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Billiards in polygons and homogeneous foliations on $\mathbf{C}^2$

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Abstract. In this article we construct a new framework for the study of polygonal billiards. To every polygonal billiard we associate a holomorphic foliation on $\mathbf{C}^2$. The dynamics of the billiard ball is linked to the directional flow of the complex vector field defining the associated holomorphic foliation.

1. *Introduction*

For simplicity the main results and ideas in this article are stated, proved and developed in the context of triangular billiards. The general case is given at the end in §5.

Let P be a triangle having angles $\lambda_1\pi$, $\lambda_2\pi$ and $\lambda_3\pi$. We avoid degenerated cases, so $0 < \lambda_j < 1$, for all j and $\sum_{j=1}^{3} \lambda_j = 1$. A frictionless point inside P describes a *billiard trajectory*, as time runs from $-\infty$ to ∞, if it moves with constant velocity in the interior of P, reflects off edges so that speed is unchanged and the angle of incidence is equal to the angle of reflection. Motion stops if the point hits the vertices of P, for reflection is not well-defined. A unique translation surface $S = S(P)$ corresponds to such a dynamical system. It is generated by P through a standard unfolding process [**15**]. A *translation surface*, is a real surface whose transition functions are translations. In this article the surface $S(P)$ is smooth and not closed, for it is supposed to be generated by a triangle with no vertices. Unless otherwise stated, *P is henceforth supposed to be with no vertices*. The billiard dynamics in P is equivalent to the geodesic flow on the unitary tangent bundle of $S(P)$. This surface has trivial holonomy, therefore one can consider, without ambiguity, the set of geodesics parallel to a given direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. This set defines a real foliation on $S(P)$, usually called *the geodesic foliation on $S(P)$ defined by the direction θ*. We denote it by D_θ.

The aim of this article is to describe the set of foliations D_θ in terms of a special class of holomorphic homogeneous foliations on $\mathbf{C}^2$ and their projections to $\mathbf{RP}(3)$.

Let $\mathcal{F}_\lambda$ be the foliation on $\mathbf{C}^2$ defined by the holomorphic 1-form ω_λ,

$$\frac{\omega_\lambda}{z_1 z_2 (z_2 - z_1)} = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \lambda_3 \frac{d(z_2 - z_1)}{z_2 - z_1}, \tag{1}$$

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where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. This foliation is homogeneous, i.e. invariant under the natural action of the homothetic transformation group $\{T_k(z_1, z_2) := k(z_1, z_2) \mid k \in \mathbf{C}^*\}$. The foliation $\mathcal{F}_\lambda$ presents an isolated singularity at the origin. It leaves three complex lines invariant through the origin. This set of complex lines is commonly called the *tangent cone* of the foliation; it is described by the equation $z_1 z_2 (z_2 - z_1) = 0$. In the complement of the tangent cone any two leaves are diffeomorphic, for there always exists a homothecy whose restriction sends one leaf into the other. Every leaf of $\mathcal{F}_\lambda$ in the complement of the tangent cone will be called a *generic leaf* and denoted $\mathcal{L}$. Up to diffeomorphism, $\mathcal{F}_\lambda$ presents three kinds of leaves: a point, $\mathbf{C}^*$ and a generic leaf $\mathcal{L}$.

THEOREM 1. *If the submodule* $\mathrm{Res}(\lambda) := \{(n_1, n_2, n_3) \in \mathbf{Z}^3 \mid \sum_j n_j \lambda_j = 0\}$ *of* $\mathbf{Z}^3$ *is equal to zero, then the generic leaf of* $\mathcal{F}_\lambda$ *is homeomorphic to a plane with a countable set of handles. This topological surface is also known as the* Loch Ness *monster, see Figure 1.*

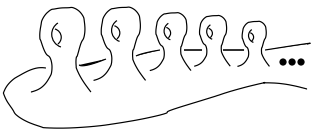


FIGURE 1. Loch Ness monster.

This nomenclature can be found in [**6**]. When the angles of the triangle are rational multiples of π, the generic leaf of $\mathcal{F}_\lambda$ is homeomorphic to $S \setminus E$, where S is a finite genus orientable surface and $E \subseteq S$ is a finite set of points [**10**].

From (1), we deduce that the integral curves of the holomorphic vector field

$$X_\lambda := z_1(\lambda_3 z_2 + \lambda_2(z_2 - z_1))\partial/\partial z_1 + z_2(\lambda_3 z_1 + \lambda_1(z_1 - z_2))\partial/\partial z_2 \tag{2}$$

are the leaves of $\mathcal{F}_\lambda$. Let $\mathcal{F}_{\lambda,0}$ be the real foliation on $\mathbf{C}^2$ defined by the integral curves of the real analytic vector field $\mathrm{Re}(X_\lambda)$. This vector field is quadratic and homogeneous. The foliation $\mathcal{F}_{\lambda,0}$ has an isolated singularity at the origin and, for every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$, the restriction $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is a real foliation with no singularities. We describe this foliation in terms of the billiard game inside the triangle P.

The vector field X_λ induces a natural translation surface structure on every generic leaf of $\mathcal{F}_\lambda$. Charts are given by the local solutions $\varphi : \mathbf{C} \ni t \longrightarrow \mathcal{L}$ to the equation $\varphi^* X_\lambda = \partial/\partial t$. Our main result states the following.

THEOREM 2. *Every generic leaf of* $\mathcal{F}_\lambda$ *is isomorphic, as translation surface, to the surface* $S(P)$.

Let $\rho \in \mathbf{R}^*$ and $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. We denote by $\rho e^{2\pi i}\mathcal{L}$ the image of the restriction to $\mathcal{L}$ of the homothecy $T_{\rho e^{2\pi i}}$. From the preceding theorem we deduce the following.

COROLLARY 1. *For every generic leaf* $\mathcal{L} \in \mathcal{F}_\lambda$ *there exists a direction* $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ *such that the foliation* $\mathcal{F}_{\lambda,0|\mathcal{L}}$ *is holomorphically conjugated to the geodesic foliation defined by the direction* θ *on the translation surface* $S(P)$. *Moreover, if* $\mathcal{F}_{\lambda,0|\mathcal{L}}$ *is holomorphically*

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conjugated to the geodesic foliation D_θ *on* $S(P)$*, then, for every* $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$ *and* $\rho \in \mathbf{R}^*$*, the foliation* $\mathcal{F}_{\lambda|\rho e^{i\theta'}\mathcal{L}}$ *is holomorphically conjugated to the geodesic foliation* $D_{\theta+\theta'}$ *on the translation surface* $S(P)$.

In particular, we obtain results concerning the existence of periodic orbits.

THEOREM 3. *Let* $\lambda \in \mathbf{Q}^3$*, then* periodic orbits *in* $\mathcal{F}_{\lambda,0}$ *are dense in* $\mathbf{C}^2$.

As for billiards, the main application of the dictionary contained in this article is the topological classification of the translation surfaces generated by non-rational 'generic' triangles *with no vertices*.

1.1. *Billiards on* $\mathbf{RP}(3)$. Using the canonical projection $\Pi_{\mathbf{RP}(3)} : \mathbf{C}^2 \setminus 0 \longrightarrow \mathbf{RP}(3)$, we associate with the holomorphic foliations $\mathcal{F}_\lambda$ and $\mathcal{F}_{\lambda,0}$ singular foliations on the projective space $\mathbf{RP}(3)$, $\mathcal{G}_\lambda$ and $\mathcal{G}_{\lambda,0}$, with respective real codimension one and two. Any two leaves $\mathcal{L}, \mathcal{L}' \in \mathcal{G}_\lambda$ in the complement of the singular locus of $\mathcal{G}_\lambda$ are diffeomorphic. As with $\mathcal{F}_\lambda$, such leaves will be called *generic* leaves. Under additional hypothesis for λ, the generic leaves of $\mathcal{G}_\lambda$ and $\mathcal{F}_\lambda$ are diffeomorphic. In such a case we will say that λ is *not strongly resonant*. The diffeomorphism is achieved by the canonical projection $\Pi_{\mathbf{RP}(3)}$. In this context, for every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$, the projection $\Pi_{\mathbf{RP}(3)}$ conjugates the foliation $\mathcal{F}_{\lambda,0|\mathcal{L}}$ to the restriction of $\mathcal{G}_{\lambda,0}$ to $\Pi_{\mathbf{RP}(3)}(\mathcal{L})$. For strong resonant parameters λ, the projection $\Pi_{\mathbf{RP}(3)}$,when restricted to a generic leaf of $\mathcal{F}_\lambda$, defines a two-sheeted covering over its image that semiconjugates $\mathcal{F}_{\lambda,0|\mathcal{L}}$ to the restriction of $\mathcal{G}_{\lambda,0}$ to $\Pi_{\mathbf{RP}(3)}(\mathcal{L})$.

Every generic leaf in $\mathcal{G}_\lambda$ inherits a translation surface structure from the corresponding generic leaf on $\mathcal{F}_\lambda$.

COROLLARY 2. *Every generic leaf of* $\mathcal{G}_\lambda$ *is isomorphic, as translation surface, to the surface* $S(P)$.

This result is just a projective version of Theorem 2. In particular, for every generic leaf $\mathcal{L} \in \mathcal{G}_\lambda$ there exist a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ such that the foliation $\mathcal{G}_{\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction θ on the translation surface $S(P)$. For every leaf $\Pi_{\mathbf{RP}(3)}(\mathcal{L}') = \mathcal{L} \in \mathcal{G}_\lambda$, $\mathcal{L}' \in \mathcal{F}_\lambda$ generic, we denote $\rho e^{i\theta}\mathcal{L} := \Pi_{\mathbf{RP}(3)}(\rho e^{i\theta}\mathcal{L}')$. We deduce as well that if $\mathcal{G}_{\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction θ on $S(P)$, then for every $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$, $\mathcal{G}_{\lambda,0|e^{i\theta'}\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta + \theta'$ on $S(P)$.

We recall that the phase space of the billiard is naturally identified with $P \times \mathbf{R}/2\pi\mathbf{Z}$ [**1, 14**]. From [**9**], we know that the billiard flow in the phase space of a typical polygon is ergodic. Using this fact we deduce the following.

THEOREM 4. *There exist a dense* G_δ *set in the space of triangles for which almost every (Lebesgue) point in* $\mathbf{RP}(3)$ *is contained in a leaf of* $\mathcal{G}_{\lambda,0}$ *which is dense in* $\mathbf{RP}(3)$.

We begin this paper with a general survey of quadratic homogeneous holomorphic foliations of $\mathbf{C}^2$ having an isolated singularity and a reduced tangent cone. In §2.1 we prove Theorem 1. In §3, Theorem 2 is proved and later, in §4, we show the main

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applications of this result to polygonal billiards and homogeneous foliations, including the proof Theorem 4. Finally, in §5, we show that the main results obtained for triangles and quadratic homogeneous foliations remain valid for general polygons.

2. *Homogeneous foliations on* $\mathbf{C}^2$

Homogeneous holomorphic foliations on $\mathbf{C}^2$ constitute the framework which we use for studying polygonal billiards. We will consider holomorphic homogeneous foliations on $\mathbf{C}^2$ having an isolated singularity and given by a Pffafian form $\omega = A_1\, dz_1 + A_2\, dz_2$, where $A_j \in \mathbf{C}[z_1, z_2]$ are homogeneous polynomials having the same degree and such that $\gcd(A_1, A_2) = 1$. The holomorphic foliation defined by ω presents only $\nu + 1$ invariant complex lines through the singularity, if and only if $z_1A_1 + z_2A_2$ is an homogeneous irreducible polynomial of degree $\nu + 1$. The zero locus of this polynomial is called the *tangent cone* of the foliation and the natural number ν is the *degree* of the foliation. As a consequence of the decomposition of rational fractions in one variable, ω can be written in the same form as (1) except that all $\lambda_j \in \mathbf{C}^*$, $j = 1, 2, 3$, and $\sum_{j=1}^{3} \lambda_j$ equals a given normalization constant $K \in \mathbf{Z}$ [**3**]. Thus, $\mathcal{F}_\lambda$ is a special type of quadratic homogeneous foliation: the one for which $\{\lambda_j \pi\}_{j=1}^{3}$ are the angles of a non-degenerated triangle P and the normalization constant is equal to one. We observe that

$$F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2} (z_2 - z_1)^{\lambda_3} \tag{3}$$

is a first integral for $\mathcal{F}_\lambda$. This expression is usually multivalued. Let $\sim_\lambda$ be the equivalence relation induced by the action of $(\mathbf{Z}^3, +)$ on $\mathbf{C}^*$, defined by $\mathbf{Z}^3 \times \mathbf{C} \ni ((n_1, n_2, n_3), z) \rightsquigarrow e^{2\pi i \sum_j n_j \lambda_j} z$. As proved in [**11**], the normalization $\sum_{j=1}^{3} \lambda_j = 1$, implies that the fibers $F_\lambda^{-1}(k)$, $k \in \mathbf{C}^*/\sim_\lambda$ are the (connected) leaves of $\mathcal{F}_\lambda$ in the complement of the tangent cone. A straightforward computation shows that if $k, k' \in \mathbf{C}^*$, the homothecy $T_{k/k'}$ defines a diffeomorphism between $F_\lambda^{-1}(k)$ and $F_\lambda^{-1}(k')$. In particular, each leave in $\mathcal{F}_\lambda$ is invariant under every homothecy $T_{e^{2\pi i \sum_j n_j \lambda_j}}$. We will call every leaf of $\mathcal{F}_\lambda$ in the complement of the tangent cone a *generic leaf*.

2.1. *Topology of generic leaves.*

As studied in [**2**], the generic leaf of a generic holomorphic homogeneous foliation on $\mathbf{C}^2$ is dense in $\mathbf{C}^2$. This is not the case for the foliation $\mathcal{F}_\lambda$ associated to the billiard in the triangle.

Let $\Pi : \widetilde{\mathbf{CP}(2)} \longrightarrow \mathbf{CP}(2)$ be the complex surface obtained by blowing up the origin of the affine chart $\mathbf{C}^2 = \{(z_1, z_2)\}$ of $\mathbf{CP}(2)$. This surface is well known: it is Hirzebruch's first surface. The foliation $\mathcal{F}_\lambda$ extends to a foliation $\widetilde{\mathcal{F}_\lambda}$ on $\widetilde{\mathbf{CP}(2)}$. Let π be the projection of $\widetilde{\mathbf{CP}(2)}$ onto the exceptional divisor $\mathbf{CP}(1)$. In the complement of $\Pi^{-1}(z_1 z_2 (z_2 - z_1) = 0)$, the foliation $\widetilde{\mathcal{F}_\lambda}$ is transversal to the fibration defined by π. Using the first integral, F_λ, one can calculate explicitly the generators of the holonomy group of $\widetilde{\mathcal{F}_\lambda}$, relative to this fibration [**4**].

PROPOSITION 1. [**3**] *If* $\lambda_j = p_j/q \in \mathbf{Q}$, $\gcd(p_1, p_2, p_3, q) = 1$, *then the closure in* $\widetilde{\mathbf{CP}(2)}$ *of every generic leaf* $\mathcal{L} \in \widetilde{\mathcal{F}_\lambda}$ *is an algebraic curve defined by* $z_1^{p_1} z_2^{p_2} (z_2 - z_1)^{p_3} = k$, $k \in \mathbf{C}^*$. *On the other hand, if for some* $j = 1, 2, 3$ *the parameter* λ_j *is irrational, then*

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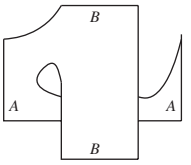


FIGURE 2. The domain U in $\mathcal{L}$.

the closure in $\widetilde{\mathbf{CP}(2)}$ *of every generic leaf* $\mathcal{L} \in \widetilde{\mathcal{F}_\lambda}$ *is a real Levi-flat singular manifold of dimension* 3 *given by* $\Pi^{-1}(|z_1|^{\lambda_1}|z_2|^{\lambda_2}|z_2 - z_1|^{\lambda_3} = k)$, $k \in \mathbf{R}^*$.

Definition 1. [**12**] A real surface S is of *infinite genus* if there is no bounded subset S' such that $S \setminus S'$ is of genus zero.

Proof of Theorem 1. We calculate the *ideal boundary* of the generic leaf $\mathcal{L}$. The ideal boundary of a real topological surface is a topological invariant [**12**]. The Loch Ness monster is, up to homeomorphism, the only infinite-genus orientable topological surface whose ideal boundary is just a point.

Let $\widetilde{U}$ be the complement in $\mathbf{CP}(1)$ of three infinitesimal neighborhoods, each one containing only one point in $\{0, 1, \infty\}$. We define U as the complement on $\widetilde{U}$ of two non-intersecting line segments $A =]\infty, 0[$ and $B =]1, \infty[$ contained in $\mathbf{CP}(1)$. We identify U with Figure 2.

Let γ_2, γ_3 be the generators of $\pi_1(\mathbf{CP}(1) \setminus \{0, 1, \infty\}, 1/2)$ given by the circles of radius $1/2$ and centers at 0 and 1, respectively. The monodromy action of the covering, $\pi_{|\mathcal{L}} : \mathcal{L} \longrightarrow E$, is given by the holonomy group of $\mathcal{F}_\lambda$ relative to the fibration defined by the projection π. This group is the image of the holonomy representation

$$\mathrm{hol} := \pi_1(\mathbf{CP}(1) \setminus \{0, 1, \infty\}, 1/2) \longrightarrow \mathrm{Diff}(\mathbf{C}, 0). \tag{4}$$

Using the first integral F_λ, we deduce that $\mathrm{hol}(\gamma_j) = z \rightsquigarrow e^{2\pi i \lambda_j} z$, $j = 2, 3$. Here, $z \in \mathbf{C}$ is the variable of the fiber $\pi^{-1}(t)$, with $t \in \mathbf{CP}(1) \setminus \{0, 1, \infty\}$. Let m be a word in these two generators. We denote by U_m the lifting of the simply connected domain U to the leaf beginning at a fixed point in the fiber $\pi^{-1}(1/2)$ and following the path given by the word m. Let $[,]$ be the commutator of two loops in the fundamental group. We remark that $U_{[\gamma_2,\gamma_3]}$ is homeomorphic to the open ring $1 < |z| < 2$. This implies that $U_{[\gamma_2,\gamma_3][\gamma_2^{-1},\gamma_3]}$ is homeomorphic to a torus to which we have removed a disc. The gluing process is shown in Figure 3, which depicts the handles contained in the generic leaf $\mathcal{L}$.

Since $\mathrm{Res}(\lambda)$ is equal to zero, the action of the free torsion group $\{\mathrm{hol}\gamma_j^n\}_{n\in\mathbf{Z}}$, $j = 2, 3$ fixed, on the fibers of the projection π defines a countable family of non-intersecting handles. Consider, for instance, $\{U_{([\gamma_2,\gamma_3][\gamma_2^{-1},\gamma_3])\gamma_j^{10n}}\}_{n\in\mathbf{Z}}$. This implies that the generic leaf $\mathcal{L}$ has infinite genus.

We affirm that, for every compact set $K' \subset \mathcal{L}$, there exists a compact set $K' \subset K \subset \mathcal{L}$ such that $\mathcal{L} \setminus K$ is connected. This implies that the ideal boundary of the generic leaf is only a point. Indeed, the base space of the covering π admits a retraction to the set of generators $\{\gamma_2, \gamma_3\}$. This retraction induces a retraction on the covering constructed from $\widetilde{U}$ onto a subspace homeomorphic to the Cayley graph of $\mathbf{Z} \times \mathbf{Z}$ that we denote Γ.

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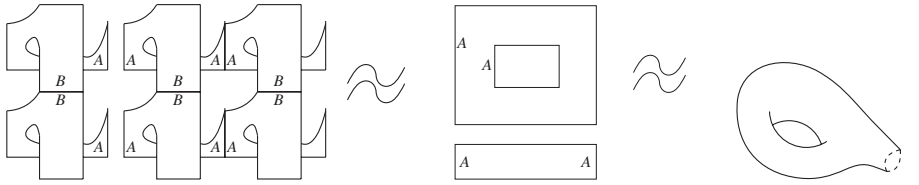


FIGURE 3. $U_{[\gamma_2,\gamma_3][\gamma_2^{-1},\gamma_3]}$ defines a handle in $\mathcal{L}$.

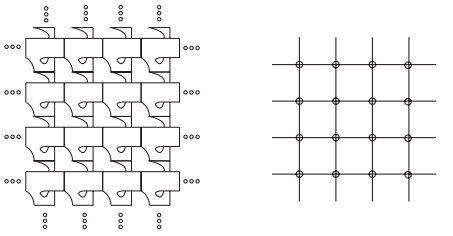


FIGURE 4. The leaf $\mathcal{L}$ has only one end.

We endow this graph with the topology induced by the retraction. Clearly, every compact set in Γ is contained in a compact set $K \subset \Gamma$ such that $\Gamma \setminus K$ is connected. The Figure 4 depicts the retraction of $\mathcal{L}$ onto the graph Γ. □

2.2. *Homogeneous foliations on* $\mathbf{RP}(3)$. Let $R_{\mathbf{R}}$ be the radial vector field on $\mathbf{R}^4$ and $z_j = x_j + \sqrt{-1}y_j$, for $j = 1, 2$. The real integrable homogeneous 1-form

$$i_{X_{R_{\mathbf{R}}}} i_{\mathrm{Re}(X_\lambda)} i_{\mathrm{Im}(X_\lambda)}\, dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \tag{5}$$

defines a real codimension-one singular foliation $\mathcal{G}_\lambda$ on the projective space $\mathbf{RP}(3)$. The projection of the real cones, $\{\rho\mathcal{L} \mid \rho \in \mathbf{R}^*, \mathcal{L} \in \mathcal{F}_\lambda \setminus 0\}$, to the projective space $\mathbf{RP}(3)$ are the leaves of $\mathcal{G}_\lambda$. A leaf of this foliation is called *generic* if and only if it originates from the projection of a real cone whose basis is a generic leaf of $\mathcal{F}_\lambda$. Every generic leaf $\Pi_{\mathbf{RP}(3)}(\mathcal{L})$ inherits a translation surface structure from $\mathcal{L}$. The singular locus of the foliation $\mathcal{G}_\lambda$, which we denote by $\mathrm{Sing}(\mathcal{G}_\lambda)$, is given by three projective real lines, $\mathbf{RP}(1)$, corresponding to the projection of the irreducible components of the tangent cone of $\mathcal{F}_\lambda$. Any two leaves of $\mathcal{G}_\lambda$ in $\mathbf{RP}(3) \setminus \mathrm{Sing}(\mathcal{G}_\lambda)$ are diffeomorphic. The homogeneous real vector field $\mathrm{Re}(X_\lambda)$ defines a singular codimension-two real foliation $\mathcal{G}_{\lambda,0}$ on $\mathbf{RP}(3)$. A direct calculation on $\mathrm{Re}(X_\lambda)$ restricted to each complex line in the tangent cone of $\mathcal{F}_\lambda$ shows that the singular locus of $\mathcal{G}_{\lambda,0}$ is given, in homogeneous coordinates $[x_1 : y_1 : x_2 : y_2]$, by the points $p_1 := [0 : 0 : 1 : 0]$, $p_2 := [1 : 0 : 0 : 0]$ and $p_3 := [1 : 0 : 1 : 0]$. Let C_j denote the projective real line in $\mathrm{Sing}(\mathcal{G}_\lambda)$ containing the point $p_j \in \mathrm{Sing}(\mathcal{G}_{\lambda,0})$. For every $\mathcal{L} \in \mathcal{G}_\lambda \setminus \mathrm{Sing}(\mathcal{G}_\lambda)$, the restriction $\mathcal{G}_{\lambda,0|\mathcal{L}}$ is a real foliation with no singularities.

Definition 2. A point $\lambda \in \mathbf{R}^3$ in the hyperplane $\sum \lambda_j = 1$ is *strongly resonant* if and only if there exists a non-trivial solution, $(n_1, n_2, n_3) \in \mathbf{Z}^3$, to the equation $\sum n_j\lambda_j \cong 1/2 (\mathrm{mod}\, \mathbf{Z})$.

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When λ is strongly resonant, the holonomy of the foliation $\widetilde{\mathcal{F}_\lambda}$ relative to the fibration defined by the projection π contains the involution $z \rightsquigarrow -z$. This implies that, for strongly resonant parameters, the restriction of the projection, $\Pi_{\mathbf{RP(3)}}$, to a generic leaf of $\mathcal{F}_\lambda$ defines a two-sheeted covering of its image.

LEMMA 1. *If λ is not strongly resonant, then, for every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$, the real foliations $\mathcal{F}_{\lambda,0|\mathcal{L}}$ and $\mathcal{G}_{\lambda,0|\Pi_{\mathbf{RP}(3)}(\mathcal{L})}$ are conjugated.*

Proof. If λ is a non strongly resonant point, then any real line through the origin on $\mathbf{C}^2$ intersects every generic leaf of $\mathcal{F}_\lambda$ in exactly one point. This implies that $\Pi_{\mathbf{RP}(3)|\mathcal{L}} := \mathcal{L} \longrightarrow \Pi_{\mathbf{RP}(3)}(\mathcal{L})$ is an isomorphism for the corresponding translation structures. □

PROPOSITION 2. *The point $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is not rational if and only if every generic leaf of $\mathcal{G}_\lambda$ is dense in* $\mathbf{RP}(3)$.

Proof. Let $\mathbf{RP}(1) \longrightarrow \mathbf{RP}(3) \longrightarrow \mathbf{S}^2$ be the fibration naturally defined by Hopf's fibration. We identify $\mathbf{S}^2$ with the exceptional divisor in $\widetilde{\mathbf{CP}(2)}$. Any generic leaf of $\mathcal{G}_{\lambda,0}$ is transversal to this fibration in the complement of the fibers over the points in $\mathbf{S}^2$ given by $t = 0$, 1 or ∞. The holonomy group of $\mathcal{G}_{\lambda,0}$ relative to this fibration is generated by the maps (in local coordinates) $\mathbf{RP}(1) \ni [1 : s] \rightsquigarrow [1 : s + 2\pi\lambda_j]$, $j = 1, 2, 3$. □

In the homogeneous coordinates of $\mathbf{RP}(3)$, the analytic multivalued function $J := F_\lambda/\overline{F_\lambda}$ is a first integral for the foliation $\mathcal{G}_\lambda$. In particular, the differential 1-form dJ/J defines this foliation. Poincaré's linearization theorem implies that, in a small neighbourhood of every point in the singular locus of $\mathcal{G}_\lambda$, this differential form is conjugated to $y\,dx - x\,dy$. This is just a manifestation of the Kupka phenomenon. In other words, the local model for the foliation, $\mathcal{G}_\lambda$, in a point of its singular locus is given by an *open book*.

In order to understand better the global geometry of this open book, let $\Pi_C : \widetilde{\mathbf{RP}(3)} \longrightarrow \mathbf{RP}(3)$ be the projection map from the algebraic variety $\widetilde{\mathbf{RP}(3)}$ that results from blowing up $\mathbf{RP}(3)$ at a singular locus of the foliation $\mathcal{G}_\lambda$. Each exceptional divisor $\mathbf{T}_j := \Pi_C^{-1}(C_j)$ is homeomorphic to a torus $\mathbf{RP}(1) \times \mathbf{RP}(1)$. The closed 1-form $\Pi_C^*\, dJ/J$ defines a real codimension-two non-singular foliation $\widetilde{\mathcal{G}_\lambda}$. Up to diffeomorphism, this foliation has only one leaf, which we also call *generic*; such a generic leaf is not homeomorphic to the generic leaf of the foliation $\mathcal{G}_\lambda$.

PROPOSITION 3. *The restriction of $\widetilde{\mathcal{G}_\lambda}$ to each torus $\mathbf{T}_j$ defines a* linear *foliation.*

Proof. In homogeneous coordinates, $\lambda_1 \arctan(y_1/x_1) + \lambda_2 \arctan(y_2/x_2) + \lambda_3 \arctan((y_2 - y_1)/(x_2 - x_1))$ is a first integral for the foliation $\mathcal{G}_\lambda$. Consider local coordinates $y_1 = tx_1$ and $y_2 = 1$. Then, the intersection of a generic leaf in $\widetilde{\mathcal{G}_{\lambda,0}}$ with the exceptional divisor $\mathbf{T}_2$ is given by

$$\lambda_1 \arctan(t) + \left(\sum_{j\geq 2} \lambda_j\right) \arctan\left(\frac{1}{x_2}\right) = k$$

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for a fixed $k \in \mathbf{R}^*$. The variables $\theta_1 = \arctan(t)$ and $\theta_2 = \arctan(1/x_2)$ parametrize the torus $\mathbf{T}_2$. The normalization, $\sum_j \lambda_j = 1$, implies that the slope of the lines in $\widetilde{\mathcal{G}_\lambda} \cap \mathbf{T}_j$ is given by $(1 - \lambda_j)/\lambda_j$. □

The real foliation $\mathcal{G}_{\lambda,0}$ defines a real singular foliation $\widetilde{\mathcal{G}_{\lambda,0}}$ on $\widetilde{\mathbf{RP}(3)}$. These two foliations are isomorphic in the complement of $\bigcup_{j=1}^3 \mathbf{T}_j$. The singular locus of $\widetilde{\mathcal{G}_{\lambda,0}}$ is $\{\Pi_C^{-1}(p_j)\}_{j=1}^3$. Each connected component of this singular locus is homeomorphic to a circle, $\mathbf{RP}(1)$, contained in an exceptional divisor, $\mathbf{T}_j$. From the preceding proposition, we conclude that the singular locus of $\widetilde{\mathcal{G}_{\lambda,0}}$, restricted to the torus, is formed by apparent singularities. Therefore $\widetilde{\mathcal{G}_{\lambda,0}} \cap \mathbf{T}_j$ is a line foliation whose leaves are (1) dense if and only if λ_j is irrational or (2) periodic if and only if λ_j is rational.

PROPOSITION 4. *In a small neighbourhood of each singular point $p_j \in \mathrm{Sing}(\mathcal{G}_{\lambda,0})$, the real foliation $\mathcal{G}_{\lambda,0}$ is analytically conjugated to the foliation defined by the integral curves of the linear vector field*

$$\lambda_j x \partial/\partial x - y\partial/\partial y - z\partial/\partial z. \tag{6}$$

Proof. In the coordinates $z_2 = tz_1$, of $\widetilde{\mathbf{CP}(2)}$, the foliation $\widetilde{\mathcal{F}_\lambda}$ is given by the integral curves of $z_1(\lambda_2 z_1 - (\lambda_2 + \lambda_3)tz_1\partial/\partial z_1 + t(t-1)\partial/\partial t)$. For $t \sim 0$, this vector field is analytically conjugated to $\widetilde{Y} := z_1(\lambda_2 z_1 \partial/\partial z_1 + t\partial/\partial t)$. This vector field is the pullback via $\Pi : \widetilde{\mathbf{CP}(2)} \longrightarrow \mathbf{CP}(2)$ of the holomorphic vector field defined, in the affine chart $\mathbf{C}^2 \hookrightarrow \mathbf{CP}(2)$, by $Y := \lambda_2 z_1^2 \partial/\partial z_1 + (1 + \lambda_2) z_1 z_2 \partial/\partial z_2$. Let $\mathrm{Re}(Y) = A_1\partial/\partial x_1 + B_1\partial/\partial x_1 + A_2\partial/\partial x_2 + B_2\partial/\partial y_2$. This real homogeneous vector field defines a real codimension-two singular foliation on $\mathbf{RP}(3)$. In the local coordinates $x_1 = 1$, this foliation is given by the integral curves of

$$\begin{aligned}\mathrm{Re}(Y) - A_1 R_{\mathbf{R}} = \lambda_2(y_1 + y_1^3)\partial/\partial y_1 - x_2 + y_1((1-\lambda_2)y_2 \\ + \lambda_2 y_1 x_2)\partial/\partial x_2 - y_2 + y_1((\lambda_2 - 1)x_2 + \lambda_2 y_2 y_1)\partial/\partial y_2.\end{aligned} \tag{7}$$

The real foliation defined by $\mathrm{Re}(Y) - A_1 R_{\mathbf{R}}$ is analytically conjugated to $\mathcal{G}_{\lambda,0}$ in a neighbourhood of $x_2 = y_2 = 0$; in particular, in a neighbourhood of the point p_2, corresponding to the origin $x_2 = y_2 = y_1 = 0$. □

From Proposition 4 we notice, for each singular point p_j, $j = 1, 2, 3$, the existence of an invariant, two-dimensional, real manifold given in local coordinates by $x = 0$ (see equation (6)). We denote this invariant manifold $\mathcal{V}_j$. The restriction of $\mathcal{G}_{\lambda,0}$ to this invariant manifold is conjugated to a *radial foliation*.

3. *Proof, main result*

The invariant complex lines through the origin in $\mathcal{F}_\lambda$ determine three points in the exceptional divisor in $\widetilde{\mathbf{CP}(2)}$. In local coordinates $z_2 = tz_1$, this points are given by $t = 0, 1, \infty$. Let E be the complement of the points in the exceptional divisor, then E is homeomorphic to the sphere with three punctures. Every generic leaf $\mathcal{L} \in \widetilde{\mathcal{F}_\lambda}$ defines a covering space

$$\pi_{|\mathcal{L}} : \mathcal{L} \longrightarrow E. \tag{8}$$

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Without loss of generality we suppose that the generic leaf $\mathcal{L}$ is given by $z_1(t) = t^{-\lambda_2}(t-1)^{-\lambda_3}$. A direct calculation shows that the projection of $X_{\lambda|\mathcal{L}}$ to the line $(0, t)$ is given by $z_1 t(t-1)\partial/\partial t$. Let

$$T : t \rightsquigarrow \int_0^t \xi^{\lambda_2 - 1}(\xi - 1)^{\lambda_3 - 1}\, d\xi. \tag{9}$$

Then $(z_1(t), t) \longrightarrow T$ defines a chart for the translation surface structure on $\mathcal{L}$, for it locally rectifies the vector field $X_{\lambda|\mathcal{L}}$. The integral in (9) defines a Schwarz–Christoffel transformation for P, that is, a conformal bijective map sending $\Im(t) > 0$ onto the interior of the triangle of angles $\lambda_j \pi$, $j = 1, 2, 3$. Let H_λ be the group of isometries of $\mathbf{R}^2$ generated by those reflections whose axes contain the sides of P. After gluing common sides, the surface $S(P)$ results by identifying copies of P in $\bigsqcup_{h \in H_\lambda} hP$ differing by a translation. Consider the parametrizations $\gamma_2(s) := e^{2\pi i s}/2$ and $\gamma_3(s) = \gamma_2(s) + 1$. We assume that $T(1/2)$ is real and $\lim_{t \to 0} T(t) = 0$. Let T_m denote the analytic continuation of (9) with respect to a loop $m \in \pi_1(E, (1/2))$. Schwartz's reflection principle implies $T_{\gamma_2} = e^{2\pi \lambda_j} T$ and $T_{\gamma_3} = e^{2\pi i \lambda_j} T + T_0$ for a certain $T_0 \neq 0$. We deduce that every translation in $\mathcal{H}_\lambda$ is of the form T_m, where m is a loop in $\pi_1(E, (1/2))$ acting trivially on the fiber $\Pi^{-1}(1/2)$ and *vice versa*. This implies that the translation structure defined of $\mathcal{L}$ by T corresponds to the translation surface structure of $S(P)$.

To deduce corollary 1, consider $z_1(t) = \rho e^{i\theta} t^{-\lambda_2}(t-1)^{-\lambda_3}$ in the preceding discussion. □

Remark. Any element in $\{z \longrightarrow \pm z + z_0 \mid z_0 \in \mathbf{C}\}$ is called a *half-translation* [**7**]. We define $S_{1/2}(P)$ to be the quotient of $S(P)$ defined by identifying any two points differing by a half-translation. When λ is a strongly resonant point, the natural projection $S(P) \longrightarrow S_{1/2}(P)$ is a double cover and, for every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$ the image of the covering $\mathcal{L} \longrightarrow \Pi_{\mathbf{RP}(3)}(\mathcal{L})$ has a natural half-translation structure. From the preceding proof we deduce that, if λ is a strongly resonant point, then $S_{1/2}(P)$ and $\Pi_{\mathbf{RP}(3)}(\mathcal{L})$ are isomorphic as half-translation surfaces.

4. *Applications*

We recall some facts concerning the phase space of the billiard game. The state of any point in the triangular billiard P is completely determined by its position and the direction of its velocity. Therefore, the phase space of this dynamical system is naturally identified with $P \times \mathbf{R}/2\pi\mathbf{Z}$, and this set will henceforth be endowed with the natural product topology. We assume that the point in the interior of $P \times \{\theta\}$ describes a trajectory parallel to the direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ until it reaches the boundary at a point (ξ, θ). Recall that P is supposed to have no vertices. The point (ξ, θ) 'jumps' to the point (ξ, θ'), where θ and θ' are related by the law 'angle of reflection is equal to the angle of incidence'. In this way, the billiard trajectories define a 'flow' in the phase space usually called *the billiard flow*. The Lebesgue measures on P and $\mathbf{R}/2\pi\mathbf{Z}$ define a natural product measure on the phase space of P.

The linear parts of the three reflections of the plane, in respect of the lines containing the sides of the triangle P, generate a subgroup of $\mathbf{O}(2)$, which we denote $\mathbf{G}$. For every direction θ we let $\mathbf{G}\theta$ be the orbit of θ under the natural action of $\mathbf{G}$ over $\mathbf{R}/2\pi\mathbf{Z}$. Clearly,

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for every $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ the set $\bigcup_{\beta\in\mathbf{G}\theta} P \times \{\beta\}$ is invariant under the billiard flow. The identification of any two points $(\xi, \beta) \sim (\xi, \beta')$, $\xi \in \partial P$, for which the directions β and β' are related by the reflection law defines an invariant surface Σ_θ. The billiard flow defines a real non-singular foliation B_θ on this surface. This foliation is formed by all billiard trajectories for which the velocity of the ball is, eventually, parallel to the direction θ.

Definition 3. Let θ_1, θ_2 and θ_3 be the directions of the sides of the triangle P. We call every element in $\bigcup_{j=1}^{3} \mathbf{G}\theta_j$ a *singular direction* of the billiard in P.

Let X be a topological space and $\mathrm{int}(X)$ its interior. For every $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, there exists a natural embedding $\mathrm{int}(\Sigma_\theta) \hookrightarrow S(P)$, conjugating B_θ to the geodesic foliation D_θ. In particular, closed billiard trajectories are sent into closed geodesics. For a non-singular direction, this embedding extends to the whole surface Σ_θ and defines a conjugation between B_θ and the geodesic foliation D_θ, in the whole surface $S(P)$ [**8**].

LEMMA 2. *Let $M_1 := \{e^{i\theta} F_\lambda^{-1}(1)\}_{\theta\in\mathbf{R}/2\pi\mathbf{Z}}$. Then the points in M_1 defining a periodic orbit in $\mathcal{F}_{\lambda,0}$ form a dense set if and only if the points in the phase space $P \times \mathbf{R}/2\pi\mathbf{Z}$ defining a periodic trajectory of the billiard flow form a dense set.*

Proof. Let $\widetilde{M_1} := \Pi^{-1}(M_1)$ be the image of M_1 in $\widetilde{\mathbf{CP}(2)}$. We define, in local coordinates $z_2 = tz_1$, the set $M_1^+ := \{(z_1, t) \in \widetilde{M_1} \mid \mathrm{Im}(t) > 0\}$. The 'parametrization' $\{\mathrm{Im}(t) > 0\} \times \mathbf{R}/2\pi\mathbf{Z} \ni (t, \theta) \rightsquigarrow (e^{i\theta} t^{-\lambda_2}(t-1)^{-\lambda_3}, t)$ induces a differentiable structure of three-dimensional real manifold on M_1. Analogously, $\mathrm{Im}(t) < 0$ defines M_1^-. The restriction of the foliations $\widetilde{\mathcal{F}_\lambda}$ and $\widetilde{\mathcal{F}_{\lambda,0}}$ to M_1 define two real foliations that we note $\mathcal{F}_+$ and $\mathcal{F}_{+,0}$.

Let f_+ denote the restriction of the Schwarz–Christoffel transformation, (9), to $\mathrm{Im}(t) > 0$. Let $\mathcal{L} \in \mathcal{F}_+$ be a leaf and π the projection of $\widetilde{\mathbf{CP}(2)}$ onto the exceptional divisor. Then, the restriction of the composition $f_+ \circ \pi$ conjugates $\mathcal{F}_{+,0|\mathcal{L}}$ to the foliation defined by the billiard flow on $P \times \{\theta\} \subset P \times \mathbf{R}/2\pi\mathbf{Z}$, for some direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. In this way, we obtain a diffeomorphism $\widetilde{f_+} : M_1^+ \longrightarrow \mathrm{int}(P) \times \mathbf{R}/2\pi\mathbf{Z}$. A point $\eta \in M_1^+$ is contained in a periodic orbit of $\widetilde{\mathcal{F}_0}$ if and only if $\widetilde{f_+}(\eta)$ is contained in a trajectory of the billiard flow. Reasoning in a similar way, we find that the density of points in the phase space that define a periodic trajectory implies the density of points in M_1^- that define a periodic orbit and *vice versa*. □

If periodic orbits of $\mathcal{F}_{\lambda,0}$, restricted to M_1, form a dense set in M_1, then periodic orbits of $\mathcal{F}_{\lambda,0}$ form a dense set in $\mathbf{C}^2$. This follows from the fact that, for every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$ and $\rho \in \mathbf{R}^*$, the foliations $\mathcal{F}_{\lambda,0|\mathcal{L}}$ and $\mathcal{F}_{\lambda,0|\rho\mathcal{L}}$ are conjugated. We remark that $\Pi_{\mathbf{RP}(3)} : M_1 \longrightarrow \mathbf{RP}(3) \setminus \mathrm{Sing}(\mathcal{G}_\lambda)$ is a double covering. This implies that if periodic orbits of $\mathcal{F}_{\lambda,0}$, restricted to M_1, are dense in M_1 then periodic orbits of $\mathcal{G}_{\lambda,0}$ are dense in $\mathbf{RP}(3)$, even if λ is a strongly resonant. We next need the following theorem.

THEOREM 5. [**1, 13**] *Let P be a right or rational triangle. Then periodic orbits of the billiard flow in the phase space $\mathbf{R}/2\pi\mathbf{Z}$ are dense.*

From this theorem, using Lemma 2, we deduce the following.

COROLLARY 3. *Let P be a right or rational triangle. Then periodic orbits of $\mathcal{G}_{\lambda,0}$ and $\mathcal{F}_{\lambda,0}$ are dense in $\mathbf{RP}(3)$ and $\mathbf{C}^2$ respectively.*

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For every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$, periodic orbits in $\mathcal{F}_{\lambda,0|\mathcal{L}}$ are never isolated. This is just a consequence of the commutativity, when considering the usual Lie bracket, of the real vector fields $\mathrm{Re}(X_\lambda)$ and $\mathrm{Im}(X_\lambda)$. Such is also the case for billiard trajectories: a periodic trajectory parallel to a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ is always included into a strip of periodic parallel trajectories [**10**]. The following example illustrates the general case.

Fagnano's triangle. In every acute triangle $\triangle ABC$, the triangle whose vertices are the endpoints of the altitudes from the vertices A, B and C, defines a closed billiard trajectory. This trajectory is usually called Fagnano's trajectory in honor to Giulio C. Fagnano.

COROLLARY 4. (Fagnano) *Let P be an acute triangle, then $\mathcal{F}_{\lambda,0}$ presents at least one (non-isolated) periodic orbit.*

A simple drawing shows that the obstruction for enlarging the strip containing Fagnano's trajectory is composed by singular billiard trajectories 'joining' two vertices of the triangle. Such billiard trajectories are usually called *generalized diagonals*. Later, we will describe the role of these singular trajectories when considered as leaves of $\mathcal{G}_{\lambda,0}$.

A foliation on a manifold M is called *minimal* if every leaf is dense in M. If, for a given direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, the billiard on a rational triangle P presents no generalized diagonal, then the foliation B_θ, on the invariant surface Σ_θ, is minimal [**10**].

COROLLARY 5. *If $(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Q}^3$, then the set of leaves $\mathcal{L} \in \mathcal{F}_\lambda$ where $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is not minimal has Lebesgue measure zero.*

For rational triangles, the Lebesgue measure of the plane induces a natural probability measure on the invariant surfaces Σ_θ of the billiard flow. In [**9**], Kerckhoff, Masur and Smillie prove the following milestone theorem.

THEOREM 6. ([**9**] (Triangular case)) *Let P be rational. Then for almost every direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ the restriction of the billiard flow to Σ_θ is uniquely ergodic.*

From a remark, made independently by A. Katok and M. Boshernitzan, one can deduce the following.

COROLLARY 6. [**9**] *There is a dense G_δ set in the space of triangles consisting of polygons for which the billiard flow is ergodic.*

Proof. (Theorem 4 §1) Consider a trajectory l' of the billiard flow dense in the phase space passing through a point $(p, \theta) \in P \times \mathbf{R}/2\pi\mathbf{Z}$. Let $\widetilde{f_+} : M_1^+ \longrightarrow \mathrm{int}(P) \times \mathbf{R}/2\pi\mathbf{Z}$ be the diffeomorphism defined in the proof of Lemma 2 and $l \in \mathcal{F}_{\lambda,0}$ the leaf passing through the point $\widetilde{f_+}^{-1}(p, \theta)$. Suppose that l avoids an open set U in $\widetilde{M}_1$, and that this set, without loss of generality, is contained in either M_1^+ or M_1^-. Then, using either $\widetilde{f_+}$ or $\widetilde{f_-}$, we deduce that l' must avoid an open set in the phase space. This is a contradiction. □

4.1. *Generalized diagonals in $\mathcal{G}_{\lambda,0}$.* In the following paragraphs, we describe the set of leaves in $\mathcal{G}_{\lambda,0}$ intersecting the local invariant manifolds $\mathcal{V}_j$, for $j = 1, 2, 3$, in terms of billiard trajectories in P.

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Definition 4. Let l be a non-compact leaf of a foliation in a (real or complex) manifold M. A point $p \in M$ is called an extremity of l if and only if there exists a parametrization, $\gamma :]0, +\infty[\longrightarrow l$, such that $\lim_{s\to+\infty} \gamma(s) = p$. We say that the extremity p is an analytic extremity if and only if the closure of l in a neighborhood of p is analytic.

Let $l \in \mathcal{G}_{\lambda,0} \setminus \mathrm{Sing}(\mathcal{G}_\lambda)$ be a leaf having the singular point p_j as an extremity. Then, it intersects a local invariant manifold $\mathcal{V}_j$. Let $\widetilde{l \cap \mathcal{V}_j}$ be the image of $\Pi^{-1}_{\mathbf{RP}(3)}(l \cap \mathcal{V}_j)$ in $\widetilde{\mathbf{CP}(2)}$ via Π^{-1}. The projection of $\widetilde{l \cap \mathcal{V}_j}$ to the exceptional divisor in $\widetilde{\mathbf{CP}(2)}$ has an extremity in $t = 0, 1, \infty$. A direct calculation in local coordinates, say $z_2 = tz_1$, shows that the Schwarz–Christoffel transformation defined by formula (9) sends $t = 0, 1, \infty$ over a vertex in P. Thus, if $l \subset \mathcal{L} \in \mathcal{G}_\lambda$ and $\mathcal{G}_{\lambda,0|\mathcal{L}}$ is analytically conjugated (or semiconjugated for a strong resonant λ) to the geodesic foliation D_θ in the translation surface $S(P)$, then the geodesic determined by the leaf l is sent into the polygon by the projection $S(P) \longrightarrow P$ over a singular billiard trajectory, that is, a trajectory meeting a vertex. Conversely, every such singular trajectory in the billiard lifts up to a geodesic in $S(P)$, forming part of a geodesic foliation D_θ. If this foliation on the translation surface is conjugated (or semiconjugated for a strong resonant λ) to $\mathcal{G}_{\lambda,0|\mathcal{L}}$, for a certain $\mathcal{L} \in \mathcal{G}_\lambda$, then the leaf $l \in \mathcal{G}_{\lambda,0}$, determined by the singular billiard trajectory, has an extremity in the singular locus of $\mathcal{G}_{\lambda,0}$.

THEOREM 7. *Every non-compact leaf of $\mathcal{G}_{\lambda,0}$ has either* 0, 1 *or* 2 analytic *extremities in the singular locus. The set of leaves in $\mathcal{G}_{\lambda,0}$ having two extremities is countable and each singularity in $\mathcal{G}_{\lambda,0}$ is an analytic extremity for a countable set of this kind of leaves.*

Proof. From Proposition 4, we deduce that extremities of leaves in $\mathcal{G}_{\lambda,0}$ are analytic. Every leaf of $\mathcal{G}_{\lambda,0} \setminus \mathrm{Sing}(\mathcal{G}_\lambda)$ having two extremities corresponds to a *generalized diagonal* of the billiard on the triangle P. The set of generalized diagonals having an extremity in a fixed vertex is countable [**10**]. As a consequence, the reunion of all leaves in $\mathcal{G}_{\lambda,0} \setminus \mathrm{Sing}(\mathcal{G}_\lambda)$ intersecting $\bigcup_{j=1}^{3} \mathcal{V}_j$ do not form a global invariant variety. □

Remark. Let $l \in \widetilde{\mathcal{G}_{\lambda,0}}$ be a leaf visiting a neighbourhood $U_j \setminus \mathbf{T}_j$ of the torus $\mathbf{T}_j$, but not having an extremity in $\Pi_C^{-1}(p_j)$. We recall that $\widetilde{\mathcal{G}_{\lambda,0|\mathbf{T}_j}}$ is a line foliation. Then, the leaf l describes a finite number of 'turns' around the torus $\mathbf{T}_j$ before leaving the neighbourhood U_j. As we have seen in Proposition 3, this number of turns is bounded by a constant depending only on the 'slope', $(1 - \lambda_j)/\lambda_j$. In terms of billiards, this phenomenon corresponds to the fact that every billiard trajectory on P, visiting a neighbourhood of a vertex *and not having the vertex as an extremity*, makes a finite number of reflections before leaving the neighbourhood. This number of reflections is bounded by $\lceil \lambda_j \rceil$, where $\lceil . \rceil$ notes the ceiling function. Therefore, one can think of the line foliation $\widetilde{\mathcal{G}_{\lambda,0|\mathbf{T}_j}}$ *as the foliation representing the billiard dynamics at a vertex of the triangle*, see Figure 5.

5. *General polygons*

Polygons are plane figures whose boundaries are formed by a finite number of sequential line segments. We say that a polygon P is *simple* if the only points belonging to two edges are the vertices of P. Henceforth *general polygon* will mean *bounded simple polygon*. The

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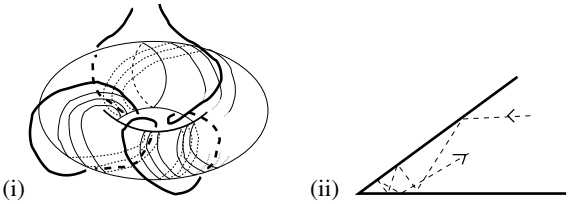


FIGURE 5. The line foliation $\widetilde{\mathcal{G}_{\lambda,0}}_{|\mathbf{T}_j}$ (i) and the billiard dynamics near a vertex (ii).

aim of this section is to set the basis for constructing a dictionary between the billiard game in a general polygon and a special class of holomorphic homogeneous foliations on $\mathbf{C}^2$.

Remark. Henceforward the general polygon P *is supposed to have no vertices*.

As in the triangular case, the billiard dynamics in P is equivalent to the geodesic flow on the unitary tangent bundle of a translation surface $S = S(P)$ 'generated' by P through a standard unfolding process [**15**]. We keep the notation D_θ for the geodesic foliation defined on $S(P)$ by the choice of a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$.

5.1. *General polygons and Schwarz–Christoffel transformations.* Suppose that the general polygon $P \subset \mathbf{C}$ has $\nu + 1$ sides and interior angle $\lambda_j\pi$ at a vertex w_j, $j = 1, \ldots, \nu + 1$. Then, there are constants c, $C \in \mathbf{C}$ and real numbers $\{a_j\}_{j=1}^{\nu+1}$, with $a_j < a_{j+1}$, such that

$$f(t) = c + C \int^t \prod_{j=1}^{\nu+1} (\xi - a_j)^{\lambda_j - 1}\, d\xi \tag{10}$$

is a conformal representation of $\mathrm{Im}(t) > 0$ into the interior of P satisfying $\lim_{t\to a_j} f(t) = w_j$. Conformal representations of this kind are usually called Schwarz–Christoffel transformations. The points a_j are called the *prevertices* of the map (10). Their determination is known as the Schwarz–Christoffel parameter problem [**5**].

5.2. *Homogeneous foliations.* Consider a homogeneous holomorphic foliation on $\mathbf{C}^2$ with an isolated singularity and leaving $\nu + 1$, $\nu \geq 2$, different complex lines invariant through the singularity. We suppose that this set of invariant lines is given by the zeros of the polynomial $P_{\nu+1} = \prod_{j=1}^{\nu+1}(z_2 - b_j z_1)$, $b_j \in \mathbf{C}^*$, and set $b = (b_1, \ldots, b_{\nu+1})$. From [**3**], this foliation is given by the holomorphic 1-form:

$$\frac{\omega_{b,\mu}}{P_{\nu+1}} = \sum_{j=1}^{\nu+1} \mu_j \frac{d(z_2 - b_j z_1)}{z_2 - b_j z_1}, \tag{11}$$

where $\mu = (\mu_1, \ldots, \mu_{\nu+1})$ is a point of the affine hyperplane $\{(\mu_1, \ldots, \mu_{\nu+1}) \in \mathbf{C}^n \mid \sum_{j=1}^{\nu+1} \mu_j = K,\ \mu_j \neq 0\ \forall j\}$ and $K \in \mathbf{Z}^*$ is an arbitrary normalization constant whose choice does not affect the foliation defined by (11). We remark that

$$F_{b,\mu}(z_1, z_2) = \prod_{j=1}^{\nu+1} (z_2 - b_j z_1)^{\mu_j} \tag{12}$$

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is a first integral for the foliation defined by $\omega_{b,\mu}$. We denote this foliation $\mathcal{F}_{b,\mu}$. This foliation presents, up to diffeomorphism, three leaves: the singular point, $\nu+1$ punctured complex lines through the origin and a generic leaf, which we will denote $\mathcal{L}$ as well. Let $X_{b,\mu}$ be the holomorphic vector field given by $i_{X_{b,\mu}}\, dz_1 \wedge dz_2 = \omega_{b,\mu}$. The integral curves of $X_{b,\mu}$ are the leaves of $\mathcal{F}_{b,\mu}$. As for the triangular case, the vector field $X_{b,\lambda|\mathcal{L}}$ defines a translation structure on every generic leaf $\mathcal{L} \in \mathcal{F}_{b,\lambda}$. Let $\mathcal{F}_{b,\mu,0}$ be the real foliation on $\mathbf{C}^2$ defined by the integral curves of the vector field $\mathrm{Re}(X_{b,\mu})$.

5.3. *Dictionary, general version.* Let P be a polygon having $\nu+1$ sides and interior angle $\lambda_j\pi$ at a vertex w_j, $j=1,\ldots,\nu+1$. Suppose that (10) is a conformal representation of $\mathrm{Im}(t)>0$ into the interior of P. In this section we mimic what has been previously done for triangles, in order to establish the relation between the holomorphic 1-form $\omega_{b,\mu}$ whose parameters are $\mu_j=\lambda_j$ and $b_j=a_j$, $j=1,\ldots,\nu+1$ and the billiard game on P.

Definition 5. We call $\mathcal{F}_{a,\lambda}$ the *polygonal foliation associated to the billiard game on the polygon* P.

Remarks. The Schwarz–Christoffel parameter problem has three degrees of freedom [**5**]. Henceforward we decide that $a_1=\infty$, $a_2=0$ and $a_3=1$ in (10). In the triangular case, this choice leads to the expression in (1).

If the polygon P has $\nu+1$ sides, then the parameter λ of the foliation $\mathcal{F}_{a,\lambda}$ satisfies $\sum_{j=1}^{\nu+1}\lambda_j=\nu-1$. In the coordinates $z_2=tz_1$, the fibers of (12) are given by $z_1^{\nu-1}=kt^{-\lambda_2}(t-1)^{-\lambda_3}\prod_{j=4}^{\nu+1}(t-a_j)^{-\lambda_j}$, $k\in\mathbf{C}^*$. Because of this, each fiber of the first integral (12) has at most $\nu-1$ connected components. The subgroup of the homothecy group of $\mathbf{C}^2$,

$$H_{\nu-1} := \{T_\eta(z_1,z_2)=\eta(z_1,z_2) \mid \eta^{\nu-1}=1\}, \tag{13}$$

acts transitively on the set of connected components of each fiber $F_{a,\lambda}^{-1}(k)$, $k\in\mathbf{C}^*$. We observe that $H_{\nu-1}$ is the group of homothecies preserving $X_{a,\lambda}$, that is, satisfying $X_{a,\lambda}(T_\eta(z_1,z_2))=DT_\eta X_{a,\lambda}(z_1,z_2)$. We will call $H_{\nu-1}$ the *group of automorphisms of the vector field* $X_{a,\lambda}$.

5.4. *Proof, main result (revisited).* The notation and language in the following paragraphs emulate those of §3.

Let E be the complement in the exceptional divisor of $\widetilde{\mathbf{CP}(2)}$ of $\{\infty,0,1,a_4,\ldots,a_{\nu+1}\}$ and $z_2=tz_1$. Every generic leaf $\mathcal{L}\in\widetilde{\mathcal{F}_{a,\lambda}}$ defines a covering space $\pi_{|\mathcal{L}}:\mathcal{L}\longrightarrow E$. Let $\Sigma_{a,\lambda}$ denote the $(\nu+1)$-punctured sphere obtained by identifying two copies of the polygon P in their common sides. We note $f:E\longrightarrow\Sigma_{a,\lambda}$ the analytic continuation of the Schwarz–Christoffel transformation

$$T := t \rightsquigarrow \int_0^t \xi^{\lambda_2-1}(\xi-1)^{\lambda_3-1}\prod_{j=4}^{\nu+1}(\xi-a_j)^{\lambda_j-1}\,d\xi, \quad \mathrm{Im}(t)>0 \tag{14}$$

to E. Let $\mathcal{L}\in\widetilde{\mathcal{F}_{a,\lambda}}$ be a generic leaf of the foliation on $\widetilde{\mathbf{CP}(2)}$ induced by $\mathcal{F}_{a,\lambda}$. A straightforward calculation shows that, in the local coordinates $z_2=tz_1$, the projection

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of the vector field $\widetilde{X_{a,\lambda|\mathcal{L}}}$ to E is given by the branches of $kt^{1-\lambda_2}(t-1)^{1-\lambda_3}\prod_{j=4}^{\nu+1}(t-a_j)^{1-\lambda_j}\partial/\partial t$, where $k \in \mathbf{C}^*$ is a constant depending on the leaf $\mathcal{L}$. Therefore, (14) defines a chart for the translation structure on $\mathcal{L}$ induced by $X_{a,\lambda|\mathcal{L}}$. The map f lifts to a translation surface isomorphism $\widetilde{f}: \mathcal{L} \longrightarrow S(P)$ if and only if the analytic continuation of f along a loop $m \in \pi_1(E, *)$ for which $\mathrm{hol}(m) = \mathrm{Id}_{\mathbf{C}}$ (see (4)) induces a translation on $T(\Im(t) > 0) \simeq P$ and *vice versa*. This is not always the case. Indeed, let $\gamma_1, \ldots, \gamma_{\nu+1}$ be simple loops generating $\pi_1(E, *)$. The normalization condition $\sum_{j=1}^{\nu+1} = \nu - 1$ implies that the holonomy group of $\mathcal{F}_{a,\lambda}$ is generated by the linear diffeomorphism $z \longrightarrow e^{((2\pi i\lambda - j)/(\nu-1))}z$, $j = 1, \ldots, \nu+1$. On the other hand, Schawrtz's reflection principle implies that the analytic continuation T_{γ_j} is, without loss of generality, of the form $T \longrightarrow e^{2\pi i\lambda_j}T + T_j$ for a certain $T_j \in \mathbf{C}^*$. Therefore, in some 'non-typical' cases, the generic leaf of the foliation $\mathcal{F}_{a,\lambda}$ and the translation surface $S(P)$ are not even homeomorphic.

Example. The translation surface generated by any rectangular figure has genus one, whereas the closure in $\mathbf{CP}(2)$ of the generic leave of the holomorphic homogeneous foliation $\mathcal{F}_{a,\lambda}$, having as first integral $F_{a,\lambda}(z_1, z_2) = z_1^{1/2}z_2^{1/2}(z_2 - z_1)^{1/2}(z_2 + z_1)^{1/2}$, has genus three.

Definition 6. A polygon having angles $\{\lambda_j\pi\}_{j=1}^{\nu+1}$ is called *reasonable* if and only if every integer of the form $\sum_{j=1}^{\nu+1} n_j\lambda_j$, $n_j \in \mathbf{Z}$, for all j, is congruent to $\nu - 1$ modulo $\mathbf{Z}$.

Clearly, every triangle is reasonable, whereas rectangles are not. Reasonable polygons form the natural context in which Theorem 2 remains valid. From §3, we deduce the following.

THEOREM 8. *Let P be a reasonable polygon. Then, every generic leaf $\mathcal{L} \in \mathcal{F}_{a,\lambda}$ is isomorphic, as translation surface, to the surface $S(P)$.*

COROLLARY 7. *Let P be a reasonable polygon. Then, for every generic leaf $\mathcal{L} \in \mathcal{F}_{a,\lambda}$ there exists a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ such that the foliation $\mathcal{F}_{a,\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction θ on the translation surface $S(P)$. Furthermore, for every $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$ and $\rho \in \mathbf{R}^*$, $\mathcal{F}_{a,\lambda,0|\rho e^{i\theta'}\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta + (\nu - 1)\theta'$ on $S(P)$.*

Remarks.

(i) The factor $(\nu - 1)\theta'$ appears because the projection of the vector field $\widetilde{X_{a,\lambda}}_{|\rho e^{i\theta'}\mathcal{L}}$ to the exceptional divisor E is given, when $\Pi(\mathcal{L}) = F_{a,\lambda}^{-1}(k)$, by the branches of $(\rho\mathbf{e}^{\mathbf{i}\theta'})^{\nu-\mathbf{1}}kt^{1-\lambda_2}(t-1)^{1-\lambda_3}\prod_{j=4}^{\nu+1}(t-a_j)^{1-\lambda_j}\partial/\partial t$.

(ii) Let P be a polygon and $\mathcal{F}_{a,\lambda}$ the corresponding polygonal foliation. Let $H(\mathcal{F}_{a,\lambda})$ be the homothecy subgroup formed by those T_k such that $z \rightsquigarrow kz$ is an element of the holonomy group of the foliation $\widetilde{\mathcal{F}_{a,\lambda}}$, relative to the fibration defined by π. Then, *the polygon P is reasonable if and only if the intersection of $H(\mathcal{F}_{a,\lambda})$ with $H_{\nu-1}$, the group of automorphisms of the vector field $X_{a,\lambda}$, is trivial.*

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5.4.1. *Non-reasonable case.* Let H_λ be the intersection of the homothecy subgroups $H_{\nu-1}$ and $H(\mathcal{F}_{a,\lambda})$. Consider the action

$$\mathbf{C}^2 \times H_\lambda \ni ((z_1, z_2), T_k) \rightsquigarrow T_k(z_1, z_2) \tag{15}$$

and let $\mathbf{C}^2_\lambda$ be the quotient of $\mathbf{C}^2$ by this action. This quotient is a (complex) ruled surface with an isolated singularity. We denote the natural projection as $\Pi_\lambda : \mathbf{C}^2 \longrightarrow \mathbf{C}^2_\lambda$. The foliation $\mathcal{F}_{a,\lambda}$ defines a foliation on $\mathbf{C}^2_\lambda$, which we denote $\widehat{\mathcal{F}_{a,\lambda}}$. The projection via Π_λ of any generic leaf of $\mathcal{F}_{a,\lambda}$ is called generic as well. Every generic leaf of $\widehat{\mathcal{F}_{a,\lambda}}$ inherits a translation surface structure. Due to the fact that H_λ contains only automorphisms of the holomorphic vector field $X_{a,\lambda}$, the foliation $\mathcal{F}_{a,\lambda,0}$ defines a real foliation on $\mathbf{C}^2_\lambda$, which we denote $\widehat{\mathcal{F}_{a,\lambda,0}}$.

COROLLARY 8. *Let P be a non-reasonable polygon. Then, every generic leaf $\mathcal{L} \in \widehat{\mathcal{F}_{a,\lambda}}$ is isomorphic, as translation surface, to the surface $S(P)$.*

Let $\mathcal{L}' \in \mathcal{F}_{a,\lambda}$ be generic and $\mathcal{L} = \Pi_\lambda(\mathcal{L}')$. We define as well $k\mathcal{L} := \Pi_\lambda(k\mathcal{L}')$. Then we have the following.

COROLLARY 9. *Let P be a non-reasonable polygon. Then, for every generic leaf $\mathcal{L} \in \widehat{\mathcal{F}_{a,\lambda}}$ there exists a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ such that the foliation $\widehat{\mathcal{F}_{a,\lambda,0|\mathcal{L}}}$ is holomorphically conjugated to the geodesic foliation defined by the direction θ on the translation surface $S(P)$. Furthermore, for every $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$ and $\rho \in \mathbf{R}^*$, $\widehat{\mathcal{F}_{a,\lambda,0|\rho e^{i\theta'}\mathcal{L}}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta + (\nu - 1)\theta'$ on $S(P)$.*

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