101.43 On the least distance from a point to a line

In this Note we will find a formula in areal coordinates for this least distance and deduce a closely related formula for the least distance between two parallel lines. Our elementary treatment will be followed by examples which may be checked by routine trigonometry.

The coordinates

With $\Delta ABC = \Delta$ to denote the area of the triangle of reference, a point $P[x, y, z]$ has actual areal coordinates $x = \dfrac{\Delta PBC}{\Delta}$, $y = \dfrac{\Delta PCA}{\Delta}$, $z = \dfrac{\Delta PAB}{\Delta}$ $(x + y + z = 1)$. We also use the cotangent notation $\alpha = \cot A$, $\beta = \cot B, \gamma = \cot C$ $(\beta\gamma + \gamma\alpha + \alpha\beta = 1), \Sigma = \alpha + \beta + \gamma, \Pi = \alpha\beta\gamma$.

In addition, we have the cotangent metric for the squared distance between two points $P[x, y, z]$ and $P'[x', y', z']$, namely

$$\frac{PP'^2}{2\Delta} = \alpha(x - x')^2 + \beta(y - y')^2 + \gamma(z - z')^2, \tag{1}$$

The method

Actual coordinates for the respective meets of the line $lx + my + nz = 0$ with AC, AB are

$$V\left[\frac{n}{n - l}, 0, -\frac{l}{n - l}\right], \qquad W\left[\frac{-m}{l - m}, \frac{l}{l - m}, 0\right]$$

by inspection. Then the metric (1) leads easily to

$$\frac{VW^2}{2\Delta} = \frac{l^2\left(\alpha(m - n)^2 + \beta(n - l)^2 + \gamma(l - m)^2\right)}{((n - l)(l - m))^2}$$

for the distance VW where $\alpha(m - n)^2 + \beta(n - l)^2 + \gamma(l - m)^2 = \phi$, say.

So, for the point $P[X, Y, Z]$ not on the line $lx + my + nz = 0$, the area ratio $\Delta PVW : \Delta ABC$ is given by the absolute value of the determinant

$$\begin{vmatrix} X & Y & Z \\ \dfrac{n}{n - l} & 0 & \dfrac{-l}{n - l} \\ \dfrac{-m}{l - m} & \dfrac{l}{l - m} & 0 \end{vmatrix}$$

which reduces to

$$\left|\frac{l(lX + mY + nZ)}{(n - l)(l - m)}\right|.$$

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Hence the required perpendicular distance p is given by the length of the altitude from P of the triangle PVW, leading to the formula

$$p^2 = \frac{2\Delta\,(lX + mY + nZ)^2}{\phi}. \tag{2}$$

Parallel lines

The line $lx + my + nz = \Theta$ or $(l-\Theta)x + (m-\Theta)y + (n-\Theta)z = 0$ is parallel to $lx + my + nz = 0$ by dint of the standard condition for parallelism

$$\begin{vmatrix} 1 & 1 & 1 \\ l & m & n \\ l-\Theta & m-\Theta & n-\Theta \end{vmatrix} = 0,$$

so the particular point $P[X, Y, Z]$ lies on the line specified by $\theta = lX + mY + nZ$.

Hence the least distance between the two parallel lines represented by the equations $lx + my + nz = 0, \theta$ is given by

$$d^2 = \frac{2\Delta\theta^2}{\phi}. \tag{3}$$

Examples

Before examining formulae (2, 3) in special cases, we will need the result $\Delta = 2R^2 \sin A \sin B \sin C = \dfrac{2R^2}{\Sigma - \Pi}$ where R is the circumradius.

(i) The altitude from vertex $A[1, 0, 0]$ of the triangle of reference is normal to the line $x = 0$.

Hence $l = 1, \quad m = n = 0, \quad lX + mY + nZ = 1$ and $\phi = \beta + \gamma = \dfrac{\sin A}{\sin B \sin C}$, leading to

$$p^2 = 4R^2 \sin^2 B \sin^2 C \quad \text{or} \quad p = 2R \sin B \sin C.$$

(ii) Given the orthocentre $H[\beta\gamma, \gamma\alpha, \alpha\beta]$, the equation $\beta y - \gamma z = 0$ for the altitude in (i) may be found using the determinant

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = 0.$$

Thus $l = 0, m = \beta, n = -\gamma$ and the parallel line passing through the circumcentre $O\left[\frac{1}{2}(1 - \beta\gamma), \frac{1}{2}(1 - \gamma\alpha), \frac{1}{2}(1 - \alpha\beta)\right]$ leads to $\theta = \frac{1}{2}(\beta - \gamma)$ where $\phi = \beta + \gamma$ again.

Thus $d^2 = R^2(\beta - \gamma)^2 \sin^2 B \sin^2 C$ whence $d = R\left|\sin(B - C)\right|$.

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(iii) The median from A has equation $y - z = 0$ and $\theta^2 = 1$ prescribes the parallel lines through B and C. Thus $l = 0$, $m = -n = 1$ and $\phi = 4\alpha + b + \gamma$ lead to

$$d^2 = \frac{4R^2}{(\Sigma - \Pi)(\Sigma + 3\alpha)}.$$

Note finally that more algebra would be incurred by using the method of Lagrange multpliers to derive results (2,3) above ([1]).

Reference

1. J. A. Scott, On the Lagrange method for constrained critical points, *Math. Gaz.* **91** (March 2007) pp. 81-85.

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101.44 On a trivariate Fibonacci identity

Introduction

An especially beautiful aspect of the mathematics associated with the Fibonacci sequence and its generalisations is the wide variety of approaches that may be adopted when formulating a proof of a particular result. Bearing this in mind, we focus our attention here on the following known trivariate identity:

$$F_{a+b+c} = F_{a+1}F_{b+1}F_{c+1} + F_aF_bF_c - F_{a-1}F_{b-1}F_{c-1}, \tag{1}$$

which appears in [1, 2, 3, 4].

As far as we have been able to ascertain, this identity first appeared in the literature in the form of a problem and its solution in [1, 4]. The method of solution employed in this case was proof by induction. Then, several decades later, a solution was given using matrix methods [2]. It is also possible to demonstrate the truth of (1) by algebraic means. Indeed, utilising Binet's formula [5, 6, 7, 8] for the nth Fibonacci number, several pages of tedious algebraic manipulations lead to the result. It is also possible to obtain (1) by using the following identity [3, 5]:

$$F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}. \tag{2}$$

On applying this twice, we obtain (1) with a considerable degree of algebraic simplification.

An alternative approach is to seek a proof that includes a combinatorial aspect. We have not, however, found such a proof in the literature. Consequently, our aim in this paper is to accompany and extend the already-published material on (1) by providing a proof that incorporates a combinatorial element.

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