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EXCHANGE MOVES AND NONCONJUGATE BRAID REPRESENTATIVES OF KNOTS

REIKO SHINJO AND ALEXANDER STOIMENOW

Abstract. We prove that for $n \geqslant 4$, every knot has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move.

§1. Overview

The *braid groups* B_n were introduced in the 1930s in the work of Artin [2]. An element $b \in B_n$ is called an *n-braid*. Alexander [1] related braids to *knots and links* in real 3-dimensional space, by means of a *closure* operation $\hat{}$. In that realm, it became important to understand the *braid representatives* of a given link L, that is, those b with $L = \hat{b}$. *Markov's theorem* relates these representatives by two moves, the *conjugacy* in the braid group, and *(de)stabilization*, which passes between $b \in B_n$ and $b\sigma_n^{\pm 1} \in B_{n+1}$ (see, e.g., [14]). Markov's moves and braid group algebra have become fundamental in Jones' pioneering work [11] and its later continuation toward quantum invariants. Conjugacy is, starting with Garside's [9], and later many others' work, now relatively well group-theoretically understood. In contrast, the effect of (de)stabilization on conjugacy classes of braid representatives of a given link is in general difficult to understand. Only in very special situations can these conjugacy classes be well described, for example, [5].

In this paper we are concerned with the question when infinitely many conjugacy classes of n-braid representatives of a given knot occur. Birman and Menasco [4] introduced a move called *exchange move* (see Section 2.2), and proved that it necessarily underlies the switch between many conjugacy classes of braid representatives of L. We prove here that it is also sufficient for generating infinitely many such classes.

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THEOREM 1.1. *Let K be a knot and $n \geqslant 4$. Then K has infinitely many conjugacy classes of n-braid representatives if and only if it has one admitting an exchange move.*

Among the different braid representatives of a link L the one with the fewest strands is called a *minimal braid*. The number of strands of a minimal braid is called the *braid index* $b(L)$ of L. Obviously it makes sense to consider only $n \geqslant b(L)$.

Some nonconjugate braids close to isotopic links. Birman had observed [3] that stabilizations of different signs are nonconjugate, because of different exponent sum. However, it is well known now that for any link L and $n \geqslant b(L)$, there are only finitely many exponent sums of n-braid representatives of L. This is a consequence of Birman–Menasco's Theorem 2.1 stated below, but also, much more directly, of the Morton–Franks–Williams inequality for the skein polynomial. Therefore, the exponent sum is (not very surprisingly) too weak to distinguish infinitely many conjugacy classes. It was also known from [5] that only finitely many conjugacy classes occur when $n \leqslant 3$.

In the case $n > b(L)$ of nonminimal braids, Morton [13] discovered an infinite sequence of conjugacy classes of 4-braids with closure being the unknot. Further examples were exhibited more recently by Fukunaga [7, 8] and Shinjo [18]. For every knot, there are obvious (stabilized) nonminimal braid representatives admitting an exchange move. Thus Theorem 1.1 can always be applied. Shinjo obtained this special case of the theorem in her previous paper [19].

In the case $n = b(L)$ of minimal braids, Birman had conjectured that there would always be a single conjugacy class of minimal braids representing a link. However, Murasugi and Thomas [16] disproved Birman's conjecture, exhibiting some counterexamples in[1] B_4. Our result can be seen, for knots, as such a general construction. It is also a precursor to a follow-up paper [20], where we discuss in more detail the case of links (for which also special cases were known [21, 22]), and extend our main result.

[1] They claim also such examples in higher B_n, but this is not justified from their proof, which uses the homomorphism $B_4 \to B_3$.

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§2. Preliminaries

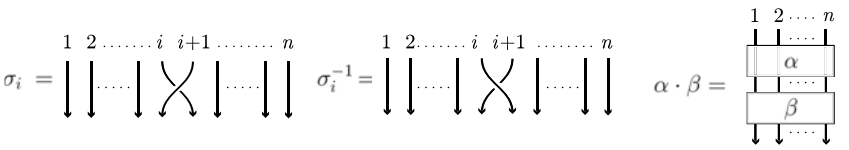
2.1 Braids and closures

DEFINITION 1. The *braid group* B_n on n strands can be defined by generators and relations as

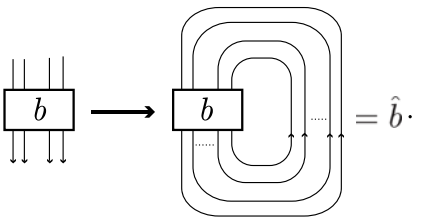
$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_j\sigma_i\sigma_j = \sigma_i\sigma_j\sigma_i & |i-j| = 1 \end{array} \right\rangle.$$

The σ_i are called *Artin standard generators*. An element $b \in B_n$ is an *n-braid*.

There is a graphical representative of braids, where in σ_i strands i and $i+1$ cross, and multiplication is given by stacking. (We number strands from left to right and orient them downward.)



The *closure* $\hat{b}$ of a braid b is a knot or link (with orientation) in S^3:



There is a permutation homomorphism of B_n,

$$\pi : B_n \to S_n, \quad \text{given by } \pi(\sigma_i) = (i, i+1).$$

(The permutation on the right is a transposition.) We call $\pi(b)$ the *braid permutation* of b. We call b a *pure braid* if $\pi(b) = Id$.

Let b be an n-braid with numbered endpoints as in Figure 1. Suppose that b has its strings connected as follows: 1 to i_1, 2 to $i_2, \ldots, n$ to i_n, that is, $\pi(b)(k) = i_k$. Then we write

$$\pi(b) = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

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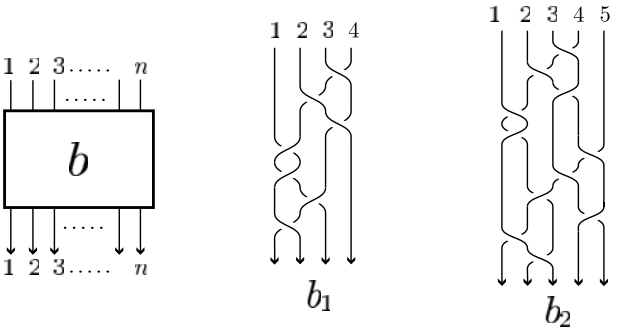


Figure 1.
An n-braid.

For example the braid b_1 in Figure 1 has the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 3 \end{pmatrix},$$

where (1 2 4 3) means a cyclic permutation. The braid b_2 in the figure has the permutation (1 3 5)(2 4). We remark that when the closure of b is a k-component link, the braid permutation of b is a product of k disjoint cycles. (We will be mainly interested in $k = 1$.) The length of each cycle is equal to the number of strings of b which make up a component of $\hat{b}$.

When we choose a (nonempty) subset C of $\{1, \ldots, n\}$ whose elements form a subset of the cycles of $\pi(b)$, we can define a *subbraid* b' of b by choosing only strings numbered in C. For subbraids b' and b'' of b one can define the *(strand) linking number* $lk(b', b'')$ by the linking number $lk(\hat{b}', \hat{b}'')$ between sublinks of $\hat{b}$. For example, in b_2 of Figure 1, we have $lk(b_2', b_2'') = 0$, where $\hat{b}_2'$ and $\hat{b}_2''$ are the two components of $\hat{b}_2$. (The subbraid b_2' comprises the strings starting at the top as number $1, 3, 5$, and b_2'' strings $2, 4$.)

2.2 Exchange moves

Let

$$\Delta_n^2 = (\sigma_1 \cdot \ldots \cdot \sigma_{n-1})^n$$

be the (right-handed) full twist on n strands. The *center* of B_n (elements that commute with all B_n) is infinite cyclic and generated by Δ_n^2. Let similarly

$$\Delta_{[i,j]}^2 = (\sigma_i \cdot \ldots \cdot \sigma_{j-1})^{j-i+1}$$

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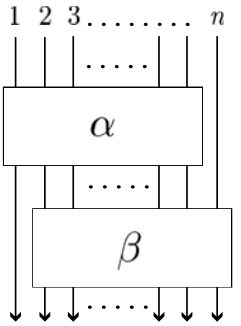


Figure 2.
The n-braid b.

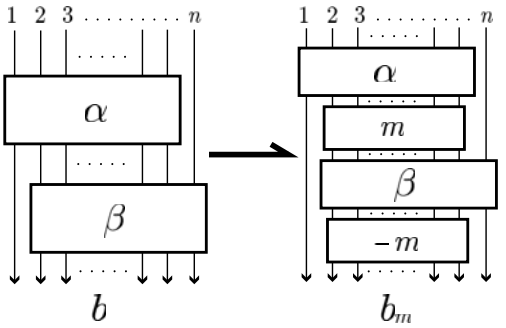


Figure 3.
The braid b_m.

be the restricted full twist on strands i to j. Let also for $1 \leqslant i < j \leqslant n$,

$$B_{i,j} := \langle \sigma_i, \ldots, \sigma_{j-1} \rangle \tag{1}$$

be the subgroup of B_n of braids operating on strands $i \ldots, j$. Where ambiguity is avoided (as indicated by diagrams we draw), we can identify $B_{i,j} \simeq B_{j-i+1}$.

We say that $b \in B_n$ *admits an exchange move*, if b is as illustrated in Figure 2, where $\alpha \in B_{1,n-1}$, $\beta \in B_{2,n}$, and $n \geqslant 4$.

An *exchange move* [4] is the transformation between the braids b and b_m shown in Figure 3. Here m is some nonzero integer, and the boxes labeled $\pm m$ represent the full twists $\Delta^{\pm 2m}_{[2,n-1]}$ respectively, acting on the middle $n-2$ strands. (Thus a positive number of full twists are understood to be right full twists, and $-m$ full twists mean m full left-handed twists.)

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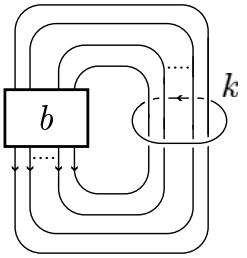


Figure 4.
The axis-addition link of b.

Of course, no nontrivial braid on 2 strands admits an exchange move, and all exchange moves on 3 strands are trivial, so that we naturally assume $n \geqslant 4$ throughout. There is another, more common, way to describe the exchange move, namely by

$$\alpha\beta \longleftrightarrow \alpha\kappa^m\beta\kappa^{-m}, \quad \text{where } \kappa = (\sigma_1 \cdot \ldots \cdot \sigma_{n-2})(\sigma_{n-2} \cdot \ldots \cdot \sigma_1). \tag{2}$$

This description is equivalent to the previous one, because $\kappa \cdot \Delta^2_{[2,n-1]} = \Delta^2_{[1,n-1]}$, and this element commutes with α. A further equivalent formulation of the move, for $m = \pm 1$, is

$$\sigma_1\beta\sigma_1^{-1}\beta' \longleftrightarrow \sigma_1^{-1}\beta\sigma_1\beta',$$

with $\beta, \beta' \in B_{2,n}$. This form is less convenient for out treatment, so we will not use it below.

The exchange move underlies the switch between conjugacy classes with the same closure link, in a universal way.

THEOREM 2.1. (Birman–Menasco [4]) *The n-braid representatives of a given link decompose into a finite number of classes under the combination of exchange moves and conjugacy.*

2.3 Axis link and Conway polynomial

DEFINITION 2. The *axis (addition) link* of a braid b, denoted by L_b, is the oriented link consisting of the closure of b and an unknotted curve k, the axis of the closed braid, as in Figure 4.

We remark that the axis-addition links of conjugate braids are ambient isotopic. Thus for a proof of nonconjugacy we study invariants of the axis link. As such an invariant we employ the *Conway polynomial* ∇. It takes

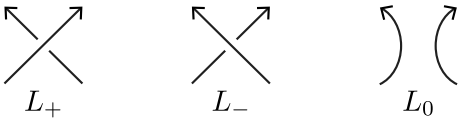
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values in $\mathbb{Z}[z]$ and is given by the value 1 on the unknot and the relation

$$\nabla(L_+) - \nabla(L_-) = z\nabla(L_0). \tag{3}$$

This relation involves three links with diagrams

(4)



differing just at one crossing. They are called a *skein triple*. We write $[P]_m$ for the coefficient of z^m in $P \in \mathbb{Z}[z]$, and more shortly $a_m = [\nabla]_m$ for the coefficient of z^m in the Conway polynomial. It is well known that for an n-component link L, all coefficients of ∇ in z-degree m vanish when $m < n-1$ or $m+n$ is even.

We denote the *linking number* of two components of L by $lk(\cdot,\cdot)$. Now we recall a formula, given by Hoste [10], which expresses the lowest nontrivial coefficient a_{n-1} of $\nabla(L)$ in terms of component linking numbers.

THEOREM 2.2. (see [10]) *Let $L = L_1 \cup \cdots \cup L_p$ be a p-component link of components $L_1, \ldots, L_p$. Let $l_{k,m} = lk(L_k, L_m)$. Then the coefficient a_{p-1} of the Conway polynomial in degree $p-1$ is*

$$a_{p-1}(L) = \sum_T \prod_{(k,m)\in T} l_{k,m}. \tag{5}$$

In this formula, the sum ranges over spanning trees T of the complete graph G on the vertex set $\{1, \ldots, p\}$, and (k,m) denotes the edge in G connecting the k-th and m-th vertex.

Proof of Theorem 1.1. The 'only if' part in Theorem 1.1 immediately follows from Theorem 2.1. We start now the proof of the 'if' part, which will extend over several sections until the end of the paper.

In order to exhibit braids b_m in Figure 3 as nonconjugate, we follow the approach in [19], and evaluate the second coefficient of ∇ on the axis-addition link of b. We show that an exchange move alters this coefficient except in a situation described in the following proposition.

PROPOSITION 2.3. *Let $n \geqslant 4$, and K be a knot represented as the closure of an n-braid b admitting an exchange move as in Figure 3. Write the braid permutation $\pi(b) = (x_1, x_2, \ldots, x_n)$, where we fix the cyclic ambiguity of x_i by letting the cycle end on $x_n = n$. Then, unless $n = 2n'+1$ (for $n' \in \mathbb{N}$) and $x_{n'+1} = 1$, all links L_{b_m}, for b_m as in Figure 3, are distinguished by a_3.*

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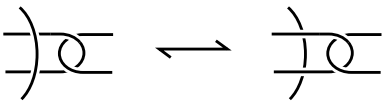


Figure 5.
A delta move.

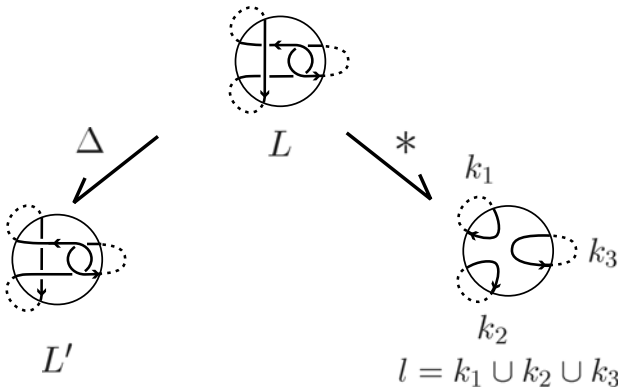


Figure 6.
Three links related by local moves.

In the next section, we will be concerned with proving this proposition. The remaining, and more complicated, cases will be settled in Section 4 by looking at the axis-addition link L_{b^2} of the square of b.

§3. Proof of Proposition 2.3

First, we give a lemma needed later. A *delta move* is a local move defined in [15], and this move is equivalent to the move in Figure 5. We consider the delta move on the left-hand side in Figure 6, where the dotted arcs show how the strands connect.

In a way similar to the proof of Lemma 2.2 in [19], we can prove the following lemma using Theorem 2.2. (We remind that the linking number and i-th coefficient of the Conway polynomial are written as $lk(\cdot,\cdot)$ and $a_i(\cdot)$, respectively.)

LEMMA 3.1. *Let L, L' and $l = k_1 \cup k_2 \cup k_3$ be oriented links related by the local moves as in Figure 6. Then $a_3(L) - a_3(L') = lk(k_2, k_3) - lk(k_3, k_1)$.*

Now we are ready to start the proof of the first partial case of Theorem 1.1.

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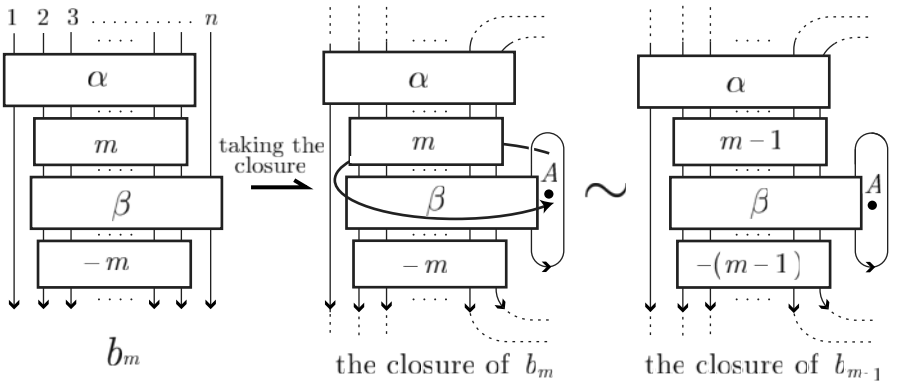


Figure 7.
Braids with the same closure.

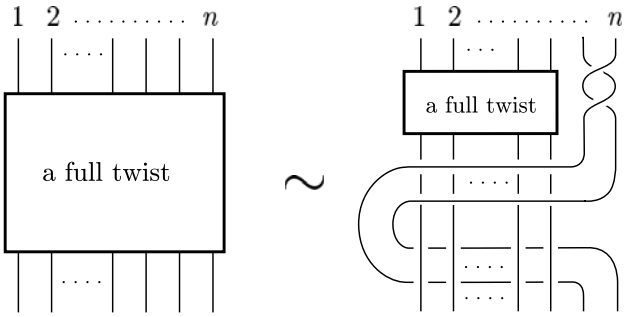


Figure 8.
A full twist of n-strings.

Proof of Proposition 2.3. We set $b_0 = b$. From a braid b, we construct an infinite sequence of braids $\{b_m, m \in \mathbb{Z}\}$ as in Figure 3, where m and $-m$ represent m and $-m$ full twists, respectively. The closures of b_m and b_{m-1} ($m \in \mathbb{N}$) are related by ambient isotopy as in Figure 7, where A denotes the braid axis. This means that all braids in the sequence close to K.

Since a full twist of n strings can be deformed as in Figure 8 up to ambient isotopy, the axis-addition link L_{b_m} of b_m, which is the leftmost diagram in Figure 9, can be deformed into the rightmost link in the figure, still denoted by L_{b_m}. Here k is the component corresponding to the braid axis and the boxes m and $-m$ represent m-full twists and $-m$-full twists, respectively.

Then there are sequences of links $L_{b_m} = L^0, L^1, L^2, \ldots, L^{n-1} = L_{b_{m-1}}$ and $l^0, l^1, l^2, \ldots, l^{n-1}$ such that L^{i+1} resp. l^i are obtained from L^i by the delta move Δ_i resp. the move $*_i$, both illustrated in Figures 10 ($i = 0$) and 11

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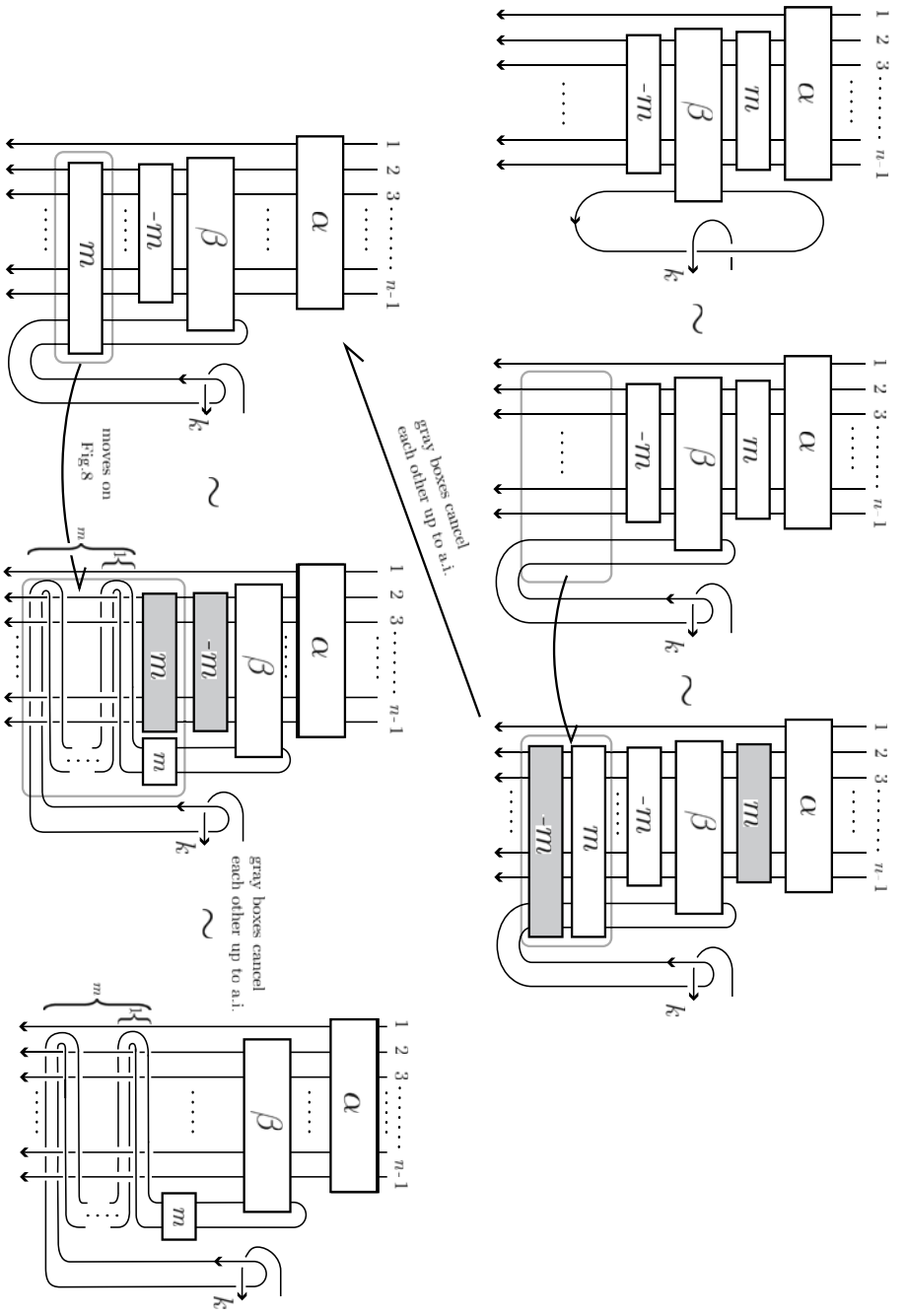


Figure 9.
Deforming L_{b_m} by ambient isotopy ('a.i.').

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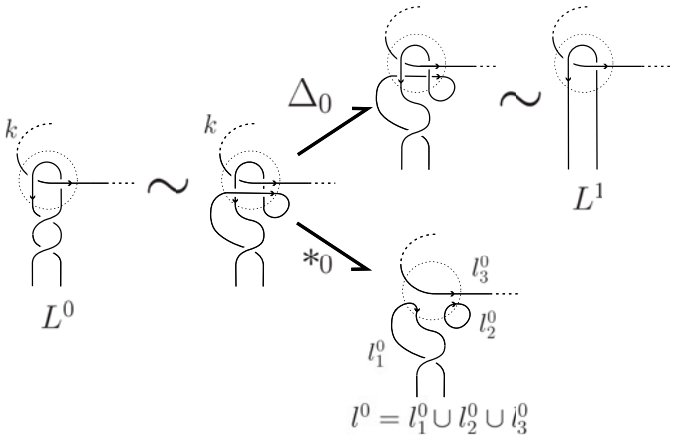


Figure 10.
The moves Δ_0 and $*_0$.

$(i = 1, \ldots, n-2)$. In particular, there are $n-1$ delta moves transforming L_{b_m} into $L_{b_{m-1}}$: the first is chosen to undo a full twist in the box of m on the right diagram of Figure 9, and the other $n-2$ undo one full winding of the band below that box.

By Lemma 3.1, the change in a_3 resulting from Δ_0 can be obtained as follows:

$$a_3(L^1) - a_3(L^0) = lk(l_1^0, l_3^0) - lk(l_2^0, l_3^0) = n - 1,$$

where $l^0 = l_1^0 \cup l_2^0 \cup l_3^0$ is the 3-component link illustrated in Figure 10.

Next we consider the change in a_3 resulting from Δ_i illustrated in the Figure 11 $(i = 1, 2, \ldots, n-2)$. Let S_{L^i} (resp. S_{l^i}) be a part of L^i (resp. l^i) as in the left (resp. right) diagram of Figure 12. Namely S_{L^i} and S_{l^i} are the unions of n strings and an unknotted component. Some of these $(n-1)$ strings of S_{l^i} belong to l_1^i and the other belong to l_2^i. The numbers of strings determine $lk(l_1^i, l_3^i)$ and $lk(l_2^i, l_3^i)$.

By considering how $S_{l^i} - l_3^i$ has its strings connected, permutations of the n down going strings can be assigned to S_{L^i} and S_{l^i}, similarly to a braid permutation. We call these the permutations of S_{L^i} and S_{l^i}. Note that the permutation assigned to S_{L^i} is the same as the braid permutation $\pi(b)$ of b. Since $l^i - l_3^i$ is a 2 component link, the permutation of S_{l^i} consists of 2 cycles. To determine the length of these cycles, we observe that the move $*_i$ corresponds to taking the product of a transposition $(n-i, n)$ with the permutation of S_{L^i}.

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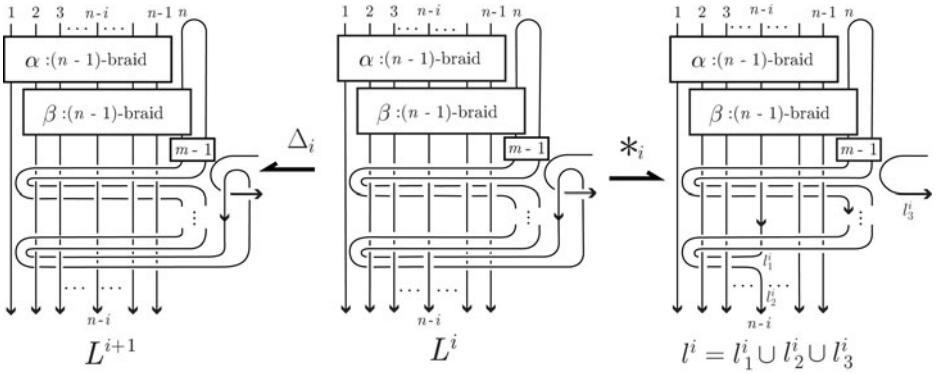


Figure 11.
The local moves on L^i.

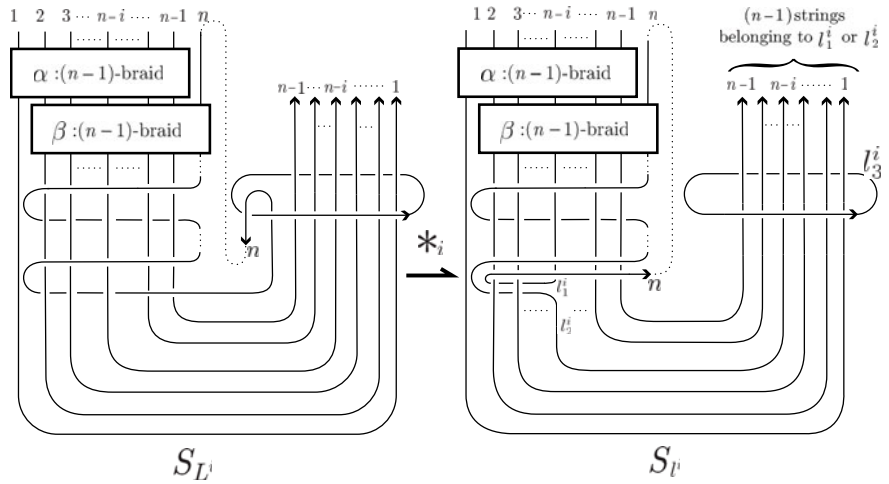


Figure 12.

Take $j = j(i)$ so that $n - i = x_j$. Then

$$(x_j, n)(x_1, x_2, \ldots, x_{n-1}, n) = (x_1, x_2, \ldots, x_{j-1}, n)(x_j, \ldots, x_{n-2}, x_{n-1}).$$

The cyclic permutations $(x_1, x_2, \ldots, x_{j-1}, n)$ and $(x_j, \ldots, x_{n-2}, x_{n-1})$ correspond to l_1^i and l_2^i, respectively. Remark that the string of S_{l^i} with lower end point n belongs to l_1^i, and it does not contribute now to $lk(l_1^i, l_3^i)$. By Lemma 3.1,

$$\begin{aligned} a_3(L^{i+1}) - a_3(L^i) &= lk(l_1^i, l_3^i) - lk(l_2^i, l_3^i) \\ &= (j-1) - (n-j) = 2j - n - 1. \end{aligned}$$

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Suppose that $x_l = 1$. Then

$$\begin{aligned} a_3(L_{b_{m+1}}) - a_3(L_{b_m}) &= \{\text{the change in } a_3 \text{ resulting from } \Delta_0\} \\ &\quad + \sum_{i=1}^{n-2} \{\text{the change in } a_3 \text{ resulting from } \Delta_i\} \\ &= (n-1) + \sum_{j=1}^{n-1} (2j - n - 1) - (2l - n - 1) \\ &= -2l + n + 1. \end{aligned}$$

The difference $-2l + n + 1$ is a constant which does not depend on m. If it is nonzero, the sequence $\{a_3(L_{b_p}), p \in \mathbb{N}\}$ forms an arithmetic progression with nonzero common difference. When n is even, $-2l + n + 1$ is odd. This means that $-2l + n + 1 \neq 0$. When n is odd, namely $n = 2n' + 1$ for some $n' \in \mathbb{N}$, then $-2l + n + 1 = 2(n' - l + 1)$. Unless $l = n' + 1$, we have $-2l + n + 1 \neq 0$. The equation $l = n' + 1$ means that $x_{n'+1} = 1$. Therefore $a_3(L_{b_{m+1}}) - a_3(L_{b_m})$ is nonzero and independent of m, unless $n = 2n' + 1$ $(n' \in \mathbb{N})$ and $x_{n'+1} = 1$. This completes the proof of Proposition 2.3. □

§4. Remaining knot cases

From now on we assume that $n = 2n' + 1$ and $\pi(b) = (x_1, x_2, \ldots, x_{n-1}, n)$ with $x_{n'+1} = 1$. To prove that b_m are nonconjugate, we look at b_m^2: if two braids are conjugate, so are their squares. Note that, when n is odd and $\pi(b)$ is a cycle, so is $\pi(b^2)$. Thus $L_{b_m^2}$ are again 2-component links. We show the following:

PROPOSITION 4.1. *Let b be an n-braid admitting an exchange move. If $n = 2n' + 1 > 3$ odd and $\pi(b) = (x_1, x_2, \ldots, x_{n-1}, n)$ with $x_{n'+1} = 1$, then $a_3(L_{b_m^2})$ is a quadratic polynomial in m with nonzero quadratic term.*

In particular, there are at most two $L_{b_m^2}$ whose a_3 is equal to some fixed value, and so at most two of b_m are conjugate. Thus with Proposition 4.1, the proof of Theorem 1.1 will be complete.

Proof of Proposition 4.1. Let us first simplify the form of α and β in Figure 2.

First, every permutation of $2, \ldots, n-1$ applied on either side of $\pi(\alpha)$ can be realized by a braid which can be moved into β. Thus we can achieve

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that $\pi(\alpha) = (1, 2)$. So

$$\alpha = \alpha' \cdot \sigma_1^{-1} \tag{6}$$

for some pure braid α' on strands $1, \dots, n-1$.

Then $\pi(\beta) = (x_1', x_2', \dots, x_{n-2}', n)$ is a cycle with $x_{n'}' = 2$ and $n > x_j' > 2$ otherwise. Now, any such permutation of these $x_j' \neq 2, n$ can be realized by conjugating $\pi(\beta)$ with a permutation of $3, \dots, n-1$. Thus similarly β can be conjugated by a braid on strands $3, \dots, n-1$ (and leaving the other three strands separate).

Since we achieved that $\pi(\alpha)$ fixes all of $3, \dots, n-1$, the permutation of $x_j' \neq 2, n$ in $\pi(\beta)$ can be achieved by a conjugation of $b = \alpha \cdot \beta$, at the cost of multiplying α by some pure braid on strands $1, \dots, n-1$, which we can absorb into α' of (6). Note that in this argument, we make use of the following property: a conjugation by a braid on strands $3, \dots, n-1$ commutes with a full twist on strands $2, \dots, n-1$, and hence also with the exchange move. Thus if we obtain infinitely many nonconjugate braids by iterated exchange move on the so rearranged braid b, we obtain these also from the initial braid b.

This means that we can assume that we can write

$$\beta = \beta' \cdot \beta_0,$$

for some pure braid β' on strands $2, \dots, n$, as long as β_0 is a braid on strands $2, \dots, n$ with $\pi(\beta)$ being a cycle with $x_{n'}' = 2$ when $x_{n-1}' = n$. In the following we choose and fix

$$\beta_0 = \sigma_3^{-1}\sigma_5^{-1} \cdot \dots \cdot \sigma_{n-2}^{-1} \cdot \sigma_2^{-1}\sigma_4^{-1} \cdot \dots \cdot \sigma_{n-1}^{-1}.$$

LEMMA 4.2. *We have $a_3(L_{b^2_m}) - a_3(L_{b^2}) = Am^2 + Bm + C$ for some (rational) numbers A, B, C (depending, a priori, on n and b). Moreover, A does not depend on the braids α', β' in the presentation*

$$b = \alpha' \cdot \sigma_1^{-1} \cdot \beta' \cdot \beta_0. \tag{7}$$

Proof. The axis link of $L_{b^2_m}$ can be built and simplified similarly to Figure 9. The procedure is shown in Figure 13. In this case we involve the up going strand also on the left of b. Now we can cancel the full twists on $n-2$ strands in b^2_m by creating pairs of bands that circle, in the opposite way, around the middle $n-2$ strands. See Figure 14. It shows the case

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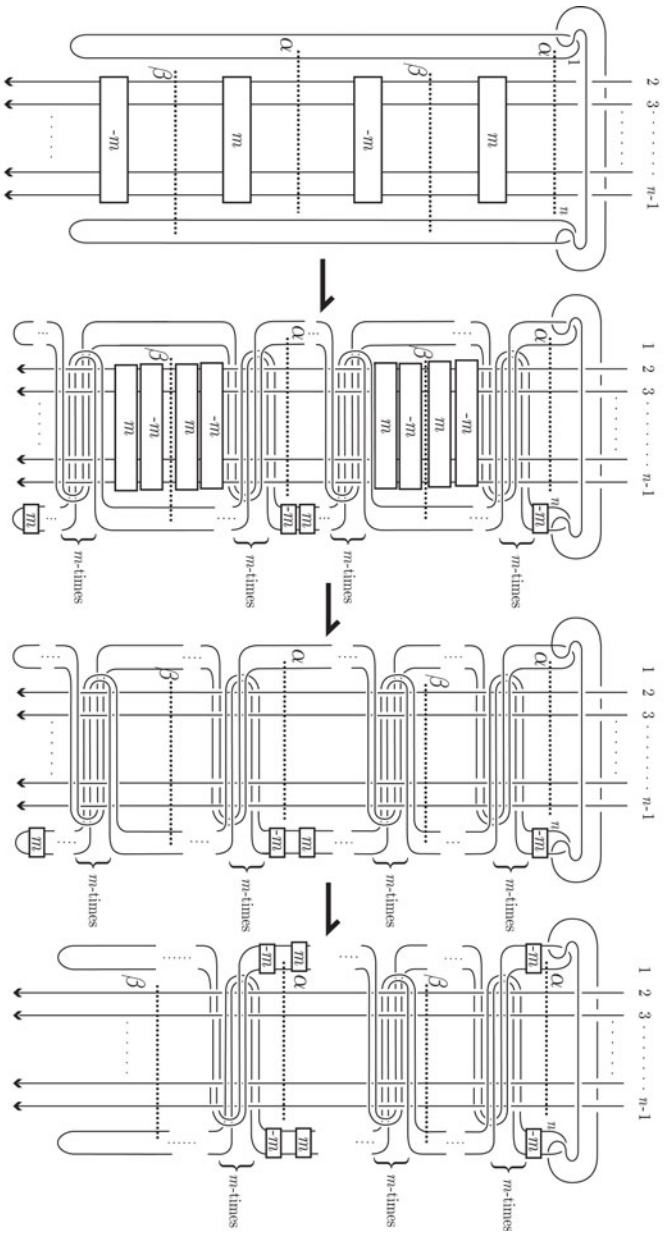


Figure 13.
Building $L_{b^2_m}$. The move from the first to the second diagram consists in inserting two groups of m full twists and two groups of $-m$ full twists on all $n+2$ strands, and then grouping out the subtwists on the central $n-2$ strands into a part denoted by a box.

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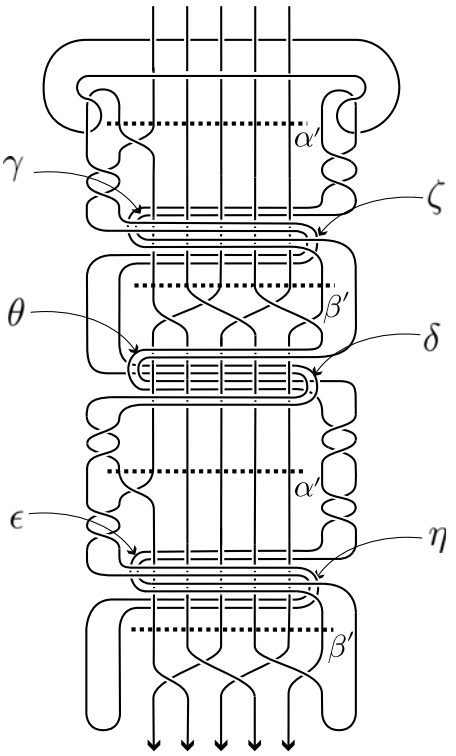


Figure 14.
$L_{b_m^2}$ (for $m = 1$).

$n = 7$ and $m = 1$. (One of the pairs of circling bands, the one at the bottom, untangles, so we have 3 such pairs.) We indicate the braids α' and β' just by a dashed line, showing where they have to be inserted.

Now the bands δ and ζ cancel, and η trivializes. Then, θ and ε cancel by a half-turn (and all their internal twists cancel; see Figure 15), but to cancel them further, we need to move the band past $\delta\zeta$ in the encircled region of Figure 16. (For general m, the parts δ and ζ will have $|m| - 1$ full turns of the band around the other $n - 2$ strings in the opposite direction.)

Next, γ can be deleted at the cost of changing a_3 by a quantity linear in m (whose linear terms may depend on n, α' and β'). This can be seen from Lemma 3.1, in the way we applied it in Section 3. It must be realized that, in spite of the bands δ and ζ in the lower part of the figure, the linking number of l_1^i and l_2^i with l_3^i does not depend on m. Thus the change of a_3 under undoing one full twist of γ does not depend on m either.

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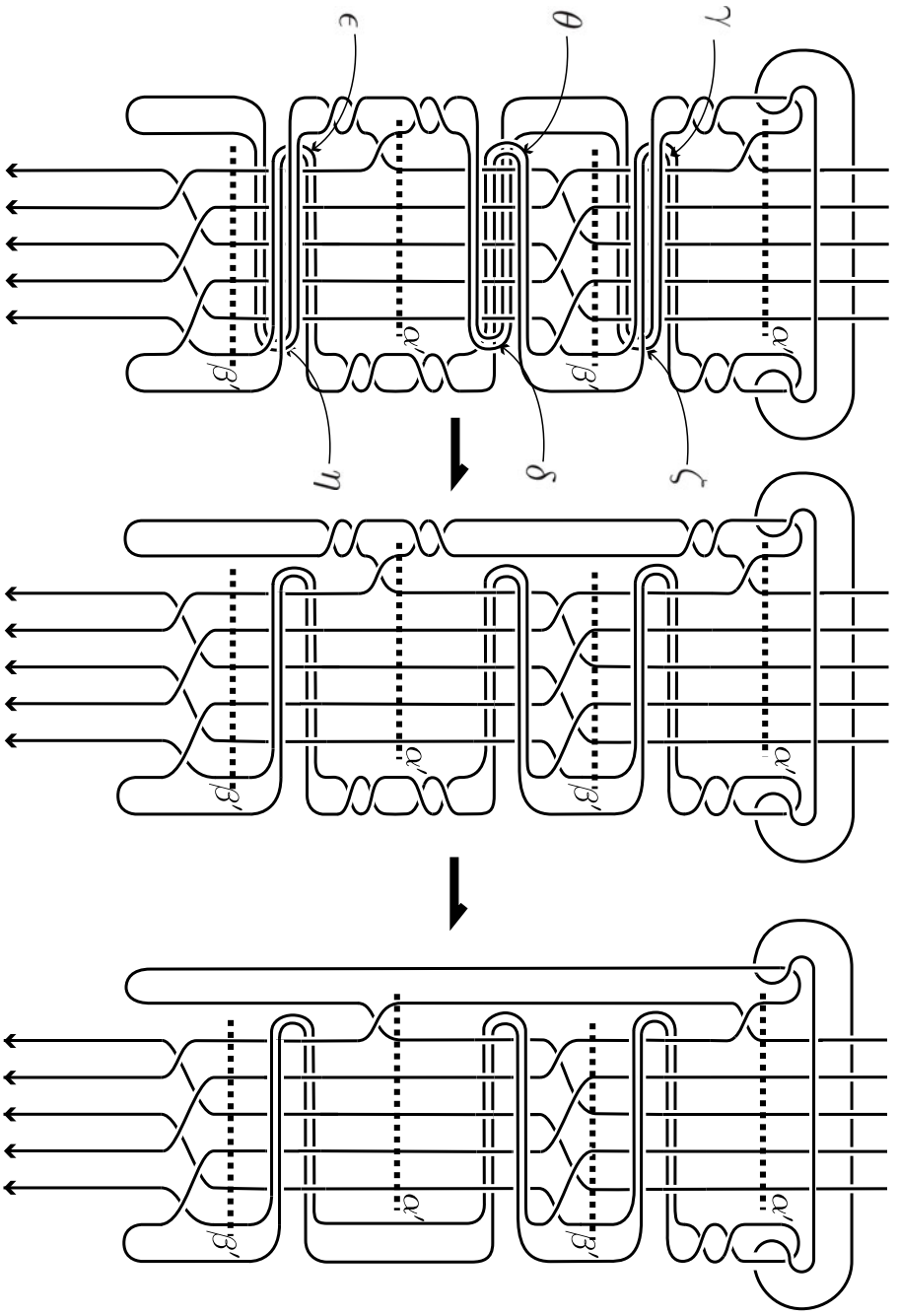


Figure 15.
Simplifying $L_{b_m^2}$ (for $m = 1$).

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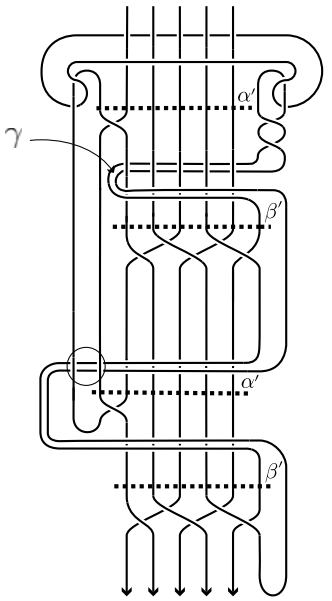


Figure 16.
Simplified $L_{b_m^2}$ (for $m = 1$).

This means that, for the purpose of proving Lemma 4.2, we can disregard the band γ, and so we assume that it is trivial. Then we obtain from $L_{b_m^2}$ the links K_m as shown (for $m = 1, 2$ and $n = 7$) in Figure 17.

Using the relation (3), we can write

$$a_3(K_{m+1}) - a_3(K_m) = a_2(L_1) - a_2(L_2) - a_2(L_3) + a_2(L_4), \tag{8}$$

where $L_i = L_{m,i}$ are 3-component links obtained from K_{m+1} by changing some and smoothing exactly one of the 4 crossings in the encircled part. (Apply the skein relation to these 4 crossings in any order.)

Now, it is easy to observe that among the 3 linking numbers between the components of each L_i, only one (the one not involving the braid axis) depends, linearly, on m (a dependence which holds for either signs of m), and α' and β' affect all 3 linking numbers only by some additive constant.

It follows then from Theorem 2.2 that $a_3(K_{m+1}) - a_3(K_m)$ is a linear expression in m with a linear term independent on α' and β'. By inductive iteration, we obtain the claim of Lemma 4.2. □

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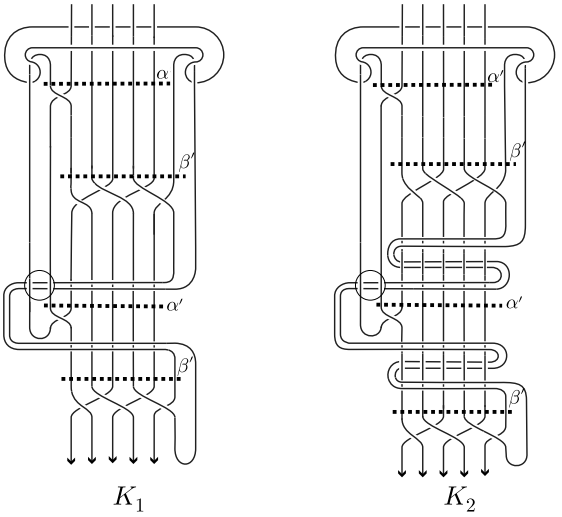


Figure 17.
K_m.

With Lemma 4.2, for the proof of Proposition 4.1, it is legitimate to assume that α' and β' are trivial, and (7) becomes

$$\beta_0 = \sigma_1^{-1}\sigma_3^{-1} \cdot \ldots \cdot \sigma_{n-2}^{-1} \cdot \sigma_2^{-1}\sigma_4^{-1} \cdot \ldots \cdot \sigma_{n-1}^{-1}.$$

It is not difficult to evaluate the quadratic coefficient A in the lemma for this special case.

Now,

$$2A = a_3(K_1) + a_3(K_{-1}) - 2a_3(K_0),$$

and so it is enough to show that the right-hand side of this equation, call it D_n, does not vanish for any odd $n \geqslant 5$.

Again, one can express $a_3(K_1) - a_3(K_0)$ and $a_3(K_0) - a_3(K_{-1})$ using (8). Next, observe that, essentially because the action of $\pi(b^2)$ on intermediate strands is to shift by 4 to left or right, the replacement of any odd $n \geqslant 5$ by $n + 4k$ alters the component linking numbers of the links L_i in (8) by multiples of k. It follows then that D_{n+4k} is a certain quadratic expression in $k \geqslant 0$ for $n = 5$ and $n = 7$. To determine these expressions, one can make a direct calculation using (8) and Theorem 2.2. This is, however, somewhat tedious and error-prone. Thus we also used a different method for verification.

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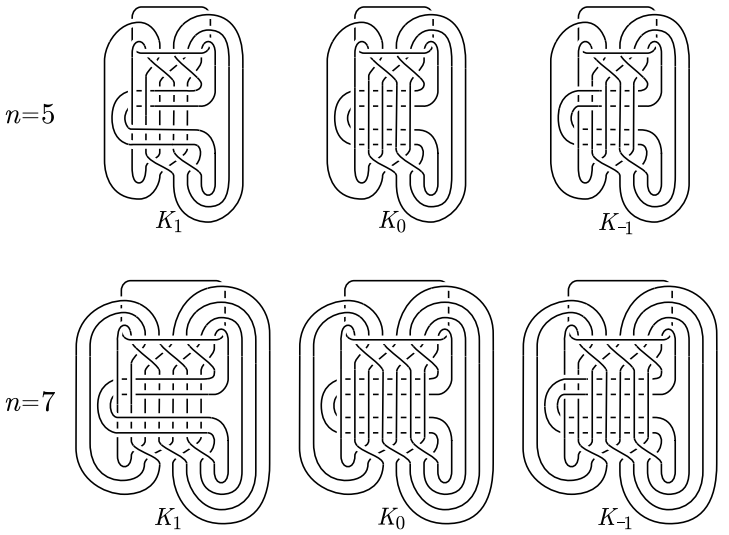


Figure 18.
K_m for $|m| \leqslant 1$ and $n = 5, 7$.

We drew, as in Figure 18, the links $K_{\pm 1}$ and K_0 for $k = 0, 1, 2$ in either case (i.e., $n = 5, 7, \ldots, 15$), and calculated $c_i = a_3(K_i)$ by computer.

The most complicated diagrams have 118 crossings, but it took a total of 10.5 seconds to evaluate a_3 on all 18 diagrams using the skein polynomial truncation algorithm of [23]. The result is shown below:

n	5	9	13	7	11	15
c_1	-13	-38	-59	38	137	312
c_0	5	30	91	14	55	140
c_{-1}	-17	-46	-71	46	149	328

From this one determines that

$$D_n = \begin{cases} -40 - 72k - 32k^2 & \text{if } n = 5 + 4k \\ 56 + 88k + 32k^2 & \text{if } n = 7 + 4k. \end{cases}$$

This is never zero for any $k \geqslant 0$. (It vanishes, however, for $k = -1$, which is in nice accordance with the triviality of the cases $n = 1, 3$.) With this the proof of Proposition 4.1 is concluded.

This also completes the proof of Theorem 1.1. □

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§5. Table

It is a consequence of Theorem 1.1 that any composite knot K of braid index $b(K) \geqslant 4$ has infinitely many nonconjugate minimal braid representatives. (This insight was also obtained in [22].)

In the tables of prime knots K of crossing number $c(K) \leqslant 10$ [17, appendix], we found that most knots of $b(K) \geqslant 4$, except 7 knots for $c(K) = 9$ and 15 knots for $c(K) = 10$, have a minimal representative admitting an exchange move. So these knots have infinitely many conjugacy classes of minimal braid representatives.

The below list gives 4-braid representatives admitting an exchange move for 95 knots of braid index 4 (up to mirroring) in Rolfsen's tables. An integer $i > 0$ means σ_i, an $i < 0$ stands for σ_{-i}^{-1}. Note that an n-braid word admits an exchange move up to cyclic permutation if and only if it has no (cyclic) subword of the form

$$\sigma_1^{\pm 1} \cdot \ldots \cdot \sigma_{n-1}^{\pm 1} \cdot \ldots \cdot \sigma_1^{\pm 1} \cdot \ldots \cdot \sigma_{n-1}^{\pm 1} \cdot \ldots \tag{9}$$

We shifted, as in [6], indices down by 1 for $10_{163}, \ldots, 10_{166}$ to discard Perko's duplication. Thus, for example, the knot written below as 10_{162} is Rolfsen's 10_{163}.

A further mistake in Rolfsen's tables is that therein 10_{83} and 10_{86} are swapped: the Conway notation and Alexander polynomial for each one refers to the diagram of the other. The convention taken here is that we interchange Conway notations and Alexander polynomials to fix the mismatch (as in the corrected reprinting of Rolfsen's book), and *not* the diagrams (as, e.g., in [12]).

6_1	-3-3-2112-13-2	8_{15}	-3-3-2-2-3-1-12-1
7_2	-3-3-3-1-1-213-2	9_4	-3-3-3-3-3-1-1-213-2
7_4	-3-3-2-1-1-213-2	9_7	-3-3-3-3-2-23-1-1-21
7_6	-3-3-3-12-123-2	9_{10}	-3-3-2-2-2-1-1-213-2
7_7	-3-3-21-21322	9_{11}	-3-3-3-3-12-1-32
8_4	-3-3-3-1211-32	9_{13}	-3-3-3-3-2-1-1-213-2
8_6	-3-3-3-31-213-2	9_{17}	-3-12-1222-32
8_8	-3-3-31-2113-2	9_{18}	-3-3-3-2-2-1-1-213-2
8_{11}	-3-3-2-21-213-2	9_{20}	-3-3-3-1-12-1-32
8_{13}	-3-3-21-2113-2	9_{22}	-3-12-12-3222
8_{14}	-3-3-3-21-213-2	9_{23}	-3-3-3-2-2-1-12-13-2

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9_{24} -3-3-12-1-3222
9_{26} -3-3-3-12-12-32
9_{27} -3-3-12-122-32
9_{28} -3-3-1-12-1-322
9_{30} -3-3-12-12-322
9_{31} -3-3-1-12-12-32
9_{32} -3-3-12-1-32-32
9_{33} -3-12-1-32-322
9_{36} -3-3-3-12-1-3-32
9_{42} -3-3-3-12-1332
9_{43} -3-3-31-21-3-3-2
9_{44} -3-3-31-2133-2
9_{45} -3-3-2-2-3-12-12
9_{49} -3-3-2-2-3-3-12-13-2
10_{6} -3-3-3-3-3-31-213-2
10_{8} -3-3-3-3-3-1211-32
10_{12} -3-3-3-3-31-2113-2
10_{14} -3-3-3-3-3-21-213-2
10_{15} -3-3-3-3-1211-322
10_{19} -3-3-3-3-12112-32
10_{21} -3-3-2-2-2-21-213-2
10_{22} -3-3-3-31-21113-2
10_{23} -3-3-2-2-21-2113-2
10_{25} -3-3-3-3-2-21-213-2
10_{26} -3-3-3-121122-32
10_{27} -3-3-3-3-21-2113-2
10_{32} -3-3-3-21-21113-2
10_{39} -3-3-3-2-2-21-213-2
10_{40} -3-3-3-2-21-2113-2
10_{49} -3-3-3-3-2-2-3-1-12-1
10_{50} -3-3-2-21-2-2-213-2
10_{51} -3-3-2-21-2-2113-2
10_{52} -3-3-3-12112-3-32
10_{54} -3-3-3-1211-3-322
10_{56} -3-3-3-21-2-2-213-2
10_{57} -3-3-3-21-2-2113-2
10_{61} -3-3-3-1211-3-3-32
10_{65} -3-3-21-2-2-2113-2
10_{66} -3-3-3-2-2-2-3-1-12-1
10_{72} -3-3-3-3-2-23-21-21
10_{76} -3-3-3-3-2-2-231-21
10_{77} -3-3-3-3-2-231-211
10_{80} -3-3-3-2-2-3-3-1-12-1
10_{83} -3-3-21-2-21-213-2
10_{84} -3-3-31-2-21-213-2
10_{86} -3-3-21-21-2113-2
10_{87} -3-3-31-21-2113-2
10_{90} -3-31-2-211-213-2
10_{93} -3-3-1211-32-3-32
10_{102} -3-3-1211-32-322
10_{103} -3-31-2-21-2-213-2
10_{108} -3-3-1211-3-32-32
10_{114} -3-31-21-21-213-2
10_{128} -3-3-3-2-2-3-12-1-3-2
10_{129} -3-3-3223-12-132
10_{130} -3-3-3-12112332
10_{131} -3-3-3-2-1-1-2133-2
10_{132} -3-3-3-1-1-21113-2
10_{133} -3-3-3-2-23-12-13-2
10_{134} -3-3-3-2-2-3-3-1-1-21
10_{135} -3-3-3-2112213-2
10_{140} -3-3-3-1-1-21333-2
10_{142} -3-3-2-2-3-3-12-1-3-2
10_{144} -3-3-2-2-3-1211-32
10_{150} -3-3-3-2-2-233-12-1
10_{151} -3-3-3-12-122-13-2
10_{153} -3-3-31-21-3-3222
10_{154} -3-3-2-2-3-1-2-2-1-12
10_{156} -3-3-3-12-133-23-2
10_{158} -3-3-31-213-2322
10_{160} -3-3-3-2-2-32-31-21
10_{162} -3-3-2-2-32-3-1211
10_{163} -3-3-2-23-233-12-1

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The 22 knots of braid index 4 for which we could not find a 4-braid admitting an exchange move are:

9_{29},	9_{34},	9_{38},	9_{40},	9_{46},	9_{47},
9_{48},	10_{92},	10_{95},	10_{98},	10_{111},	10_{113},
10_{117},	10_{119},	10_{121},	10_{122},	10_{136},	10_{145},
10_{146},	10_{147},	10_{164},	10_{165}.		

Among the 5- and 6-braid knots in the table, all have a minimal braid admitting an exchange move. This can be checked from the representatives given in [24]. It is easy to see that the (cyclic) shape (9) (with $n = 5, 6$) can be avoided after possibly applying some commutativity relations.

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