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GENERALIZED LOCAL *Tb* THEOREMS FOR SQUARE FUNCTIONS

ANA GRAU DE LA HERRÁN AND STEVE HOFMANN

Abstract. A local *Tb* theorem is an L^2 boundedness criterion by which the question of the global behavior of an operator is reduced to its local behavior, acting on a family of test functions b_Q indexed by the dyadic cubes. We present two versions of such results, in particular, treating square function operators whose kernels do not satisfy the standard Littlewood–Paley pointwise estimates. As an application of one version of the local *Tb* theorem, we show how the solvability of the Kato problem (which was implicitly based on local *Tb* theory) may be deduced from this general criterion.

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§1. *Introduction, history and preliminaries.* The *Tb* theorem, like its predecessor, the $T1$ theorem, is an L^2 boundedness criterion, originally proved by McIntosh and Meyer [**28**], and by David *et al* [**10**] in the context of singular integrals, but later extended by Semmes [**31**] to the setting of "square functions". The latter arise in many applications in complex function theory and in partial differential equations (PDEs), and may be viewed as singular integrals taking values in a Hilbert space.

The essential idea of *Tb*- and $T1$-type theorems is that they reduce the question of L^2 boundedness to verifying the behavior of an operator on a single test function b (or even the constant function 1). The "local" versions that we

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have obtained are related to previous work of Christ [7], who proved the first local *Tb* theorem in the singular integral setting. The term "local" in this context refers to the fact that, instead of one globally defined testing function b, one is allowed to test the operator locally, say on each dyadic cube, with a local testing function that is adapted to that cube. The advantage here, in applications, is the additional flexibility that one gains: it may be easier to verify "good" behavior of the operator locally when the testing functions are allowed to vary. The point is that sometimes particular properties of the operator may be exploited to verify the appropriate testing criterion.

Variations and extensions of Christ's result, in the setting either of singular integrals or of square functions, are due to Nazarov *et al* [30], Hytönen and Martikainen [23], Auscher *et al* [3], Hofmann [15, 16], Auscher and Yang [6], Auscher and Routin [4], Hytönen and Nazarov [24], Tan and Yan [33], Lacey and Martikainen [25, 26] and Martikainen and Mourgoglou [27]. Typically, in the aforementioned works, some pointwise size and regularity conditions were imposed on the kernels of the class of operators (singular integrals or square functions) under consideration. Our contribution in the present work is the replacement of pointwise kernel conditions by weaker size and regularity conditions that are expressed in terms of appropriate L^p averages.

We note that this sort of "local *Tb*" criterion, in the square function setting, lies at the heart of the solution of the Kato square root problem, and was already implicit there; see [2, 18, 21] (see also [5, 31] for related results); the connection to local *Tb* theory is discussed in the survey article [17]. One of the aims of the present paper is to make this connection totally explicit: that is, we prove a general version of the local *Tb* theorem for square functions from which the solution of the Kato problem follows directly. In particular, this requires that one replace pointwise size and smoothness conditions on the kernel by appropriate integral decay and orthogonality conditions.

Moreover, we further generalize the local *Tb* theorem for square functions to allow for relaxed size conditions on the testing functions b_Q (i.e., scale-invariant L^p bounds, with $p > 1$, rather than $p = 2$). This has been done previously in the setting of scalar testing functions and pointwise kernel bounds, in [16], and, for general non-doubling measures, in [27]. In particular, we extend the result of [16] in two ways, allowing for matrix-valued testing functions b_Q, and for the removal of pointwise kernel conditions. In this context, we also mention a companion paper to this one [12], in which we prove a local *Tb* theorem for square functions, with vector-valued (i.e., $\mathbb{C}^N$-valued) testing functions, which has application to the theory of layer potentials. We remark that another extension of the main result of [16], of a different sort, to the setting of open sets with Ahlfors–David regular boundaries, has been presented by the first-named author of the present paper and Mourgoglou [13], and has applications to problems that connect the behavior of the harmonic measure for domains with quantitative rectifiability properties of the boundary (see [19, 20]).

1.1. *Summary of results.* We state here the main results that we prove in this paper and provide some explanatory context. In these results, $\{\Theta_t\}$ will be a

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family of operators mapping $L^2(\mathbb{R}^n, \mathbb{C}^m)$, $m \geqslant 1$, uniformly into $L^2(\mathbb{R}^n, \mathbb{C})$, for which we seek to prove square function bounds. Further notation and terminology will be explained in detail in the sequel.

THEOREM 1.1. *Let $\{\Theta_t\}_{t>0}$ be as in Hypothesis 3.4, and suppose that there exist positive constants $C_0 < \infty$, $\delta > 0$, an exponent $p > r$ and a system $\{b_Q\}$ of complex $m \times m$ matrix-valued functions indexed by dyadic cubes $Q \subset \mathbb{R}^n$, such that for each dyadic cube Q:*

$$\int_{\mathbb{R}^n} |b_Q(x)|^p \, dx \leqslant C_0 |Q|,$$

$$\int_Q \left(\mathcal{A}_Q^{\mathrm{dyad}}(\Theta_t b_Q)(x)\right)^p \, dx \leqslant C_0 |Q|,$$

$$\delta |\xi|^2 \leqslant Re\left(\xi \cdot \fint_Q b_Q(x)\, dx \bar{\xi}\right) \quad \textit{for all } \xi \in \mathbb{C}^m.$$

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\, dt}{t} \leqslant C \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Remark. Note that we assume here that we are given a family $\{b_Q\}$ of matrix-valued testing functions, indexed by the dyadic cubes, each of whose elements belongs to L^p, with $p > 1$. Moreover, "Hypothesis 3.4" entails that certain integral size and smoothness conditions hold for Θ_t. Thus, the point of Theorem 1.1 is that it allows for a weaker size condition (L^p, with $p > 1$) on the testing functions, in the vector-valued setting (i.e., with matrix-valued testing functions), and without pointwise kernel estimates. Previous such results had required either L^2 testing conditions for the family $\{b_Q\}$ or had been restricted to the scalar-valued setting and to Θ_t for which pointwise kernel bounds hold; see, e.g., [**14**, **16**, **17**, **26**, **27**]. In the case of Theorem 1.1, the range of p allowed for the testing functions is constrained by the range of "hypercontractive" estimates enjoyed by Θ_t. This theorem is proved in §3.

THEOREM 1.2. *Let $\{\Theta_t\}_{t>0}$ and the subspace $H \subset L^2(\mathbb{R}^n, \mathbb{C}^m)$ be as in Definition 4.1, and suppose that there exist positive constants $C_0 < \infty$, $\delta > 0$ and a system $\{b_Q\} \subset H^m$, indexed by dyadic cubes $Q \subset \mathbb{R}^n$, such that for each dyadic cube Q:*

$$\int_{\mathbb{R}^n} |b_Q(x)|^2 \, dx \leqslant C_0 |Q|,$$

$$\int_0^{\ell(Q)} \int_Q |\Theta_t b_Q(x)|^2 \frac{dx\, dt}{t} \leqslant C_0 |Q|,$$

$$\delta |\xi|^2 \leqslant Re\left(\xi \cdot \fint_Q b_Q(x)\, dx \bar{\xi}\right) \quad \textit{for all } \xi \in \mathbb{C}^m.$$

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\, dt}{t} \leqslant C \|f\|^2_{L^2(\mathbb{R}^n)} \quad \textit{for all } f \in H. \tag{1.3}$$

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Remark. Here, quasi-orthogonality is assumed to hold only on a subspace H of $L^2(\mathbb{R}^n, \mathbb{C}^m)$, and one then obtains (1.3) for $f \in H$. Algebraically, this requires that the testing functions b_Q belong to H^m, a subspace of $m \times m$ matrix-valued L^2 which entails that the column vectors of the matrix-valued testing functions belong to H. This theorem is proved in §4.

THEOREM 1.4 [2]. *Let L be a uniformly elliptic (i.e., accretive) divergence form operator with complex coefficients. Then, for all $h \in \dot{L}^2_1(\mathbb{R}^n)$, we have*

$$\|\sqrt{L}h\|_{L^2(\mathbb{R}^n)} \leqslant C\|\nabla h\|_{L^2(\mathbb{R}^n)},$$

with C depending only on n, λ and Λ.

Remark. We show that the solution of the Kato square root problem may be deduced as a consequence of Theorem 4.7. As mentioned above, local *Tb* theory was implicit in the solution of the Kato problem. Theorems 4.7 and 5.1 make this connection completely explicit. Of course, the essential techniques of the proof remain that of [2, 18, 21]. This theorem is proved in §5.

1.2. *Notation.*

- We shall use the letters c, C to denote positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems. We shall also write $A \lesssim B$ and $A \approx B$ to mean, respectively, that $A \leqslant CB$ and $0 < c \leqslant A/B \leqslant C$, where the constants c and C are as above, unless explicitly noted. Moreover, if we want to specify any particular dependency of the constant, we will denote it by a subscript or by $C(\cdot)$, e.g., C_n or $C(n)$ is a constant that depends on dimension n.
- We denote points in $\mathbb{R}^{n+1}$ by $(x, t) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ (we use the notational convention that $x_{n+1} = t$) or, sometimes, for convenience, by a capital letter X.
- We set $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ and $\partial\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times \{0\}$.
- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e., $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$.
- The letter Q, and sometimes the letter P, will be used to denote a cube in $\mathbb{R}^n$.
- We let $\mathbb{D}$ denote the collection of all closed dyadic cubes in $\mathbb{R}^n$, and let $\mathbb{D}_k$ denote the grid of dyadic cubes of side length 2^{-k}.
- For a Borel set $A \subset \mathbb{R}^n$, a Borel measure μ defined on $\mathbb{R}^n$ and a Borel measurable function f, we set $\fint_A f\, d\mu = (1/\mu(A)) \int_A f\, d\mu$.
- For a Lebesque measurable set $A \subset \mathbb{R}^n$, we let $|A|$ denote its Lebesgue measure.
- Let $q \in [1, \infty]$; we denote by $q' \in [1, \infty]$ the number such that we have $1/q + 1/q' = 1$, where as usual we define $1/\infty := 0$.
- For a Banach space X, we let $\mathcal{B}(X)$ denote the space of bounded linear operators on X.

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1.3. *Some standard definitions.*

Definition 1.5. We define the Hardy–Littlewood maximal operator $\mathcal{M}$, acting on $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, by

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy,$$

where $B_r(x)$ is the ball centered at x of radius r.

Definition 1.6. We say that P_t is a nice approximate identity if P_t is an operator of convolution type, with a smooth, compactly supported kernel Φ. That means that for a function $f : \mathbb{R}^n \to \mathbb{C}$,

$$P_t f = \Phi_t * f \quad \text{with } \Phi_t = t^{-n}\Phi\left(\frac{x}{t}\right), \qquad \int \Phi(x)\, dx = 1, \Phi_t \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

In particular, then, $P_t 1 = 1$ and

$$(P_t f)(x) \leqslant C\mathcal{M} f(x).$$

Definition 1.7. For $0 < s \leqslant 1$, the homogeneous Sobolev space $\dot{L}_1^2$ is the completion of C_0^∞ with respect to the norm $\|f\|_{\dot{L}_1^2} := \|\nabla f\|_{L^2}$.

In the sequel, we shall use the following pair of results from [1].

LEMMA 1.8 [1, Lemma 3.11]. *Suppose that Θ_t is an operator satisfying*

$$\|\Theta_t(f\mathbb{1}_{2^{k+1}Q\setminus 2^k Q})\|_{L^2(Q)} \leqslant C2^{-k(n+2)/2}\|f\|_{L^2(2^{k+1}Q\setminus 2^k Q)}, \quad k \geqslant 0, \tag{1.9}$$

whenever $t \approx \ell(Q)$, and that $\|\Theta_t\|_{2->2} \leqslant C$. Let $b \in L^\infty(\mathbb{R}^n)$, and let $\mathcal{A}_t$ denote a self-adjoint averaging whose kernel $\varphi_t(x)$ satisfies $|\varphi_t(x)| \leqslant Ct^{-n}\mathbb{1}_{\{|x|<Ct\}}$, $\varphi_t \geqslant 0$, $\int \varphi_t(x)\, dx = 1$. Then

$$\sup_{t>0} \|(\Theta_t b)\mathcal{A}_t f\|_{L^2(\mathbb{R}^n)} \leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

LEMMA 1.10 [1, Lemma 3.5]. *Suppose that $\{R_t\}_{t\in\mathbb{R}}$ is a family of operators satisfying*

$$\|R_t(f\mathbb{1}_{2^{k+1}Q\setminus 2^k Q})\|_{L^2(Q)} \leqslant C2^{-k(n+2+\beta)/2}\|f\|_{L^2(2^{k+1}Q\setminus 2^k Q)}, \quad k \geqslant 0, \tag{1.11}$$

for all $t \approx \ell(Q)$ and for some $\beta > 0$. Suppose also that $\sup_t \|R_t\|_{2\to 2} \leqslant C$ and that $R_t 1 = 0$ for all $t > 0$. Then, for $h \in \dot{L}_1^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |R_t h|^2 \leqslant Ct^2 \int_{\mathbb{R}^n} |\nabla_x h|^2. \tag{1.12}$$

If, in addition, $\|R_t\nabla_x\|_{2\to 2} \leqslant C/t$, then also

$$\iint_{\mathbb{R}^{n+1}_+} |R_t f(x)|^2 \frac{dx\, dt}{t} \leqslant C\|f\|_2^2. \tag{1.13}$$

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Remark 1.14. Lemmas 3.11 and 3.5 of [1] are stated for operators satisfying the off-diagonal decay estimates (1.9) and (1.11), respectively, for all $t \lesssim \ell(Q)$, but the proof in [1] actually requires only that off-diagonal decay bounds hold with $t \approx \ell(Q)$. Also, [1, Lemma 3.5] is stated under the hypothesis that (1.11) holds with $\beta \geqslant 2$, but the proof given there actually requires only that $\beta > 0$, as we assume here.

§2. *A fundamental lemma.* In this section, we present a fundamental lemma whose proof follows very closely the analogous step in the solution of the Kato problem [2, 18]. We include the proof here for the sake of self-containment.

For any cube Q, let

$$R_Q := Q \times (0, \ell(Q))$$

denote the standard Carleson box above Q, and let A_t denote the usual dyadic averaging operator, i.e.,

$$A_t f(x) := \fint_{Q(x,t)} f(y)\, dy, \tag{2.1}$$

where $Q(x, t)$ is the minimal dyadic cube containing x with side length at least t.

We shall work with cones in $\mathbb{C}^m$ (which we identify with $\mathbb{R}^{2m}$), having vertex at the origin. Given a unit vector $\nu \in \mathbb{C}^m$ and $\alpha > 0$, we let $\Gamma^\alpha(\nu)$ denote the cone of aperture α and central axis ν, i.e.,

$$\Gamma^\alpha(\nu) := \left\{ z \in \mathbb{C}^m : \left| \frac{z}{|z|} - \nu \right| < \alpha \right\}. \tag{2.2}$$

Sometimes, when working with a fixed cone, we shall simply write Γ^α, leaving the direction vector ν implicit. As usual, let 1_{Γ^α} denote the indicator function of Γ^α.

LEMMA 2.3. *Given a dyadic cube Q, we suppose that b_Q is an $m \times m$ matrix-valued function, with complex entries, satisfying*

$$\int_{\mathbb{R}^n} |b_Q(x)|^p\, dx \leqslant C_0 |Q| \tag{2.4}$$

and

$$\delta |\xi|^2 \leqslant \mathcal{R}e\left(\xi \cdot \fint_Q b_Q(x)\, dx \bar{\xi} \right) \quad \textit{for all } \xi \in \mathbb{C}^m. \tag{2.5}$$

Then there are positive constants η and ϵ, depending only on δ and C_0, such that, given any cone $\Gamma^\epsilon = \Gamma^\epsilon(\nu)$, we may find a family $\{Q_j\}$ of non-overlapping dyadic subcubes of Q satisfying

$$\sum_j |Q_j| \leqslant (1 - \eta)|Q| \tag{2.6}$$

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and, moreover, for any measurable $\zeta : \mathbb{R}^{n+1}_+ \to \mathbb{C}^m$,

$$|\zeta(x,t)|^2 1_{\Gamma^\epsilon}(\zeta(x,t)) \leqslant 4|\zeta(x,t)A_t b_Q(x)|^2 \quad \textit{for all } (x,t) \in R_Q \setminus \Big(\bigcup_j R_{Q_j}\Big). \tag{2.7}$$

Proof of Lemma 2.3. As noted above, the proof follows that of the analogous step in the solution of the Kato square root problem ([**2**, **18**], but see also [**21**]). We first construct an appropriate family of non-overlapping dyadic subcubes, using a stopping time argument.

Without loss of generality (by renormalizing), we may assume that $\delta \equiv 1$ in (2.5) (of course, this changes C_0, depending on δ). Let Q be as in the hypotheses of the lemma, and fix a cone $\Gamma^\epsilon = \Gamma^\epsilon(\nu)$. We subdivide Q dyadically and select a family $\{Q_j\}$ of non-overlapping dyadic subcubes of Q, which are maximal with respect to the property that at least one of the following conditions holds:

$$\frac{1}{|Q_j|}\int_{Q_j} |b_Q(x)|\,dx \geqslant \frac{1}{8\epsilon} \quad \text{(type I)}, \tag{2.8}$$

$$\mathcal{R}e\Big(\nu \cdot \fint_{Q_j} b_Q(x)\,dx\,\bar{\nu}\Big) \leqslant \frac{3}{4} \quad \text{(type II)}, \tag{2.9}$$

where ν is the direction vector of the cone $\Gamma^\epsilon = \Gamma^\epsilon(\nu)$.

Having constructed the family, let us first verify that it satisfies the required condition (2.6). Define $E := Q\setminus(\bigcup_j Q_j)$. Then, from condition (2.5), since $\delta \equiv 1$, and taking $\xi = \nu$, we have

$$\begin{aligned} |Q| &\leqslant \mathcal{R}e\Big(\nu \cdot \int_Q b_Q(x)\,dx\,\bar{\nu}\Big) \\ &= \mathcal{R}e\Big(\nu \cdot \int_E b_Q(x)\,dx\,\bar{\nu}\Big) + \mathcal{R}e\sum_j\Big(\nu \cdot \int_{Q_j} b_Q(x)\,dx\,\bar{\nu}\Big) := S + T. \end{aligned}$$

Since ν is a unit vector, using condition (2.4) and Hölder's inequality, we get

$$S \leqslant \int_E |b_Q(x)|\,dx \leqslant |E|^{1/p'}\Big(\int_Q |b_Q(x)|^p\,dx\Big)^{1/p} \leqslant C|E|^{1/p'}|Q|^{1/p}.$$

For the second part we separate the family of subcubes $\{Q_j\}$ into two cases: the ones that satisfy the type I condition and the ones that satisfy the type II condition (the same subcube can satisfy both conditions at the same time; in this case we arbitrarily assign them to be of type I). We have

$$\begin{aligned} T &= \mathcal{R}e\sum_{j,\text{ type I}}\Big(\nu \cdot \int_{Q_j} b_Q(x)\,dx\,\bar{\nu}\Big) \\ &\quad + \mathcal{R}e\sum_{j,\text{ type II}}\Big(\nu \cdot \int_{Q_j} b_Q(x)\,dx\,\bar{\nu}\Big) =: I + II. \end{aligned}$$

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Set

$$B_1 := \bigcup_{k,\text{ type I}} Q_k, \qquad B_2 := \bigcup_{k,\text{ type II}} Q_k.$$

For the type I subcubes we apply Hölder's inequality and condition (2.4) to obtain

$$\begin{aligned} |I| &\leqslant \sum_{j,\text{ type I}} \int_{Q_j} |b_Q(x)|\,dx = \int_{B_1} |b_Q(x)|\,dx \\ &\leqslant \left(\int_Q |b_Q(x)|^p\,dx\right)^{1/p} |B_1|^{1/p'} \lesssim |Q|^{1/p}|B_1|^{1/p'}. \end{aligned} \tag{2.10}$$

For the measure of B_1, by the definition of type I and (2.4), we have that

$$\begin{aligned} |B_1| &= \sum_{j,\text{ type I}} |Q_j| \leqslant \sum_{j,\text{ type I}} 8\epsilon \int_{Q_j} |b_Q(x)|\,dx \\ &= 8\epsilon \int_{B_1} |b_Q(x)|\,dx \leqslant 8\epsilon |B_1|^{1/p'} \left(\int_Q |b_Q(x)|^p\,dx\right)^{1/p} \\ &\lesssim 8\epsilon |B_1|^{1/p'}|Q|^{1/p}, \end{aligned}$$

whence it follows that

$$|B_1| \lesssim \epsilon^p |Q|.$$

Combining this bound with (2.10), and choosing ϵ small enough, we have

$$|I| \leqslant C\epsilon^{p/p'}|Q| \leqslant \tfrac{1}{8}|Q|. \tag{2.11}$$

By the definition of the type II cubes, we have

$$|II| := \left|\mathcal{R}e \sum_{j,\text{ type II}} \left(\nu \cdot \int_{Q_j} b_Q(x)\,dx\,\bar{\nu}\right)\right| \leqslant \frac{3}{4} \sum_{j,\text{ type II}} |Q_j| \leqslant \frac{3}{4}|Q|.$$

Combining our estimates, we obtain

$$|Q| \leqslant S + T \leqslant C|E|^{1/p'}|Q|^{1/p} + \tfrac{1}{8}|Q| + \tfrac{3}{4}|Q|,$$

whence it follows that $|Q| \leqslant C|E| =: \eta^{-1}|E|$. Thus,

$$\sum_j |Q_j| = |Q\backslash E| = |Q| - |E| \leqslant (1-\eta)|Q|,$$

so condition (2.6) holds.

Let us now proceed to verify (2.7). We set

$$E_Q^* := R_Q \setminus \left(\bigcup_j R_{Q_j}\right).$$

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We shall prove first that

$$|z \cdot A_t b_Q(x)\bar{\nu}| \geqslant \tfrac{1}{2}|z| \quad \text{if } z \in \Gamma^\epsilon \text{ and } (x,t) \in E_Q^*, \tag{2.12}$$

where we recall that ν is the unit direction vector for Γ^ϵ. Indeed, if $(x,t) \in E_Q^*$, then $Q(x,t)$ is not contained in any selected cube Q_j and therefore the dyadic average $A_t b_Q$ satisfies the opposite inequalities to those in (2.8) and (2.9). Thus, by the triangle inequality, for any unit vector $\omega \in \mathbb{C}^m$, we have

$$|\omega \cdot A_t b_Q(x)\bar{\nu}| \geqslant |\nu \cdot A_t b_Q(x)\bar{\nu}| - |(\omega - \nu)A_t b_Q(x)\bar{\nu}| \geqslant \frac{3}{4} - |(\omega - \nu)|\frac{1}{8\epsilon}.$$

If we choose $\omega = z/|z|$, with $z \in \Gamma^\epsilon$, then $|\omega - \nu| < \epsilon$ (by definition of Γ^ϵ), so that

$$\left|\frac{z}{|z|} \cdot A_t b_Q(x)\bar{\nu}\right| \geqslant \frac{3}{4} - \frac{\epsilon}{8\epsilon} > \frac{1}{2},$$

which yields (2.12). We now apply (2.12) with $z = \zeta(x,t) \in \Gamma^\epsilon$, so that

$$|\zeta(x,t)|^2 \leqslant 4|\zeta(x,t) \cdot A_t b_Q(x)\bar{\nu}|^2.$$

Since $\bar{\nu}$ is a unit vector, we obtain (2.7) and thus also the conclusion of Lemma 2.3. □

§3. *Local Tb theorem without pointwise kernel bounds, version 1: proof of Theorem 1.1.* As mentioned in the introduction, in lieu of pointwise conditions on the kernel of Θ_t, we shall substitute L^2 or L^q conditions. For testing functions in L^p, with $p < 2$, this appears to require that we work with conical (local) square functions rather than vertical ones.

We begin by introducing some auxiliary operators.

Definition 3.1. We say that a family of convolution operators $\{Q_s\}_{s>0}$† is a *CLP family ("Calderón–Littlewood–Paley" family)* if for some $\sigma > 0$ and some $\psi \in L^1(\mathbb{R}^n)$, with $|\psi(x)| \lesssim (1+|x|)^{-n-\sigma}$, we have

$$Q_s f = s^{-n}\psi(\cdot/s) * f \qquad \text{and} \qquad |\widehat{\psi}(\xi)| \leqslant C \min(|\xi|, |\xi|^{-\sigma}),$$
$$\sup_{s>0}(\|Q_s f\|_{L^2(\mathbb{R}^n)} + \|s\nabla Q_s f\|_{L^2(\mathbb{R}^n)}) \leqslant C\|f\|_{L^2(\mathbb{R}^n)},$$
$$\int_{\mathbb{R}^n}\int_0^\infty |Q_s f(x)|^2 \frac{ds\,dx}{s} \leqslant C\|f\|^2_{L^2(\mathbb{R}^n)},$$
$$\int_0^\infty Q_s^2 \frac{ds}{s} = I,$$

where convergence to the identity in the last formula is in the strong operator topology on $\mathcal{B}(L^2)$.

† We apologize for the fact that the letter "Q" will sometimes be used to denote a cube, and sometimes to denote an operator. However, in each case, this notation is "traditional". To avoid the possibility of confusion, let us note that the Littlewood–Paley operators will always appear with a subscript "s" or "t" (e.g., Q_s), whereas these two particular letters will never be used as subscripts for a cube Q.

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We also introduce some additional notation.

- Given a dyadic cube Q, we let $\mathbb{D}(Q)$ denote the collection of all dyadic subcubes of Q (including, of course, Q itself).
- Given a cube Q, we let $U_Q := Q \times (\ell(Q)/2, \ell(Q)]$ be the "Whitney box" above Q.
- Given $x \in \mathbb{R}^n$, we define the "dyadic cone" with vertex at x by

$$\gamma^{\mathrm{dyad}}(x) := \bigcup_{Q \in \mathbb{D}: x \in Q} U_Q \tag{3.2}$$

and, given a dyadic cube Q, for $x \in Q$, we define the "truncated dyadic cone" with vertex at x, of height $\ell(Q)$, by

$$\gamma_Q^{\mathrm{dyad}}(x) := \bigcup_{Q' \in \mathbb{D}(Q): x \in Q'} U_{Q'}. \tag{3.3}$$

HYPOTHESIS 3.4. *We consider a family of operators $\{\Theta_t\}_{t>0}$, acting on functions valued in $\mathbb{C}^m$, $m \geqslant 1$, so that $\Theta_t := (\Theta_t^1, \Theta_t^2, \ldots, \Theta_t^m)$ and, for $f = (f^1, f^2, \ldots, f^m) \in L^2(\mathbb{R}^n, \mathbb{C}^m)$, we set*

$$\Theta_t f := \sum_{j=1}^m \Theta_t^j f^j.$$

We also define the action of Θ_t on an $m \times m$ matrix-valued function $b = (b_{jk})$ in the obvious way, i.e., $\Theta_t b = ((\Theta_t b)_1, (\Theta_t b)_2, \ldots, (\Theta_t b)_m)$ is a $\mathbb{C}^m$ valued function, with

$$(\Theta_t b)_k := \sum_{j=1}^m \Theta_t^j b_{jk}.$$

We suppose that Θ_t satisfies the following properties.

(a) *(Uniform L^2 bounds and off-diagonal decay in L^2.)*

$$\sup_{t>0} \|\Theta_t f\|_{L^2(\mathbb{R}^n)} \leqslant C \|f\|_{L^2(\mathbb{R}^n)}, \tag{3.5}$$

$$\|\Theta_t f_j\|_{L^2(Q)} \leqslant C 2^{-j(n+2+\beta)/2} \|f_j\|_{L^2(2^{j+1}Q \setminus 2^j Q)}, \quad \ell(Q) \leqslant t \leqslant 2\ell(Q) \tag{3.6}$$

for some $\beta > 0$, where $f_j := f \mathbb{1}_{2^{j+1}Q \setminus 2^j Q}$.

(b) *(Quasi-orthogonality in L^2.) For some (hence every) CLP family $\{Q_s\}$, there is a $\beta > 0$ for which we have*

$$\|\Theta_t Q_s h\|_{L^2(\mathbb{R}^n)} \leqslant C \left(\frac{s}{t}\right)^\beta \|h\|_{L^2(\mathbb{R}^n)} \quad \text{for all } s \leqslant t. \tag{3.7}$$

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(c) *("Hypercontractive" off-diagonal decay.) There are some* $1 < r < 2$ *and some* $\mu > n/r$ *(*$\mu = n/r + \varepsilon$ *for some* $\varepsilon > 0$*) such that*

$$\left(\int_{Q^*} |\Theta_t(f\mathbb{1}_{S_j(Q)})(y)|^2\,dy\right)^{1/2} \leqslant C2^{-j\mu}t^{-n(1/r-1/2)}\left(\int_{S_j(Q)} |f(y)|^r\,dy\right)^{1/r} \quad \text{for all } j \geqslant 0,\ \ell(Q) < t \leqslant 2\ell(Q), \tag{3.8}$$

where $S_0(Q) = 16Q$ *and* $S_j(Q) = 2^{j+4}Q\backslash 2^{j+3}Q$, $j \geqslant 1$.

(d) *(Improved integrability.) There is an exponent* $q > 2$ *such that*

$$\sup_{t>0} \|\Theta_t f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^q(\mathbb{R}^n)}. \tag{3.9}$$

Remark 3.10. If (3.6) holds for all $t > 0$ (not just for $t \approx \ell(Q)$), as is often the case in applications, then hypothesis (d) is redundant, and in fact (3.9) holds for all $q > 2$; indeed, one has the pointwise bound

$$\mathcal{M}(\Theta_t f) \lesssim (\mathcal{M}(|f|^2))^{1/2}.$$

We leave the details to the interested reader.

Remark 3.11. We observe that, for example, (b)–(d) hold, with $\theta_t = t\partial_t P_t$, where $P_t = \mathrm{e}^{-t\sqrt{-\Delta}}$ is the usual Poisson semigroup, and that (a) holds with $\beta = 0$, for the same operator. We may obtain a positive value of β in (a) by considering higher order derivatives of P_t. As a practical matter, when considering square functions arising in PDE applications, it is often a fairly routine matter to pass to higher order derivatives. The advantage of the present formulation of our conditions is that these conditions may continue to hold in the absence of pointwise kernel bounds. In PDE applications, (d) is typically obtained as a consequence of higher integrability estimates of "Meyers" type (cf. [**29**]).

We shall need to work with some "dyadic conical" square functions and their local analogues.

Definition 3.12. Given F with domain $\mathbb{R}^{n+1}_+$, we define the *dyadic conical square function* of F by

$$\mathcal{A}^{\mathrm{dyad}}(F)(x) := \left(\iint_{\gamma^{\mathrm{dyad}}(x)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

Similarly, we define the *truncated dyadic conical square function* relative to a dyadic cube Q by

$$\mathcal{A}^{\mathrm{dyad}}_Q(F)(x) := \left(\iint_{\gamma^{\mathrm{dyad}}_Q(x)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

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Remark 3.13. We observe that $\mathcal{A}_Q^{\mathrm{dyad}}(F)$ is dominated by the standard local square function

$$\mathcal{A}_Q^{\alpha}(F)(x) := \left(\int_0^{\ell(Q)} \int_{|x-y|<\alpha t} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2},$$

assuming that the aperture α is chosen large enough, as may be seen from the definition of the dyadic cones and simple geometric considerations.

The main result proved in this section is Theorem 1.1, which we restate here for the reader's convenience.

THEOREM 3.14. *Let $\{\Theta_t\}_{t>0}$ be as in Hypothesis 3.4, and suppose that there exist positive constants $C_0 < \infty$, $\delta > 0$, an exponent $p > r$ and a system $\{b_Q\}$ of complex $m \times m$ matrix-valued functions indexed by dyadic cubes $Q \subset \mathbb{R}^n$, such that for each dyadic cube Q:*

$$\int_{\mathbb{R}^n} |b_Q(x)|^p \, dx \leqslant C_0 |Q|, \tag{3.15}$$

$$\int_Q (\mathcal{A}_Q^{\mathrm{dyad}}(\Theta_t b_Q)(x))^p \, dx \leqslant C_0 |Q|, \tag{3.16}$$

$$\delta |\xi|^2 \leqslant Re\left(\xi \cdot \fint_Q b_Q(x)\, dx \bar{\xi} \right) \quad \text{for all } \xi \in \mathbb{C}^m. \tag{3.17}$$

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\,dt}{t} \leqslant C \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.18}$$

Remark 3.19. We observe that by Remark 3.13, we could replace $\mathcal{A}_Q^{\mathrm{dyad}}$ by $\mathcal{A}_Q^{\alpha}$, with α sufficiently large, in hypothesis (3.16).

In order to prove Theorem 3.14, we shall use the following generalization of the Christ–Journé $T1$ theorem for square functions.

THEOREM 3.20 ($T1$ Theorem). *Suppose that Θ_t satisfies conditions (3.5)–(3.7) as above, and also the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dx\,dt}{t} \leqslant C. \tag{3.21}$$

Then we have the square function estimate

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\,dt}{t} \leqslant C \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.22}$$

Remark 3.23. Here, the constant function 1 should be interpreted in the matrix-valued sense, i.e., as the $m \times m$ identity matrix.

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We defer the proof of Theorem 3.20 to an appendix (Appendix A).

Proof of Theorem 3.14. Following [**2**, **18**], we first make a preliminary reduction. Recalling the definition of the cones $\Gamma^\alpha = \Gamma^\alpha(v) \subset \mathbb{C}^m$ (see (2.2)), we fix $\epsilon > 0$ as in Lemma 2.3, and we cover $\mathbb{C}^m$ by cones of aperture ϵ, enumerating these cones as $\Gamma_1^\epsilon, \dots, \Gamma_K^\epsilon$, where $K = K(\epsilon, m)$. Since, for each cube Q,

$$\int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dx\,dt}{t} \leqslant \sum_{k=1}^K \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 1_{\Gamma_k^\epsilon}(\Theta_t 1(x)) \frac{dx\,dt}{t}, \tag{3.24}$$

and since K depends only on m and ϵ, where we have chosen the latter as in Lemma 2.3, depending only on δ and C_0, then, by Theorem 3.20 it suffices to prove that for any fixed cone $\Gamma^\epsilon = \Gamma^\epsilon(v)$, we have

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dx\,dt}{t} \leqslant C, \tag{3.25}$$

where C depends only on the various quantitative parameters in our hypotheses. To this end, we fix Γ^ϵ and we introduce a dyadic conical version of the square function restricted to Γ^ϵ:

$$\mathcal{A}^{\mathrm{dyad}}_{Q,\,\Gamma^\epsilon}(F)(x) := \left(\iint_{\gamma_Q^{\mathrm{dyad}}(x)} |F(y,t)|^2 1_{\Gamma^\epsilon}(F(y,t)) \frac{dy\,dt}{t^{n+1}} \right)^{1/2}. \tag{3.26}$$

We shall require the following pair of lemmas.

LEMMA 3.27. *Fix a dyadic cube Q, let η, ϵ and $\{Q_j\}$ be as in Lemma 2.3 and set $E := Q \setminus (\bigcup_j Q_j)$. Then*

$$\int_E (\mathcal{A}^{\mathrm{dyad}}_{Q,\,\Gamma^\epsilon}(\Theta_t 1)(x))^p \, dx \leqslant C|Q|, \tag{3.28}$$

where p is the exponent in (3.15)–(3.16) and C depends only on the various quantitative parameters in our hypotheses.

LEMMA 3.29. *Suppose that there exist $N < +\infty$ and $\beta \in (0,1)$ such that for every cube Q,*

$$|\{x \in Q : \mathcal{A}^{\mathrm{dyad}}_{Q,\,\Gamma^\epsilon}(\Theta_t 1)(x) > N\}| \leqslant (1-\beta)|Q|. \tag{3.30}$$

Then (3.25) holds.

Let us momentarily take these two lemmas for granted, and deduce the conclusion of Theorem 1.1. For convenience of notation, we set $G_Q := \mathcal{A}^{\mathrm{dyad}}_{Q,\,\Gamma^\epsilon}(\Theta_t 1)$. By our previous comments, it is enough to verify (3.25); so, by Lemma 3.29, it suffices to verify (3.30). In turn, the latter estimate follows

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from Lemma 2.3, Tchebychev's inequality and (3.28) as follows. Set $E := Q\setminus(\bigcup_j Q_j)$, as in Lemmas 2.3 and 3.27. We then have

$$\begin{aligned}|\{x \in Q : G_Q(x) > N\}| &\leqslant \sum_j |Q_j| + |\{x \in E : G_Q(x) > N\}| \\ &\leqslant (1-\eta)|Q| + \frac{1}{N^p}\int_E (G_Q(x))^p\,dx \\ &\leqslant \left[(1-\eta) + \frac{C}{N^p}\right]|Q| \leqslant (1-\beta)|Q|,\end{aligned}$$

with, say, $\beta = \eta/2$, by choice of N large enough. Here, we have used (2.6) in the second inequality and (3.28) in the third. Thus, we have reduced matters to proving Lemmas 3.27 and 3.29. We treat the latter first.

3.1. *Proof of Lemma 3.29.*

Proof. Fix (momentarily) a large M, so that $2^{-M} \in (0,1)$ and, given a dyadic cube Q, set

$$\mathbb{D}_M(Q) := \{Q' \in \mathbb{D}(Q) : 2^{-M} \leqslant \ell(Q') \leqslant 2^M\}$$

(of course, the upper bound on $\ell(Q')$ is relevant only if $\ell(Q) > 2^M$ and, trivially, $\mathbb{D}_M(Q)$ is empty if $\ell(Q) < 2^{-M}$). Define a truncated dyadic cone

$$\gamma^{\rm dyad}_{Q,M}(x) := \bigcup_{Q'\in\mathbb{D}_M(Q):x\in Q'} U_{Q'}.$$

For each dyadic cube Q, set

$$G_{Q,M}(x) := \left(\iint_{\gamma^{\rm dyad}_{Q,M}(x)} |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y))\frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

Thus, by definition, $G_{Q,M} \leqslant G_Q := \mathcal{A}^{\rm dyad}_{Q,\Gamma^\epsilon}(\Theta_t 1)$ and we note that $G_{Q,M} \to G_Q$, as $M \to \infty$, by monotone convergence. Now let N, β be as in the hypotheses. Setting

$$\Omega_{N,M} = \Omega_{N,M}(Q) := \{x \in Q : G_{Q,M} > N\}, \tag{3.31}$$

we have that

$$|\Omega_{N,M}| \leqslant |\{x \in Q : G_Q > N\}| \leqslant (1-\beta)|Q|, \tag{3.32}$$

by hypothesis. We set

$$K(M) := \sup_Q \frac{1}{|Q|}\int_Q (G_{Q,M}(x))^2\,dx.$$

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We observe that, for each fixed M, $K(M)$ is finite, by the truncation of the cones defining $G_{Q,M}$. Our goal is to show that

$$K_0 := \sup_{M<\infty} K(M) < \infty. \tag{3.33}$$

Indeed, in that case, letting $M \to \infty$, we then obtain by monotone convergence that, for every cube Q,

$$\begin{aligned} K_0|Q| &\geqslant \int_Q (G_Q(x))^2\,dx = \int_Q \iint_{\gamma_Q^{\mathrm{dyad}}(x)} |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dy\,dt}{t^{n+1}}\,dx \\ &= \int_Q \sum_{Q'\in\mathbb{D}(Q)} 1_{Q'}(x) \iint_{U_{Q'}} |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dy\,dt}{t^{n+1}}\,dx \\ &= \sum_{Q'\in\mathbb{D}(Q)} \int_{\ell(Q')/2}^{\ell(Q')} \int_{Q'} |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y)) \left(t^{-n}\int_{Q'} dx\right) \frac{dy\,dt}{t} \\ &\approx \int_0^{\ell(Q)} \int_Q |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dy\,dt}{t}, \end{aligned}$$

whence the desired bound (3.25) follows.

Thus, it suffices to prove (3.33). We fix a cube Q and let $\Omega_{N,M}$ be the set defined in (3.31). We may restrict our attention to the non-trivial case that $2^{-M} \leqslant \ell(Q)$. Our first task is to construct a family of stopping time cubes‡ covering $\Omega_{N,M}$, modulo a set of measure zero. We proceed as follows. Set $F_{N,M} := Q\backslash\Omega_{N,M}$ and observe that $F_{N,M}$ has positive measure, by (3.32); thus, in particular, the interior of Q (which we denote $\mathrm{int}(Q)$) meets $F_{N,M}$. We subdivide Q dyadically, stopping the first time that we obtain a cube whose interior misses $F_{N,M}$. In this way, we obtain a family $\{P_k\}$ of dyadic subcubes of Q, which are maximal with respect to containment of $\mathrm{int}(P_k)$ in $\Omega_{N,M}$. Obviously, $\bigcup_j \mathrm{int}(P_k) \subset \Omega_{N,M}$, and we wish to establish the opposite containment, up to a set of measure zero. More precisely, let Z denote the set of measure zero consisting of the union of the boundaries of all cubes in $\mathbb{D}(Q)$. We claim that

$$(\Omega_{N,M}\backslash Z) \subset \bigcup_k \mathrm{int}(P_k).$$

Suppose not: then there is a point $x_0 \in \Omega_{N,M}$ which does not belong to the interior of any P_k, nor to the boundary of any dyadic cube. Thus, the interior of every $Q' \in \mathbb{D}(Q)$, which contains x_0, must meet $F_{N,M}$ (otherwise, one such cube would have been a selected P_k). In particular, if Q_0 is the unique cube in $\mathbb{D}(Q)$, with side length 2^{-M-1}, which contains x_0, then there is a point x_1 in $F_{N,M}\cap\mathrm{int}(Q_0)$. Notice that for $Q' \in \mathbb{D}_M(Q)$, we have that $x_0 \in Q'$ if and only if $x_1 \in Q'$, since in either case we must have $Q_0 \subset Q'$. Consequently, $\gamma_{Q,M}^{\mathrm{dyad}}(x_0) =$

‡ We cannot just use a standard Whitney covering: since we are working with dyadic cones in the definition of $G_{Q,M}$, it is unclear whether the set $\Omega_{N,M}$ is open.

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$\gamma^{\mathrm{dyad}}_{Q,M}(x_1)$ and, therefore, $G_{Q,M}(x_0) = G_{Q,M}(x_1)$, which contradicts that $x_0 \in \Omega_{N,M}$ and $x_1 \in Q\backslash\Omega_{N,M}$. We have therefore shown that

$$\Omega_{N,M} = \bigcup_k P_k,$$

up to a set of measure zero, so, by (3.32), we have

$$\sum_k |P_k| \leqslant (1-\beta)|Q|. \tag{3.34}$$

With this covering in hand, we proceed to the proof of (3.33). For convenience of notation, we let $\widetilde{\mathbb{D}}_M(Q)$ denote the collection of all $Q' \in \mathbb{D}_M(Q)$ that are not contained in any P_k and, given $x \in Q$, we set

$$\mathbb{D}_M(Q,x) := \{Q' \in \mathbb{D}_M(Q) : x \in Q'\},$$
$$\widetilde{\mathbb{D}}_M(Q,x) := \{Q' \in \widetilde{\mathbb{D}}_M(Q) : x \in Q'\}.$$

Recalling the definition of the dyadic cones $\gamma^{\mathrm{dyad}}_{Q,M}$, we observe that for $x \in P_k$, we have

$$\begin{aligned}\gamma^{\mathrm{dyad}}_{Q,M}(x) &:= \bigcup_{Q'\in\mathbb{D}_M(Q,x)} U_{Q'} = \left(\bigcup_{Q'\in\mathbb{D}_M(P_k,x)} U_{Q'}\right) \cup \left(\bigcup_{Q'\in\widetilde{\mathbb{D}}_M(Q,x)} U_{Q'}\right)\\ &=: \gamma^{\mathrm{dyad}}_{P_k,M}(x) \cup \tilde{\gamma}^{\mathrm{dyad}}_{Q,M}(x).\end{aligned}$$

We then make the corresponding definitions

$$G_{P_k,M}(x) := \left(\iint_{\gamma^{\mathrm{dyad}}_{P_k,M}(x)} |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dy\,dt}{t^{n+1}}\right)^{1/2}$$

and

$$\widetilde{G}_{Q,M}(x) := \left(\iint_{\tilde{\gamma}^{\mathrm{dyad}}_{Q,M}(x)} |\Theta_t 1(y)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(y)) \frac{dy\,dt}{t^{n+1}}\right)^{1/2}.$$

Then, by definition, for $x \in P_k$, we have

$$(G_{Q,M}(x))^2 = (G_{P_k,M}(x))^2 + (\widetilde{G}_{Q,M}(x))^2$$

and, also, by the stopping time construction,

$$\widetilde{G}_{Q,M}(x) = \widetilde{G}_{Q,M}(x_1) \leqslant G_{Q,M}(x_1) \leqslant N \tag{3.35}$$

for some $x_1 \in F_{N,M} = Q\backslash\Omega_{N,M}$. Consequently,

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$$\begin{aligned}\int_Q (G_{Q,M}(x))^2\,dx &= \int_{F_{N,M}} (G_{Q,M}(x))^2\,dx + \sum_k \int_{P_k} (G_{Q,M}(x))^2\,dx\\ &\leqslant N^2|Q| + \sum_k \int_{P_k} (G_{P_k,M}(x))^2\,dx + \sum_k \int_{P_k} (\widetilde{G}_{Q,M}(x))^2\\ &\leqslant N^2|Q| + K(M)\sum_k |P_k| + N^2\sum_k |P_k|\\ &\leqslant 2N^2|Q| + K(M)(1-\beta)|Q|,\end{aligned}$$

where in the last two inequalities we have used the definition of $K(M)$ and (3.35) and then (3.34). Dividing by $|Q|$, and then taking the supremum over all dyadic Q, we obtain

$$K(M) \leqslant 2N^2 + K(M)(1-\beta) \implies K(M) \leqslant 2N^2/\beta. \qquad \square$$

3.2. *Proof of Lemma 3.27.*

Proof. We fix a dyadic cube Q, a cone

$$\Gamma^\epsilon = \left\{z \in \mathbb{C}^m : \left|\frac{z}{|z|} - \nu\right| < \epsilon\right\},$$

with aperture ϵ, and a unit direction vector $\nu \in \mathbb{C}^m$. Let A_t denote the dyadic averaging operator as defined above in (2.1).

Assuming the hypotheses of Theorem 1.1, by Lemma 2.3 above, we may then construct the required family $\{Q_j\}$ of dyadic subcubes of Q, satisfying (2.6), as well as

$$|\Theta_t 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) \leqslant 4|\Theta_t 1(x) A_t b_Q(x)|^2 \quad \text{for all } (x,t) \in E_Q^*, \tag{3.36}$$

where $E_Q^* := R_Q \backslash (\bigcup_j R_{Q_j})$.

We note that for $x \in E = Q\backslash(\bigcup_j Q_j)$, if $x \in Q' \in \mathbb{D}(Q)$, then Q' is not contained in any Q_j. Thus, for $x \in E$, by definition of the dyadic cones, we have $\gamma_Q^{\mathrm{dyad}}(x) \subset E_Q^*$. Consequently, by (3.36),

$$\int_E (\mathcal{A}^{\mathrm{dyad}}_{Q,\Gamma^\epsilon}(\Theta_t 1))^p\,dx \lesssim \int_Q \left(\iint_{\gamma_Q^{\mathrm{dyad}}(x)} |\Theta_t 1(y) A_t b_Q(y)|^2 \frac{dy\,dt}{t^{n+1}}\right)^{p/2} dx.$$

We now use the Coifman–Meyer trick to write

$$\begin{aligned}\Theta_t 1 A_t b_Q &= ((\Theta_t 1)A_t - (\Theta_t 1)A_t P_t)b_Q + ((\Theta_t 1)A_t P_t - \Theta_t)b_Q + \Theta_t b_Q\\ &=: R_t^{(1)} b_Q + R_t^{(2)} b_Q + \Theta_t b_Q,\end{aligned} \tag{3.37}$$

where as usual P_t denotes a nice approximate identity operator (cf. Definition 1.6 above). By (3.16), we obtain (3.28) for the contribution of the term $\Theta_t b_Q$.

To treat the two "remainder terms" $R_t^{(1)} b_Q$ and $R_t^{(2)} b_Q$, we first recall the following known result in the theory of weights (see, e.g., [**9**, Corollary 3.14]).

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PROPOSITION 3.38. *Let T be a sublinear operator satisfying*

$$T : L^2(\mathbb{R}^n, v) \to L^2(\mathbb{R}^n, v) \quad \text{for all } v \in A_{2/r}.$$

Then $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, for $p > r$, with $1 < r < 2$ fixed as in Hypothesis 3.4(c).

By the proposition and (3.15), we are left to prove, for $i = 1, 2$, that

$$\int_{\mathbb{R}^n} |\mathcal{A}^{\mathrm{dyad}}(R_t^{(i)} f)(x)|^2 v(x)\, dx \leqslant C \int_{\mathbb{R}^n} |f(x)|^2 v(x)\, dx, \quad v \in A_{2/r}. \tag{3.39}$$

By definition of the dyadic cones used to construct $\mathcal{A}^{\mathrm{dyad}}$, the left-hand side of (3.39) equals

$$\int_{\mathbb{R}^n} \sum_{Q \in \mathbb{D}} 1_Q(x) \iint_{U_Q} |R_t^{(i)} f(y)|^2 \frac{dy\, dt}{t^{n+1}} v(x)\, dx$$
$$\leqslant \int_0^\infty \int_{\mathbb{R}^n} t^{-n} \int_{|x-y|<Ct} |R_t^{(i)} f(y)|^2\, dy v(x)\, dx \frac{dt}{t}.$$

Thus, by a standard orthogonality argument (see, e.g., [**22**, §3]), it suffices to prove that for some $\beta_1 > 0$ (depending on the weight v), we have

$$\int_{\mathbb{R}^n} t^{-n} \int_{|x-y|<Ct} |R_t^{(i)} Q_s h(y)|^2\, dy v(x)\, dx$$
$$\leqslant C \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\beta_1} \int_{\mathbb{R}^n} |h(x)|^2 v(x)\, dx, \tag{3.40}$$

where $\{Q_s\}_{s>0}$ is a CLP family as in Definition 3.1, which in addition satisfies the weighted L^2 estimate

$$\int_0^\infty \int_{\mathbb{R}^n} |Q_s f(x)|^2 v(x) \frac{dx\, dt}{t} \leqslant C \int_{\mathbb{R}^n} |f(x)|^2 v(x)\, dx, \quad v \in A_{2/r}$$

(we note that, in fact, the latter bound holds more generally for all v in the larger class A_2, provided that Q_s as in Definition 3.1 is given by convolution with a sufficiently nice kernel $\psi_s(x) = s^{-n}\psi(x/s)$, say for ψ in the Schwartz class of smooth, rapidly decaying functions). To prove (3.40), we first show that the estimate holds in the unweighted case $v \equiv 1$, i.e., that for some $\beta_0 > 0$, we have

$$\int_{\mathbb{R}^n} t^{-n} \int_{|x-y|<Ct} |R_t^{(i)} Q_s h(y)|^2\, dy\, dx \approx \int_{\mathbb{R}^n} |R_t^{(i)} Q_s h(y)|^2\, dy$$
$$\lesssim \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\beta_0} \int_{\mathbb{R}^n} |h(x)|^2\, dx. \tag{3.41}$$

In fact, more generally, we shall prove the following result.

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LEMMA 3.42. *Let H be a subspace of $L^2(\mathbb{R}^n, \mathbb{C}^m)$, $m \geqslant 1$, and suppose that Θ_t satisfies (3.5), (3.6) and, for every $h \in H$, (3.7). Let $R_t^{(i)}$, $i = 1, 2$, be defined as in (3.37). Then there is some $\beta_0 > 0$ such that the bound (3.41) holds for all $h \in H$ and for $i = 1, 2$.*

We remark that for our purposes at present, we simply take H to be all of $L^2(\mathbb{R}^n, \mathbb{C}^m)$, but we record the more general version stated above, as well as the following corollary, for future reference.

COROLLARY 3.43. *Under the hypotheses of Lemma 3.42, for $i = 1, 2$, we have the square function bound*

$$\int\!\!\int_{\mathbb{R}^{n+1}_+} |R_t^{(i)} h(x)|^2 \frac{dx\,dt}{t} \leqslant C\|h\|_2^2 \quad \textit{for all } h \in H.$$

The corollary follows from the lemma by a standard orthogonality argument of "Schur's lemma" type; see, e.g., [**22**, §3]. We omit the details.

Proof of Lemma 3.42. We note that by (3.5) and (3.6), we may invoke Lemma 1.8 (which restates [**1**, Lemma 3.11]) to deduce that $\Theta_t 1$ is well defined and satisfies

$$\sup_{t>0}(\|(\Theta_t 1)A_t\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} + \|(\Theta_t 1)A_t P_t\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}) \leqslant C, \qquad (3.44)$$

i.e., (3.5) holds for $(\Theta_t 1)A_t$ and $(\Theta_t 1)A_t P_t$. Moreover, the kernels of A_t and P_t are each supported in $\{|x - y| \leqslant Ct\}$, so that $(\Theta_t 1)A_t$ and $(\Theta_t 1)A_t P_t$ satisfy (3.6) trivially. Thus, each $R_t^{(i)}$ satisfies (3.5) and (3.6). Thus, since in addition we have that $R_t^{(i)} 1 = 0$ for $i = 1, 2$, we may invoke Lemma 1.10 (which restates [**1**, Lemma 3.5]) to obtain that

$$\|R_t^{(i)} Q_s h\|_2 \lesssim t\|\nabla Q_s h\|_2 \quad \text{for all } h \in L^2.$$

The latter bound in turn yields (3.41), in the case $t \leqslant s$, with $\beta_0 = 2$, by properties of Q_s (cf. Definition 3.1).

Consider now the case $s \leqslant t$ of (3.41). Since A_t is a projection operator, we have

$$R_t^{(1)} = (\Theta_t 1)A_t(A_t - P_t),$$

so that the desired bound for this term, for all $h \in L^2$, follows from (3.44) and the well-known fact (see, e.g., [**2**, p. 644]) that there is some $\mu > 0$ such that

$$\|A_t Q_s\|_{L^2\to L^2} + \|P_t Q_s\|_{L^2\to L^2} \lesssim \left(\frac{s}{t}\right)^{\mu}, \quad s \leqslant t.$$

Similarly, the case $s \leqslant t$ of (3.41) follows for $R_t^{(2)}$ from the latter fact and the fact that (3.7) holds for all $h \in H$. □

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With (3.41) in hand, we return to the proof of (3.40), and we make the following claim.

CLAIM 3.45.

$$\int_{\mathbb{R}^n} t^{-n} \int_{|x-y|<Ct} |R_t^{(i)} Q_s h(y)|^2 \, dy \tilde{v}(x) \, dx \leqslant C \int_{\mathbb{R}^n} |h(x)|^2 \tilde{v}(x) \, dx \quad \text{for all } \tilde{v} \in A_{2/r}, \tag{3.46}$$

with $i = 1, 2$.

Interpolating with change of measure [**32**] between (3.41) and (3.46), we get (3.40). Indeed, for each $v \in A_{2/r}$, there exists $\tau > 0$ such that $v^{1+\tau} \in A_{2/r}$, so we choose $\tilde{v}(x) = v^{1+\tau}(x)$. To our knowledge, the idea of using interpolation with change of measure in this way first appeared in the paper [**11**]. Modulo the proof of the claim, the proof of Lemma 3.27 is now complete. □

Proof of Claim 3.45. Define $\tilde{h}(x) = Q_s h(x)$. By properties of Q_s (it is controlled by the maximal operator) and since $A_{2/r} \subset A_2$, we have $\|\tilde{h}\|_{L^2_{\tilde{v}}(\mathbb{R}^n)} \leqslant C\|h\|_{L^2_{\tilde{v}}(\mathbb{R}^n)}$. Thus, it is enough to prove that

$$\int_{\mathbb{R}^n} t^{-n} \int_{|x-y|<Ct} |R_t \tilde{h}(y)|^2 \, dy \tilde{v}(x) \, dx \leqslant C \int_{\mathbb{R}^n} \tilde{h}(x)^2 \tilde{v}(x) \, dx.$$

For $t > 0$, let $\mathbb{D}(t)$ denote the grid of dyadic cubes $P \subset \mathbb{R}^n$ with length $\ell(P) \in (t/2, t]$, and let P^* denote the concentric dilate $P^* = \kappa \cdot P$ for some large κ. Recall that $S_0(P) = 16P$ and $S_j(P) := 2^{j+4}P \backslash 2^{j+3}P$, $j \geqslant 1$. By Hypothesis 3.4(c) and since $\tilde{v} \in A_{2/r}$ (which gives us an $L^{2/r}_{\tilde{v}}$ bound for the maximal function), we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} t^{-n} \int_{|x-y|<Ct} |R_t^{(i)} \tilde{h}(y)|^2 \, dy \tilde{v}(x) \, dx \right)^{1/2} \\
&\leqslant C \left(\sum_{P \in \mathbb{D}(t)} \int_P \frac{1}{|P^*|} \int_{P^*} |R_t^{(i)} \tilde{h}(y)|^2 \, dy \tilde{v}(x) \, dx \right)^{1/2} \\
&\leqslant C \sum_{j=0}^{\infty} \left(\sum_{P \in \mathbb{D}(t)} \int_P \frac{1}{|P^*|} \int_{P^*} |R_t^{(i)} (\tilde{h} \mathbb{1}_{S_j(P)})(y)|^2 \, dy \tilde{v}(x) \, dx \right)^{1/2} \\
&\leqslant C \sum_{j=0}^{\infty} \left(\sum_{P \in \mathbb{D}(t)} \int_P \frac{1}{|P^*|} 2^{-2j\mu} t^{-n(2/r-1)} \right. \\
&\qquad \left. \times \left(\int_{S_j(P)} |\tilde{h}(y)|^r \, dy \right)^{2/r} \tilde{v}(x) \, dx \right)^{1/2},
\end{aligned} \tag{3.47}$$

where, in the last step, we have used that by (3.44), we may apply (3.8) to $R_t^{(i)}$, since A_t is a projection, and since A_t and P_t have compactly supported kernels.

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In turn, since $|P^*| \approx t^n$ and since $\mu = n/r + \varepsilon$, the last expression in (3.47) is comparable to

$$\begin{aligned}
&\sum_{j=0}^{\infty}\Bigg(\sum_{P\in\mathbb{D}(t)}\int_P 2^{-2j\epsilon}2^{-2jn/r}t^{-2n/r}\tilde{v}(x)\Bigg(\int_{S_j(P)}|\tilde{h}(y)|^r\,dy\Bigg)^{2/r}dx\Bigg)^{1/2}\\
&\quad\approx \sum_{j=0}^{\infty}\Bigg(\sum_{P\in\mathbb{D}(t)}\int_P 2^{-2j\epsilon}\tilde{v}(x)\Bigg(\fint_{S_j(P)}|\tilde{h}(y)|^r\,dy\Bigg)^{2/r}dx\Bigg)^{1/2}\\
&\quad\lesssim \sum_{j=0}^{\infty}2^{-j\epsilon}\Bigg(\sum_{P\in\mathbb{D}(t)}\int_P(\mathcal{M}(|\tilde{h}(x)|^r))^{2/r}\tilde{v}(x)\,dx\Bigg)^{1/2}\\
&\quad\lesssim \sum_{j=0}^{\infty}2^{-j\epsilon}\Bigg(\int_{\mathbb{R}^n}|\tilde{h}(x)|^2\tilde{v}(x)\,dx\Bigg)^{1/2}\approx\Bigg(\int_{\mathbb{R}^n}|\tilde{h}(x)|^2\tilde{v}(x)\,dx\Bigg)^{1/2},
\end{aligned}$$

where, in the next-to-last step, we have used that the maximal operator $\mathcal{M}$ is bounded on $L^{2/r}_{\tilde{v}}$, since $\tilde{v}\in A_{2/r}$. Thus, Claim 3.45 is now established. □

The proof of Theorem 3.14 is now complete. □

§4. *Local Tb theorem without pointwise kernel bounds, version 2: proof of Theorem 1.2.* In this section, we present a version of the local *Tb* theorem for square functions, which may be applied directly to solve the Kato square root problem. We continue to use the notation introduced in the previous section, but we shall modify our hypotheses on the operators Θ_t.

HYPOTHESIS 4.1. *We consider a family of operators $\{\Theta_t\}_{t>0}$, acting on functions valued in $\mathbb{C}^m$, $m\geqslant 1$, so that $\Theta_t := (\Theta_t^1, \Theta_t^2, \dots, \Theta_t^m)$ and, for $f = (f^1, f^2, \dots, f^m)\in L^2(\mathbb{R}^n, \mathbb{C}^m)$, we set*

$$\Theta_t f := \sum_{j=1}^m \Theta_t^j f^j.$$

We also define the action of Θ_t on an $m\times m$ matrix-valued function $b = (b_{jk})$ in the obvious way, i.e., $\Theta_t b = ((\Theta_t b)_1, (\Theta_t b)_2, \dots, (\Theta_t b)_m)$ is a $\mathbb{C}^m$-valued function, with

$$(\Theta_t b)_k := \sum_{j=1}^m \Theta_t^j b_{jk}.$$

We suppose that Θ_t satisfies the following properties.

(a) *(Uniform L^2 bounds and off-diagonal decay in L^2.)*

$$\sup_{t>0}\|\Theta_t f\|_{L^2(\mathbb{R}^n)}\leqslant C\|f\|_{L^2(\mathbb{R}^n)}, \tag{4.2}$$

$$\|\Theta_t f_j\|_{L^2(Q)}\leqslant C2^{-j(n+2+\beta)/2}\|f_j\|_{L^2(2^{j+1}Q\setminus 2^jQ)},\quad \ell(Q)\leqslant t\leqslant 2\ell(Q) \tag{4.3}$$

for some $\beta>0$, where $f_j := f\mathbb{1}_{2^{j+1}Q\setminus 2^jQ}$.

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(b) *(Quasi-orthogonality on a subspace of L^2.) There exists a subspace H of $L^2(\mathbb{R}^n, \mathbb{C}^m)$ such that for some (hence every) CLP family $\{Q_s\}$ (cf. Definition 3.1), there is a $\beta > 0$ for which we have*

$$\|\Theta_t Q_s h\|_{L^2(\mathbb{R}^n)} \leqslant C\left(\frac{s}{t}\right)^{\beta} \|h\|_{L^2(\mathbb{R}^n)} \quad \text{for all } h \in H, \text{ for all } s \leqslant t. \tag{4.4}$$

Remark 4.5. Let us note that it is only in the quasi-orthogonality condition (b) that we restrict to a subspace $H \subset L^2(\mathbb{R}^n, \mathbb{C}^m)$; in condition (a), we suppose that the operator is allowed to act on general C^m-valued functions in L^2.

Definition 4.6. Given a subspace $H \subset L^2(\mathbb{R}^n, \mathbb{C}^m)$, we let H^m denote the subspace of $L^2(\mathbb{R}^n, \mathbb{M}^m)$ (i.e., complex $m \times m$ matrix-valued L^2), consisting of L^2 matrix-valued functions $b = (b_{jk})$ for which each column vector $b^k := (b_{1k}, b_{2k}, \ldots, b_{mk})$ belongs to H.

The main result proved in this section is Theorem 1.2, which we restate here for the reader's convenience.

THEOREM 4.7. *Let $\{\Theta_t\}_{t>0}$ and the subspace $H \subset L^2(\mathbb{R}^n, \mathbb{C}^m)$ be as in Hypothesis 4.1, and suppose that there exist positive constants $C_0 < \infty$, $\delta > 0$ and a system $\{b_Q\} \subset H^m$, indexed by dyadic cubes $Q \subset \mathbb{R}^n$, such that for each dyadic cube Q:*

$$\int_{\mathbb{R}^n} |b_Q(x)|^2 \, dx \leqslant C_0 |Q|, \tag{4.8}$$

$$\int_0^{\ell(Q)} \int_Q |\Theta_t b_Q(x)|^2 \frac{dx\, dt}{t} \leqslant C_0 |Q|, \tag{4.9}$$

$$\delta |\xi|^2 \leqslant Re\left(\xi \cdot \fint_Q b_Q(x)\, dx \bar{\xi}\right) \quad \text{for all } \xi \in \mathbb{C}^m. \tag{4.10}$$

Then

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\, dt}{t} \leqslant C \|f\|^2_{L^2(\mathbb{R}^n)} \quad \text{for all } f \in H. \tag{4.11}$$

The proof of Theorem 4.7 will be based upon the following generalization of the Christ–Journé $T1$ theorem for square functions.

THEOREM 4.12 ($T1$ theorem). *Suppose that Θ_t satisfies conditions (4.2)–(4.4) as above, and also that we have the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{\ell(Q)} |\Theta_t 1(x)|^2 \frac{dx\, dt}{t} \leqslant C. \tag{4.13}$$

Then we have the square function estimate

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\, dt}{t} \leqslant C \|f\|^2_{L^2(\mathbb{R}^n)} \quad \text{for all } f \in H. \tag{4.14}$$

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As above, we view "1" in this context as the $m \times m$ identity matrix. We defer the proof of Theorem 4.12 to Appendix A.

Proof of Theorem 4.7. The proof is a variant of our previous arguments and, in turn, is based upon the proof of the Kato square root conjecture [**2**, **18**, **21**]. The fact that the testing functions belong to L^2 allows for some simplifications, as compared to the proof of Theorem 3.14: in particular, we shall not require Lemmas 3.27 and 3.29. On the other hand, the restriction to the subspace H does not cause significant additional difficulty.

By Theorem 4.12, it suffices to establish (4.13). As in the previous section, we choose an $\epsilon > 0$ small enough that Lemma 2.3 holds, and we cover $\mathbb{C}^m$ by a family of cones $\{\Gamma_k^\epsilon\}_{1 \leqslant k \leqslant K(\epsilon, M)}$ of aperture ϵ. It suffices to establish the uniform estimate

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Theta_t 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dx\, dt}{t} \leqslant C \tag{4.15}$$

for any fixed cone Γ^ϵ, since in that case we may apply the latter estimate to each of the cones Γ_k^ϵ, and then sum in k to obtain (4.13).

Let us then fix a cube Q and a cone Γ^ϵ. We invoke Lemma 2.3 to obtain a constant $\eta > 0$ and a family of non-overlapping cubes $\{Q_j\} \subset \mathbb{D}(Q)$ satisfying

$$\sum_j |Q_j| \leqslant (1 - \eta)|Q| \tag{4.16}$$

and

$$\begin{aligned} |\Theta_t 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) &\leqslant 4|\Theta_t 1(x) A_t b_Q(x)|^2 \\ \text{for all } (x,t) \in E_Q^* &:= R_Q \setminus \Big(\bigcup_j R_{Q_j} \Big), \end{aligned} \tag{4.17}$$

where we recall that for any cube Q, $R_Q := Q \times (0, \ell(Q))$ is the corresponding Carleson box. Now, by a well-known John–Nirenberg-type lemma for Carleson measures (see, e.g., [**17**, Lemma 1.37]§), the left-hand side of (4.15) is bounded by a constant times

$$\sup_Q \frac{1}{|Q|} \iint_{E_Q^*} |\Theta_t 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dx\, dt}{t}.$$

In turn, by (4.17), we have

$$\iint_{E_Q^*} |\Theta_t 1(x)|^2 1_{\Gamma^\epsilon}(\Theta_t 1(x)) \frac{dx\, dt}{t} \leqslant 4 \iint_{R_Q} |\Theta_t 1(x) A_t b_Q(x)|^2 \frac{dx\, dt}{t},$$

§ To avoid possible confusion, we remark that Lemma 3.29 is also of John–Nirenberg type, and some variant of it could be used here as well; however, [**17**, Lemma 1.37] is simpler to use in the present setting, with $p = 2$.

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so it suffices to show that the last expression is no larger than a uniform constant times $|Q|$. To this end, as usual we write

$$\begin{aligned}\Theta_t 1 A_t b_Q &= ((\Theta_t 1)A_t - (\Theta_t 1)A_t P_t)b_Q + ((\Theta_t 1)A_t P_t - \Theta_t)b_Q + \Theta_t b_Q \\ &=: R_t^{(1)} b_Q + R_t^{(2)} b_Q + \Theta_t b_Q,\end{aligned}$$

where as above P_t denotes a nice approximate identity operator (cf. Definition 1.6). By (4.9), the contribution of $\Theta_t b_Q$ immediately satisfies the desired bound. We may handle the contributions of the two "remainder" terms $R_t^{(1)} b_Q$ and $R_t^{(2)} b_Q$ by invoking (4.8) and Corollary 3.43, which in the present setting is applied only to $h \in H$ (recall that, in particular, each column vector of b_Q belongs to H, by hypothesis). □

§5. *Application to the Kato square root problem.* Let A be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda|\xi|^2 \leqslant \mathcal{R}e\langle A(x)\xi, \xi\rangle := \sum_{i,j} A_{ij}(x)\xi_j\overline{\xi_i}, \quad \|A\|_\infty \leqslant \Lambda$$

for all $x \in \mathbb{R}^n$, all $\xi \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leqslant \Lambda < \infty$. We define the divergence form operator

$$Lu := -\mathrm{div}(A(x)\nabla u),$$

which we interpret in the usual weak sense via a sesquilinear form.

The accretivity condition above enables one to define an accretive square root $\sqrt{L}$. The main result proved in this section is Theorem 1.4, which we restate here for the reader's convenience. Of course, the original proof appears in [**2**].

THEOREM 5.1 [**2**]. *Let L be a divergence form operator defined as above. Then, for all $h \in \dot{L}_1^2(\mathbb{R}^n)$, we have*

$$\|\sqrt{L}h\|_{L^2(\mathbb{R}^n)} \leqslant C\|\nabla h\|_{L^2(\mathbb{R}^n)},$$

with C depending only on n, λ and Λ.

Proof. In [**5**], it is shown that the conclusion of Theorem 5.1 is equivalent to the square function estimate

$$\int\!\!\int_{\mathbb{R}^{n+1}_+} |\Theta_t \nabla h|^2 \frac{dx\,dt}{t} \leqslant C_{n,\lambda,\Lambda} \int_{\mathbb{R}^n} |\nabla h|^2\,dx,$$

where $\Theta_t := te^{-t^2 L}\,\mathrm{div}\,A$. Thus, to prove this theorem, it is enough to verify the conditions of Theorem 4.7 for this operator Θ_t, with $m = n$, and with

$$H := \{\nabla h :\ h \in \dot{L}_1^2(\mathbb{R}^n, \mathbb{C}^n)\},$$

a subspace of $L^2(\mathbb{R}^n, \mathbb{C}^n)$.

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(a) It follows from standard semigroup theory for divergence form elliptic operators that the family $\{te^{-t^2L}\,\mathrm{div}\}_{t>0}$ satisfies uniform L^2 bounds, as well as L^2 off-diagonal estimates (for the latter, just dualize the well-known "Gaffney estimates" for $t\nabla e^{-t^2L}$). Thus, since $A \in L^\infty(\mathbb{R}^n)$, we have condition (a) for Θ_t.

(b) We choose $\{Q_s\}_{s>0}$ of convolution type satisfying the required conditions of Definition 3.1, and in addition we choose the kernel $\psi \in C_0^\infty$. We note that $Q_s\nabla F = \nabla Q_s F$. Let $\nabla F \in H$, where (by definition) $F \in \dot{L}_1^2(\mathbb{R}^n)$. Then F is equivalent (modulo constants) to the realization $F = I_1 f$, where $f \in L^2(\mathbb{R}^n)$, with $\|f\|_2 \approx \|\nabla F\|_2$, and $I_1 = (-\Delta)^{-1/2}$ is the standard Riesz potential of order 1. We then have

$$\begin{aligned} te^{-t^2L}\,\mathrm{div}\,AQ_s\nabla F &= te^{-t^2L}\,\mathrm{div}\,A\nabla Q_s F \\ &= -tLe^{-t^2L}Q_s F \\ &= -\frac{1}{t}t^2Le^{-t^2L}Q_s I_1 f \\ &= \frac{-s}{t}t^2Le^{-t^2L}\left(\frac{1}{s}Q_s I_1\right)f. \end{aligned}$$

Using the fact that $t^2Le^{-t^2L} : L^2 \to L^2$ and $(1/s)Q_s I_1 : L^2 \to L^2$, uniformly in t and s, respectively, we obtain condition (b).

Finally, we need to find a family of b_Q indexed by cubes Q satisfying the required conditions. In [**2**, **18**, **21**], it is proved that such a family exists, with b_Q of the form

$$b_Q = \nabla F_Q := e^{-\varepsilon\ell(Q)^2L}(\varphi_Q),$$

with $\varphi_Q(x) := (x - x_Q)\eta_Q$, where $\varepsilon > 0$ is a number chosen suitably small depending only on n, λ and Λ, and where x_Q denotes the center of Q and η_Q is a smooth cut-off function which is 1 on $4Q$ and vanishes outside of $5Q$. We refer the reader to [**2**] for details of the proofs of the required properties. □

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A. Appendix. Proof of the generalized Christ–Journé T1 theorem for square functions. In this appendix, we give the proofs of Theorems 3.20 and 4.12. In fact, it will suffice to prove the latter, since taking $H = L^2$ then yields the former. The proof will follow that of the $T1$ theorem of [**8**]; indeed, the lack of pointwise kernel bounds, and the fact that our square function acts only on elements of H, do not present serious obstacles.

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Proof of Theorem 4.12. We consider

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t h(x)|^2 \frac{dx\,dt}{t},$$

with $h \in H$, where H is a subspace of $L^2(\mathbb{R}^n, \mathbb{C}^m)$. Let us set

$$d\mu(x,t) := |\Theta_t 1(x)|^2 \frac{dx\,dt}{t},$$

so that, by hypothesis, we have the Carleson measure estimate

$$\|\mu\|_{\mathcal{C}} := \sup_Q \frac{1}{|Q|} \iint_{R_Q} d\mu(x,t) < \infty.$$

We follow the familiar idea of Coifman–Meyer to write

$$\Theta_t h = (\Theta_t - (\Theta_t 1) A_t P_t) h + (\Theta_t 1) A_t P_t h =: R_t h + (\Theta_t 1) P_t h,$$

where, as usual, A_t is the dyadic averaging operator and P_t is a nice approximate identity. By Carleson's embedding lemma and the well-known non-tangential estimate for $A_t P_t h$,

$$\iint_{\mathbb{R}^{n+1}_+} |(\Theta_t 1) A_t P_t h(x)|^2 \frac{dx\,dt}{t} \lesssim \|\mu\|_{\mathcal{C}} \|h\|_2^2,$$

as desired (we remark that for this term, the estimate holds for all $h \in L^2(\mathbb{R}^n, \mathbb{C}^m)$). Moreover,

$$\iint_{\mathbb{R}^{n+1}_+} |R_t h(x)|^2 \frac{dx\,dt}{t} \lesssim \|h\|_2^2 \quad \text{for all } h \in H,$$

by Corollary 3.43; indeed, the present R_t is precisely the same (up to a minus sign) as $R_t^{(2)}$ considered in (3.37). □

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Ana Grau De La Herrán,
Department of Mathematics and Statistics,
University of Helsinki,
Finland
E-mail: ana.grau@helsinki.fi

Steve Hofmann,
Department of Mathematics,
University of Missouri,
Columbia, MI 65211,
U.S.A.
E-mail: hofmanns@missouri.edu

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