{h}_{\Pi_2^o}(u) + (s(\Pi_1) - s(\Pi_2)) \cdot u\|^2 \\
&= \|\tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o}\|^2 + 2\langle (\tilde{h}_{\Pi_1^o}(u) - \tilde{h}_{\Pi_2^o}(u)), (s(\Pi_1) - s(\Pi_2)) \cdot u \rangle \\
&\quad + \|(s(\Pi_1) - s(\Pi_2)) \cdot u\|^2 \\
&= \|\tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o}\|^2 + \kappa_n |s(\Pi_1) - s(\Pi_2)|^2,
\end{aligned}$$

which contains the desired conclusion. □

§5. *A distance concept for normal bundles.* Corresponding to the distance for parallel classes introduced in §3, we define here a distance concept for normal bundles. Moreover, we prove a proposition concerning the behavior of this distance with respect to translations and the Steiner point.

Let $\mathbf{N}_1$, $\mathbf{N}_2$ be the respective normal bundles of two convex bodies K_1, K_2. Both of these normal bundles are defined on the set $U = \mathcal{R}(K_1) \cap \mathcal{R}(K_2)$ that differs from S^{n-1} only by a null set. If $u \in U$ and $N_\nu(u)$ denotes the normal from $\mathbf{N}_\nu$ of direction u let $\rho(N_1, N_2; u)$ denote the (minimal) distance between $N_1(u)$ and $N_2(u)$. Since $\rho(N_1, N_2; u)$ is for every fixed $u \in U$ a class function we may also denote it by $\rho(\mathbf{N}_1, \mathbf{N}_2; u)$. We now define the distance between the normal bundles $\mathbf{N}_1$ and $\mathbf{N}_2$ by

$$\Delta(\mathbf{N}_1, \mathbf{N}_2) = \frac{1}{\sqrt{\sigma_n}} \|\rho(\mathbf{N}_1, \mathbf{N}_2; u)\|. \tag{1}$$

Note that $\rho(\mathbf{N}_1, \mathbf{N}_2; u)$ is a square integrable function of u since it is a continuous function on U, due to the fact that the support points of K_ν belonging to regular directions depend continuously on u (see [1, §16]).

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As the following proposition shows, the distance $\Delta(\mathbf{N}_1, \mathbf{N}_2)$ has a noteworthy property with respect to translations of Π_ν that is similar to that regarding δ formulated in Proposition 5.

PROPOSITION 6. *Let Π_1^o and Π_2^o be two parallel classes having the same Steiner point, and let Π_1 and Π_2 be respective translates of Π_1^o and Π_2^o. If $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_1^o$, $\mathbf{N}_2^o$ are the corresponding normal bundles then*

$$\Delta(\mathbf{N}_1, \mathbf{N}_2)^2 = \Delta(\mathbf{N}_1^o, \mathbf{N}_2^o)^2 + \frac{n-1}{n}|s(\Pi_1) - s(\Pi_2)|^2.$$

Hence, the minimum of $\Delta(\mathbf{N}_1, \mathbf{N}_2)$, considered as a function of Π_1 and Π_2, is $\Delta(\mathbf{N}_1^o, \mathbf{N}_2^o)$, and it is attained exactly if the respective Steiner points of Π_1 and Π_2 coincide.

Proof. The proof requires some preliminary remarks that will also be useful for the proof of the theorem in the next section.

First we note the following known fact (see the proof of the relation (13) in [3]). If grad F signifies the gradient of F of a real-valued function on S^{n-1}, i.e., the restriction to S^{n-1} of the usual gradient on $\mathbf{R}^n \setminus \{o\}$ of the constant radial extension of F, then grad h_{K_ν} exists almost everywhere and is the vector from the orthogonal projection of o onto the support plane $H(u)$ to the support point of K belonging to the direction u. Consequently, we have

$$\rho(N_1, N_2; u) = |\mathrm{grad}(h_{K_1}(u) - h_{K_2}(u))| = |\mathrm{grad}(\tilde{h}_{\Pi_1}(u) - \tilde{h}_{\Pi_2}(u))|$$

and in view of equation (1) one obtains for the corresponding normal bundles that

$$\Delta(\mathbf{N}_1, \mathbf{N}_2) = \frac{1}{\sqrt{\sigma_n}}\|\mathrm{grad}(\tilde{h}_{\Pi_1} - \tilde{h}_{\Pi_2})\|. \tag{2}$$

A convex body will be said to be *smooth* if it is strictly convex and has a twice continuously differentiable support function. Smoothness is obviously an invariant property, and we may therefore talk about smooth parallel classes. If K_1 and K_2 are convex bodies then well-known approximation theorems show that there exist sequences K_1^j, K_2^j $(j = 1, 2, \ldots)$ of smooth convex bodies that converge (in the Hausdorff or L_2-metric) to, respectively, K_1 and K_2 (see [1, p. 36]). Moreover, it is easily shown that for such sequences and any $u \in \mathcal{R}(K_1) \cap \mathcal{R}(K_2)$ the support points of K_1^j and K_2^j belonging to this direction converge to the corresponding support points of K_1 and K_2 (cf. the proof in [2, Lemma 2.3.4]). It follows that the sequence $\rho(\mathbf{N}(K_1^j), \mathbf{N}(K_2^j); u)$ converges to $\rho(\mathbf{N}_1, \mathbf{N}_2; u)$. Because of the "bounded convergence theorem" this implies that $\Delta(\mathbf{N}(K_1^j), \mathbf{N}(K_2^j))$ converges to $\Delta(\mathbf{N}_1, \mathbf{N}_2)$. Hence, it suffices to prove the above proposition for smooth parallel classes. In this case two statements are known to hold: if

$$h_{K_\nu}(u) \sim \sum_{m=0}^{\infty} \Phi_{m,\nu}(u) \tag{3}$$

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is the expansion of h_{K_ν} in terms of spherical harmonics, where $\Phi_{m,\nu}$ has order m, then $\Phi_{0,\nu} = \frac{1}{2}\bar{w}(K_\nu)$, $\Phi_{1,\nu} = s(K_\nu)\cdot u$; and

$$\|\mathrm{grad}(h_{K_1} - h_{K_2})\|^2 = \sum_{m=1}^{\infty} m(n+m-2)\|\Phi_{m,1} - \Phi_{m,2}\|^2 \tag{4}$$

where

$$\|\Phi_{1,1} - \Phi_{1,2}\|^2 = \kappa_n |s(K_1) - s(K_2)|^2. \tag{5}$$

See [2, p. 75] for proofs and more details.

In terms of the corresponding class functions, we obtain from equations (2), (4) and (5) that

$$\begin{aligned}\sigma_n \Delta(\mathbf{N}_1, \mathbf{N}_2)^2 &= \|\mathrm{grad}(\tilde{h}_{\Pi_1} - \tilde{h}_{\Pi_2})\|^2 \\ &= \sum_{m=1}^{\infty} m(n+m-2)\|\Phi_{m,1} - \Phi_{m,2}\|^2 \end{aligned} \tag{6}$$

and

$$\begin{aligned}\sigma_n \Delta(\mathbf{N}_1^o, \mathbf{N}_2^o)^2 &= \|\mathrm{grad}(\tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o})\|^2 \\ &= \sum_{m=2}^{\infty} m(n+m-2)\|\Phi_{m,1} - \Phi_{m,2}\|^2. \end{aligned} \tag{7}$$

Proposition 6 is now an immediate consequence of equations (5), (6), and (7). □

§6. *The determination of parallel classes by their normal bundles.* This section contains our main result, which is essentially the converse of Proposition 1. Actually it not only answers the question of whether the equality $\mathbf{N}(\Pi_1) = \mathbf{N}(\Pi_2)$ implies that $\Pi_1 = \Pi_2$, but it also contains sharp inequalities that provide a solution to the associated stability problem of this relationship. In other words, it contains estimates of the mutual distance between Π_1 and Π_2 (or between Π_1^o and Π_2^o) if the deviation of the corresponding normal bundles $\mathbf{N}(\Pi_1)$ and $\mathbf{N}(\Pi_2)$ (or $\mathbf{N}(\Pi_1^o)$ and $\mathbf{N}(\Pi_2^o)$) is given. This result is formulated in the following theorem.

THEOREM. *Let Π_1 and Π_2 be two parallel classes of $\mathcal{K}^n$. If Π_1^o, Π_2^o are the respective translates with coinciding Steiner points, and if $\mathbf{N}_1$, $\mathbf{N}_2$ are the corresponding normal bundles of Π_1, Π_2, then*

$$(n-1)\delta(\Pi_1, \Pi_2)^2 + (n+1)\delta(\Pi_1^o, \Pi_2^o)^2 \le \Delta(\mathbf{N}_1, \mathbf{N}_2)^2. \tag{8}$$

In particular, this shows that

$$\delta(\Pi_1, \Pi_2) \le \frac{1}{\sqrt{n-1}}\Delta(\mathbf{N}_1, \mathbf{N}_2) \tag{9}$$

and

$$\delta(\Pi_1^o, \Pi_2^o) \le \frac{1}{\sqrt{2n}}\Delta(\mathbf{N}_1, \mathbf{N}_2). \tag{10}$$

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Equality holds in (8) and (10) if and only if the modified support functions $\tilde{h}_{\Pi_1^o}$ *and* $\tilde{h}_{\Pi_2^o}$ *differ only by a spherical harmonic of order two, and in (9) exactly if* $\Pi_1^o = \Pi_2^o$.

Proof. We only have to show inequality (8) and the statements about the occurrence of equality in (8), (9), and (10).

Let $K_\nu \in \Pi_\nu$, and let the sequence K_ν^j $(j = 1, 2, \ldots)$ be defined as in the proof of Proposition 6. If Π_ν^j is the parallel class containing K_ν^j then the relation (3) can be written in the form

$$\tilde{h}_{\Pi_\nu^j} \sim \sum_{m=1}^{\infty} \Phi_{m,\nu}^j.$$

Taking into account the relation corresponding to equation (4) we obtain

$$\|\operatorname{grad}(\tilde{h}_{\Pi_1^j} - \tilde{h}_{\Pi_2^j})\|^2 = \sum_{m=1}^{\infty} m(n+m-2)\|\Phi_{m,1}^j - \Phi_{m,2}^j\|^2.$$

Thus, observing equation (2) and writing $\mathbf{N}(K_\nu^j) = \mathbf{N}_\nu^j$ we find that for every positive integer k

$$\sigma_n \Delta(\mathbf{N}_1^j, \mathbf{N}_2^j)^2 \geq \sum_{m=1}^{k} m(n+m-2)\|\Phi_{m,1}^j - \Phi_{m,2}^j\|^2.$$

Noting that for fixed m and ν we have $\lim_{j\to\infty} \Phi_{m,\nu}^j(u) = \Phi_{m,\nu}(u)$ (cf. [**2**, p. 128]) and first letting $j \to \infty$ and then $k \to \infty$ we deduce that

$$\sigma_n \Delta(\mathbf{N}_1, \mathbf{N_2})^2 \geq \sum_{m=1}^{\infty} m(n+m-2)\|\Phi_{m,1} - \Phi_{m,2}\|^2. \tag{11}$$

Applied to $\mathbf{N}_\nu^o = \mathbf{N}(\Pi_\nu^o)$ this shows that

$$\sigma_n \Delta(\mathbf{N}_1^o, \mathbf{N}_2^o)^2 \geq \sum_{m=2}^{\infty} m(n+m-2)\|\Phi_{m,1} - \Phi_{m,2}\|^2. \tag{12}$$

Since part (b) of Proposition 4 together with Parseval's equation implies that

$$\sigma_n \delta(\Pi_1, \Pi_2)^2 = \sum_{m=1}^{\infty} \|\Phi_{m,1} - \Phi_{m,2}\|^2$$

and

$$\sigma_n \delta(\Pi_1^o, \Pi_2^o)^2 = \sum_{m=2}^{\infty} \|\Phi_{m,1} - \Phi_{m,2}\|^2$$

one obtains

$$\begin{aligned}\sigma_n \Delta(\mathbf{N}_1, \mathbf{N}_2)^2 &\geq (n-1)\|\Phi_{1,1} - \Phi_{1,2}\|^2 + 2n \sum_{m=2}^{\infty} \|\Phi_{m,1} - \Phi_{m,2}\|^2 \\ &= (n-1) \sum_{m=1}^{\infty} \|\Phi_{m,1} - \Phi_{m,2}\|^2 + (n+1) \sum_{m=2}^{\infty} \|\Phi_{m,1} - \Phi_{m,2}\|^2\end{aligned}$$

and inequality (8) follows.

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If equality holds in (8) this proof shows that it is necessary that $\|\Phi_{m,1} - \Phi_{m,2}\| = 0$ whenever $m \geq 3$. Hence in this case

$$\tilde{h}_{\Pi_1} - \tilde{h}_{\Pi_2} = (\Phi_{1,1} - \Phi_{1,2}) + (\Phi_{2,1} - \Phi_{2,2}),$$

and since

$$(\tilde{h}_{\Pi_1} - \tilde{h}_{\Pi_2}) - (\Phi_{1,1} - \Phi_{1,2}) = \tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o}$$

it follows that $\tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o}$ must be a spherical harmonic of order two. If equality holds in (9) it follows from (8) that $\delta(\Pi_1^o, \Pi_2^o) = 0$ and therefore $\Pi_1^o = \Pi_2^o$. Finally, if equality holds in (10) then equality must hold in (8) for $\Pi_\nu = \Pi_\nu^o$ and it follows as before that $\tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o}$ must be a spherical harmonic of order two.

Observing that in the case when $\tilde{h}_{\Pi_1^o} - \tilde{h}_{\Pi_2^o}$ is a spherical harmonic of order two equality holds in (11) and (12) and therefore in (8), one easily checks that in all cases equality does hold if the listed necessary conditions are satisfied. □

We now formulate three corollaries that are immediate consequences of the above theorem. Inequality (9) shows that $\Delta(\Pi_1, \Pi_2) = 0$ implies that $\delta(\Pi_1, \Pi_2) = 0$. Hence, in combination with Propositions 1 and 4, we immediately obtain the following result.

COROLLARY 1. *Two convex bodies in* $\mathbf{R}^n$ *have the same normal bundle if and only if one of the bodies is a parallel body of the other.*

Next let us consider the case in the above theorem when $K \in \mathcal{K}^n$, Π_1 is the parallel class that contains K, and Π_2 is the class containing a single point p. (We view p both as a point and as a convex body.) In this case the parallel class $\Pi_2 = \Pi(p)$ consists of all balls centered at p and $\mathbf{N}(p)$ consist of all lines passing through p. Hence, if B_p denotes the n-dimensional ball centered at p of radius $\bar{w}(K)/2$ then, according to Proposition 1, $\delta(\Pi_1, \Pi_2) = d(K, B_p)$ and $\delta(\Pi_1^o, \Pi_2^o) = d(K, B_{s(K)})$, and our theorem yields the following corollary.

COROLLARY 2. *If* $p \in \mathbf{R}^n$ *and* B_p *is defined as above then*

$$(n-1)d(K, B_p)^2 + (n+1)d(K, B_{s(K)})^2 \leq \Delta(\mathbf{N}(K), \mathbf{N}(p))^2.$$

In particular, if all normals of K pass through a given point p then K must be the ball B_p.

As before, this inequality implies that

$$d(K, B_p) \leq \frac{1}{\sqrt{n-1}} \Delta(\mathbf{N}(K), \mathbf{N}(p))$$

and

$$d(K, B_{s(K)}) \leq \frac{1}{\sqrt{2n}} \Delta(\mathbf{N}(\mathbf{K}), \mathbf{N}s(K)).$$

Inequalities of this type have already been proved in [**3**], where also some consequences regarding the equilibrium of a physical body on a horizontal surface are discussed.

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Although the relation $\mathbf{N}(K_1) = \mathbf{N}(K_2)$ does not in general imply that $K_1 = K_2$, one may deduce this equality if some additional conditions are satisfied. The following corollary deals with two such cases. A boundary point of a convex body K is said to be *singular* if it is contained in more than one support plane.

COROLLARY 3. *Assume that* $\mathbf{N}(K_1) = \mathbf{N}(K_2)$. *If, in addition,* $\bar{w}(K_1) = \bar{w}(K_2)$ *or if each of the boundaries of* K_1 *and* K_2 *contains at least one singular point then* $K_1 = K_2$.

Indeed, $\mathbf{N}(K_1) = \mathbf{N}(K_2)$ implies that $K_1 \approx K_2$, and from this together with $\bar{w}(K_1) = \bar{w}(K_2)$ and Proposition 2 it follows that $K_1 = K_2$; if the boundaries of K_1 and K_2 have singular points then none of these bodies can be a parallel body of the other and one obtains again that $K_1 = K_2$.

Since every polytope, every (bounded) cone or cylinder, and every convex body in $\mathbf{R}^n$ of dimension less than n has singular points, one obtains that within such classes the bodies are uniquely determined by their normal bundles.

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